Generating some large filters of quasiorder lattices

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Dedicated to Professor Vilmos Totik on his retirement (2023) and forthcoming seventieth birthday (2024); see also Remark 5.1 in Section 5.

Abstract. For a poset $(P; \leq)$, the quasiorders (AKA preorders) extending the poset order "$\leq$" form a complete lattice F, which is a filter in the lattice of all quasiorders of the set P. We prove that if the poset order "$\leq$" is small, then F can be generated by few elements.

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1. Introduction

Before stating the main result, Theorem 2.4, we survey the previously known results on generation of quasiorder lattices. First of all, we recall some concepts and notations. Reflexive and transitive relations of a set X are called *quasiorders* or (in a significant portion of the literature) *preorders* (of X). For a set X, we will denote by $\mathrm{Quo}(X)$ the complete lattice of quasiorders of X; the lattice order "$\leq$" in $\mathrm{Quo}(X)$ is the set inclusion relation "$\subseteq$". A subset Y of a complete lattice L is a *complete-generating set* if no proper complete sublattice of L includes Y. Furthermore, we say that L is *n-generated as a complete lattice* if L has an at most n-element complete-generating set. For a finite lattice, this concept is equivalent to the existence of an at most n-element generating set in the usual sense. E.g., if $|L| \leq n$, then L is n-generated. As a convention for this paper, when we write that "generated" or "generating set", we always mean "generated as a complete lattice" or "complete-generating set" even when this is not emphasized.

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To recall the first result on small generating sets of Quo(X) from Chajda and Czédli [2], let $\beth_0 := \aleph_0$ and, for any integer $n \in \mathbb{N}_0 = \{0, 1, 2, \dots\}$, let $\beth_{n+1} := 2^{\beth_n}$. Let $\beth_\omega := \sup\{\beth_n : n \in \mathbb{N}_0\}$. With these notations, if $|X| < \beth_\omega$, then Quo(X) is 6-generated (but note that the six-element complete-generating set given in [2] has an additional property).

A cardinal number $\eta > \aleph_0$ is *strongly inaccessible* if for any cardinal λ, $\lambda < \eta \Rightarrow 2^\lambda < \eta$ and, in addition, for any set I of cardinals smaller than η, $|I| < \eta \Rightarrow \sup I < \eta$. We call a cardinal number λ *accessible* if there exists no strongly inaccessible cardinal number η such that $\eta \le \lambda$. For example, the finite cardinals and $\beth_\lambda$ for $\lambda \le \omega$ are accessible. We know from Kuratowski's result, see [16] and see also Levy [17] for a secondary source that if ZFC has a model, then it also has a model in which all cardinal numbers are accessible.

As the next step after Chajda and Czédli [2], Takách [21] extended the result of [2] to every set X of accessible cardinality. About two decades later, Dolgos [13] proved that Quo(X) is 5-generated for any set X with size (=cardinality) $|X| \le \aleph_0$, and Kulin [15] extended this result by proving that Quo(X) is 5-generated for any set X with an accessible cardinality. Not much later, with the exception of $|X| \in \{4, 6, 8, 10\}$, Czédli [6] proved that if X is a set with $|X| \le \aleph_0$, then Quo(X) is 4-generated. Furthermore, [6] also proved that for any set X with at least three elements, Quo(X) is not 3-generated. Soon thereafter, Czédli and Kulin [10, Theorem 3.5] reduced the number of exceptions and permitted all infinite sets X with accessible cardinalities. We summarize these results as follows.

Lemma 1.1 (Czédli [6]**, Czédli and Kulin** [10]**).** *If X is an at least two-element set of an accessible cardinality and this cardinality is different from* 4*, then the complete lattice* Quo(X) *is* 4*-generated as a complete lattice. For $|X| \ge 3$,* Quo(X)*, as a complete lattice, is not* 3*-generated. For $|X| = 4$,* Quo(X) *is* 5*-generated.*

For $|X| = 4$, we do not know whether Quo(X) is 4-generated or not. Our understanding becomes less comprehensive when we stipulate that two out of the four generators are comparable. What we know from Czédli and Kulin [10, Theorem 3.5] and Ahmed and Czédli [1, Remark 3.4] is that whenever X is a set such that $|X|$ is an accessible cardinal but $|X| \notin \{0, 1, 2, 4, 5, 7, 8, 9, 10, 12\}$, then Quo($X$) has a 4-element generating set that is not an antichain.

