

LENGTH-PRESERVING EXTENSIONS OF A SEMIMODULAR LATTICE BY LOWERING A JOIN-IRREDUCIBLE ELEMENT

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Dedicated to the memory of my father, György

ABSTRACT. We extend the bijective correspondence between finite semimodular lattices and Faigle geometries to an analogous correspondence between semimodular lattices of finite lengths and a larger class of geometries. As the main application, we prove that if e is a join-irreducible element of a semimodular lattice L of finite length and $h < e$ in L such that e does not cover h , then e can be “lowered” to a covering of h by taking a length-preserving semimodular extension K of L but not changing the rest of join-irreducible elements. With the help of our “lowering construction”, we prove a general theorem on length-preserving semimodular extensions of semimodular lattices. This theorem implies earlier results proved by G. Grätzer and E. W. Kiss (1986), M. Wild (1993), and G. Czédli and E. T. Schmidt (2010) on extensions to geometric lattices, and an unpublished result of E. T. Schmidt. Our approach offers shorter proofs of these results than the original ones.

1. INTRODUCTION AND GOALS

This paper is intended to be self-contained for readers familiar with the rudiments of lattice theory, including that any semimodular lattice of finite length satisfies the *Jordan–Hölder chain condition* (JHCC), that is, any two maximal chains in an interval have the same length. The paper adheres to the convention that

every lattice occurring here is a semimodular lattice of finite length (1.1)

even when this is not repeated. For brevity, we often call these lattices (1.1)-lattices.

1.1. First goal: extending the concept of Faigle geometries. For a (1.1)-lattice L , $J(L)$ will always stand for the poset (partially ordered set) of (nonzero) join-irreducible elements of L . A finite semimodular lattice L is well described by a finite geometry on $J(L)$, which we call the *Faigle geometry* associated with L ; see Faigle [11], and see also Czédli [6] for a recent revisiting. By extending U. Faigle’s 1980 result, our first goal is to define a larger class of geometries that are in a “canonical ” bijective correspondence with (1.1)-lattices.

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1.2. Second goal: lowering a join-irreducible element. For a (1.1)-lattice L , let $\text{At}(L) := \{a \in L : 0 \prec a\}$ denote the *set of atoms* of L . If, in addition to (1.1), $J(L) = \text{At}(L)$, then L is a *geometric lattice*. By a theorem for Dilworth, see [3, page 125], each finite lattice L can be embedded into a finite geometric lattice K^* . Wild [19] observed that, *implicitly*, that Dilworth’s *proof* yields a length-preserving embedding if L is semimodular; see pages 125–131 in P. Crawley and R.P. Dilworth [3]. Prior to M. Wild’s observation, Grätzer and Kiss [14] proved in 1986 that each finite semimodular lattice L has a *length-preserving embedding* (in other words, a *cover-preserving $\{0, 1\}$ -embedding*) into a finite geometric lattice K^* . In 1993, Wild [19] went further by constructing a length-preserving geometric extension K^* of L with the additional property that $|\text{At}(K^*)| = |J(L)|$. In 2010, Czédli and Schmidt [8] extended Wild’s result to (1.1)-lattices. The proofs of these results are either long and complicated or, as in Wild [19], rely on results from matroid theory. Our second goal is to construct K^* in easier steps, by lowering only one non-atom join-irreducible element and lowering it just a little at each step. Theorem 4.3 asserts that such a lowering is always possible.

1.3. Third goal: applications and properties of the lowering construction. The advantage of our lowering construction is that it is simpler than the “big jump” from L to the geometric lattice K^* . Due to this fact, we prove Theorem 5.1, which is a general result on length-preserving extensions of semimodular lattices. Also, we present some lemmas and examples to give more insight into our construct. The two theorems of the paper imply the mentioned results on extensions into geometric lattices, an unpublished result of E. T. Schmidt on higher dimensional rectangular extensions, and a recent result from Czédli and Molkhasi [7]. Our approach to these results is shorter than the original ones except possibly for Wild [19]. Even the classical result that $\text{length}(L) = |J(L)|$ for a finite distributive lattice L becomes a trivial consequence of our theorems.

1.4. Outline. Section 2 generalizes Faigle geometries in (2.3) and proves their simplest properties, see Lemma 2.4. Proposition 3.2 in Section 3 establishes a canonical bijective correspondence between our geometries and (1.1)-lattices. Section 4 gives the exact meaning of lowering a join-irreducible element, see Definition 4.1, presents a related construction in Definition 4.2, and proves Theorem 4.3 to ensure that our construction always gives a lowering. Section 5 proves a general result, Theorem 5.1, on length-preserving extensions of (1.1)-lattices. Section 6 derives some earlier results from our theorems. Section 7 proves some properties of the lowering construction and derives two earlier results as corollaries.

2. A GENERALIZATION OF FAIGLE GEOMETRIES

There is an important but rarely used bijective correspondence between finite semimodular lattices and Faigle geometries; see Faigle [11], in which these geometries are called proper geometries, and see Czédli [6], where these geometries are revisited. Mostly, we go after [6] rather than Faigle [11]. This section is self-contained, so the reader need not look into [6] and Faigle [11]. The plan for this section is simple. Starting from a (1.1)-lattice L and motivated by Faigle geometries, we define a geometry $\text{Geom}(L)$ in a natural way. We prove some properties of $\text{Geom}(L)$ and make these properties axioms. In the next step, we prove some

consequences of the axioms. Finally, we prove that $\text{Geom}(L)$ determines L up to isomorphism.

Next, we mention or introduce some notation. An element p of a (1.1)-lattice L belongs to the poset $J(L) = (J(L), \leq)$ of *join-irreducible elements* of L if p has exactly one lower cover, which we denote by p_{*L} . For a poset $P = (P, \leq_P)$ and $u \in P$, we let

$$\begin{aligned} \downarrow_P u &:= \{x \in P : x \leq u\}, \quad \Downarrow_P u := \{x \in P : x < u\}, \text{ and,} \\ \text{dually, } \uparrow_P u &:= \{x \in P : x \geq u\}, \quad \Uparrow_P u := \{x \in P : x > u\}. \end{aligned} \quad (2.1)$$

Note that if, say $u \in P_1$ and P_1 is a subposet of P_2 then, say, $\Downarrow_{P_1} u$ can be different from $\Downarrow_{P_2} u$. Therefore, we rarely omit the subscript when using notation (2.1).

Definition 2.1. For a semimodular lattice L of finite length, let P_L be the poset $J(L)$ and let $\mathcal{F}_L := \{P_L \cap \downarrow_L x : x \in L\}$. Then $\mathcal{F}_L^* := (P_L, \mathcal{F}_L)$, also denoted by $\text{Geom}(L)$, is the *geometry* of L . The elements of \mathcal{F} are called the *flats* of L .

For a set U , we denote the *powerset* $\{X : X \subseteq U\}$ of U by $\text{Pow}(U)$. For a chain C , $\text{length}(C)$ is defined by $|C| = 1 + \text{length}(C)$. For a poset P , $\text{length}(P) := \sup\{\text{length}(C) : C \text{ is a chain in } P\}$. If $X \subseteq P$ such that for every $u \in X$ we have that $\downarrow_P u \subseteq X$, then X is called a *down-set* of P . For $x, y \in P$, $x \prec_P y$ denotes that y covers x in P . Most ingredients of the following lemma and its proof are extracted from Faigle [11] and Czédli [6].

Lemma 2.2. For a semimodular lattice L of finite length, let $\mathcal{F}^* = (P, \mathcal{F})$ stand for $\text{Geom}(L) = (P_L, \mathcal{F}_L)$; see Definition 2.1. Then \mathcal{F}^* has the following properties.

- (FL) P is a poset, $\mathcal{F} \subseteq \text{Pow}(P)$, and the poset (\mathcal{F}, \subseteq) is of finite length.
- (F \cap) $P \in \mathcal{F}$ and \mathcal{F} is \cap -closed, that is, for all $\mathcal{U} \subseteq \mathcal{F}$, we have that $\bigcap \mathcal{U} \in \mathcal{F}$;
- (F \downarrow) Every member of \mathcal{F} is a down-set of P ;
- (Pr) $\emptyset \in \mathcal{F}$ and for each $u \in P$, both $\downarrow_P u$ and $\Downarrow_P u$ belong to \mathcal{F} ; and
- (CP) For any $q \in P$ and $X \in \mathcal{F}$ such that $q \notin X$ and $\Downarrow_P q \subseteq X$, there exists a $Y \in \mathcal{F}$ such that $X \prec_{\mathcal{F}} Y$ and $q \in Y$.

Proof. Since L is of finite length, it is a complete lattice. Hence, we can apply the rule $\bigcap\{P \cap \downarrow_L x_i : i \in I\} = P \cap \downarrow_L \bigwedge\{x_i : i \in I\}$ to obtain that (F \cap) holds in \mathcal{F}^* . Trivially, so does (F \downarrow). Since

$$\text{every } z \in L \text{ is of the form } \bigvee_L (P \cap \downarrow_L z), \quad (2.2)$$

it follows that (\mathcal{F}, \subseteq) is isomorphic to (L, \leq) . Thus, \mathcal{F}^* satisfies (FL). Clearly, $\emptyset = P \cap \downarrow_L 0 \in \mathcal{F}$ and, for any $u \in P$, $\downarrow_P u = P \cap \downarrow_L u \in \mathcal{F}$. By the join-irreducibility of u , $\Downarrow_P u = P \cap \Downarrow_L u = P \cap \downarrow_L u_{*L} \in \mathcal{F}$, whence \mathcal{F}^* satisfies (Pr). To deal with (CP), assume that $X \in \mathcal{F}$ is witnessed by $X = P \cap \downarrow_L x$. Let $u \in P \setminus X$ such that $\Downarrow_P u \subseteq X$. From $\downarrow_P u = P \cap \downarrow_L u_{*L}$, we obtain that $u_{*L} \leq x$ while $u \not\leq_L x$ since $u \notin X$. Hence, $x = x \vee u_{*L} \prec_L x \vee u := y$ by semimodularity. Then $u \in Y := P \cap \downarrow_L y$, and $X \prec_{\mathcal{F}} Y$ since $(\mathcal{F}, \subseteq) \cong (L, \leq)$. Hence, (CP) holds in \mathcal{F}^* , completing the proof. \square

Unless otherwise specified explicitly, the meaning of a *geometry* in this paper is always the following.

$$\begin{aligned} &\text{A geometry is a pair } \mathcal{F}^* = (P, \mathcal{F}) \text{ satisfying} \\ &(\text{FL}), (\text{F}\cap), (\text{F}\downarrow), (\text{Pr}), \text{ and } (\text{CP}). \end{aligned} \quad (2.3)$$

If confusion threatens or just for emphasis, we can occasionally say that \mathcal{F}^* is a *geometry in the sense of* (2.3). From now on, we call (FL), (F \cap), (F \downarrow), (Pr), and

(CP) *axioms* rather than properties. Note that the acronyms (Pr) and (CP), taken from Czédli [6], and (FL) come from “principal”, “covering property”, and “finite length, respectively. For a geometry \mathcal{F}^* , it follows from (Pr) that \mathcal{F} is finite if and only if so is P ; in this case we say that the geometry is finite. The following remark follows trivially from Czédli [6].

