Affine matching of two sets of points in arbitrary dimensions

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Abstract

In many applications of computer vision, image processing, and remotely sensed data processing, an appropriate matching of two sets of points is required. Our approach assumes one–to–one correspondence between these sets and finds the optimal global affine transformation that matches them. The suggested method can be used in arbitrary dimensions. A sufficient existence condition for a unique transformation is given and proven.

1 Introduction

Many applications lead to the following mathematical problem: Two corresponding sets of points $\{p_i\}$ and $\{q_i\}$ $(i = 1, 2, \ldots, n)$ are given in the k–dimensional Euclidean space $\mathbb{R}^k$, and the transformation $T : \mathbb{R}^k \to \mathbb{R}^k$ is to be found that gives the minimal mean squared error

$$\sum_{i=1}^{n} \|T(q_i) - p_i\|^2.$$

The dimension k is usually 2 or 3. Some solutions have been proposed for this problem assuming rigid–body transformation (i.e., where only rotations and translations are allowed) [1, 3, 6, 7, 13], affine transformation (i.e., which maps straight lines to straight lines, parallelism is preserved, but angles can be altered) [8], and non-linear transformation (i.e., which can map straight lines to curves) [2, 5, 8]. In [10], a solution is proposed when the correspondence between the point sets is unknown, assuming affine transformation. It is mentioned, that if the correspondence was known, a simpler solution is possible e.g., using least squares method, but neither such a method nor a sufficient existence condition for unique solution is given or referenced.

In this paper, we present a method for solving the problem assuming affine transformation, which can be used in arbitrary dimensions. The method is described in Section 2. We state and prove a sufficient existence condition for a unique solution in Section 3. A related open problem concerning degeneracy is presented in Section 4.

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2 Method for affine matching of two sets of points

Given a matrix

$$\mathcal{T} = \begin{pmatrix} t_{11} & t_{12} & \cdots & t_{1k} & t_{1,k+1} \\ t_{21} & t_{22} & \cdots & t_{2k} & t_{2,k+1} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ t_{k1} & t_{k2} & \cdots & t_{kk} & t_{k,k+1} \\ 0 & 0 & \cdots & 0 & 1 \end{pmatrix},$$

it determines an affine transformation $T : \mathbb{R}^k \to \mathbb{R}^k$ as follows: For $x = (x_1, \ldots, x_k)$ and $y = (y_1, \ldots, y_k)$ in $\mathbb{R}^k$ we have $y = T(x)$ if and only if

$$\begin{pmatrix} y_{i1} \\ y_{i2} \\ \vdots \\ y_{ik} \\ 1 \end{pmatrix} = \begin{pmatrix} t_{11} & t_{12} & \cdots & t_{1k} & t_{1,k+1} \\ t_{21} & t_{22} & \cdots & t_{2k} & t_{2,k+1} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ t_{k1} & t_{k2} & \cdots & t_{kk} & t_{k,k+1} \\ 0 & 0 & \cdots & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} x_{i1} \\ x_{i2} \\ \vdots \\ x_{ik} \\ 1 \end{pmatrix}.$$

Note that homogeneous coordinates are used. Each affine transformation T can uniquely be represented in this form [4]. The transformation has $k \cdot (k+1)$ degrees of freedom according to the non–constant matrix elements.

Let us fix an affine transformation $T : \mathbb{R}^k \to \mathbb{R}^k$ and the corresponding $\mathcal{T}$ as above. Let $\{p_i\}$ and $\{q_i\}$ be two sets of n points, where

$$\begin{aligned} p_i &= (p_{i1}, p_{i2}, \ldots, p_{ik}) \in \mathbb{R}^k \quad \text{and} \\ q_i &= (q_{i1}, q_{i2}, \ldots, q_{ik}) \in \mathbb{R}^k \quad (i = 1, 2, \ldots, n). \end{aligned}$$

Let $\{p'_i\}$ be a set of n points in $\mathbb{R}^k$, where $p'_i = T(q_i)$ $(i = 1, 2, \ldots, n)$. Define the merit function $\mathcal{S}$ of $k \cdot (k+1)$ variables as follows:

$$\mathcal{S}(t_{11}, \ldots, t_{k,k+1}) = \sum_{i=1}^{n} \|p'_i - p_i\|^2 = \sum_{i=1}^{n} \sum_{j=1}^{k} (t_{j1} \cdot q_{i1} + \ldots + t_{jk} \cdot q_{ik} + t_{j,k+1} - p_{ij})^2.$$

which is generally regarded as the *matching error*.