For the lattice Equ(X) of all equivalence relations of a set X, there are results similar to Lemma 1.1 and the paragraph following it; see Czédli [4], [5], [7], [8], Czédli and Oluoch [12], Dolgos [13], Strietz [19]–[20], and Zádori [22]. Most of the results and papers mentioned so far are based on Zádori's excellent method given in [22], in which one of the theorems asserts that the lattice Equ(X) is 4-generated for every finite set X.

Recently, Czédli [8] and, mainly, [9] have suggested (but not elaborated) cryptographic protocols based on large lattices generated by few elements. This idea is also among our motivations, since a straightforward induction shows that for $n \in \mathbb{N}^+ = \{1, 2, 3, \dots\}$, the quasiorder lattice $\mathrm{Quo}(\{1, \dots, n\})$ has at least 4^{n-1} elements, so it is large.

2. Concepts, notations, and the main result

First, we recall or introduce the concepts and notations that are necessary to formulate the main result. As is usual in lattice theory, a lattice $(L; \vee, \wedge)$ or $(L; \leq)$ is denoted simply by its underlying set, L. However, as a poset and its underlying set will often play different roles, we make a notational difference between them as follows. A poset $(X; \leq)$ or $(X; \mu)$ will be denoted by the corresponding blackboard bold letter $\mathbb{X}$, and the underlying set of a poset $\mathbb{Y}$ is the corresponding italic letter Y. Sometimes, we write $u \in \mathbb{X}$ or $Y \subseteq \mathbb{X}$ instead of $u \in X$ or $Y \subseteq X$, respectively.

For an element u of a lattice L, we denote the *principal filter* $\{x \in L : u \leq x\}$ by $\mathrm{fil}(u)$ or, if confusion threatens, by $\mathrm{fil}_L(u)$. Similarly, $\mathrm{idl}(u) = \mathrm{idl}_L(u)$ stands for the *the principal ideal* $\{x \in L : x \leq u\}$.

Posets with more than one element are said to be *nontrivial* while the singleton poset is *trivial*. The *length* of a finite n-element chain is $n-1$. For $n \in \mathbb{N}_0$, a *poset is of length* n if it has a chain of length n but it has no chain of length $n+1$. If $\mathbb{P} = (P; \leq)$ is a poset and it is of length n for some $n \in \mathbb{N}_0$, then we say that $\mathbb{P}$ is *of finite length*.

We always assume that our poset $\mathbb{P}$ is of finite length; in this case, the covering relation $\prec$ of $\mathbb{P}$ determines the (partial) order $\leq$ of $\mathbb{P}$. While the Hasse diagram of $\mathbb{P}$ is a directed graph (every edge is directed upwards but this is not indicated), this diagram can also be considered an undirected graph, the *graph of the poset* $\mathbb{P}$. The edges of this graph are the two-element subsets $\{x, y\}$ of $\mathbb{P}$ such that $x \prec y$ or $y \prec x$. We say that the poset $\mathbb{P}$ is a *forest* if its graph contains no circle of positive length. In this case, the (*connectivity*) *components* of $\mathbb{P}$ (that is, the blocks of the least equivalence of $\mathbb{P}$ that collapses every edge of $\mathbb{P}$) are called *trees*. For a poset $\mathbb{X}$, we denote the *set of maximal elements* as $\mathrm{Max}(\mathbb{X})$ and the *set of minimal elements* as $\mathrm{Min}(\mathbb{X})$.

Next, we define the functions and the parameters that occur in the main result, Theorem 2.4; most of them will be denoted by self-explanatory mnemonics[1]. The parameters are illustrated by Figures 1 and 2 together with the corresponding Examples 2.6 and 2.7, respectively; note that some ingredients of these multi-purpose figures will be defined later. The main result has some easy-to-read consequences with less functions and parameters; see Corollary 4.2 and Examples 2.6, 2.7, and 4.3.

