Independent joins of tolerance factorable varieties

Ivan Chajda, Gábor Czédli, and Radomír Halaš

Dedicated to Béla Csákány on his eightieth birthday

Abstract. Let ***Lat*** denote the variety of lattices. In 1982, the second author proved that ***Lat*** is *strongly tolerance factorable*, that is, the members of ***Lat*** have quotients in ***Lat*** modulo tolerances, although ***Lat*** has proper tolerances. We did not know any other nontrivial example of a strongly tolerance factorable variety. Now we prove that this property is preserved by forming independent joins (also called products) of varieties. This enables us to present infinitely many strongly tolerance factorable varieties with proper tolerances. Extending a recent result of G. Czédli and G. Grätzer, we show that the tolerances of these varieties are exactly the homomorphic images of their congruences. Our observation that (strong) tolerance factorability is not necessarily preserved when passing from a variety to an equivalent one leads to an open problem.

Basic concepts. Given an algebra $\mathcal{A} = (A, F)$, a binary reflexive, symmetric, and compatible relation $T \subseteq A \times A = A^2$ is called a *tolerance* on $\mathcal{A}$. The set of tolerances of $\mathcal{A}$ is denoted by $\mathrm{Tol}(\mathcal{A})$. A tolerance which is not a congruence is called *proper*. By a *block* of a tolerance T we mean a maximal subset B of A such that $B^2 \subseteq T$. Let $\mathrm{Block}(T)$ denote the set of all blocks of T. It follows from Zorn's lemma that $\mathrm{Block}(T)$ determines T since, for $X \subseteq A$, $X^2 \subseteq T$ iff $X \subseteq U$ for some $U \in \mathrm{Block}(T)$. Therefore, for each n, each n-ary $f \in F$, and all $B_1, \dots, B_n \in \mathrm{Block}(T)$, there exists a $B \in \mathrm{Block}(T)$ such that

$$\{f(b_1, \dots, b_n) : b_1 \in B_1, \dots, b_n \in B_n\} \subseteq B. \tag{1}$$

We say that $\mathcal{A}$ is *T-factorable* if, for each n, each n-ary $f \in F$ and all $B_1, \dots, B_n \in \mathrm{Block}(T)$, the block B in (1) is uniquely determined. In this case, we define $f(B_1, \dots, B_n) := B$, and we call the algebra $(\mathrm{Block}(T), F)$ the *quotient algebra* $\mathcal{A}/T$ of $\mathcal{A}$ *modulo the tolerance* T. If $\mathcal{A}$ is T-factorable for all $T \in \mathrm{Tol}(A)$, then we say that $\mathcal{A}$ is *tolerance factorable.* In what follows, we focus on the following properties of varieties; $\boldsymbol{\mathcal{V}}$ denotes a variety of algebras. The *tolerances* of $\boldsymbol{\mathcal{V}}$ are understood as the tolerances of algebras of $\boldsymbol{\mathcal{V}}$.

(P1) $\boldsymbol{\mathcal{V}}$ is *tolerance factorable* if all of its members are tolerance factorable.

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(P2) $\boldsymbol{\mathcal{V}}$ is *strongly tolerance factorable* if it is tolerance factorable and, for all $\mathcal{A} \in \boldsymbol{\mathcal{V}}$ and all $T \in \mathrm{Tol}(\mathcal{A})$, $\mathcal{A}/T \in \boldsymbol{\mathcal{V}}$.

(P3) *The tolerances of* $\boldsymbol{\mathcal{V}}$ *are the images of its congruences* if for each $\mathcal{A} \in \boldsymbol{\mathcal{V}}$ and every $T \in \mathrm{Tol}(\mathcal{A})$, there exist an algebra $\mathcal{A}' \in \boldsymbol{\mathcal{V}}$, a congruence $\boldsymbol{\theta}$ of $\mathcal{A}'$ and a surjective homomorphism $\varphi\colon \mathcal{A}' \to \mathcal{A}$ such that $T = \{(\varphi(a), \varphi(b)) : (a, b) \in \boldsymbol{\theta}\}$.

(P4) $\boldsymbol{\mathcal{V}}$ has proper tolerances if at least one of its members has a proper tolerance.

Term equivalence, in short, *equivalence*, of varieties was introduced by W. D. Neumann [8]. (He called it rational equivalence.) Instead of recalling the technical definition, we mention that the variety of Boolean algebras is equivalent to that of Boolean rings. The variety of sets (with no operations) is denoted by ***Set***.