Remark 2.3. Out of geometries in the sense of (2.3), the finite ones are the same as Faigle geometries, which were called proper geometries in Faigle [11].

Before establishing a bijective correspondence between (1.1)-lattices and geometries in the sense of (2.3), it is reasonable to explore some properties of these geometries. Observe that axiom (F \cap) means that \mathcal{F} is a *closure system*, in other words, a *Moore family* on P . As usual in case of closure systems, we can take the *closure map* (in other word, the *closure operator*) $\text{cl}_{\mathcal{F}}$ associated with \mathcal{F} as follows.

$$\text{cl}_{\mathcal{F}}: \text{Pow}(P) \rightarrow \text{Pow}(P) \quad \text{is defined by} \quad X \mapsto \bigcap \{Y \in \mathcal{F} : X \subseteq Y\}. \quad (2.4)$$

It is well known (and easy to see) that for any $X \subseteq Y \in \text{Pow}(P)$, we have that $X \subseteq \text{cl}_{\mathcal{F}}(X) = \text{cl}_{\mathcal{F}}(\text{cl}_{\mathcal{F}}(X)) \subseteq \text{cl}_{\mathcal{F}}(Y)$; we will frequently use these properties of closure maps implicitly. For Faigle geometries, the following lemma is known from Czédli [6] and, apart from a slight difference in definitions, also from Faigle [11].

Lemma 2.4. *Let $\mathcal{F}^* = (P, \mathcal{F})$ be a geometry in the sense of (2.3), and assume that $X, Z \in \mathcal{F}$. Then*

- (DC) $X \prec_{\mathcal{F}} Z \iff (\exists u \in P \setminus X) (\downarrow_P u \subseteq X \text{ and } Z = \text{cl}_{\mathcal{F}}(\{u\} \cup X))$. Also,
- (UC) *in (CP), Y is uniquely determined. Furthermore,*
- (CE) $X \preceq_{\mathcal{F}} Z \iff (\exists u \in P) (\downarrow_P u \subseteq X \text{ and } Z = \text{cl}_{\mathcal{F}}(\{u\} \cup X))$.

The acronyms above come from “description of covering”, “unique cover”, and “covers or equals”, respectively. Note that in this paper “ \subset ” always means the conjunction of “ \subseteq ” and “ \neq ”.

Proof of Lemma 2.4. If both Y_1 and Y_2 satisfied the requirements of (CP) and $Y_1 \neq Y_2$, then $Y_1 \cap Y_2 \in \mathcal{F}$, $X \prec_{\mathcal{F}} Y_1$, and $X \prec_{\mathcal{F}} Y_2$ would lead to $q \in Y_1 \cap Y_2 = X$, a contradiction. Thus, we conclude (UC).

To prove the “ \Rightarrow ” part of (DC), assume that $X \prec_{\mathcal{F}} Z$. Since P is of finite length, we can take a minimal element u in $Z \setminus X$. Then $\downarrow_P u \subseteq X$ by the minimality of u , and $u \notin X$. Using that $X \prec_{\mathcal{F}} Z$ and $X \subset \{u\} \cup X \subseteq \text{cl}_{\mathcal{F}}(\{u\} \cup X) \subseteq \text{cl}_{\mathcal{F}}(Z) = Z$, we conclude that $Z = \text{cl}_{\mathcal{F}}(\{u\} \cup X)$, as required. Conversely, to prove “ \Leftarrow ” part, assume the existence of a $u \in P \setminus X$ such that $\downarrow_P u \subseteq X$ and $Z = \text{cl}_{\mathcal{F}}(\{u\} \cup X)$. By (CP) and (UC), there is a unique $Y \in \mathcal{F}$ such that $u \in Y$ and $X \prec_{\mathcal{F}} Y$. Since $X \subset \{u\} \cup X \subseteq \text{cl}_{\mathcal{F}}(\{u\} \cup X) \subseteq \text{cl}_{\mathcal{F}}(Y) = Y$, we obtain that $X \prec_{\mathcal{F}} Y = \text{cl}_{\mathcal{F}}(\{u\} \cup X) = Z$, as required. This proves (DC). Finally, (CE) is a trivial consequence of (DC) since $u \in X \iff \text{cl}_{\mathcal{F}}(\{u\} \cup X) = X$. \square

3. CANONICAL CORRESPONDENCE

This section elaborates the correspondence between (1.1)-lattices and geometries in the sense of (2.3). For geometries $\mathcal{F}_1^* = (P_1, \mathcal{F}_1)$ and $\mathcal{F}_2^* = (P_2, \mathcal{F}_2)$, by an *isomorphism* $\bar{\mu}: \mathcal{F}_1^* \rightarrow \mathcal{F}_2^*$ we mean a map $\bar{\mu}: \mathcal{F}_1 \rightarrow \mathcal{F}_2$ defined by $\bar{\mu}(X) := \{\mu(u) : u \in X\}$ where $\mu: P_1 \rightarrow P_2$ is a poset isomorphism. Next, we define to maps; we will show later that their codomains are correctly given.

Definition 3.1. Let \mathbf{G}^c and \mathbf{L}^f be the class of geometries in the sense of (2.3) and that of (1.1)-lattices, respectively. (The superscript “c” comes from “class”.) We consider Geom , which is given in Definition 2.1, a map $\text{Geom}: \mathbf{L}^f \rightarrow \mathbf{G}^c$. Also, we define a map $\text{Lat}: \mathbf{G}^c \rightarrow \mathbf{L}^f$ by the rule that for $\mathcal{F}^* = (P, \mathcal{F}) \in \mathbf{G}^c$, we let $\text{Lat}(\mathcal{F}^*) := (\mathcal{F}, \subseteq)$, it is the *lattice* associated with \mathcal{F}^* .

Proposition 3.2. *For $L \in \mathbf{L}^f$ and $\mathcal{F}^* = (P, \mathcal{F}) \in \mathbf{G}^c$, we have that $\text{Geom}(L) \in \mathbf{G}^c$, $\text{Lat}(\mathcal{F}^*) \in \mathbf{L}^f$, $\text{Lat}(\text{Geom}(L)) \cong L$, and $\text{Geom}(\text{Lat}(\mathcal{F}^*)) \cong \mathcal{F}^*$.*

Before the proof, note that Proposition 3.2 allows us to say that (1.1)-lattices and geometries in the sense of (2.3) are *canonically equivalent*. Also, we can use $\text{Geom}(L)$ if $L \in \mathbf{L}^f$ if given, and we can define an $L \in \mathbf{L}^f$ by defining an $\mathcal{F}^* \in \mathbf{G}^c$ and letting $L := \text{Lat}(\mathcal{F}^*)$. It will frequently be implicit that we pass from geometries to lattices and back. The advantage of using geometries is well shown by the proof of Theorem 4.3 since a poset defined by a geometry is automatically a lattice and it is semimodular.

Proof of Proposition 3.2. If $L \in \mathbf{L}^f$, then $\text{Geom}(L) \in \mathbf{G}^c$ by Lemma 2.2. Next, let $\mathcal{F}^* \in \mathbf{G}^c$; we need to show that $\text{Lat}(\mathcal{F}^*) = (\mathcal{F}, \subseteq)$ belongs to \mathbf{L}^f . Since the members of any closure system are known to form a complete lattice, $\text{Lat}(\mathcal{F}^*)$ is a complete lattice. Note that for $X, Y \in \text{Lat}(\mathcal{F}^*)$, that is, for $X, Y \in \mathcal{F}$, we have that $X \wedge Y = X \cap Y$ and $X \vee Y = \text{cl}_{\mathcal{F}}(X \cup Y)$. By (FL), $\text{Lat}(\mathcal{F}^*)$ is of finite length. To prove that it is semimodular, assume that $X, Y \in \mathcal{F}$ such that $X \wedge Y \prec_{\mathcal{F}} X$; we need to show that $Y \prec_{\mathcal{F}} X \vee Y$. Clearly, $Y \subset X \vee Y$. (DC), see Lemma 2.4, allows us to pick a $u \in P \setminus (X \cap Y)$ such that $\downarrow_P u \subseteq X \cap Y$ and $X = \text{cl}_{\mathcal{F}}(\{u\} \cup (X \cap Y))$. Observe that $X \vee Y = \text{cl}_{\mathcal{F}}(X \cup Y) = \text{cl}_{\mathcal{F}}(\text{cl}_{\mathcal{F}}(\{u\} \cup (X \cap Y)) \cup Y) = \text{cl}_{\mathcal{F}}(\{u\} \cup (X \cap Y) \cup Y) = \text{cl}_{\mathcal{F}}(\{u\} \cup Y)$. If $u \in Y$, then $X = \text{cl}_{\mathcal{F}}(\{u\} \cup (X \cap Y)) \subseteq \text{cl}_{\mathcal{F}}(Y) = Y$ leads to $X = X \wedge Y$, a contradiction. Hence, $u \notin Y$. Also, $\downarrow_P u \subseteq X \cap Y \subseteq Y$, whence (DC) yields that $Y \prec_{\mathcal{F}} \text{cl}_{\mathcal{F}}(\{u\} \cup Y) = X \vee Y$, as required. This shows that $\text{Lat}(\mathcal{F}^*) \in \mathbf{L}^f$. By (2.2), it is clear that the map

$$\varphi_{\text{lat}}: L \mapsto \text{Lat}(\text{Geom}(L)) \text{ defined by } x \mapsto J(L) \cap \downarrow_L x$$

is a lattice isomorphism. Using the notation $\mathcal{F}'^* = (P', \mathcal{F}') := \text{Geom}(\text{Lat}(\mathcal{F}^*))$, let

$$\overline{\varphi}_{\text{geo}}: \mathcal{F}^* \rightarrow \mathcal{F}'^* \text{ where } \varphi_{\text{geo}}: P \rightarrow P' \text{ is defined by } u \mapsto \downarrow_{P'} u.$$