[1]We have selected mnemonics that can be conveniently located in the PDF file of the paper using the search feature of Acrobat Reader or some browsers. For example, the search for “ce(” or “f4”finds the first occurrence of $\mathrm{ce}(\mathbb{P})$ or f_4, respectively

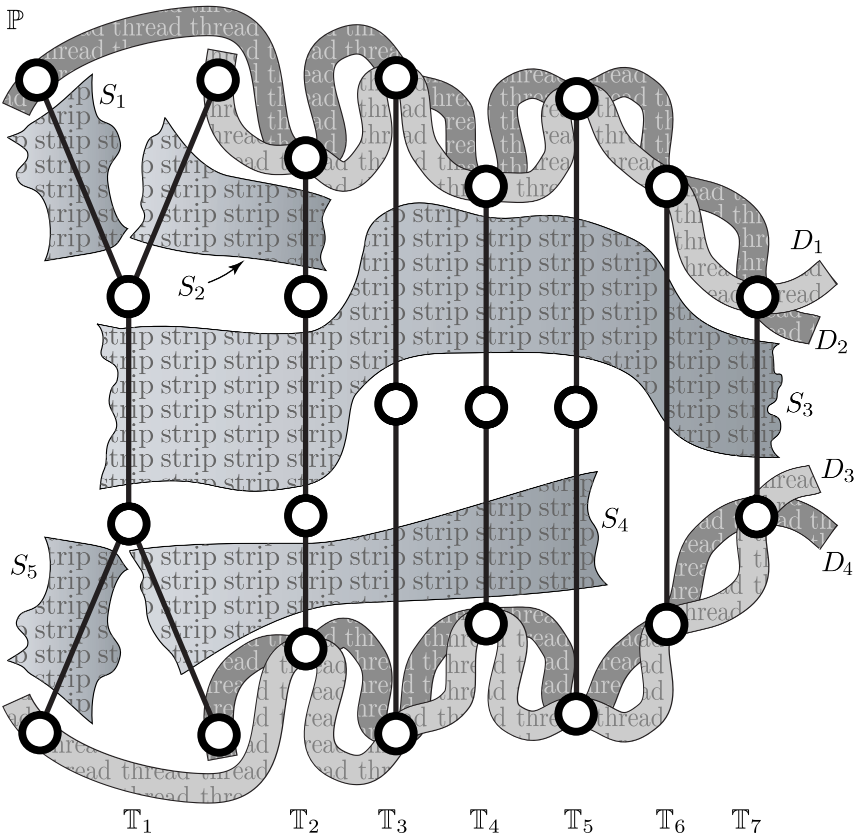

FIGURE 1. A forest $\mathbb{P}$; see Example 2.6.

Definition 2.1. For a positive integer n, let $\mathrm{LASp}(n)$ denote[2] the smallest $k \in \mathbb{N}^+ := \{1, 2, 3, \dots\}$ such that

$$n \leq \binom{k}{\lfloor k/2 \rfloor} \tag{2.1}$$

where $\lfloor k/2 \rfloor$ denotes the (lower) integer part of $k/2$, and let $\mathrm{LASp}(0) := 0$. For some values of n, $\mathrm{LASp}(n)$ is given in the following tables.

[2] The acronym LASp comes from the Left Adjoint of Sperner's function, which is a functor from the categorified poset $(\{0, 1, 2, \dots\}; \leq)$ to itself; see Czédli [9] for details.

n	$\mathrm{LASp}(n)$
$0,\dots,3$	n
$4,\dots,6$	4
$7,\dots,10$	5
$11,\dots,20$	6
$21,\dots,35$	7
$36,\dots,70$	8
$71,\dots,126$	9
$127,\dots,252$	10
$253,\dots,462$	11
$92\,379,\dots,184\,756$	20

n	$\mathrm{LASp}(n)$
10^{10}	37
10^{20}	70
10^{30}	104
10^{40}	137
10^{50}	171
10^{60}	204
10^{70}	237
10^{80}	271
10^{90}	304
10^{100}	337

Definition 2.2. We define a function $f_4\colon \{\text{accessible cardinals}\}\setminus\{0\}\to\mathbb{N}_0$ by the rules $f_4(1):=0$, $f_4(2):=2$, $f_4(3):=4$, $f_4(4):=5$ and, for any accessible cardinal $x\geq 5$, $f_4(x):=4$. (The subscript is only a reminder to the fact[3] that $f_4(x)=4$ for most cardinal numbers x.)