Motivation and the target. Besides ***Lat*** and ***Set***, no other strongly tolerance factorable variety with proper tolerances has been known since 1982. Our initial goal was to find some other ones. We prove that independent joins, see later, preserve each of the properties (P1)–(P4). This enables us to construct infinitely many, pairwise non-equivalent, strongly tolerance factorable varieties with proper tolerances. Also, we show that if a variety is strongly tolerance factorable, then its tolerances are the images of its congruences, but the converse implication fails. Finally, we show that (strong) tolerance factorability is not always preserved when passing from a variety to an equivalent one, and we raise an open problem based on this fact.

Independent joins. Let $n \in \mathbb{N} = \{1, 2, \dots\}$, and let $\boldsymbol{\mathcal{V}}_1, \dots, \boldsymbol{\mathcal{V}}_n$ be varieties of the same type. These varieties are called *independent* if there exist an n-ary term t in their common type such that, for $i = 1, \dots, n$, $\boldsymbol{\mathcal{V}}_i$ satisfies the identity $t(x_1, \dots, x_n) = x_i$. In this case, the join $\boldsymbol{\mathcal{V}}$ of the varieties $\boldsymbol{\mathcal{V}}_1, \dots, \boldsymbol{\mathcal{V}}_n$ is called an *independent join* (in the lattice of all varieties of a given type). This concept was introduced by G. Grätzer, H. Lakser, and J. Płonka [5]. Independent joins of varieties are also called (direct) *products*.

Proposition 1 (W. Taylor [10], G. Grätzer, H. Lakser, and J. Płonka [5]). *Assume that a variety* $\boldsymbol{\mathcal{V}}$ *is the independent join of its subvarieties* $\boldsymbol{\mathcal{V}}_1, \cdots, \boldsymbol{\mathcal{V}}_n$.

(1) *Every algebra* $\mathcal{A} \in \boldsymbol{\mathcal{V}}$ *is* (*isomorphic to*) *a product* $\mathcal{A}_1 \times \cdots \times \mathcal{A}_n$ *with* $\mathcal{A}_1 \in \boldsymbol{\mathcal{V}}_1, \dots, \mathcal{A}_n \in \boldsymbol{\mathcal{V}}_n$. *These* A_i *are uniquely determined up to isomorphism.*

(2) *If* B *is a subalgebra of* $\mathcal{A} = \mathcal{A}_1 \times \cdots \times \mathcal{A}_n$ *considered above, then there exist subalgebras* B_i *of* $\mathcal{A}_i$ $(i = 1, \dots, n)$ *such that* $B = B_1 \times \cdots \times B_n$.

(3) *Every tolerance* T *of* $\mathcal{A}$ *is the form of* $T_1 \times \cdots \times T_n$ *such that* T_i *is a tolerance of* $\mathcal{A}_i$ *for* $i = 1 \dots, n$. *If* T *is a congruence, then so are the* T_i.

Although (3) in [10] is stated only for congruences, the one-line argument "regard T as a subalgebra of $\mathcal{A}_1^2 \times \cdots \times \mathcal{A}_n^2$ and apply (2)" of [10] also works if T is a tolerance rather than a congruence.

Results and examples. The properties (P1)–(P4) are not independent from each other and from congruence permutability. It is well-known, for example from J. D. H. Smith [9], that congruence permutable varieties do not have proper tolerances. Obviously, a variety without proper tolerances is strongly tolerance factorable and its tolerances are the images of its congruences. Also, we present the following statement, which generalizes the result of G. Czédli and G. Grätzer [4].

Proposition 2. *If a variety is strongly tolerance factorable, then its tolerances are the images of its congruences.*

The statements of this section will be proved in the next one. In connection with Proposition 2, we also prove the following easier statement, which asserts that every tolerance (in a variety) is the image of some congruence (possibly outside the variety).

Proposition 3. *For every algebra $\mathcal{A}$ and each tolerance $T \in \mathrm{Tol}(A)$, there exist an algebra $\mathcal{A}'$, a congruence $\boldsymbol{\theta}$ of $\mathcal{A}'$ and a surjective homomorphism $\varphi\colon \mathcal{A}' \to \mathcal{A}$ such that $T = \{(\varphi(a), \varphi(b)) : (a, b) \in \boldsymbol{\theta}\}$.*

Tolerance factorability does not imply strong tolerance factorability. For example, let $\boldsymbol{\mathcal{V}}$ be a nontrivial proper subvariety of the variety $\boldsymbol{Lat}$ of all lattices. We know from G. Czédli [3] that $\boldsymbol{Lat}$ is strongly tolerance factorable; see also G. Grätzer and G. H. Wenzel [6] for an alternative proof. Consequently, $\boldsymbol{\mathcal{V}}$ is tolerance factorable. However, it is not strongly tolerance factorable by G. Czédli [3, Theorem 3].