In a straightforward but somewhat tedious way, by the same argument as in the proof of Czédli [6, Theorem 2.5(D)], it follows that $\overline{\varphi}_{\text{geo}}$ is an isomorphism. \square

4. LOWERING A JOIN-IRREDUCIBLE ELEMENT

In addition to geometries in the sense of (2.3), the basic concept of the paper is given in the following definition; this concept is visually illustrated by Figures 1 and 3. Recall that for $e \in J(L)$, the unique lower cover of e in L is denoted by e_{*L} . For $e' \in J(K)$, we write e'_{*K} rather than e'_{*K} or $(e')_{*K}$. When we intend to “lower” e to a new join-irreducible element e' of an extended lattice K , to be constructed, we would like to decide where e' should be in K . A reasonable way to do so is to specify which element of L should be e'_{*K} .

Definition 4.1. For a (1.1)-lattice L , $e \in J(L)$, and a lattice K , we say that K is *obtained from L by lowering e* if the following hold.

- (i) K is also a (1.1)-lattice, L is a sublattice of K , and $\text{length}(K) = \text{length}(L)$.

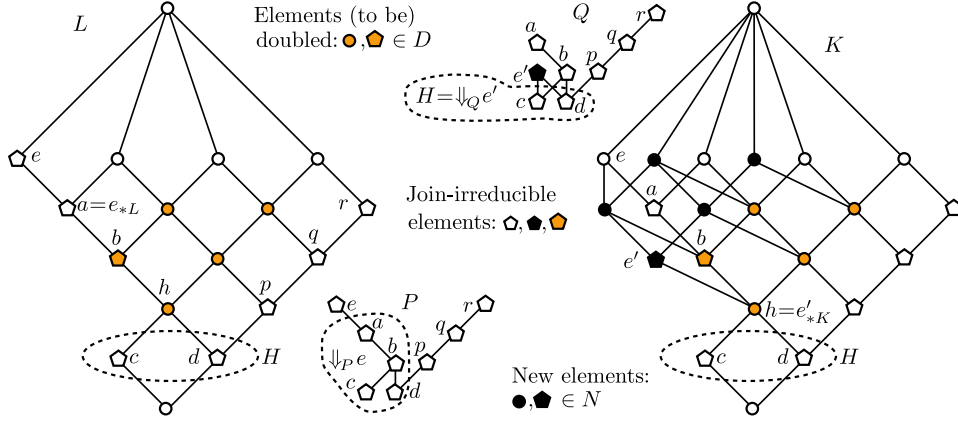


FIGURE 1. Illustrating the lowering construction $L(e_{*L}h)$; see Definition 4.2

- (ii) With the notation $P := J(L)$ and $Q := J(K)$, we have that $P \setminus Q = \{e\}$ and $Q \setminus P = \{e'\}$ for a unique $e' \in Q$.
- (iii) With $h := e'_{*K}$, we have that $h \in L$ and $h < e_{*L}$.

If we want to say more about the relation between K and L , we use the terminology that K is an *extension of L that lowers e to a cover of h* .

For example, L and K in Figure 1 show how we can lower $e \in J(L)$ to a cover of $h \in L$; the dotted ovals, P , and Q are not relevant at present. Remark 4.5 will point out why L and K in the figure are semimodular lattices. If e is an atom (that is, $0 \prec_L e$), then $h < e_{*L} = 0$ is impossible and e cannot be lowered. We are going to prove that if $e \in J(L)$ is *not* an atom, then there exists an extension K of L that lowers e to a cover of h . First, in harmony with Figure 1, we describe a construction. The set of atoms of L will be denoted by $\text{At}(L)$; note that $\text{At}(L) \subseteq J(L)$.

Definition 4.2. Let L be a semimodular lattice of finite length, $e \in J(L) \setminus \text{At}(L)$, $h \in L$, and $h < e_{*L}$. We define a lattice $L(e_{*L}h)$ as follows. With

$$D := \{x \in \uparrow_L h : (\forall y \in L) (x \leq_L y \Rightarrow e \not\leq y)\}. \text{ Naturally, we assume} \quad (4.1)$$

$$\text{that } N := \{x^\blacktriangle : x \in D\} \text{ is disjoint from } L, \text{ and we let } K := L \cup N. \quad (4.2)$$

For $y = x^\blacktriangle \in N$, we often denote x by y^∇ . We define a relation \leq_K on K by

$$x \leq_K y \stackrel{\text{def}}{\iff} \begin{cases} x, y \in L \text{ and } x \leq_L y & \text{or} \\ x, y \in N \text{ and } x^\nabla \leq_L y^\nabla & \text{or} \\ x \in L, y \in N, \text{ and } x \leq_L y^\nabla & \text{or} \\ x \in N, y \in L, \text{ and } x^\nabla \vee_L e \leq_L y. & \end{cases} \quad (4.3)$$

Finally, we define $L(e_{*L}h)$ as (K, \leq_K) . (It needs a proof that it is a lattice.)

For $x \in D$, x^\blacktriangle is a new element, the “lifted variant” of x . The set D is the set of elements that will be “doubled” while N is the set of the “new” elements; this explains their notations. Based on Definitions 4.1 and 4.2, we are in the position to formulate the first of the two theorems of the paper.

Theorem 4.3. Assume that L is a semimodular lattice of finite length, e is a join-irreducible element of L that is not an atom of L , and $h \in L$ is less than the unique

lower cover e_{*L} of e in L . Then, in the sense of Definition 4.1, $K := L(e_{*L} \searrow h)$ from Definition 4.2 is an extension of L that lowers e to a cover of h .

The assumption that $e \notin \text{At}(L)$ is only for emphasis; indeed, this assumption follows from another one, $h < e_{*L}$. Although most applications of Theorem 4.3 do not need the following lemma, we formulate it here since we can economically prove it jointly with the theorem.

Lemma 4.4. *With the notation and assumptions of Theorem 4.3, Definitions 4.1, and Definition 4.2, we have that $e' := h^\blacktriangle$ and*

$$x \wedge_K y = \begin{cases} x \wedge_L y, & \text{if } x, y \in L, \\ (x^\nabla \wedge_L y^\nabla)^\blacktriangle, & \text{if } x, y \in N, \\ x = (x^\nabla \wedge_L y)^\blacktriangle, & \text{if } x \in N \text{ and } e \leq y \in L, \\ x^\nabla \wedge_L y, & \text{if } x \in N \text{ and } e \not\leq y \in L, \\ y = (x \wedge_L y^\nabla)^\blacktriangle, & \text{if } e \leq x \in L \text{ and } y \in N, \\ x \wedge_L y^\nabla, & \text{if } e \not\leq x \in L \text{ and } y \in N \end{cases} \quad (4.4)$$

for any $x, y \in K = L(e_{*L} \searrow h)$.

Before proving Theorem 4.3 and the lemma above, note the following.

Remark 4.5. The theory of planar semimodular lattices, see the appendix of (the arXiv version of) Czédli [6] with a four dozen element list of publications on this topic, was crucial to find an appropriate construction. Furthermore, it follows immediately from the structural description given in Czédli [5] or in Czédli and Schmidt [9] that L in each of our figures is a semimodular lattice. (The semimodularity of K in the same figure follows from Lemma 7.7.)

Proof of Theorem 4.3 and Lemma 4.4. Let $\mathcal{F}^* = (P, \mathcal{F}) = \text{Geom}(L) = (P_L, \mathcal{F}_L)$ be the geometry of L , see Definition 2.1. In addition to the usual lattice theoretical tools, the canonical correspondence provided by Proposition 3.2 allows us to translate some lattice theoretical objects like (4.1)–(4.2) to the geometrical language, and we can benefit from the axioms (FL)–(CP) as well as from (DC)–(CE) from Lemma 2.4. Let $H := P \cap \downarrow_L h \in \mathcal{F}$; note that $h = \bigvee_L H$. The translation of (4.1) is trivial but we have to use (4.3) to rewrite (4.2) into (4.6) below. (The role of (4.3) is to guarantee that the set theoretical inclusion “ \subseteq ” corresponds to “ \leq_K ”.)

$$\mathcal{D} := \{X \in \mathcal{F} : H \subseteq X \text{ and } (\forall Y \in \mathcal{F}) (X \preceq_{\mathcal{F}} Y \Rightarrow e \notin Y)\}, \quad (4.5)$$

$$\mathcal{N} := \{\{e\} \cup X : X \in \mathcal{D}\} \quad \text{and} \quad \mathcal{G} := \mathcal{F} \cup \mathcal{N}. \quad (4.6)$$

Observe that (CE) from Lemma 2.4 allows us to rewrite the definition of \mathcal{D} to

$$\mathcal{D} := \{X \in \mathcal{F} : H \subseteq X \text{ and } (\forall u \in P) (\downarrow_P u \subseteq X \Rightarrow e \notin \text{cl}_{\mathcal{F}}(\{u\} \cup X))\}. \quad (4.7)$$

For convenience, let us agree that for $X \in \text{Pow}(P)$, we let

$$X^{+e} := \{e\} \cup X \quad \text{and} \quad X^{-e} := X \setminus \{e\}. \quad (4.8)$$

Note that for $X \in \mathcal{N}$ and $Y \in \mathcal{D}$, we have that $X^{-e} \in \mathcal{D}$ and $Y^{+e} \in \mathcal{N}$. The canonical correspondence between L and \mathcal{F}^* will often be used implicitly. As opposed to Definition 4.1(ii), we let $R := P = J(L)$; at present, it is only an underlying set. We define a relation \leq_R on R by

$$y <_R x \stackrel{\text{def}}{\iff} \begin{cases} y <_P x \neq e, & \text{or} \\ y \in H \text{ and } x = e. \end{cases} \quad (4.9)$$

If $x, y, z \in R$ such that $x <_R y <_R z$, then we easily obtain that $x <_R z$; either since $e \notin \{x, y, z\}$, or since $H \in \mathcal{F}$ is a down-set in P and $H \subseteq \downarrow_P e$. Hence, $R = (R, \leq_R)$ is a poset. With this poset and \mathcal{G} from (4.6), we let $\mathcal{G}^* := (R, \mathcal{G})$; we are going to prove that \mathcal{G}^* is a geometry in the sense of (2.3). Observe that

$$\text{if } X \in \mathcal{F} \text{ and } \downarrow_P e \subseteq X, \text{ then } X \notin \mathcal{D}. \quad (4.10)$$

Indeed, $H \subseteq \downarrow_P e$, $|\downarrow_P e \setminus \downarrow_P e| = |\{e\}| = 1$ implies that $\downarrow_P e \prec_{\mathcal{F}} \downarrow_P e$, and $\downarrow_P e \prec_{\mathcal{F}} \downarrow_P e \ni e$ gives that $\downarrow_P e \notin \mathcal{D}$. Hence, (4.10) follows from (4.5).

