Point–Based Registration Assuming Affine Motion

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Abstract. Registration is a fundamental task in image processing. Its purpose is to find a geometrical transformation that relates the points of an image to their corresponding points of another image. The determination of the optimal transformation depends on the types of variations between the images. In this paper we propose a robust method based on two sets of points representing the images. One–to–one correspondence is assumed between these two sets. Our approach finds global affine transformation between the sets of points and can be used in any arbitrary dimension $k \geq 1$. A sufficient existence condition for a unique solution is given and proven. Our method can be used to solve various registration problems emerged in numerous fields, including medical image processing, remotely sensed data processing, and computer vision.
Keywords: registration problem; matching sets of points

1 Introduction

There is an increasing number of applications that require accurate aligning of one image with another taken from different viewpoints, by different imaging devices, or at different times. The geometrical transformation is to be found that maps a *floating image data set* in precise spatial correspondence with a *reference image data set*. This process of alignment is known as *registration*, although other words, such as *co–registration*, *matching*, and *fusion*, are also used. Examples of systems where image registration is a significant component include aligning images from different medical modalities for diagnosis, matching a target with a real–time image of a scene for target recognition, monitoring global land usage using satellite images, and matching stereo images to recover shape for autonomous navigation [6, 10].

The registration technique for a given task depends on the knowledge about the characteristics of the type of variations. Registration methods can be viewed as different combinations of choices for the following four components [6]:

- *Search space* is determined by the type of transformation we have to consider, i.e., what is the class of transformations that is capable of aligning

the images. Some widely used types are *rigid-body*, when translations and rotations are allowed only, *affine*, which maps parallel lines to parallel lines, and *nonlinear*, which can transform straight lines to curves.
– *Feature data set* describes what kind of image properties are used in matching.
– *Similarity measure* is a function of the transformation parameters which shows how well the floating and the reference image fit. The task of registration is to optimize this function.
– *Search strategy* determines what kind of optimization method to use.

Fig. 1 explains the major steps of a general registration process.

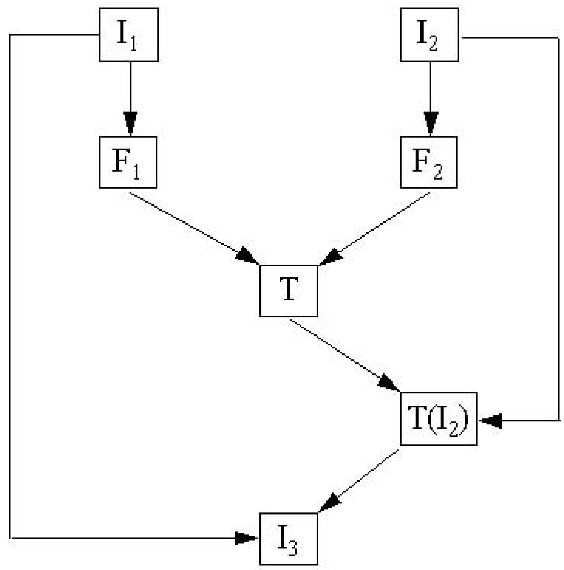

Fig. 1. Major steps of a general registration process. Feature data sets F_1 and F_2 are extracted from reference image I_1 and floating image I_2, respectively. Transformation T is calculated using F_1 and F_2. I_2 is aligned to I_1 by applying T. A brand new image I_3 can be calculated by fusing I_1 and $T(I_2)$

A general and robust solution for registration problems is selecting points as features. A general point–based method consists of three steps. First, the points are identified, then points in the floating image are corresponded with points in the reference image, finally a spatial mapping is determined. Point–based methods can be either *interactive* or *automatic*. Using an interactive point–based method, usually few pairs of points (4–20) are identified and corresponded by the user. Methods of this type are available for rigid–body [1] and nonlinear [4, 9] problems. Automatic determination of the features usually results huge amount of points. In this case finding correspondences can be rather difficult (eg., the number of elements of the point sets is not necessarily the same) and require a special algorithm. Widely used methods are *head-hat* method [13], *hierarchical Chamfer matching* [2, 5], and *iterative closest point* [3] method. These are used mainly for rigid–body problems, but extension to more general transformations is easy.

In this paper we propose an interactive point-based method.

2 Affine Method for Aligning Two Sets of Points

In this Section we propose a robust method based on identified pairs of points, which assumes affine motion between the images. Let $k \geq 1$ denote the dimension of the images and let n be the number of pairs of points.