Our main achievement is the following statement.

Theorem 4. *Assume that a variety $\boldsymbol{\mathcal{V}}$ is the independent join of its subvarieties $\boldsymbol{\mathcal{V}}_1, \dots, \boldsymbol{\mathcal{V}}_n$. Consider one of the properties*

(1) *strong tolerance factorability,*
(2) *tolerance factorability,*
(3) *the tolerances of the variety are the images of its congruences.*

If this property holds for all the $\boldsymbol{\mathcal{V}}_i$, then it also holds for $\boldsymbol{\mathcal{V}}$.

Now we are ready to give several examples for strongly tolerance factorable varieties with proper tolerances. It would be easy to give such examples by taking varieties equivalent to $\boldsymbol{Lat}$. (For example, we could replace the binary join by the n-ary operation $f(x_1, \dots, x_n) := x_1 \vee x_2$.) Hence we will give pairwise non-equivalent varieties even if Example 8 implies the surprising fact that strong tolerance factorability is not necessarily preserved when passing from a variety to an equivalent one.

For $2 \le n \in \mathbb{N}$ and $1 \le i \le n$, let $\boldsymbol{\mathcal{S}}_i^{(n)}$ be the variety consisting of all algebras (X, f_n) such that X is a set and f_n is an n-ary operation symbol inducing the i-th projection on X. That is, $\boldsymbol{\mathcal{S}}_i^{(n)}$ is of type $\{f_n\}$, and it is defined by the identity $f_n(x_1, \dots, x_n) = x_i$. Let $\boldsymbol{\mathcal{S}}^{(n)} = \boldsymbol{\mathcal{S}}_1^{(n)} \vee \dots \vee \boldsymbol{\mathcal{S}}_n^{(n)}$ and $\boldsymbol{\mathcal{S}}^{(1)} = \boldsymbol{Set}$.

Example 5. The varieties $\boldsymbol{\mathcal{S}}^{(n)}$, $n \in \mathbb{N}$, are strongly tolerance factorable and pairwise non-equivalent, and they have proper tolerances.

Notice that $\boldsymbol{\mathcal{S}}^{(2)}$ is the variety of *rectangular bands*, which are idempotent semigroups satisfying the identity $xyx = x$. See A. H. Clifford [2], who introduced this concept, and B. Jónsson and C. Tsinakis [7].

Next, consider lattices with an additional unary operation g_n that induces an automorphism of the lattice structure such that the identity $g_n^n(x) = x$ (where $g_n^n(x)$ denotes the n-fold iteration $g_n\bigl(g_n(\dots g_n(x)\dots)\bigr)$ of g_n) holds. We can call them *rotational lattices of order n*. The variety of these lattices is denoted by $\boldsymbol{RLat}_n$. Note that $\boldsymbol{RLat}_1$ is equivalent to $\boldsymbol{Lat}$ while $\boldsymbol{RLat}_2$ consists of *lattices with involution*, which were studied, for example, in I. Chajda and G. Czédli [1]. Note also that $\boldsymbol{RLat}_n \subseteq \boldsymbol{RLat}_m$ iff $n \mid m$.

Example 6. The varieties $\boldsymbol{RLat}_n$, $n \in \mathbb{N}$, are strongly tolerance factorable and pairwise non-equivalent, and they have proper tolerances. Moreover, none of them is equivalent to a variety given in Example 5.

Armed with Theorem 4, one can give some more sophisticated examples. For example, we present the following. Let h be a binary term symbol, and let $m, n \in \mathbb{N}$. We consider the type $\tau_{m,n} = \{\vee, \wedge, g_m, f_n, h\}$. Define the action of f_n and h on the algebras of $\boldsymbol{RLat}_m$ as first projections. This way these algebras become τ_{mn}-algebras and they form a variety ${}^n(\boldsymbol{RLat}_m)$. Similarly, define $\vee$, $\wedge$, g_m, and h on the members of $\boldsymbol{\mathcal{S}}^{(n)}$ as second projections. The algebras we obtain constitute a variety $(\boldsymbol{\mathcal{S}}^{(n)})^m$ of type τ_{mn}. Let $\boldsymbol{C}_{mn} = {}^n(\boldsymbol{RLat}_m) \vee (\boldsymbol{\mathcal{S}}^{(n)})^m$.