We claim that $\mathcal{F} \cap \mathcal{N} = \emptyset$. For the sake of contradiction, suppose that there is a Y belonging to $\mathcal{F} \cap \mathcal{N}$. Then $Y \in \mathcal{N}$ gives that $e \in Y$ and $Y^{-e} \in \mathcal{D}$. However, then $Y \in \mathcal{F}$ and $(F\downarrow)$ lead to $\downarrow_P e \subseteq Y$, whence $\downarrow_P e \subseteq Y^{-e} \in \mathcal{D}$ contradicts (4.10). This shows that $\mathcal{F} \cap \mathcal{N} = \emptyset$. A subset \mathcal{W}_1 of a set \mathcal{W}_2 of sets is *convex* if for all $X, Z \in \mathcal{W}_1$ and $Y \in \mathcal{W}_2$, $X \subseteq Y \subseteq Z$ implies that $Y \in \mathcal{W}_1$. We claim that

$$\begin{aligned} H \in \mathcal{D}, \mathcal{D} \text{ is a convex subset of } \mathcal{F}, \text{ and each} \\ \text{of } \mathcal{N} \text{ and } \mathcal{D} \cup \mathcal{N} \text{ is a convex subset of } \mathcal{G}. \end{aligned} \quad (4.11)$$

To show this, observe that $h < e_{*L} \prec_L e$ gives that $e \notin H$ and (by the canonical correspondence) the length of the interval $[H, \downarrow_P e]_{\mathcal{F}}$ is at least 2. If we had that $e \in Y$ and $H \prec_{\mathcal{F}} Y$ for some $Y \in \mathcal{F}$, then $(F\downarrow)$ would give that $[H, \downarrow_P e]_{\mathcal{F}}$ is a subinterval of the prime interval $[H, Y]_{\mathcal{F}}$, which would be a contradiction. Hence, $H \in \mathcal{D}$. Clearly, H is the least member of $\mathcal{D} \cup \mathcal{N}$. Thus, to show the convexity of \mathcal{D} , assume that $H \subseteq X \subseteq Y$, $X \in \mathcal{F}$, and $Y \in \mathcal{D}$. If we had a $u \in P$ with $\downarrow_P u \subseteq X$ and $e \in \text{cl}_{\mathcal{F}}(\{u\} \cup X)$, then $Y \in \mathcal{D}$ together with $\downarrow_P u \subseteq Y$ and $e \in \text{cl}_{\mathcal{F}}(\{u\} \cup Y)$ would contradict (4.7). Hence, again by (4.7), $X \in \mathcal{D}$, showing that \mathcal{D} is a convex subset of \mathcal{F} . Now assume that $X \subseteq Y \subseteq Z$, $X, Z \in \mathcal{N}$, and $Y \in \mathcal{G}$. Then $X^{-e}, Z^{-e} \in \mathcal{D}$. By $X \subseteq Y$, we have that $e \in Y$, whence $Y = (Y^{-e})^{+e}$. Also, $X^{-e} \subseteq Y^{-e} \subseteq Z^{-e}$. The convexity of \mathcal{D} gives that $Y^{-e} \in \mathcal{D}$, whereby $Y = (Y^{-e})^{+e} \in \mathcal{N}$. Thus, \mathcal{N} is convex. Next, assume that $H \subseteq X \subseteq Y$, $X \in \mathcal{G}$, and $Y \in \mathcal{D} \cup \mathcal{N}$; we need to show that $X \in \mathcal{D} \cup \mathcal{N}$. If $Y \in \mathcal{D}$, then $e \notin Y \supseteq X$ yields that $X \in \mathcal{F}$, whereby the convexity of \mathcal{D} in \mathcal{F} implies that $X \in \mathcal{D} \subseteq \mathcal{D} \cup \mathcal{N}$, as required. Hence, we can assume that $Y \in \mathcal{N}$. We can assume that $X \not\subseteq Y^{-e}$ since otherwise $X \in \mathcal{D} \subseteq \mathcal{D} \cup \mathcal{N}$ by the convexity of \mathcal{D} again. Then $e \in X$, but $X \in \mathcal{F}$ is impossible since it would lead to $\downarrow_P e \subseteq X^{-e} \subseteq Y^{-e} \in \mathcal{D}$, contradicting (4.10). Hence, $X \in \mathcal{G} \setminus \mathcal{F} = \mathcal{N} \subseteq \mathcal{D} \cup \mathcal{N}$, proving (4.11). Since $h < e_{*L}$, $e_{*L} = \bigvee_L(\downarrow_P e_{*L}) = \bigvee_L(\downarrow_P e)$, and $h = \bigvee_L H$ yield that $H \subset \downarrow_P e$, it follows immediately from (4.9) that, for $u \in R \setminus \{e\}$ and $v, w \in R$,

$$\begin{aligned} \downarrow_R u &= \downarrow_P u, \quad \downarrow_R u = \downarrow_P u, \quad \downarrow_R e = H \subset \downarrow_P e, \\ \downarrow_R e &= H^{+e} \subset \downarrow_P e, \quad \text{and} \quad v \leq_R w \Rightarrow v \leq_P w. \end{aligned} \quad (4.12)$$

Hence, \mathcal{G}^* satisfies $(F\downarrow)$ since so does \mathcal{F}^* . We know that $\text{length}(\mathcal{F}) = \text{length}(L)$ is finite. Let $\mathcal{C} \subseteq \mathcal{G}$ be a chain. Then $|\mathcal{C} \cap \mathcal{F}| \leq 1 + \text{length}(\mathcal{F})$. Since \mathcal{N} and \mathcal{D} are order isomorphic (with respect to \subseteq), $|\mathcal{C} \cap \mathcal{N}| \leq 1 + \text{length}(\mathcal{D}) \leq 1 + \text{length}(\mathcal{F})$. Hence, $|\mathcal{C}| \leq 2 + 2 \cdot \text{length}(\mathcal{F})$, showing that (FL) holds for \mathcal{G}^* .

Since $(F\cap)$ holds for \mathcal{F}^* and H is the least element of $\mathcal{D} \cup \mathcal{N}$, it follows easily from (4.11), $H \subseteq \downarrow_P e$, and the validity of $(F\downarrow)$ for \mathcal{F}^* that, for $X, Y \in \mathcal{G}$,

$$X \cap Y = \begin{cases} X \cap Y \in \mathcal{F}, & \text{if } X, Y \in \mathcal{F}, \\ (X^{-e} \cap Y^{-e})^{+e} \in \mathcal{N}, & \text{if } X, Y \in \mathcal{N}, \\ X = (X^{-e} \cap Y)^{+e} \in \mathcal{N}, & \text{if } X \in \mathcal{N} \text{ and } e \in Y \in \mathcal{F}, \\ X^{-e} \cap Y \in \mathcal{F}, & \text{if } X \in \mathcal{N} \text{ and } e \notin Y \in \mathcal{F}, \\ Y = (X \cap Y^{-e})^{+e} \in \mathcal{N}, & \text{if } e \in X \in \mathcal{F} \text{ and } Y \in \mathcal{N}, \\ X \cap Y^{-e} \in \mathcal{F}, & \text{if } e \notin X \in \mathcal{F} \text{ and } Y \in \mathcal{N}. \end{cases} \quad (4.13)$$

In particular, (4.13) implies that \mathcal{G}^* satisfies $(F\cap)$. Since $H \in \mathcal{D}$ by (4.11), we obtain from (4.12) that $\downarrow_R e \in \mathcal{G}$ and $\downarrow_R e = H^{+e} \in \mathcal{N} \subseteq \mathcal{G}$. These facts and (4.12) yield that \mathcal{G}^* satisfies (Pr).

Next, to prove that \mathcal{G}^* satisfies (CP), assume that

$$X \in \mathcal{G}, q \in R \setminus X, \text{ and } \downarrow_R q \subseteq X. \quad (4.14)$$

We need to find a $Y \in \mathcal{G}$ such that $X \prec_{\mathcal{G}} Y \ni q$. There are three cases to consider.

Case 1. In addition to (4.14), we assume that $q = e$. Since $e = q \notin X$, we have that $X \in \mathcal{F}$. If $X \in \mathcal{D}$, then $X \prec_{\mathcal{F}} X^{+e} =: Y \ni e = q$ since $|Y \setminus X| = 1$. So assume that $X \in \mathcal{F} \setminus \mathcal{D}$. We know that $H = \downarrow_R e = \downarrow_R q \subseteq X$. This fact, $e \notin X$, and (4.5) give a $Y \in \mathcal{F}$ with $X \prec_{\mathcal{F}} Y \ni e = q$. We only need to show that $X \prec_{\mathcal{G}} Y$. For the sake of contradiction, suppose that $X \subset Z \subset Y$ for some $Z \in \mathcal{N}$. Since $e \notin X \notin \mathcal{D}$, $H \subseteq X \subseteq Y^{-e} \in \mathcal{D}$ contradicts (4.11). Thus, $X \prec_{\mathcal{G}} Y$, as required.