Definition 2.3. Let $\mathbb{P}=(P;\leq)=(P;\mu)$ be a poset of finite length.

(A) Let $\mathrm{Comp}(\mathbb{P})$ stand for the set of (connectivity) *components* of $\mathbb{P}$. If $|\mathrm{Comp}(\mathbb{P})|=1$, then $\mathbb{P}$ is a *connected poset*. We say that $\mathbb{P}$ is of *finite component size* if there is a $k\in\mathbb{N}^+:=\mathbb{N}_0\setminus\{0\}$ such that for every $\mathbb{X}\in\mathrm{Comp}(\mathbb{P})$, $\mathbb{X}$ has at most k elements (that is, $|X|\leq k$). In this case, the least such k is called the *component size* of $\mathbb{P}$; we denote it by $\mathrm{csize}(\mathbb{P})$.

(B) Let $\mathrm{cmp}(\mathbb{P})=|\mathrm{Comp}(\mathbb{P})|$; it is the cardinal number of the components of $\mathbb{P}$.

(C) Let $\mathrm{Edge}(\mathbb{P}):=\{(x,y)\in P^2 : x\prec y\}$, where $\prec$ is the covering relation of $\mathbb{P}$, stand for the *set of edges* of $\mathbb{P}$.

(D) Assuming that $\mathbb{P}$ is of a finite component size, $\mathrm{csize}(\mathbb{P})^2$ is a finite upper bound on $\{|\mathrm{Edge}(\mathbb{X})| : \mathbb{X}\in\mathrm{Comp}(\mathbb{P})\}$ and so we can define the *component edge number* of $\mathbb{P}$ as $\mathrm{ce}(\mathbb{P}):=\max\{|\mathrm{Edge}(\mathbb{X})| : \mathbb{X}\in\mathrm{Comp}(\mathbb{P})\}$. (Note that even though $\mathrm{ce}(\mathbb{P})\in\mathbb{N}_0$, $\mathbb{P}$ can have infinitely many edges.)

(E) We denote by $\mathrm{Quo}^{\mathrm{e}}(\mathbb{P})$ the lattice of quasiorders of P that extend μ. Note that $\mathrm{Quo}^{\mathrm{e}}(\mathbb{P})$ is the same as the principle filter $\mathrm{fil}_{\mathrm{Quo}(P)}(\mu)$ of $\mathrm{Quo}(P)$.

(F) For a poset $\mathbb{A}=(A;\mu_A)=(A;\leq)$ and $x,y\in A$, let

$$\mathrm{qu}_\mu(y,x)\in\mathrm{Quo}^{\mathrm{e}}(\mathbb{A})$$

stand for the least member of $\mathrm{Quo}^{\mathrm{e}}(\mathbb{A})$ containing (y,x). For a finite connected poset $\mathbb{X}=(X;\mu_X)=(X;\leq)$, we define the *special parameter* $\mathrm{spp}(\mathbb{X})$ as follows. If X is a singleton, then $\mathrm{spp}(\mathbb{X}):=0$. If $\mathbb{X}$ is a chain with more than one element, then $\mathrm{spp}(\mathbb{X}):=1$. If $\mathbb{X}$ is not a chain, then $\mathrm{spp}(\mathbb{X})$ is the smallest number $k\in\mathbb{N}^+$ such that there is a $(k-1)$-element subset $Y(\mathbb{X})$ of $\mathrm{Quo}^{\mathrm{e}}(\mathbb{X})$ with the property that

$$Y(\mathbb{X})\cup\{\mathrm{qu}_\mu(y,x) : x\prec y \text{ in } \mathbb{X}\}\text{ generates }\mathrm{Quo}^{\mathrm{e}}(\mathbb{X}). \tag{2.2}$$

[3]The results of the paper would remain true if $f_4(4)$ was defined as the minimal number of generators of $\mathrm{Quo}(4)$. All we know at present is that this minimal number is 4 or 5.

We do not claim that $Y(\mathbb{X})$ above is unique but, for each $\mathbb{X}$, we always work with a fixed one satisfying (2.2).