Example 7. The varieties $\boldsymbol{C}_{mn}$, $m, n \in \mathbb{N}$, are strongly tolerance factorable and they have proper tolerances. Furthermore, $\boldsymbol{C}_{mn}$ is equivalent to $\boldsymbol{C}_{ij}$ iff $(i, j) = (m, n)$.

Note that the varieties in Example 6 are congruence distributive while those in Examples 5 and 7 satisfy no nontrivial congruence lattice identity.

Next, in the language of lattices, we consider the ternary lattice terms $t_\vee(x, y, z) = x \vee (y \wedge z)$ and $t_\wedge(x, y, z) = x \wedge (y \vee z)$. Clearly, the identities $x \vee y = t_\vee(x, y, y)$ and $x \wedge y = t_\wedge(x, y, y)$ hold in all lattices. This motivates the following definition of another variety in the language of $\{t_\vee, t_\wedge\}$ as follows. In each of the six usual laws defining $\boldsymbol{Lat}$, we replace $\vee$ and $\wedge$ by $t_\vee(x, y, y)$ and $t_\wedge(x, y, y)$. For example, the absorption law $x = x \vee (x \wedge y)$ turns into the identity $x = t_\vee\bigl(x, t_\wedge(x, y, y), t_\wedge(x, y, y)\bigr)$. The six identities we obtain this way together with the identities $t_\vee(x, y, z) = t_\vee(x, t_\wedge(y, z, z), t_\wedge(y, z, z))$ and $t_\wedge(x, y, z) = t_\wedge(x, t_\vee(y, z, z), t_\vee(y, z, z))$ define a variety, which will be denoted by $\boldsymbol{TLat}$.

Example 8. $\boldsymbol{TLat}$ is equivalent to $\boldsymbol{Lat}$. Hence the tolerances of $\boldsymbol{TLat}$ are the images of its congruences. However, $\boldsymbol{TLat}$ is not tolerance factorable.

Let $\mathcal{A} \in \boldsymbol{TLat}$ and $T \in \mathrm{Tol}(\mathcal{A})$. Although $\boldsymbol{TLat}$ is not tolerance factorable, the fact that it is equivalent to a tolerance factorable variety (which is $\boldsymbol{Lat}$)

yields a natural way of defining $\mathcal{A}/T$. Furthermore, the strong tolerance factorability of $\boldsymbol{Lat}$ implies that $\mathcal{A}/T \in \boldsymbol{TLat}$.

Since $\boldsymbol{TLat}$ is only an "artificial" variety, we raise the following problem.

Problem 9. Is there a well-known variety $\boldsymbol{\mathcal{V}}$ such that although $\boldsymbol{\mathcal{V}}$ is not tolerance factorable, it is equivalent to some tolerance factorable (possibly "artificial") variety?

Proofs.

Proof of Proposition 3. We generalize the idea of G. Czédli and G. Grätzer [4]. To avoid ambiguity, the operation on A defined by an operation symbol $f \in F$ will be denoted by f_A, and a similar notation applies for the algebra we are going to construct. Let $f \in F$ with arity n, and let $B_1, \dots, B_n \in \mathrm{Block}(T)$. By Zorn's lemma, we can choose a block $B \in \mathrm{Block}(T)$ such that (1) holds. Even if this choice is not unique, we define $f_{\mathrm{Block}(T)}(B_1, \dots, B_n) := B$. Doing this for all $(B_1, \dots, B_n) \in \mathrm{Block}(T)^n$, we obtain an operation $f_{\mathrm{Block}(T)}$ on $\mathrm{Block}(T)$, which we call a *random operation.* Armed with random operations, $\mathcal{B} := \big(\mathrm{Block}(T), F\big)$ becomes an algebra, which we call a *random quotient algebra* of $\mathcal{A}$ modulo T.

Next, let $\mathcal{C} = \mathcal{A} \times \mathcal{B}$. Denoting $\{(x, Y) \in A \times \mathrm{Block}(T) : x \in Y\}$ by D, the construction implies that $\mathcal{D} = (D, F)$ is a subalgebra of $\mathcal{C}$. (It will play the role of $\mathcal{A}'$.) Define $\boldsymbol{\theta} = \big\{\big((x_1, Y_1), (x_2, Y_2)\big) \in D^2 : Y_1 = Y_2\big\}$. As the kernel of the second projection from $\mathcal{D}$ to $\mathcal{B}$, it is a congruence on $\mathcal{D}$. The first projection $\varphi\colon \mathcal{D} \to \mathcal{A}$, $(x, Y) \mapsto x$, is a surjective homomorphism since, for every $x \in A$, $\{x\}$ extends to a block of T.