Our registration method is described by giving the following four components:

- *search space*
  Global transformation described by a $(k+1) \times (k+1)$ matrix $\mathcal{T}$ of the form

$$\mathcal{T} = \begin{pmatrix} t_{11} & t_{12} & \cdots & t_{1k} & t_{1,k+1} \\ t_{21} & t_{22} & \cdots & t_{2k} & t_{2,k+1} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ t_{k1} & t_{k2} & \cdots & t_{kk} & t_{k,k+1} \\ 0 & 0 & \cdots & 0 & 1 \end{pmatrix}$$

  is to be found. Given $\mathcal{T}$ and a point $x = (x_1, \ldots, x_k) \in \mathbb{R}^k$, the transformation sends x to $y = (y_1, \ldots, y_k) \in \mathbb{R}^k$ if and only if $(y_1, \ldots, y_k, 1)^T = \mathcal{T} \cdot (x_1, \ldots, x_k, 1)^T$ holds for the corresponding *homogeneous coordinates* [8]. Notice that each affine transformation can be described this way (Fig. 2). This kind of transformation has $k \cdot (k+1)$ degrees of freedom according to the matrix elements to be determined.
- *feature data set*
  A set of n *reference points* $\{p_1, p_2, \ldots, p_n\}$, $p_i = (p_{i1}, \ldots, p_{ik}) \in \mathbb{R}^k$, and a set of n *floating points* $\{q_1, q_2, \ldots, q_n\}$, $q_i = (q_{i1}, \ldots, q_{ik}) \in \mathbb{R}^k$, are to be identified in the reference image and the floating image, respectively (Fig. 3). We assume that q_i is corresponded to p_i $(1 \leq i \leq n)$.
- *similarity measure*
  Suppose that we get point $\overline{q}_i = (\overline{q}_{i1}, \ldots, \overline{q}_{ik})$ when point q_i is transformed by matrix $\mathcal{T}$ $(1 \leq i \leq n)$:

$$\begin{pmatrix} \overline{q}_{i1} \\ \overline{q}_{i2} \\ \vdots \\ \overline{q}_{ik} \\ 1 \end{pmatrix} = \begin{pmatrix} t_{11} & t_{12} & \cdots & t_{1k} & t_{1,k+1} \\ t_{21} & t_{22} & \cdots & t_{2k} & t_{2,k+1} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ t_{k1} & t_{k2} & \cdots & t_{kk} & t_{k,k+1} \\ 0 & 0 & \cdots & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} q_{i1} \\ q_{i2} \\ \vdots \\ q_{ik} \\ 1 \end{pmatrix}.$$

Fig. 2. Example of a 2D affine transformation: The original image (left) and the transformed one (right). Lines are mapped to lines, parallelism is preserved, but angles can be altered

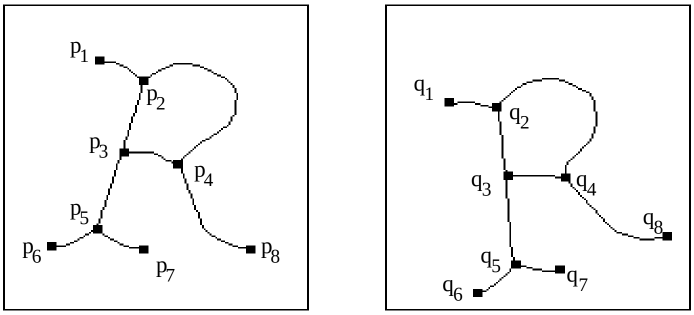

Fig. 3. Example of identified pairs of points in 2D. Eight pairs (p_i, q_i) of points are identified in the reference image (left) and in the floating image (right), respectively

Define the function $\mathcal{S}$ of $k \cdot (k+1)$ variables as follows:

$$\mathcal{S}(t_{11}, \ldots, t_{k,k+1}) = \sum_{i=1}^{n} \|\overline{q}_i - p_i\|^2 = \sum_{i=1}^{n} \sum_{j=1}^{k} (\overline{q}_{ij} - p_{ij})^2$$
$$= \sum_{i=1}^{n} \sum_{j=1}^{k} (t_{j1} \cdot q_{i1} + \ldots + t_{jk} \cdot q_{ik} + t_{j,k+1} - p_{ij})^2 .$$

It can be regarded as the matching error.