(G) Assume that $\mathrm{cmp}(\mathbb{P}) = |\mathrm{Comp}(\mathbb{P})|$ is at least 2, and let $\mathbb{M}_1$ and $\mathbb{M}_2$ be two distinct components of $\mathbb{P}$. Let $\mathbb{P}^{(0)}_{\mathbb{M}_1,\mathbb{M}_2}$ denote the poset that we obtain from $\mathbb{P}$ by omitting $\mathbb{M}_1$ and $\mathbb{M}_2$. That is, $\mathbb{P}^{(0)}_{\mathbb{M}_1,\mathbb{M}_2}$ is the subposet of $\mathbb{P}$ determined by $P^{(0)}_{\mathbb{M}_1,\mathbb{M}_2} = P \setminus (M_1 \cup M_2)$. If $\mathrm{cmp}(\mathbb{P}) = 2$ then, exceptionally, $\mathbb{P}^{(0)}_{\mathbb{M}_1,\mathbb{M}_2}$ is the "empty poset". We define the *thread number* $\mathrm{thn}(\mathbb{P}, \mathbb{M}_1, \mathbb{M}_2)$ as follows.

$$\text{If } \mathbb{P} \text{ is an antichain, then } \mathrm{thn}(\mathbb{P}, \mathbb{M}_1, \mathbb{M}_2) := 1. \tag{2.3}$$

If $\mathbb{P}$ is not an antichain and $\mathrm{cmp}(\mathbb{P}) = 2$, then

$$\mathrm{thn}(\mathbb{P}, \mathbb{M}_1, \mathbb{M}_2) := \prod_{i=1}^{2} |\mathrm{Max}(\mathbb{M}_i)| + \prod_{i=1}^{2} |\mathrm{Min}(\mathbb{M}_i)|. \tag{2.4}$$

If $\mathbb{P}$ is not an antichain and $\mathrm{cmp}(\mathbb{P}) \geq 3$, then

$$\begin{aligned} &\mathrm{thn}(\mathbb{P}, \mathbb{M}_1, \mathbb{M}_2) := \\ &\max\{|\mathrm{Max}(\mathbb{X})| : \mathbb{X} \in \mathrm{Comp}(\mathbb{P}^{(0)}_{\mathbb{M}_1,\mathbb{M}_2})\} \cdot \prod_{i=1}^{2} |\mathrm{Max}(\mathbb{M}_i)| + \\ &\max\{|\mathrm{Min}(\mathbb{X})| : \mathbb{X} \in \mathrm{Comp}(\mathbb{P}^{(0)}_{\mathbb{M}_1,\mathbb{M}_2})\} \cdot \prod_{i=1}^{2} |\mathrm{Min}(\mathbb{M}_i)|. \end{aligned} \tag{2.5}$$

Based on (2.3), (2.4), (2.5), and Part (F) of (this) Definition 2.3, we define the *combined parameter* $\mathrm{cp}(\mathbb{P})$ of $\mathbb{P}$ by the following equation.

$$\begin{aligned} \mathrm{cp}(\mathbb{P}) := \min\bigl\{ &f_4(\mathrm{cmp}(\mathbb{P})) \cdot \mathrm{thn}(\mathbb{P}, \mathbb{M}_1, \mathbb{M}_2) + \mathrm{spp}(\mathbb{M}_1) \\ &+ \mathrm{spp}(\mathbb{M}_2) : \mathbb{M}_1, \mathbb{M}_2 \in \mathrm{Comp}(\mathbb{P}) \text{ and } \mathbb{M}_1 \neq \mathbb{M}_2 \bigr\}. \end{aligned} \tag{2.6}$$

If $\mathrm{Comp}(\mathbb{P})$ contains two or more chains, then we attain the minimum in (2.6) by selecting $\mathbb{M}_1$ and $\mathbb{M}_2$ as the smallest chain components.

Based on Definitions 2.1, 2.2, and 2.3(A)–(G), we are in the position to formulate our result on the lattice $\mathrm{Quo}^{\mathrm{e}}(\mathbb{P})$ of all of quasiorders of $\mathbb{P}$ that extend $\leq$, which is also denoted by μ.