Clearly, if $\big((x_1, Y_1), (x_2, Y_2)\big) \in \boldsymbol{\theta}$, then $\{x_1, x_2\} \subseteq Y_1 = Y_2 \in \mathrm{Block}(T)$ implies that $\big(\varphi(x_1, Y_1), \varphi(x_2, Y_2)\big) = (x_1, x_2) \in T$. Conversely, if $(x_1, x_2) \in T$, then there is a $Y \in \mathrm{Block}(T)$ with $\{x_1, x_2\} \subseteq Y$, whence $(x_1, Y), (x_2, Y) \in D$, $\big((x_1, Y), (x_2, Y)\big) \in \boldsymbol{\theta}$, and $x_i = \varphi(x_i, Y)$ finally yield the equality $T = \big\{\big(\varphi(x_1, Y_1), \varphi(x_2, Y_2)\big) : \big((x_1, Y_1), (x_2, Y_2)\big) \in \boldsymbol{\theta}\big\}$. □

Proof of Proposition 2. Let $\boldsymbol{\mathcal{V}}$ be a strongly tolerance factorable variety. Then $\mathcal{B} = \big(\mathrm{Block}(T), F\big)$ in the proof above is $\mathcal{A}/T$, and it belongs to $\boldsymbol{\mathcal{V}}$. So does $\mathcal{D}$, and the second half of the previous proof applies. □

Lemma 10. *Assume that T is as in Proposition* 1(3) *and* $B \in \mathrm{Block}(T)$. *Then there exist $B_i \in \mathrm{Block}(T_i)$, $i \in \{1, \dots, n\}$, such that $B = B_1 \times \dots \times B_n$, and they are uniquely determined. Furthermore,* $\mathrm{Block}(T) = \mathrm{Block}(T_1) \times \dots \times \mathrm{Block}(T_n)$.

Proof. Let π_i denote the projection map $A \to A_i$, $(x_1, \dots, x_n) \mapsto x_i$. Define $B_i := \pi_i(B)$. First we show that $B_1 \in \mathrm{Block}(T_1)$. If $a_1, b_1 \in B_1$, then $(a_1, a_2, \dots, a_n), (b_1, b_2, \dots, b_n) \in B$ for some $a_j, b_j \in A_j$, $2 \le j \le n$. Hence $B^2 \subseteq T$ implies that $(a_1, b_1) \in T_1$. This gives that $B_1^2 \subseteq T_1$, and we obtain $B_i^2 \subseteq T_i$ for all $i \in \{1, \dots, n\}$ by symmetric arguments. Thus

$$(B_1 \times \dots \times B_n)^2 \subseteq T_1 \times \dots \times T_n = T,$$

which together with $B \in \mathrm{Block}(T)$ and the obvious $B \subseteq B_1 \times \cdots \times B_n$ implies that

$$B = B_1 \times \cdots \times B_n. \tag{2}$$

The uniqueness of the B_i is trivial. If $B_1 \subseteq C_1 \subseteq A_1$ such that $C_1^2 \subseteq T_1$, then

$$B^2 = (B_1 \times \cdots \times B_n)^2 \subseteq (C_1 \times B_2 \times \cdots \times B_n)^2 \subseteq T_1 \times \cdots \times T_n = T.$$

Hence $B \in \mathrm{Block}(T)$ yields that the inclusions above are equalities, which implies that $B_1 = C_1$. Thus $B_1 \in \mathrm{Block}(T_1)$ and $B_i \in \mathrm{Block}(T_i)$ for all i. This together with (2) proves that $\mathrm{Block}(T) \subseteq \mathrm{Block}(T_1) \times \cdots \times \mathrm{Block}(T_n)$.