Case 2. In addition to (4.14), we assume that $q \neq e$ and $X \in \mathcal{F}$. Then (4.12) gives that $\downarrow_P q = \downarrow_R q \subseteq X$. Applying (CP) to \mathcal{F}^* , we obtain a $Y \in \mathcal{F}$ such that $X \prec_{\mathcal{F}} Y \ni q$. So it suffices to show that $X \prec_{\mathcal{G}} Y$. For the sake of contradiction, suppose that this is not so, and pick a $Z \in \mathcal{N}$ such that $X \subset Z \subset Y$. We know from (4.6) that $Z^{-e} \in \mathcal{D}$. If $e \in X$, then $X \in \mathcal{F}$ and $(F\downarrow)$ give that $\downarrow_P e \subseteq X$, whence $\downarrow_P e \subseteq X^{-e} \subseteq Z^{-e} \in \mathcal{D}$ contradicts (4.10). Thus, $e \notin X$, and we have that $X \subseteq Z^{-e} \subseteq Z \subset Y$. Since $Z^{-e} \in \mathcal{D} \subseteq \mathcal{F}$ and $X \prec_{\mathcal{F}} Y$, it follows that $X = Z^{-e} \in \mathcal{D}$. But then (4.5) and $X \prec_{\mathcal{F}} Y$ lead to $e \notin Y$, contradicting $e \in Z \subseteq Y$. Thus, $X \prec_{\mathcal{G}} Y$, as required.

Case 3. In addition to (4.14), we assume that $q \neq e$ and $X \in \mathcal{N}$. By (4.12), $\downarrow_P q \subseteq X$. Observe that $e \notin \downarrow_P q$ since otherwise

$$\downarrow_P e \subseteq \downarrow_P q \stackrel{(4.12)}{=} \downarrow_R q \subseteq X \in \mathcal{N} \quad (4.15)$$

together with $e \notin \downarrow_P e$ and $X^{-e} \in \mathcal{D} \subseteq \mathcal{F}$ would give that $\downarrow_P e = \downarrow_P e \setminus \{e\} \subseteq X \setminus \{e\} = X^{-e} \in \mathcal{D}$, contradicting (4.10). Combining $e \notin \downarrow_P q$ and (4.15), we obtain that $\downarrow_P q \subseteq X^{-e} \in \mathcal{D} \subseteq \mathcal{F}$, and we know from (4.14) that $q \notin X^{-e}$. Thus, applying (CP) to \mathcal{F}^* , we can pick a $Y \in \mathcal{F}$ such that $X^{-e} \prec_{\mathcal{F}} Y \ni q$. Note that $\mathcal{D} \ni X^{-e} \prec_{\mathcal{F}} Y$ and (4.5) imply that $e \notin Y$. If $Y \in \mathcal{D}$, then (4.11) easily implies that $X \prec_{\mathcal{G}} Y^{+e} \ni q$, and Y^{+e} does the job instead of Y . Thus, we can assume that $Y \in \mathcal{F} \setminus \mathcal{D}$. Combining this with $H \subseteq X^{-e} \subset Y$ and (4.5), we can pick an $E \in \mathcal{F}$ such that $Y \preceq_{\mathcal{F}} E \ni e$. In fact, $Y \prec_{\mathcal{F}} E$ since $e \notin Y$ but $e \in E$. Observe that $q \in Y \subseteq E$, $X = \{e\} \cup X^{-e} \subseteq E \cup Y = E$, but $X \neq E$ since $q \in E \setminus X$. That is, $X \subset E \ni q$. Hence, to complete Case 3, we only need to show that $X \prec_{\mathcal{G}} E$. For the sake of contradiction, suppose that there is an $S \in \mathcal{G}$ such that

$X \subset S \subset E$. Note that $e \in S$ since $X \in \mathcal{N}$ gives that $e \in X \subset S$. There are two subcases depending on whether S is in \mathcal{F} or not. First, assume that $S \in \mathcal{F}$. Then $X^{-e} \subset X \subset S$ gives that $X^{-e} \subset S$. By (4.5), $X^{-e} \in \mathcal{D}$ and $e \in S$ exclude that $X^{-e} \prec_{\mathcal{F}} S$. Hence, $\text{length}([X^{-e}, S]_{\mathcal{F}}) \geq 2$. Combining this with $S \subset E$, we have that $\text{length}([X^{-e}, E]_{\mathcal{F}}) \geq 3$. But this contradicts the semimodularity of $\mathcal{F} \cong L$ by (JHCC) since $X^{-e} \prec_{\mathcal{F}} Y \prec_{\mathcal{F}} E$.

Second, assume that $S \in \mathcal{N}$. Then both X and S contain e since they belong to \mathcal{N} , whence $X \subset S$ turns into $X^{-e} \subset S^{-e}$. Furthermore, $S^{-e} \subset S \subset E$, and since $S^{-e} \in \mathcal{D}$, (4.5) shows that $S^{-e} \not\prec_{\mathcal{F}} E$. Combining this with $X^{-e} \subset S^{-e}$ similarly to the first subcase, we obtain that $\text{length}([X^{-e}, E]_{\mathcal{F}}) \geq 3$, and we have already seen how this leads to a contradiction. This completes Case 3.

It follows from Cases 1–3 that (CP) holds for \mathcal{G}^* and \mathcal{G}^* is a geometry. Let $K := \text{Lat}(\mathcal{G}^*)$. We are going to show that $K := L(e_{*\searrow} h)$ and it is an extension of L that lowers e to a cover of h .

By Proposition 3.2, K is a semimodular lattice of finite length. To show that $\text{length}(K) = \text{length}(L)$, it suffices to find a maximal chain in \mathcal{F} that is also a maximal chain in \mathcal{G} . First, take a maximal chain in $X_0 \prec_{\mathcal{F}} X_1 \prec_{\mathcal{F}} \dots \prec_{\mathcal{F}} X_t = \downarrow_P e$ in the interval $[\emptyset, \downarrow_P e]_{\mathcal{F}}$. Since $e \notin X_i$ for $i = 0, 1, \dots, t$, no member of \mathcal{N} can be inserted into this chain, that is, the coverings here are coverings also in \mathcal{G} . Since $|\downarrow_P e \setminus \downarrow_P e| = |\{e\}| = 1$, we have that $X_t = \downarrow_P e \prec_{\mathcal{F}} \downarrow_P e =: X_{t+1}$ and $X_t \prec_{\mathcal{G}} X_{t+1}$. Next, in the interval $[\downarrow_P e, P]_{\mathcal{F}}$, we take a maximal chain $\downarrow_P e = X_{t+1} \prec_{\mathcal{F}} \dots \prec_{\mathcal{F}} X_k = P$. It follows easily from (4.10) that no member of \mathcal{N} can be inserted, whence $X_{t+1} \prec_{\mathcal{G}} \dots \prec_{\mathcal{G}} X_k$. Consequently, $\{X_0, X_1, \dots, X_k\}$ is a maximal chain both in \mathcal{F} and \mathcal{G} , whereby $\text{length}(K) = \text{length}(L)$. It was proved by Wild [19] that

whenever L' and K' are semimodular lattices of the same finite length (4.16)
and L' is a meet-subsemilattice of K' , then L' is a sublattice of K' ;

see also Czédli [6] for an outline of the proof. (Note that Wild [19] proved even more.) Since the meet is the intersection both in $\mathcal{F} \cong L$ and $\mathcal{G} \cong K$, or because of the first line of (4.13), Wild's above-mentioned result yields that L is a sublattice of K . The validity of (4.4) follows immediately from the fact that it is only a transcript of (4.13). It is clear that $J(K)$ is

$$J(\mathcal{G}) = (\{\downarrow_R u : u \in R\}, \subseteq); \text{ in fact, for any geometry } (R, \mathcal{G}) \quad (4.17)$$

(in the sense of (2.3)), not only for our $\mathcal{G}^* = (R, \mathcal{G})$. Indeed, $\downarrow_R u$ is clearly the unique lower cover of $\downarrow_R u$, whence the set given in (4.17) consists of join-irreducible members of \mathcal{G} , and (F \downarrow) makes it clear that each member of \mathcal{G} is the join (in fact, the union) of *these* join-irreducible members of \mathcal{G} . Let Q denote the poset $J(K)$, and let $e' := \downarrow_R e$. By the canonical correspondence and (4.17), Q is order isomorphic to R . Since $\downarrow_R e = H$ is the only lower cover of e' and H corresponds to h , we have that $e'_{*K} = h$. By (4.12), $J(\mathcal{G}) \setminus \{\downarrow_R e\} = J(\mathcal{G}) \setminus \{\downarrow_P e\}$. Hence, the canonical correspondence gives that condition (ii) of Definition 4.1 holds. This completes the joint proof of Theorem 4.3 and Lemma 4.4. \square

5. A MORE GENERAL THEOREM ON LENGTH-PRESERVING EXTENSIONS

Two chains, C_1 and C_2 , in a poset are *parallel* if $x_1 \parallel x_2$ for all $x_1 \in C_1$ and $x_2 \in C_2$. (As usual, $x_1 \parallel x_2$ is the conjunction of $x_1 \not\leq x_2$ and $x_2 \not\leq x_1$.) Parallel chains are *set-theoretically disjoint*, that is, their intersection is \emptyset . (In Lemma 6.4 and thereafter, there will occur a lattice-theoretical concept of disjointness.) Every

poset is the union of a set of pairwise set-theoretically disjoint chains; for example, we can take all singleton chains but, usually, there are other possibilities, too. With the help of our $L(e_{*\searrow}h)$ construction, we are going to prove the following result.

Theorem 5.1. *Let L be a semimodular lattice of finite length. Assume that $J(L)$ is the union of a set $\{C_i : i \in I\}$ of pairwise set-theoretically disjoint chains. Then L has a length-preserving semimodular extension K such that for a set $\{C'_i : i \in I\}$ of pairwise parallel chains of $J(K)$, we have that $J(K) = \bigcup \{C'_i : i \in I\}$ and, for all $i \in I$, $|C'_i| = |C_i|$.*

In other words, if L is a (1.1)-lattice and we partition $J(L)$ into chains, then for a length-preserving semimodular extension K of L , we can partition $J(K)$ into pairwise parallel (new) chains that are of the same sizes as the original chains.