– *search strategy*
The least square solution of matrix $\mathcal{T}$ is determined by minimizing function $\mathcal{S}$. Direct matching is applied. Function $\mathcal{S}$ may be minimal if all of the partial derivatives $\frac{\partial \mathcal{S}}{\partial t_{11}}, \ldots, \frac{\partial \mathcal{S}}{\partial t_{k,k+1}}$ are equal to zero. The required $k \cdot (k+1)$

equations:

$$\frac{\partial \mathcal{S}}{\partial t_{uv}} = 2 \cdot \sum_{i=1}^{n} q_{iv} \cdot (t_{u,k+1} - p_{iu} + \sum_{l=1}^{k} t_{ul} \cdot q_{il}) = 0$$
$$(1 \leq u, v \leq k),$$

$$\frac{\partial \mathcal{S}}{\partial t_{u,k+1}} = 2 \cdot \sum_{i=1}^{n} (t_{u,k+1} - p_{iu} + \sum_{l=1}^{k} t_{ul} \cdot q_{il}) = 0$$
$$(1 \leq u \leq k).$$

We get the following system of linear equations:

$$\begin{pmatrix} \begin{matrix} a_{11} & \dots & a_{1k} & b_1 \\ \vdots & \ddots & \vdots & \vdots \\ a_{k1} & \dots & a_{kk} & b_k \\ b_1 & \dots & b_k & n \end{matrix} & & & \text{\Huge 0} \\ & \begin{matrix} a_{11} & \dots & a_{1k} & b_1 \\ \vdots & \ddots & \vdots & \vdots \\ a_{k1} & \dots & a_{kk} & b_k \\ b_1 & \dots & b_k & n \end{matrix} & & \\ & & \ddots & \\ \text{\Huge 0} & & & \begin{matrix} a_{11} & \dots & a_{1k} & b_1 \\ \vdots & \ddots & \vdots & \vdots \\ a_{k1} & \dots & a_{kk} & b_k \\ b_1 & \dots & b_k & n \end{matrix} \end{pmatrix} \cdot \begin{pmatrix} t_{11} \\ \vdots \\ t_{1k} \\ t_{1,k+1} \\ t_{21} \\ \vdots \\ t_{2k} \\ t_{2,k+1} \\ \vdots \\ t_{k1} \\ \vdots \\ t_{kk} \\ t_{k,k+1} \end{pmatrix} = \begin{pmatrix} c_{11} \\ \vdots \\ c_{1k} \\ d_1 \\ c_{21} \\ \vdots \\ c_{2k} \\ d_2 \\ \vdots \\ c_{k1} \\ \vdots \\ c_{kk} \\ d_k \end{pmatrix},$$

where

$$a_{uv} = a_{vu} = \sum_{i=1}^{n} q_{iu} \cdot q_{iv} \ ,$$
$$b_u = \sum_{i=1}^{n} q_{iu} \ ,$$
$$c_{uv} = \sum_{i=1}^{n} p_{iu} \cdot q_{iv} \ ,$$
$$d_u = \sum_{i=1}^{n} p_{iu}$$
$$(1 \leq u, v \leq k).$$

The above system of linear equations can be solved by using an appropriate numerical method. There exists a unique solution if and only if $det(M) \neq 0$, where

$$M = \begin{pmatrix} a_{11} & \dots & a_{1k} & b_1 \\ \vdots & \ddots & \vdots & \vdots \\ a_{k1} & \dots & a_{kk} & b_k \\ b_1 & \dots & b_k & n \end{pmatrix}.$$

3 Discussion

In this section we state and prove a sufficient existence condition for a unique solution.

By a hyperplane of the Euclidean space $\mathbb{R}^k$ we mean a subset of the form $\{a + x : x \in S\}$ where S is a $(k-1)$-dimensional linear subspace. Given some points $q_1, \dots, q_n$ in $\mathbb{R}^k$, we say that these points *span* $\mathbb{R}^k$ if no hyperplane of $\mathbb{R}^k$ contains them. If any $k+1$ points from $q_1, \dots, q_n$ span $\mathbb{R}^k$ then we say that $q_1, \dots, q_n$ are in general position.

Theorem. If $q_1, \dots, q_n$ span $\mathbb{R}^k$ then $\det(M) \neq 0$.