Finally, to prove the converse inclusion, assume that $U_i \in \mathrm{Block}(T_i)$ for $i = 1, \ldots, n$, and let $U = U_1 \times \cdots \times U_n$. Clearly, $U^2 \subseteq T_1 \times \cdots \times T_n = T$. By Zorn's lemma, there is a $B \in \mathrm{Block}(T)$ such that $U \subseteq B$. We already know that $B_i \in \mathrm{Block}(T_i)$ and (2) holds. This together with $U \subseteq B$ yields that $U_i \subseteq B_i$. Comparable blocks of T_i are equal, whence $U_i = B_i$, for all i. Hence $U = B \in \mathrm{Block}(T)$, proving that $\mathrm{Block}(T_1) \times \cdots \times \mathrm{Block}(T_n) \subseteq \mathrm{Block}(T)$. □

Proof of Theorem 4. Assume first that the $\boldsymbol{\mathcal{V}}_i$ are tolerance factorable. Let T be as in Proposition 1(3). Assume that s is a k-ary term in the language of $\boldsymbol{\mathcal{V}}$ and $B_1, \ldots, B_k \in \mathrm{Block}(T)$. By Lemma 10, there are uniquely determined $B_{ij} \in \mathrm{Block}(T_j)$ such that

$$B_i = B_{i1} \times \cdots \times B_{in} \quad \text{for} \quad i = 1, \ldots, k. \tag{3}$$

Assume that C is in $\mathrm{Block}(T)$ such that

$$\{s(b_1, \ldots, b_k) : b_1 \in B_1, \ldots, b_k \in B_k\} \subseteq C. \tag{4}$$

According to $\mathcal{A} = \mathcal{A}_1 \times \cdots \times \mathcal{A}_n$, we can write $b_i = (b_{i1}, \ldots, b_{in})$. Since s acts componentwise,

$$\begin{aligned} &\{s(b_1, \ldots, b_k) : b_1 \in B_1, \ldots, b_k \in B_k\} \\ &\quad = \big\{\big(s(b_{11}, \ldots, b_{k1}), \ldots, s(b_{1n}, \ldots, b_{kn})\big) : b_{ij} \in B_{ij}\big\} \\ &\quad = \{s(b_{11}, \ldots, b_{k1}) : b_{i1} \in B_{i1}\} \times \cdots \times \{s(b_{1n}, \ldots, b_{kn}) : b_{in} \in B_{in}\}. \end{aligned} \tag{5}$$

By Lemma 10, $C = C_1 \times \cdots \times C_n$ with $C_j \in \mathrm{Block}(T_j)$. Combining this with (4) and (5), we obtain that, for $j \in \{1, \ldots, n\}$,

$$\{s(b_{1j}, \ldots, b_{kj}) : b_{ij} \in B_{ij} \text{ for } i = 1, \ldots, k\} \subseteq C_j. \tag{6}$$

This implies the uniqueness of C_j since $\boldsymbol{\mathcal{V}}_j$ is tolerance factorable. Therefore, C in (4) is uniquely determined, and we obtain that $\boldsymbol{\mathcal{V}}$ is tolerance factorable.

Next, assume that the $\boldsymbol{\mathcal{V}}_i$ are strongly tolerance factorable. Observe that (6) also yields that $C_j = s(B_{1j}, \ldots, B_{kj})$ in the quotient algebra $\mathcal{A}_j/T_j$. This, together with (3) and $C = C_1 \times \cdots \times C_n$, implies that $\mathcal{A}/T$ is (isomorphic to) $\mathcal{A}_1/T_1 \times \cdots \times \mathcal{A}_n/T_n$. Since $\boldsymbol{\mathcal{V}}_j$ is strongly tolerance factorable, we conclude that $\mathcal{A}_j/T_j \in \boldsymbol{\mathcal{V}}_j \subseteq \boldsymbol{\mathcal{V}}$. Therefore $\mathcal{A}/T \in \boldsymbol{\mathcal{V}}$, proving that $\boldsymbol{\mathcal{V}}$ is strongly tolerance factorable.

Finally, if the tolerances of $\boldsymbol{\mathcal{V}}_i$ are the images of its congruences, for $i = 1, \ldots, n$, then Proposition 1 easily implies the same property of $\boldsymbol{\mathcal{V}}$. □

Proof of Example 5. Each of the $\boldsymbol{\mathcal{S}}_i^{(n)}$ is equivalent to $\boldsymbol{Set}$, whence it is easy to see that the $\boldsymbol{\mathcal{S}}_i^{(n)}$ are strongly tolerance factorable. The operation f_n witnesses that $\boldsymbol{\mathcal{S}}^{(n)} = \boldsymbol{\mathcal{S}}_1^{(n)} \vee \cdots \vee \boldsymbol{\mathcal{S}}_n^{(n)}$ is an independent join. Hence $\boldsymbol{\mathcal{S}}^{(n)}$ is strongly tolerance factorable by Theorem 4. The three-element algebra $\mathcal{A} = \big(\{a, b, c\}, f_n\big)$, where f_n acts as the first projection, belongs to $\boldsymbol{\mathcal{S}}_1^{(n)} \subseteq \boldsymbol{\mathcal{S}}^{(n)}$. Consider $T \in \mathrm{Tol}(\mathcal{A})$ determined by $\mathrm{Block}(T) = \big\{\{ab\}, \{bc\}\big\}$. This T witnesses that $\boldsymbol{\mathcal{S}}^{(n)}$ has proper tolerances.