Theorem 2.4. *Let $\mathbb{P} = (P; \leq) = (P; \mu)$ be a poset of finite component size $\mathrm{csize}(\mathbb{P})$ such that $|P|$ is an accessible cardinal. Then the following three assertions hold.*

(A) *If $\mathrm{cmp}(\mathbb{P}) \geq 2$, then $\mathrm{Quo}^{\mathrm{e}}(\mathbb{P})$ has a complete-generating set E such that*

$$|E| \leq \mathrm{cp}(\mathbb{P}) + \mathrm{ce}(\mathbb{P}). \tag{2.7}$$

(B) *If $\mathbb{P}$ is a forest and $\mathrm{cmp}(\mathbb{P}) \geq 2$, then $\mathrm{Quo}^{\mathrm{e}}(\mathbb{P})$ has a complete-generating set E such that*

$$|E| \leq \mathrm{cp}(\mathbb{P}) + \mathrm{LASp}(\mathrm{ce}(\mathbb{P})). \tag{2.8}$$

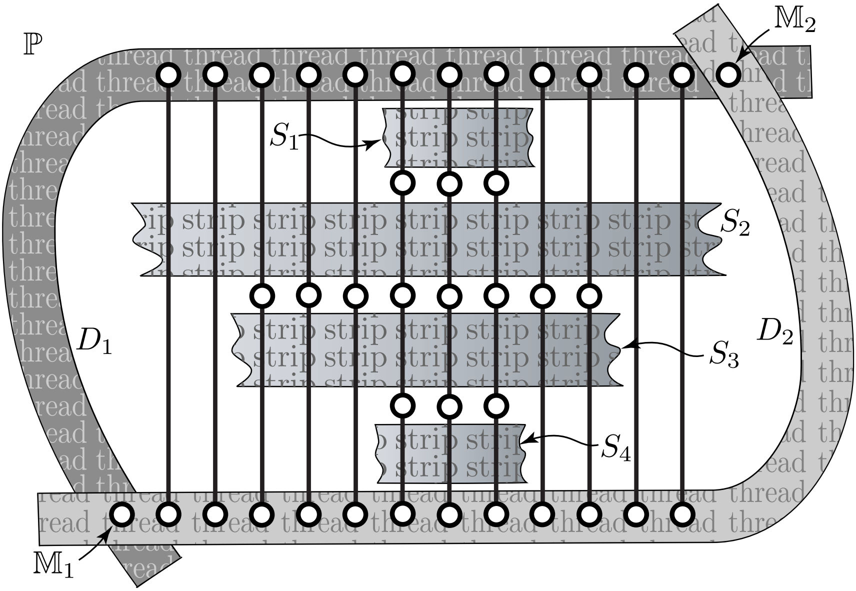

FIGURE 2. A forest $\mathbb{P}$; see Example 2.7.

(C) *If* $\mathbb{P}$ *is a finite chain, then* $\mathrm{Quo^e}(\mathbb{P})$ *is* $\mathrm{LASp}(\mathrm{ce}(\mathbb{P}))$*-generated but not* $(\mathrm{LASp}(\mathrm{ce}(\mathbb{P}))-1)$*-generated.*

As $\mathbb{P}$ is of finite component size, each of (2.7), (2.8), and $\mathrm{LASp}(\mathrm{ce}(\mathbb{P}))$ in the theorem is an integer. For the sake of completeness, we include the following remark; it is less significant than Theorem 2.4 but provides additional information.

Remark 2.5. If $\mathbb{P}$ is a finite connected poset (and so $\mathrm{cmp}(\mathbb{P})=1$), then $\mathrm{Quo^e}(\mathbb{P})$ is $(\mathrm{ce}(\mathbb{P})+\mathrm{spp}(\mathbb{P})-1)$ generated, and it is even $(\mathrm{LASp}(\mathrm{ce}(\mathbb{P}))+\mathrm{spp}(\mathbb{P})-1)$ generated if $\mathbb{P}$ is a finite tree.

We are going to prove this remark at the end of Section 3. For our figures, Theorem 2.4 asserts the following.