Remark 5.2. Lemma 6.4 will allow us to say “pairwise *lattice-theoretically disjoint* chains of $J(K)$ ” instead of “pairwise *parallel* chains of $J(K)$ ” in Theorem 5.1.

For later reference, note the following.

Remark 5.3. Let Γ be a property of (1.1) lattices preserved by taking directed unions and lowering any join-irreducible element. If Γ holds in L , then the proof below automatically shows that Γ also holds in K . If L is finite, then directed unions do not occur and it suffices that the lowering construction preserves Γ .

Proof of Theorem 5.1. Let L be a (1.1)-lattice, and denote the set (not the poset) $J(L)$ by J . We denote by $\hat{\rho} = \{(u, v) \in J \times J : u \leq_L v\}$ the ordering of $J(L)$, that is, $J(L) = (J, \hat{\rho}) = P$. For $i \in I$, let ρ_i denote the ordering of C_i , and let $\rho_* := \bigcup \{\rho_i : i \in I\}$. Since $\{C_i : i \in I\}$ is a partition of P , the relation ρ_* is an ordering of P and $\rho_* \subseteq \hat{\rho}$. Note that if $\rho_* = \hat{\rho}$, then the chains are pairwise parallel and we are ready by taking $K := L$ and $C'_i := C_i$, for $i \in I$. Let \mathcal{F}^* be the geometry $\text{Geom}(L) = ((J, \hat{\rho}), \mathcal{F})$; see Proposition 3.2. At this point, interrupting the proof of Theorem 5.1, we formulate the following auxiliary statement.

Claim 1. If $\rho_* \subset \hat{\rho}$, then there is a geometry $\mathcal{G}^* = ((J, \tau), \mathcal{G})$ such that $\rho_* \subseteq \tau \subset \hat{\rho}$ and $\mathcal{G} \supset \mathcal{F}$.

Proof of Claim 1. Since $\rho_* \subset \hat{\rho}$ and $J(L) = (J, \hat{\rho})$ is of finite length, we can pick $i, j \in I$, $e \in C_i$, and $b \in C_j$ such that $i \neq j$, $b < e$, and e is the least element of $C_i \cap \uparrow_L b$. Denote $C_i \cup \{0_L\}$ by C_i^{+0} . Let h be the largest element of $\downarrow_{C_i^{+0}} e = \{x \in C_i^{+0} : x < e\}$; in notation, $h \prec_{C_i^{+0}} e$. We claim that $h < e_{*L}$. Since e is join-irreducible, $b < e$ gives that $b \leq e_{*L}$. So if $h = 0$, then the required $h < e_{*L}$ follows from $0 < b \leq e_{*L}$. Assume that $h \neq 0$. From $h < e$ and $b < e$, we obtain that $b \leq h \vee_L b \leq e_{*L}$. Since e is the least element of $C_i \cap \uparrow_L b$, it follows that $e_{*L} \notin C_i$. But $h \in C_i$ and $h \leq e_{*L}$, whence $h < e_{*L}$. That is, in both possible cases, $h < e_{*L}$. Therefore, Proposition 3.2 and Theorem 4.3 allow us to take the geometry $\mathcal{G}^* = ((J, \tau), \mathcal{G})$ that corresponds to $K := L(e_{*\searrow}h)$. (If we look into the proof of Theorem 4.3, then Proposition 3.2 is not needed since K is constructed from \mathcal{G}^* .) By construction, $\mathcal{G} \supset \mathcal{F}$. It follows from (4.12) that $\rho_* \subseteq \tau \subset \hat{\rho}$, completing the proof of Claim 1. \square

Next, resuming the proof of Theorem 5.1, define T as the set of all geometries $\mathcal{G}^* = ((J, \tau), \mathcal{G})$ such that \mathcal{F} is a length-preserving subset of \mathcal{G} and $\rho_* \subseteq \tau \subseteq \hat{\rho}$. For members $\mathcal{G}^* = ((J, \tau), \mathcal{G})$ and $\mathcal{G}'^* = ((J, \tau'), \mathcal{G}')$ of T , we define $\mathcal{G}^* \leq \mathcal{G}'^*$ as

the conjunction of $\mathcal{G} \subseteq \mathcal{G}'$ and $\tau \supseteq \tau'$. This definition turns T into a poset (T, \leq) . Interrupting the proof again, we formulate the following statement.

Claim 2. Each chain in (T, \leq) has an upper bound in (T, \leq) .

Proof of Claim 2. Some trivial details like “ $P \in$ ” or “ $\emptyset \in$ ” will be omitted. Let $\{\mathcal{G}_i^* : i \in I\}$ be a chain in T where $\mathcal{G}_i^* = ((J, \tau_i), \mathcal{G}_i)$. We define $\mathcal{G} := \bigcup \{\mathcal{G}_i : i \in I\}$ and $\tau := \bigcap \{\tau_i : i \in I\}$. We only need to show that $\mathcal{G}^* = ((J, \tau), \mathcal{G})$ belongs to T ; then \mathcal{G}^* is clearly an upper bound of the chain $\{\mathcal{G}_i^* : i \in I\}$. Clearly, $\rho_* \subseteq \tau \subseteq \hat{\rho}$, (J, τ) is a poset, and this poset is of finite length since so is $(J, \hat{\rho})$. A *finite* chain in \mathcal{G} with more than $1 + \text{length}(\mathcal{F})$ elements (which are subsets of J) would be a chain in \mathcal{G}_i for some $i \in I$ since the union defining \mathcal{G} is directed, and this would contradict $\text{length}(\mathcal{G}_i) = \text{length}(\mathcal{F})$. Thus, (FL) holds for \mathcal{G}^* . This implies that

$$\text{any intersection of members of } \mathcal{G} \text{ is actually an intersection of finitely many of the given members.} \quad (5.1)$$

Since \mathcal{G} , as a directed union, is closed with respect to the binary intersection, (5.1) implies that \mathcal{G}^* satisfies (F \cap). Using the first of the obvious equalities

$$\downarrow_{(J, \tau)} u = \bigcap \{\downarrow_{(J, \tau_i)} u : i \in I\} \quad \text{and} \quad \downarrow_{(J, \tau)} u = \bigcap \{\downarrow_{(J, \tau_i)} u : i \in I\} \quad (5.2)$$

together with $\downarrow_{(J, \tau_i)} u \in \mathcal{G}_i \subseteq \mathcal{G}$ (coming from the validity of (Pr) for \mathcal{G}_i^*) and (F \cap) for \mathcal{G}^* , we conclude that (F \downarrow) holds for \mathcal{G}^* . Similarly, so does (Pr) by (5.2), $\{\downarrow_{(J, \tau_i)} u, \downarrow_{(J, \tau_i)} u\} \subseteq \mathcal{G}_i \subseteq \mathcal{G}$ and (F \cap).

Next, to deal with (CP), assume that $q \in J$, $X \in \mathcal{G}$, $q \notin X$, and $\downarrow_{(J, \tau)} q \subseteq X$. Pick an $i_0 \in I$ such that $X \in \mathcal{G}_{i_0}$. By (5.1), (5.2), and $\downarrow_{(J, \tau_i)} q \in \mathcal{G}_i \subseteq \mathcal{G}$, $\{i_0\}$ can be extended to a finite subset I' of I such that $\downarrow_{(J, \tau)} q = \bigcap \{\downarrow_{(J, \tau_i)} q : i \in I'\}$. Pick an $i' \in I$ such that $\mathcal{G}_{i'}^* \geq \mathcal{G}_i^*$ for all $i \in I'$. Then $\tau_{i'} \subseteq \tau_i$ and $\downarrow_{(J, \tau_i)} q \supseteq \downarrow_{(J, \tau_{i'})} q$ for all $i \in I'$. Combining this with the previous equality, $\downarrow_{(J, \tau)} q \supseteq \downarrow_{(J, \tau_{i'})} q$. Since (5.2) gives the converse inclusion, we have that $\downarrow_{(J, \tau)} q = \downarrow_{(J, \tau_{i'})} q \in \mathcal{G}_{i'}$. Thus, since $X \in \mathcal{G}_{i_0} \subseteq \mathcal{G}_{i'}$ and $q \notin X$, (CP) applied to $\mathcal{G}_{i'}$ yields a $Y \in \mathcal{G}_{i'}$ such that $q \in Y$ and Y covers X in $\mathcal{G}_{i'}$. If we had a $Z \in \mathcal{G}$ with $X \subset Z \subset Y$, then we could pick an $i'' \in I$ with $\{Z\} \cup \mathcal{G}_i \subseteq \mathcal{G}_{i''}$ and then $\text{length}(\mathcal{G}_{i'}) < \text{length}(\mathcal{G}_{i''})$ would contradict the fact that $\mathcal{G}_{i'}$ is a sublattice of $\mathcal{G}_{i''}$ by (4.16) and both $\mathcal{G}_{i'}$ and $\mathcal{G}_{i''}$ are semimodular lattices of the same length. Therefore, \mathcal{G}^* satisfies (CP), completing the proof of Claim 2. \square

Resuming the proof of Theorem 5.1, Zorn's lemma and Claim 2 allow us to take a maximal element $\mathcal{G}^* = ((J, \tau), \mathcal{G})$ of the poset T . It follows from Claim 1 that $\tau = \rho_*$. Therefore, $K := \text{Lat}(\mathcal{G}^*)$ satisfies the requirements of Theorem 5.1, completing its proof. \square

In view of Claim 1, one might feel that $\hat{\rho}$ can be reduced to an arbitrary $\tau \subset \hat{\rho}$. Even if the condition based on the partition $\{C_i : i \in I\}$ of $J(L)$ in Theorem 5.1 is not the only possibility, some condition on τ is needed; this is clear from the following example, which is followed by a proof.