Next, let $\mathcal{A}_i \in \boldsymbol{\mathcal{S}}_i^{(n)}$ be the two-element algebra, and let $\mathcal{A} = \mathcal{A}_1 \times \cdots \times \mathcal{A}_n$. Consider a k-ary term s in the type $\{f_n\}$ of $\boldsymbol{\mathcal{S}}^{(n)}$. Since $\boldsymbol{\mathcal{S}}_i^{(n)}$ is equivalent to $\boldsymbol{Set}$, s induces a projection on A_i, for $i = 1, \ldots, n$. It follows that s induces an operation on A that depends on at most n variables. On the other hand, f_n defines a term function on A that depends exactly on n variables.

That is, all terms of $\boldsymbol{\mathcal{S}}^{(n)}$ induce at most n-ary term functions on A, and some of them induce n-ary term functions. Since the same statement, with m instead of n, applies for $\boldsymbol{\mathcal{S}}^{(m)}$ for $m < n$, $\mathcal{A} = (A, f_n)$ cannot be interpreted in $\boldsymbol{\mathcal{S}}^{(m)}$. This proves that $\boldsymbol{\mathcal{S}}^{(n)}$ and $\boldsymbol{\mathcal{S}}^{(m)}$ are non-equivalent. □

Proof of Example 6. Let $\mathcal{A} = (A, \vee, \wedge, g_n) \in \boldsymbol{RLat}_n$ and $T \in \mathrm{Tol}(\mathcal{A})$. Then T is also a tolerance of the lattice reduct $(A, \vee, \wedge)$, and $\mathrm{Block}(T)$ for the lattice reduct is the same as it is for $\mathcal{A}$. We claim that, for every $B \in \mathrm{Block}(T)$,

$$g_n(B) := \{g_n(b) : b \in B\} \in \mathrm{Block}(T). \tag{7}$$

By Zorn's lemma, there is a $C \in \mathrm{Block}(T)$ such that $\{g_n(b) : b \in B\} \subseteq C$. Since $g_n^{-1} = g_n^{n-1}$ preserves T, $\{g_n^{-1}(c) : c \in C\}^2 \subseteq T$. This together with $B \subseteq \{g_n^{-1}(c) : c \in C\}$ and $B \in \mathrm{Block}(T)$ yields that $B = \{g_n^{-1}(c) : c \in C\}$. Therefore, $g_n(B) = C \in \mathrm{Block}(T)$, proving (7).

For the lattice operations, B in (1) is uniquely determined since $\boldsymbol{Lat}$ is (strongly) tolerance factorable by G. Czédli [3]. By (7), the same holds for g_n. Thus $\mathcal{A}/T$ makes sense. $(A/T, \vee, \wedge)$ is a lattice since $\boldsymbol{Lat}$ is strongly tolerance factorable. We conclude from (7) that g_n is a permutation on $\mathcal{A}/T$, whose n-th power is the identity map. Finally, assume that $B \vee C = D$ in $\mathcal{A}/T$; the case of the meet is similar. Then, by (7) and $\{b \vee c : b \in B, c \in C\} \subseteq D$,

$$\begin{aligned}\{x \vee y : x \in g_n(B),\ y \in g_n(C)\} &= \{g_n(b) \vee g_n(c) : b \in B,\ c \in C\}\\ &= \{g_n(b \vee c) : b \in B,\ c \in C\} \subseteq \{g_n(d) : d \in D\} = g_n(D).\end{aligned}$$

Hence $g_n(B) \vee g_n(C) = g_n(D)$, that is, g_n is an automorphism of $(A/T, \vee, \wedge)$. Therefore, $\boldsymbol{RLat}_n$ is strongly tolerance factorable. It has proper tolerances since so has $\boldsymbol{Lat}$, which is equivalent to the subvariety $\boldsymbol{RLat}_1$ of $\boldsymbol{RLat}_n$.