The least square solution of matrix $\mathcal{T}$ is determined by minimizing the function $\mathcal{S}$. Function $\mathcal{S}$ may be minimal if all of the partial derivatives $\frac{\partial \mathcal{S}}{\partial t_{11}}, \ldots, \frac{\partial \mathcal{S}}{\partial t_{k,k+1}}$ are equal to zero. The required $k \cdot (k+1)$ equations:

$$\begin{aligned} \frac{\partial \mathcal{S}}{\partial t_{uv}} &= 2 \cdot \sum_{i=1}^{n} q_{iv} \cdot (t_{u,k+1} - p_{iu} + \sum_{l=1}^{k} t_{ul} \cdot q_{il}) = 0 \\ & \quad (u = 1, 2, \ldots, k\,,\ v = 1, 2, \ldots, k), \end{aligned}$$

$$\begin{aligned} \frac{\partial \mathcal{S}}{\partial t_{u,k+1}} &= 2 \cdot \sum_{i=1}^{n} (t_{u,k+1} - p_{iu} + \sum_{l=1}^{k} t_{ul} \cdot q_{il}) = 0 \\ & \quad (u = 1, 2, \ldots, k). \end{aligned}$$

We get the following system of linear equations:

$$\begin{pmatrix} \begin{matrix} a_{11} & \dots & a_{1k} & b_1 \\ \vdots & \ddots & \vdots & \vdots \\ a_{k1} & \dots & a_{kk} & b_k \\ b_1 & \dots & b_k & n \end{matrix} & & & \mathbf{0} \\ & \begin{matrix} a_{11} & \dots & a_{1k} & b_1 \\ \vdots & \ddots & \vdots & \vdots \\ a_{k1} & \dots & a_{kk} & b_k \\ b_1 & \dots & b_k & n \end{matrix} & & \\ & & \ddots & \\ \mathbf{0} & & & \begin{matrix} a_{11} & \dots & a_{1k} & b_1 \\ \vdots & \ddots & \vdots & \vdots \\ a_{k1} & \dots & a_{kk} & b_k \\ b_1 & \dots & b_k & n \end{matrix} \end{pmatrix} \cdot \begin{pmatrix} t_{11} \\ \vdots \\ t_{1k} \\ t_{1,k+1} \\ t_{21} \\ \vdots \\ t_{2k} \\ t_{2,k+1} \\ \vdots \\ t_{k1} \\ \vdots \\ t_{kk} \\ t_{k,k+1} \end{pmatrix} = \begin{pmatrix} c_{11} \\ \vdots \\ c_{1k} \\ d_1 \\ c_{21} \\ \vdots \\ c_{2k} \\ d_2 \\ \vdots \\ c_{k1} \\ \vdots \\ c_{kk} \\ d_k \end{pmatrix},$$

where

$$\begin{aligned} a_{uv} = a_{vu} &= \sum_{i=1}^{n} q_{iu} \cdot q_{iv} \ , \\ b_u &= \sum_{i=1}^{n} q_{iu} \ , \\ c_{uv} &= \sum_{i=1}^{n} p_{iu} \cdot q_{iv} \ , \\ d_u &= \sum_{i=1}^{n} p_{iu} \end{aligned}$$

$$(u = 1, 2, \dots, k\,,\ v = 1, 2, \dots, k).$$

The above system of linear equations can be solved by using an appropriate numerical method [9]. There exists a unique solution if and only if $det(M) \neq 0$, where

$$M = \begin{pmatrix} a_{11} & \dots & a_{1k} & b_1 \\ \vdots & \ddots & \vdots & \vdots \\ a_{k1} & \dots & a_{kk} & b_k \\ b_1 & \dots & b_k & n \end{pmatrix}.$$

Note that if a problem is close to singular (i.e., $det(M)$ is close to 0), the method can become unstable.