Example 5.4. With L and $P = (J(L), \hat{\rho})$ given in Figure 2, there is no length-preserving semimodular extension K of L such that $J(K)$ is order-isomorphic to $P' = (J(K), \tau)$ given in on the right of the figure. If we identify x with x' for all $x \in P$, then $\tau = \hat{\rho} \setminus \{(c, e)\} \subset \hat{\rho}$.

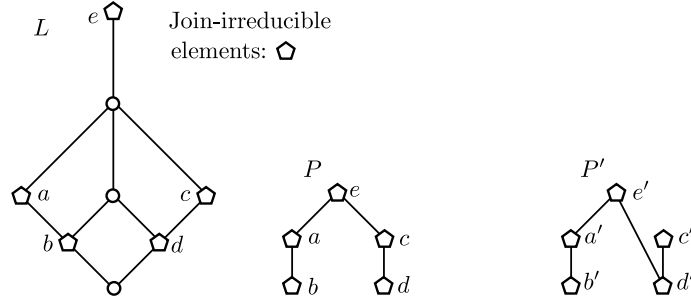


FIGURE 2. The ordering of $P = (J(L), \hat{\rho})$ cannot be reduced to $P' = (J(K), \tau)$

Proof of Example 5.4. For the sake of contradiction, suppose that K is a length-preserving semimodular extension of L such that $J(K) \cong P'$. (We do not assume that $a \in L$ corresponds to $a' \in K$, etc.) Since $e' \not\leq_K c'$ excludes that $e' = 1_K$ and $\text{length}(K) = \text{length}(L) = 4$, it follows that $0 \prec_K b' \prec_K a' \prec_K e' \prec_K 1_K$. Using that $d' \not\leq_K a' \leq_K e'$, $d' \leq_K e'$, so $a' <_K a' \vee d' \leq e'$, and $a' \prec_K e'$, and we obtain that $a' \vee d' = e'$, contradicting $e' \in J(K)$. \square

6. SOME COROLLARIES

Theorems 4.3 and 5.1, and, even Claim 1 sometimes, imply some earlier results and offer, with at most one exception, easier (usually much easier) proofs of these results than the original ones.

Corollary 6.1 (Grätzer and Kiss [14, Lemma 17]). *Each finite semimodular lattice has a length-preserving extension into a geometric lattice.*

The proof of this corollary (together with that of Theorem 4.3) is much shorter than the proof in Grätzer and Kiss [14]. The proof of the following corollary is competitive with that given in Wild [19].

Corollary 6.2 (Wild [19, Theorem 4]). *Each finite semimodular lattice L has a length-preserving extension into a geometric lattice K such that $|\text{At}(K)| = |J(L)|$.*

Proof of Corollaries 6.1 and 6.2. Apply Theorem 4.3 repeatedly. Alternatively, apply Theorem 5.1 for the partition $\{\{u\} : u \in J(L)\}$. \square

The proof of the following corollary is also much simpler than the original one.

Corollary 6.3 (Czédli and Schmidt [8, Theorem 1]). *Each semimodular lattice L of finite length has a length-preserving extension into a geometric lattice K such that $|\text{At}(K)| = |J(L)|$.*

Note that Skublics [18, Corollary 2] extended this result to some semimodular lattices that are not of finite lengths, but his result is beyond the scope of the present paper.

Proof of Corollary 6.3. Apply Theorem 5.1 for the partition $\{\{u\} : u \in J(L)\}$. \square

A poset is of *width* k if it is the union of k chains but not of fewer chains. For a (1.1)-lattice L and chains $C_1, C_2 \subseteq J(L)$, we say that C_1 and C_2 are *lattice-theoretically disjoint* if $x_1 \wedge_L x_2 = 0_L$ for all $x_1 \in C_1$ and $x_2 \in C_2$. Since $0_L \notin J(L)$,

C_1 and C_2 are parallel in this case. Since two parallel chains in $J(L)$ need not be lattice-theoretically disjoint in L , it is worth formulating the following lemma.

Lemma 6.4. *Let L be a (1.1) lattice, and let $\{C_i : i \in I\}$ be a partition of $J(L)$ into chains. Then $\{C_i : i \in I\}$ is a set of pairwise parallel chains if and only if it is a set of pairwise lattice-theoretically disjoint chains.*

Proof. Since two lattice-theoretically disjoint chains in $L \setminus \{0_L\}$ are trivially parallel, it suffices to prove the “only if” part. For the sake of contradiction, suppose that $\{C_i : i \in I\}$ consists of pairwise parallel chains but, for some $i', i'' \in I$, $i' \neq i''$ and $C_{i'}$ is not disjoint lattice-theoretically from $C_{i''}$. Take an $x' \in C_{i'}$ and an $x'' \in C_{i''}$ such that $x' \wedge_L x'' \neq 0_L$. Then we can pick a $y \in J(L)$ such that $y \leq_L x'$ and $y \leq_L x''$. There is a unique $j \in I$ such that $y \in C_j$. Since $C_{i'}$ and C_j are parallel chains, $j = i'$. But $C_{i''}$ and C_j are parallel chains, too, whereby $j = i''$. Hence, $i' = i''$, which is a contradiction. \square

Definition 6.5 (E. T. Schmidt, unpublished). A k -dimensional rectangular lattice is a finite semimodular lattice L such that $k = \text{width}(J(L))$ and $J(L)$ is the union of k pairwise lattice-theoretically disjoint chains.

Corollary 6.6 (E. T. Schmidt, unpublished). *Let L be a finite semimodular lattice, and let $k := \text{width}(J(L))$. Then L has a length-preserving extension into a k -dimensional rectangular lattice K such that $|J(K)| = |J(L)|$. Furthermore, if $\{C_1, \dots, C_n\}$ is a partition of $J(L)$ into chains, then we can choose K so that $J(K) = C'_1 \cup \dots \cup C'_n$ with pairwise lattice-theoretically disjoint chains C'_1, \dots, C'_n satisfying $|C'_1| = |C_1|, \dots, |C'_n| = |C_n|$.*

Note that the particular case $\text{width}(J(L)) = 2$ of this result is due to Grätzer and Knapp [16].

Proof of Corollary 6.6. Note in advance that Remark 5.3 also applies to this proof. By definition, $J(L)$ is the union of k chains, C_1, \dots, C_k . We can assume that these chains are pairwise set-theoretically disjoint. Indeed, if not so, then replacing C_i by $C_i \setminus C_1$ for $i = 2, \dots, k$, C_1 becomes set-theoretically disjoint from C_2, \dots, C_k . In the next step, we replace C_i by $C_i \setminus C_2$ for $i = 3, \dots, k$, and C_2 becomes set-theoretically disjoint from C_3, \dots, C_k . And so on. Now that our chains are pairwise set-theoretically disjoint, Theorem 5.1 combined with Lemma 6.4 yields a length-preserving extension of L into a (1.1)-lattice K with the required property. \square

7. SOME PROPERTIES, EXAMPLES, AND FURTHER COROLLARIES OF THE LOWERING CONSTRUCTION

For future reference and a better insight into the construct $L(e_{\ast} \searrow h)$, we prove some of its properties in this section as well as some corollaries.

Lemma 7.1. *With the assumptions and notations of Definition 4.2 and Theorem 4.3, if L is distributive, then so is $K = L(e_{\ast} \searrow h)$.*

Proof. Due to the canonical correspondence established by Proposition 3.2, we can work with $\mathcal{F}^* := \text{Geom}(L) = (P, \mathcal{F})$ and, instead of K , with $\mathcal{G}^* = (R, \mathcal{G})$ defined by (4.5), (4.7), and (4.9). By the well-known structure theorem of finite distributive lattices, see, for example, Grätzer [13, Theorem 107], \mathcal{F} is the collection of all down-sets of $P = J(L)$. By the same structure theorem and $(F \downarrow)$, it suffices to show that

every down-set of R belongs to \mathcal{G} . Let $S \subseteq R$ be a down-set of R . This means that for every $u \in S$, $\downarrow_R u \subseteq S$. If $e \notin S$, then $\downarrow_P u = \downarrow_R u \subseteq S$ for all $u \in S$ by (4.12), whence S is a down-set of P and we have that $S \in \mathcal{F} \subseteq \mathcal{G}$, as required.

Next, assume that $e \in S$. We can assume that e is a maximal element of S with respect to \leq_R or, equivalently by (4.12), with respect to \leq_P . Indeed, if there is a $u \in S$ with $e <_P u$, then (4.12) gives that $\downarrow_P e \subseteq \downarrow_P u = \downarrow_R u \subseteq S$ and, for every $v \in S \setminus \{e\}$, $\downarrow_P v = \downarrow_R v \subseteq S$, whence S is a down-set of P and we obtain that $S \in \mathcal{F} \subseteq \mathcal{G}$. So e is a maximal element of S . Let $X := S \setminus \{e\} =: S^{-e}$. Since $H = \downarrow_R e \subseteq \downarrow_R S \subseteq S$, it follows that $H \subseteq X$. Since X is a down-set of P by (4.12), $X \in \mathcal{F}$. If no cover of X in \mathcal{F} contains e , then $X \in \mathcal{D}$ by (4.5) and $S = X^{+e} \in \mathcal{N} \subseteq \mathcal{G}$, as required. So we can assume that there is a $Y \in \mathcal{F}$ such that $X \prec_{\mathcal{F}} Y \ni e$. Since $e \notin X$, we know that $\downarrow_P e \not\leq X$ in $\mathcal{F} \cong L$. But $\downarrow_P e \leq Y$ since $e \in Y$, so $X < X \vee_{\mathcal{F}} \downarrow_P e \leq Y$, whence $X \prec_{\mathcal{F}} Y$ yields that $X \vee_{\mathcal{F}} \downarrow_P e = Y$. Transposed intervals are isomorphic in a distributive lattice (and even in a modular lattice), whereby $X \wedge_{\mathcal{F}} \downarrow_P e \prec_{\mathcal{F}} \downarrow_P e$. But the only lower cover of $\downarrow_P e$ in \mathcal{F} is $\downarrow_P e$ since $\downarrow_P e$ corresponds to $e \in J(L)$ by the canonical correspondence. Hence, $X \cap \downarrow_P e = X \wedge_{\mathcal{F}} \downarrow_P e = \downarrow_P e$. Therefore, $\downarrow_P e \subseteq X$ and $\downarrow_P e \subseteq X^{+e} = S$. The inclusion $\downarrow_P e \subseteq S$ together with (4.12), which gives that $\downarrow_P u = \downarrow_R u \subseteq S$ for all $u \in S \setminus \{e\}$, imply that S is a down-set of P . Hence, $S \in \mathcal{F} \subseteq \mathcal{G}$, as required. \square

Remark 7.2. As mentioned after Definition 2.1, our construction “doubles” the set D defined in (4.1). Even if this construction shows some similarity with Alan Day’s doubling construction, see [10], the two constructions are entirely different in most of the cases. For example, Day’s doubling construction applied to (1.1) (or even to the 4-element boolean lattice) does not preserve distributivity in general, our construction does by Lemma 7.1. Furthermore, while Alan Day’s construction doubles intervals, our D is only a lower pseudo interval in the terminology of Freese, Ježek and Nation [12].