The boolean lattice with n atoms allows an automorphism of order n but no automorphism of higher order is possible. This implies easily that $\boldsymbol{RLat}_m$ is not equivalent to $\boldsymbol{RLat}_k$ if $m \neq k$. Since $\boldsymbol{RLat}_n$ is congruence distributive, it is not equivalent to $\boldsymbol{\mathcal{S}}^{(m)}$. □

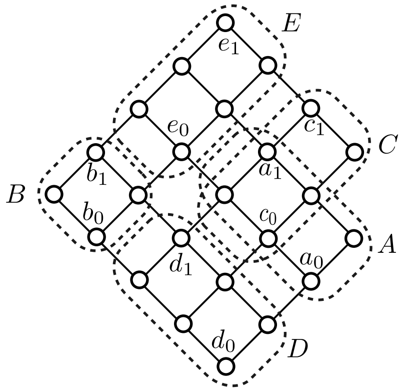

FIGURE 1. L and the blocks of T

Proof of Example 7. Since s takes care of independence, Examples 5 and 6 together with Theorem 4 yield that $\boldsymbol{C}_{mn}$ is strongly tolerance factorable and it has proper tolerances. Suppose for a contradiction that $(m, n) \neq (u, v)$ but $\boldsymbol{C}_{mn}$ is equivalent to $\boldsymbol{C}_{uv}$. If $m < u$, then the 2^m-element boolean lattice with a rotation of order m is a reduct of a member of $\boldsymbol{C}_{mn}$, while no automorphism of order u is possible on a boolean lattice of size 2^m. This contradiction implies that $m = u$. Hence $n \neq v$. Let, say, $v < n$. Take the 2^n-element $\mathcal{A} \in (\boldsymbol{\mathcal{S}}^{(n)})^m \subseteq \boldsymbol{C}_{mn}$ for which all the $\mathcal{A}_i$ in Proposition 1(1) are 2-element. Clearly, all $\boldsymbol{C}_{mn}$-terms induce unary term functions on A and also on any homomorphic image of $\mathcal{A}$. By the assumption, there is a $\boldsymbol{C}_{mv}$-structure $\mathcal{B}$ on the set A such that $\mathcal{B}$ and $\mathcal{A}$ have the same term functions. By the definition of $\boldsymbol{C}_{mv}= \boldsymbol{C}_{uv}$, $\mathcal{B}$ is (isomorphic to) $\mathcal{C} \times \mathcal{D}$, where $\mathcal{C} \in {}^{v}(\boldsymbol{RLat}_m)$ and $\mathcal{D} \in (\boldsymbol{\mathcal{S}}^{(v)})^m$. However, $\mathcal{C}$, a homomorphic image of $\mathcal{B}$, is one-element since otherwise $\mathcal{B}$ would have non-unary term functions. Hence, the term functions of $\mathcal{B}$ are the same as those of its $\boldsymbol{\mathcal{S}}^{(v)}$-reduct.

Now, we can obtain a contradiction the same way as in the proof of Example 5: $\mathcal{A}$ has an n-ary term function that depends on all of its variables while all term functions of $\mathcal{B}$ depend on at most v variables. □

Proof of Example 8. Consider the lattice L in Figure 1 as an algebra of $\boldsymbol{TLat}$. A tolerance $T \in \mathrm{Tol}(L)$ is given by its blocks $A = [a_0, a_1], \ldots, E = [e_0, e_1]$. (It is easy to check, and it follows even more easily from G. Czédli [3, Theorem 2], that T is a tolerance.) Since

$$\{t_\vee(x, y, z) : x \in A,\ y \in B,\ z \in C\} = [c_0, a_1],$$

this set is a subset of two distinct blocks, A and C. Hence $\boldsymbol{TLat}$ is not tolerance factorable. The rest is trivial. □

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Ivan Chajda

Palacký University Olomouc, Department of Algebra and Geometry, 17. listopadu 12, 771 46 Olomouc, Czech Republic
e-mail: `ivan.chajda@upol.cz`

Gábor Czédli

University of Szeged, Bolyai Institute, Szeged, Aradi vértanúk tere 1, Hungary 6720
e-mail: `czedli@math.u-szeged.hu`
URL: `http://www.math.u-szeged.hu/~czedli/`

Radomír Halaš

Palacký University Olomouc, Department of Algebra and Geometry, 17. listopadu 12, 771 46 Olomouc, Czech Republic
e-mail: `radomir.halas@upol.cz`