Proof. Suppose $\det(M) = 0$. Consider the vectors $v_j = (q_{1j}, q_{2j}, \dots, q_{nj})$ $(1 \leq j \leq k)$ in $\mathbb{R}^n$, and let $v_{k+1} = (1, 1, \dots, 1) \in \mathbb{R}^n$. With the notation $m = k+1$ observe that $M = \Big(\langle v_i, v_j \rangle\Big)_{m \times m}$ where $\langle\ ,\ \rangle$ stands for the scalar multiplication. Since the columns of M are linearly dependent, we can fix a $(\beta_1, \dots, \beta_m) \in \mathbb{R}^m \setminus \{(0, \dots, 0)\}$ such that $\sum_{j=1}^m \beta_j \langle v_i, v_j \rangle = 0$ hold for $i = 1, \dots, m$. Then

$$0 = \sum_{i=1}^m \beta_i \cdot 0 = \sum_{i=1}^m \beta_i \sum_{j=1}^m \beta_j \langle v_i, v_j \rangle = \sum_{i=1}^m \beta_i \Big\langle v_i, \sum_{j=1}^m \beta_j v_j \Big\rangle =$$
$$\Big\langle \sum_{i=1}^m \beta_i v_i, \sum_{j=1}^m \beta_j v_j \Big\rangle = \Big\langle \sum_{i=1}^m \beta_i v_i, \sum_{i=1}^m \beta_i v_i \Big\rangle,$$

whence $\sum_{i=1}^m \beta_i v_i = 0$. Therefore all the q_j, $1 \leq j \leq n$, are solutions of the following (one element) system of linear equations:

$$\beta_1 x_1 + \dots + \beta_k x_k = -\beta_m. \tag{1}$$

Since the system has solutions and $(\beta_1, \dots, \beta_m) \neq (0, \dots, 0)$, there is an $i \in \{1, \dots, k\}$ with $\beta_i \neq 0$. Hence the solutions of (1) form a hyperplane of $\mathbb{R}^k$. This hyperplane contains $q_1, \dots, q_n$. Now it follows that if $q_1, \dots, q_n$ span $\mathbb{R}^k$ then $\det(M) \neq 0$. Q.e.d.

4 Estimating Registration Error

Point–based registration might find imperfect matching due to the presence of error in localizing the points (note that points are often called *fiducials*). There are some papers dealing with the analysis of point–based registration. It is worth emphasizing that each of these papers considers only rigid–body transformations. Maurer et al. [11] proposed three types of measures of error:

- Fiducial localization error (FLE), which is the error in determining the positions of the fiducials.
- Fiducial registration error (FRE), which is the root mean square distance between corresponding points after registration. Note that point-based registration methods minimize this error measure.
- Target registration error (TRE), which is the distance between corresponding points representing ROIs (range–of–interest) after registration.

Using FRE as measure of registration accuracy is unreliable and may be misleading, thus investigations were focussed on TRE in the last decade [7, 12].

There are two important results concerning registration errors [12]:

- **Result 1.** For a fixed number of fiducials, TRE is proportional to FLE .
- **Result 2.** TRE is approximately proportional to $1/\sqrt{n}$ with n being the number of fiducials .

Fitzpatrick et al. [7] gave an exact expression for approximating TRE assuming rigid–body transformations, thus proving both **Result 1** and **Result 2**.

In this paper we examine the dependence of TRE for our affine method via using numerical simulations.

4.1 Model for Numerical Simulations

Let $\mathcal{M} = \{\,(x, y, z) \mid x, y, z \in \mathbb{R}, 0 \leq x, y, z < 256\}$ be a cube–shaped region in the 3D Euclidean space. Let $P = \{p_1, p_2, \ldots, p_n\}$ be a set of n points used for modeling the fiducials identified in the reference image, where $p_i \in \mathcal{M}$ $(1 \leq i \leq n)$. A known affine transformation $\mathrm{T}_{\mathrm{known}}$ is chosen and the set $R = \{\, r_i \mid r_i = \mathrm{T}_{\mathrm{known}} \cdot p_i,\ i = 1, \ldots, n\}$ is calculated. Set R is corrupted by an n–dimensional noise vector $(\mu_1, \ldots, \mu_n)$ whose components are random variables having σ–Gaussian distribution. This is used for modeling the FLE. The set $Q = \{\, q_i \mid q_i = r_i + \mu_i,\ i = 1, \ldots, n\}$ is constructed, where pair (p_i, q_i) of points can be regarded as a pair of corresponding fiducials used for registration. It is assumed that the FLE is identically zero in the base image. The set $S = \{\, s_j \mid s_j \in \mathcal{M},\ j = 1, \ldots, m\}$ of m points is randomly selected to represent ROIs in the reference image. Note that the same $m = 20$ target points are used for our numerical simulations . Set S is also transformed to generate set of m points $U = \{\, u_j \mid u_j = \mathrm{T}_{\mathrm{known}} \cdot s_j,\ j = 1, \ldots, m\}$. The transformation $\mathrm{T}_{\mathrm{found}}$ is determined and it is applied to the set U to calculate the set of m points $V = \{\, v_j \mid v_j = \mathrm{T}_{\mathrm{found}} u_j,\ j = 1, \ldots, m\}$.