Example 2.6. For $\mathbb{P}$ drawn in Figure 1, $\mathrm{Quo^e}(\mathbb{P})$ is 22-generated. Indeed, $\mathrm{Comp}(\mathbb{P})=\{\mathbb{T}_1,\dots,\mathbb{T}_7\}$, $\mathrm{cmp}(\mathbb{P})=7$, $\mathrm{csize}(\mathbb{P})=6$, and $\mathrm{ce}(\mathbb{P})=5$. According to the sentence right after (2.6), we can select $\mathbb{M}_1:=\mathbb{T}_6$ and $\mathbb{M}_2:=\mathbb{T}_7$. Then $\mathrm{thn}(\mathbb{P},\mathbb{M}_1,\mathbb{M}_2)=2\cdot1\cdot1+2\cdot1\cdot1=4$, $f_4(\mathrm{cmp}(\mathbb{P}))=4$, $\mathrm{spp}(\mathbb{M}_1)=1$, $\mathrm{spp}(\mathbb{M}_2)=1$, $\mathrm{cp}(\mathbb{P})=4\cdot4+1+1=18$, $\mathrm{LASp}(\mathrm{ce}(\mathbb{P}))=4$, and (2.8) gives 22. (The figure indicates some objects that will be needed only later.)

Example 2.7. For $\mathbb{P}$ drawn in Figure 2, $\mathrm{Quo^e}(\mathbb{P})$ is 12-generated. Indeed, $\mathrm{thn}(\mathbb{P},\mathbb{M}_1,\mathbb{M}_2)=1+1=2$, $f_4(\mathrm{cmp}(\mathbb{P}))=4$, $\mathrm{spp}(\mathbb{M}_1)=0=\mathrm{spp}(\mathbb{M}_2)$, $\mathrm{cp}(\mathbb{P})=8$, $\mathrm{ce}(\mathbb{P})=4=\mathrm{LASp}(\mathrm{ce}(\mathbb{P}))$, and (2.8) applies.

Remark 2.8. To explain the connection of the main result and the title of the paper, it is reasonable to say that a filter is a *large filter* if it is a principal filter $\mathrm{fil}(u)$ such that the height of u (that is, the length of $\mathrm{idl}(u)$) is small. For a set A and $\mu \in \mathrm{Quo}(A)$, let $\Theta(\mu)$ denote $\mu \cap \mu^{-1} = \{(x, y) \in A^2 : (x, y) \in \mu \text{ and } (y, x) \in \mu\}$. Then $\Theta(\mu) \in \mathrm{Equ}(A)$, allowing us to consider the *quotient set* $A/\Theta(\mu)$ and *the quotient quasiorder* $\mu/\Theta(\mu) = \{\big(a/\Theta(\mu), b/\Theta(\mu)\big) : (a, b) \in \mu\}$. Clearly, $\mu/\Theta(\mu)$ is a partial order of $A/\Theta(\mu)$. There are two paths to establish that the filters $\mathrm{fil}_{\mathrm{Quo}(A)}(\mu)$ and $\mathrm{fil}_{\mathrm{Quo}(A/\Theta(\mu))}(\mu/\Theta(\mu))$ are isomorphic lattices. First, this statement is a particular case of (the last sentence of) Czédli and Lenkehegyi [11, Theorem 1.6]. Second, a straightforward argument yields the required isomorphism. Therefore, a large filter $\mathrm{fil}_{\mathrm{Quo}(A)}(\mu)$ of $\mathrm{Quo}(A)$ is, up to isomorphism, of the form $\mathrm{Quo}^{\mathrm{e}}(\mathbb{P})$ where $\mathbb{P} := \big(A/\Theta(\mu); \mu/\Theta(\mu)\big)$.

3. Proofs

Proof of Theorem 2.4. While the meet operation in $\mathrm{Quo}(P)$ and $\mathrm{Quo}^{\mathrm{e}}(\mathbb{P})$ is the set theoretic intersection, the join in these complete lattices are described with the help of the reflexive and the transitive closure rtr as follows:

$$\begin{aligned} \bigvee\{\rho_j : j \in J\} &= \mathrm{rtr}\Big(\bigcup\{\rho_j : j \in J\}\Big) \text{ and} \\ \bigvee{}^{\mu}\{\tau_j : j \in J\} &= \mathrm{rtr}\Big(\bigcup\{\tau_j : j \in J\}\Big)\,; \end{aligned} \tag{3.1}$$

here and later, rtr is understood over P. Since $\mathrm{Quo}^{\mathrm{e}}(\mathbb{P})$, being a principal filter, is a complete sublattice of $\mathrm{Quo}(P)$, we will use $\vee$ and $\bigvee$ for the elements of $\mathrm{Quo}^{\mathrm{e}}(\mathbb{P})$. To increase readability, we partition the rest of the proof into four parts.