3 Discussion

In this section we state and prove a sufficient existence condition for a unique solution for the system of linear equations.

By a hyperplane of the Euclidean space $\mathbb{R}^k$ we mean a subset of the form $\{a + x : x \in S\}$ where S is a $(k-1)$–dimensional linear subspace. Given some points $q_1, \ldots, q_n$ in $\mathbb{R}^k$, we say that these points *span* $\mathbb{R}^k$ if no hyperplane of $\mathbb{R}^k$ contains them. If any $k+1$ points from $q_1, \ldots, q_n$ span $\mathbb{R}^k$ then we say that $q_1, \ldots, q_n$ are in *general position.*

Theorem 1. If $q_1, \ldots, q_n$ span $\mathbb{R}^k$ then $\det(M) \neq 0$.

Proof. Suppose $\det(M) = 0$. Consider the vectors $v_j = (q_{1j}, q_{2j}, \ldots, q_{nj}) \quad (1 \leq j \leq k)$ in $\mathbb{R}^n$, and let $v_{k+1} = (1, 1, \ldots, 1) \in \mathbb{R}^n$. With the notation $m = k + 1$ observe that $M = \big(\langle v_i, v_j \rangle\big)_{m \times m}$ where $\langle\ ,\ \rangle$ stands for the scalar multiplication. Since the columns of M are linearly dependent, we can fix a $(\beta_1, \ldots, \beta_m) \in \mathbb{R}^m \setminus \{(0, \ldots, 0)\}$ such that $\sum_{j=1}^m \beta_j \langle v_i, v_j \rangle = 0$ holds for $i = 1, \ldots, m$. Then

$$0 = \sum_{i=1}^m \beta_i \cdot 0 = \sum_{i=1}^m \beta_i \sum_{j=1}^m \beta_j \langle v_i, v_j \rangle = \sum_{i=1}^m \beta_i \Big\langle v_i, \sum_{j=1}^m \beta_j v_j \Big\rangle =$$

$$\Big\langle \sum_{i=1}^m \beta_i v_i, \sum_{j=1}^m \beta_j v_j \Big\rangle = \Big\langle \sum_{i=1}^m \beta_i v_i, \sum_{i=1}^m \beta_i v_i \Big\rangle,$$

whence $\sum_{i=1}^m \beta_i v_i = 0$. Therefore all the q_j, $1 \leq j \leq n$, are solutions of the following (one element) system of linear equations:

$$\beta_1 x_1 + \cdots + \beta_k x_k = -\beta_m. \tag{1}$$

Since the system has solutions and $(\beta_1, \ldots, \beta_m) \neq (0, \ldots, 0)$, there is an $i \in \{1, \ldots, k\}$ with $\beta_i \neq 0$. Hence the solutions of (1) form a hyperplane of $\mathbb{R}^k$. This hyperplane contains $q_1, \ldots, q_n$. Now it follows that if $q_1, \ldots, q_n$ span $\mathbb{R}^k$ then $\det(M) \neq 0$. Q.e.d.

4 Conclusions

In real applications, it is assumed that both $p_1, \ldots, p_n$ and $q_1, \ldots, q_n$ span $\mathbb{R}^k$. Then, if the matching error is zero (i.e., $p'_i = T(q_i) = p_i$ for $i = 1, 2, \ldots n$), the transformation is necessarily non-degenerate, i.e., $\det(\mathcal{T}) \neq 0$. Moreover, in this case the following property is fulfilled:

Observation 2. For all $I \subseteq \{1, \ldots, n\}$ with $k + 1$ elements, the p_i, $i \in I$, span $\mathbb{R}^k$ if and only if the q_i, $i \in I$, span $\mathbb{R}^k$.