TRE is formulated as follows:

$$\sqrt{\frac{1}{m}\sum_{j=1}^{m}\|s_j - v_j\|^2}.$$

We repeated the iterations 10000 times.

4.2 Results

Fig. 4 shows that TRE is proportional to FLE, for a fixed number of fiducials. Therefore, **Result 1** holds for affine transformations, too.

Fig. 5 is to demonstrate how TRE depends on the number of fiducials, for a fixed FLE. Although **Result 2** does not hold, it can be seen that the TRE is inversely proportional to the number of fiducials.

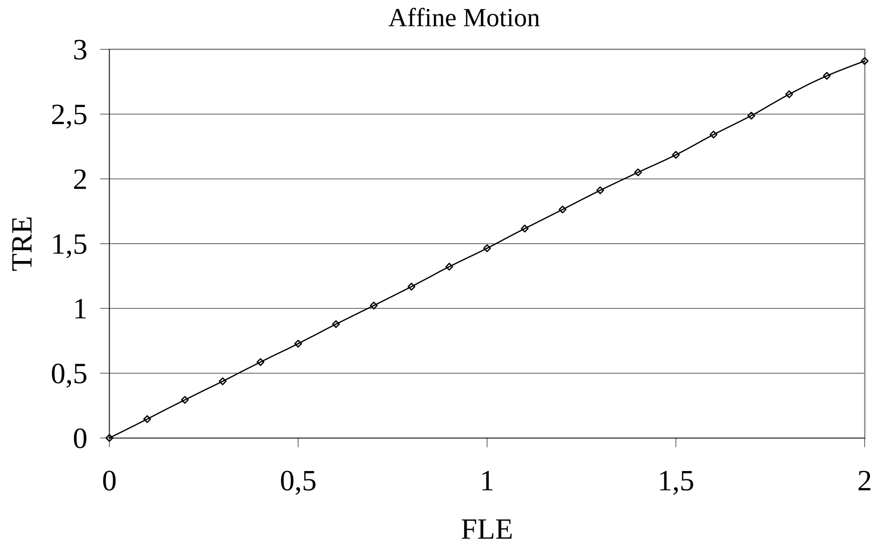

Fig. 4. TRE (for the 20 target points) as a function of FLE for 10 fiducials. It is confirmed that TRE is proportional to FLE

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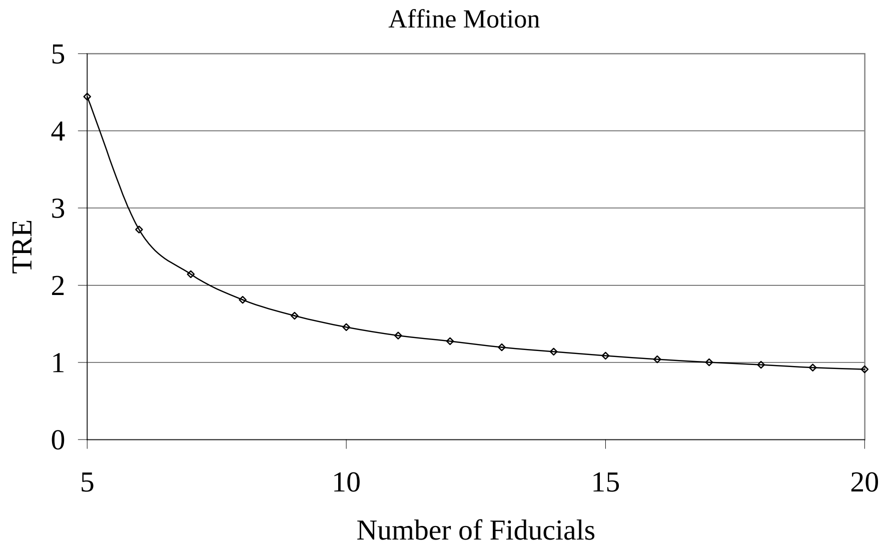

Fig. 5. TRE (for the 20 target points) as a function of the number of fiducials n, where $\sigma = 1$ Gaussian distribution is used for modelling FLE. TRE is inversely proportional to n

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