The following statement has recently been proved in Czédli and Molkhasi [7, Lemma 4.4]. The proof here is shorter.

Corollary 7.3 (Czédli and Molkhasi [7]). *Corollary 6.6 remains true if we replace “semimodular” by “distributive” at both occurrences.*

Proof. Combine Lemma 7.1 with the first sentence of the proof of Corollary 6.6. \square

Although we will soon explain the concept of join-distributivity occurring in the remark below, the reader can safely skip over these details.

Example 7.4. In general, the passage from L to $L(e_{\ast} \searrow h)$ preserves neither modularity, nor join-distributivity. This is witnessed by Figure 3, where $\{a, b, c\}$ generates a “pentagon” N_5 in $K := L(e_{\ast} \searrow h)$ on the left while it generates an M_3 on the right.

A finite semimodular lattice is *join-distributive* if it contains no M_3 (five-element nondistributive modular lattice) as a sublattice. Equivalently, if it contains no M_3 as a cover-preserving sublattice. Note that Grätzer and Knapp [15] called join-distributive planar semimodular lattices *slim*. Join-distributive lattices were introduced in many papers in many different ways and with different names; see Proposition 2.1 in Czédli [4] for surveying eight possible definitions, Proposition 6.1 in Czédli and Adaricheva [1] for two additional definitions, and see Monjardet [17] and Adaricheva, Gorbunov, and Tumanov [2] for further information.

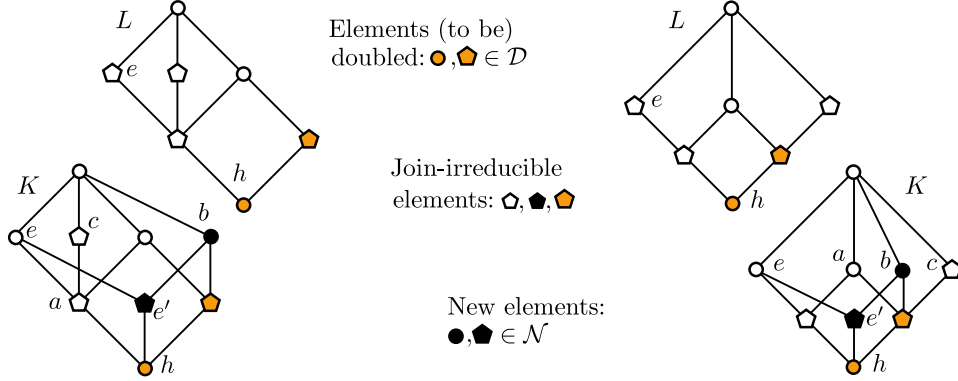


FIGURE 3. Neither modularity, nor join-distributivity is preserved

The following statement is well known; we only present it here to show an interesting easy application of our results.

Corollary 7.5. *The length of a finite distributive lattice L is $|\mathbf{J}(L)|$.*

Proof. Theorem 5.1 applied for the trivial partition $\{\{u\} : u \in \mathbf{J}(L)\}$, Remark 5.3, and Lemma 7.1 yield an extension of L into a boolean lattice K such that $|\mathbf{J}(K)| = |\mathbf{J}(L)|$ and $\text{length}(K) = \text{length}(L)$. Since K is isomorphic to the powerset lattice $(\text{Pow}(\mathbf{J}(K)), \subseteq)$, it is trivial that $\text{length}(K) = |\text{At}(K)| = |\mathbf{J}(K)|$. The equalities listed so far imply the corollary. \square

We continue the paper with three lemmas that give a better understanding of the $L(e_{*\searrow}h)$ construction.

Lemma 7.6. *With the assumptions and notations of Definition 4.2 and Theorem 4.3, D and $D \cup N$ are convex subsets of $K = L(e_{*\searrow}h)$ and h is their common smallest element. Also, D is a convex subset of L and N is a convex subset of K .*

Proof. Since L is a cover-preserving sublattice of K , the lemma is just a transcript of (4.11) by the canonical correspondence. \square

A possible way to understand $L(e_{*\searrow}h)$ is to derive its Hasse diagram from that of L . By describing the covering relation in $L(e_{*\searrow}h)$, the following lemma is useful.

Lemma 7.7. *With the assumptions and notations of Definition 4.2 and Theorem 4.3, we have that for any $x, y \in K = L(e_{*\searrow}h)$,*

$$x \prec_K y \iff \begin{cases} x, y \in L \text{ and } x \prec_L y & \text{or} \\ x, y \in N \text{ and } x^\nabla \prec_L y^\nabla & \text{or} \\ x \in L, y \in N, \text{ and } x = y^\nabla & \text{or} \\ x \in N, y = x^\nabla \vee_L e, \text{ and } \text{length}([x^\nabla, x^\nabla \vee_L e]_L) = 2. \end{cases} \quad (7.1)$$

Proof. For $x, y \in L$, (7.1) is clear, because L is a cover-preserving sublattice of K . The second line of (7.1) follows from the second line of (4.3) since N is a convex subset of K by Lemma 7.6. For $x \in L$ and $y \in N$, the statement is trivial by the third line of (4.3).

So, in the rest of the proof, we assume that $x \in N$ and $y \in L$. Note that $x \neq x^\nabla \vee_L e \in L$. First, assume that $x \prec_K y$. By the last line of (4.3), the least element

of $\uparrow_K x \cap L$ is $x^\nabla \vee_L e$, whence $y = x^\nabla \vee_L e$. Since $x^\nabla \prec_K x$ and $x \prec_K y = x^\nabla \vee_L e$, (JHCC) yields that $\text{length}([x^\nabla, x^\nabla \vee_L e]_L) = \text{length}([x^\nabla, x^\nabla \vee_L e]_K) = 2$, as required.

Second, assume that $y = x^\nabla \vee_L e$ and $\text{length}([x^\nabla, x^\nabla \vee_L e]_L) = 2$. Using the last line of (4.3) and that $x \neq x^\nabla \vee_L e$, we have that $x \prec_K x^\nabla \vee_L e = y$. Since $\text{length}([x^\nabla, y]_K) = \text{length}([x^\nabla, y]_L) = 2$ and $x^\nabla \prec_K x \prec_K y$, (JHCC) implies that $x \prec_K y$, completing the proof. \square

Lemma 7.8. *With the assumptions and notations of Definition 4.2 and Theorem 4.3, we have that for any $x, y \in K = L(e_{\ast} \searrow h)$,*

$$x \vee_K y = \begin{cases} x \vee_L y, & \text{if } x, y \in L, \\ (x^\nabla \vee_L y^\nabla)^\blacktriangle, & \text{if } x, y \in N \text{ and } x^\nabla \vee_L y^\nabla \in D, \\ x^\nabla \vee_L y^\nabla \vee_L e, & \text{if } x, y \in N \text{ and } x^\nabla \vee_L y^\nabla \notin D, \\ (x^\nabla \vee_L y)^\blacktriangle, & \text{if } x \in N, y \notin N, \text{ and } x^\nabla \vee_L y \in D, \\ x^\nabla \vee_L y \vee_L e, & \text{if } x \in N, y \notin N \text{ and } x^\nabla \vee_L y \notin D, \\ (x \vee_L y^\nabla)^\blacktriangle, & \text{if } x \notin N, y \in N, \text{ and } x \vee_L y^\nabla \in D, \\ x \vee_L y^\nabla \vee_L e, & \text{if } x \notin N, y \in N \text{ and } x \vee_L y^\nabla \notin D. \end{cases} \quad (7.2)$$

Of course, “ $\in N$ ” and “ $\notin N$ ” in (7.2) are equivalent to “ $\notin L$ ” and “ $\in L$ ”, respectively.

Proof of Lemma 7.8. Since we already know from Theorem 4.3 that L is a sublattice of K , the first line of (7.2) is clear. The rest of (7.2) follows by straightforward considerations and since x and y play a symmetrical role. Hence, we only give the details for the fifth row of (7.2), where $x \in N$, $y \notin N$, and $z := x^\nabla \vee_L y \notin D$. We need to show that $z \vee_L e$ is the least upper bound of $\{x, y\}$ in K . The inequality $x \leq_K z \vee_L e$ is clear by the fourth line of (4.3) while $y \leq_K z \vee_L e$ follows from $y \leq_L z$. Hence, $z \vee_L e$ is an upper bound of $\{x, y\}$ in K . Let $t \in K$ be another upper bound. We claim that $t \notin N$. For the sake of contradiction, suppose that $t \in N$. Then (4.3) yields that $x^\nabla \leq_L t^\nabla$ and $y \leq_L t^\nabla$. Hence, $x^\nabla \leq_L z = x^\nabla \vee_L y \leq_L t^\nabla$. Thus, Lemma 7.6 gives that $z \in D$, which is a contradiction showing that $t \notin N$. Now $x \leq_K t$ and (4.3) gives that $x^\nabla \vee_L e \leq_L t$. Also, $y \leq_K t$ gives $y \leq_L t$. Hence, $z \vee_L e = x^\nabla \vee_L y \vee_L e \leq_L t$, whereby $x \vee_K y = z \vee_L e$, as required. \square

8. DECLARATIONS

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8.2. Conflicts of Interest / Competing Interests. Not applicable.

8.3. Availability of data and material. Not applicable.

8.4. Code availability. Not applicable.

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