This raises the question whether the transformation is necessarily non–degenerete in general or when Observation 2 holds or at least when Observation 2 "strongly" holds in the following computational sense: each simplex with vertices in $\{p_1, \ldots, p_n\}$ or with vertices in $\{q_1, \ldots, q_n\}$ has a large volume (k–dimensional measure) compared with its edges.

Surprisingly, all these questions have a negative answer, for we have the following three dimensional example.

Example 3. With $n = 5$ and $k = 3$ let $q_1 = (0, 0, 24)$, $q_2 = (24, 0, 0)$, $q_3 = (0, 24, 0)$, $q_4 = (0, 0, 0)$, and $q_5 = (-24, -48, 16)$. These five points determine five tetrahedra with reasonably large volumes, the smallest of them being 1536, the volume of the tetrahedron

(q_2, q_3, q_4, q_5). Let $p_1 = (0,0,0)$, $p_2 = (3,0,0)$, $p_3 = (0,3,0)$, $p_4 = (0,0,3)$, $p_5 = (3,3,3)$, these are some vertices of a cube, so the tetrahedra they determine are at least of volume 9/2. Yet,

$$\mathcal{T} = \begin{pmatrix} 2 & -6 & -6 & 12 \\ -9 & -1 & -9 & 18 \\ 0 & 0 & 0 & 8 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

which is degenerate.

Experience shows that in real applications the choice of points always guarantees that the transformation is non–degenerate [11, 12]. However, from theoretical point of view the following open problem is worth raising: Find a meaningful sufficient condition to ensure non–degeneracy.

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References

[1] Arun, K.S., T.S. Huang, S.D. Blostein, Least squares fitting of two 3-D point sets, *IEEE Trans. Pattern Analysis and Machine Intelligence* **9** (1987), 698–703.

[2] Bookstein, F.L., Principal warps: thin–plate splines and the decomposition of deformations, *IEEE Trans. Pattern Analysis and Machine Intelligence* **11** (1989), 567–585.

[3] Faugeras, O.D., M. Hebert, A 3-D recognition and positioning algorithm using geometrical matching between primitive surfaces, Proc. Int. Joint Conf. Artificial Intelligence, Karlsruhe, 1983, 996–1002.

[4] Foley, J.D., A. van Dam, S.K. Feiner, J.F. Hughes, *Computer Graphics — Principles and practice*, Addison–Wesley Publishing Company, Reading, Massachusetts 1991.

[5] Fornefett, M., K. Rohr, H.S. Stiehl, Radial basis functions with compact support for elastic registration of medical images, Proc. Int. Workshop on Biomedical Image Registration, Bled, 1999, 173–185.

[6] Horn, B.K.P., Closed-form solution of absolute orientation using unit quaternions, *J. Opt. Soc. Amer. A* **4** (1987), 629–642.

[7] Horn, B.K.P., Closed-form solution of absolute orientation using orthonormal matrices, *J. Opt. Soc. Amer. A* **5** (1987), 1127–1135.

[8] Maguire, D., M.F. Goodchild, D.W. Rhind (eds.), *Geographical information systems — Principles and Applications*, Longman Scientific and Technical, 1991.

[9] Press, W.H., B.P. Flannery, S.A. Teukolsky, W.T. Vetterling, *Numerical Recipes in C: The Art of Scientific Computing*, Cambridge University Press, 1992.

[10] Spinzak, J., M. Werma, Affine Point Matching, *Pattern Recognition Letters* **4** (1994), 337–339.

[11] Tanács, A., K. Palágyi, A. Kuba, Medical image registration based on interactively defined anatomical landmark points, *Int. J. Machine Graphics & Vision* **7** (1998), 151–158.

[12] Tanács, A., K. Palágyi, A. Kuba, Target Registration Error of Point–Based Methods Assuming Rigid–Body and Linear Motions, Proc. Int. Workshop on Biomedical Image Registration, Bled, 1999, 223–233.

[13] Umeyama, S., Least-squares estimation of transformation parameters between two point patterns, *IEEE Trans. Pattern Analysis and Machine Intelligence* **13** (1991), 376–380.