Part 1. For a set $\gamma \subseteq P^2$, let $\mathrm{quo}(\gamma)$ and $\mathrm{qu}_\mu(\gamma)$ stand for the least quasiorder in $\mathrm{Quo}(P)$ and in $\mathrm{Quo}^{\mathrm{e}}(\mathbb{P})$, respectively, that includes γ. For $\gamma = \{(y, x)\}$, we write $\mathrm{quo}(y, x)$ and, as in Definition 2.3(C), $\mathrm{qu}_\mu(y, x)$. Note that

$$\begin{aligned} \gamma^+ = \mathrm{qu}_\mu(\gamma) &= \mu \vee \mathrm{quo}(\gamma) \text{ and, in particular,} \\ \mathrm{qu}_\mu(x, y) &= \mathrm{qu}_\mu(\{(x, y)\}) = \mu \vee \mathrm{quo}(x, y). \end{aligned} \tag{3.2}$$

For a subset X of P, we define

$$\mathrm{Quo}^{\mathrm{e}}_X(\mathbb{P}) := \{\mathrm{rtr}(\mu \cup \rho) : \rho \in \mathrm{Quo}(X)\} = \{\rho^+ : \rho \in \mathrm{Quo}(X)\};$$

it is a subposet of $\mathrm{Quo}^{\mathrm{e}}(\mathbb{P})$. For a subset X of P, ∇_X denotes the "full relation" X^2 of X; then (3.2) defines $\nabla_X^+ \in \mathrm{Quo}^{\mathrm{e}}(\mathbb{P})$. Observe that

$$\text{if } \mathbb{X} \in \mathrm{Comp}(\mathbb{P}), \text{ then } \nabla_X^+ = \mu \cup \nabla_X. \tag{3.3}$$

Since $(p, q) \in \nabla_T \subseteq \nabla_T^+ \in \mathrm{Quo}^{\mathrm{e}}(\mathbb{P})$, (3.3) implies the validity of the following "independence statement".

Claim 3.1. If $\mathbb{X} \in \mathrm{Comp}(\mathbb{P})$, $p, q \in X$, and $x, y \in P \setminus X$, then $(x, y) \in \mathrm{qu}_\mu(p, q)$ if and only if $x \le y$. More generally, if $X \subseteq P$, $\rho \subseteq X^2$, and $x, y \in P \setminus X$, then $(x, y) \in \mathrm{qu}_\mu(\rho)$ if and only if $x \le y$.

Next, let $\mathbb{T} \in \mathrm{Comp}(\mathbb{P})$. For $\rho \in \mathrm{Quo}^{\mathrm{e}}(\mathbb{T})$, $\rho \subseteq T^2 \subseteq P^2$, whereby $\mathrm{qu}_\mu(\rho)$ makes sense. Hence, we can define a map $\psi_{\mathbb{T}} \colon \mathrm{Quo}^{\mathrm{e}}(\mathbb{T}) \to \mathrm{Quo}^{\mathrm{e}}(\mathbb{P})$ by $\rho \mapsto \mathrm{qu}_\mu(\rho)$. Using (3.1) or Claim 3.1, it follows easily that

$$\psi_{\mathbb{T}} \text{ defined above is a complete lattice embedding.} \tag{3.4}$$

Claim 3.2. If $\emptyset \ne X \subseteq P$ such that X has an at most one-element intersection with each component of $\mathbb{P}$, then $\varphi_X \colon \mathrm{Quo}(X) \to \mathrm{Quo}^{\mathrm{e}}(\mathbb{P})$ defined by $\rho \mapsto \mathrm{rtr}(\mu \cup \rho)$ is a complete lattice embedding and $\mathrm{Quo}^{\mathrm{e}}_X(\mathbb{P}) = \varphi_X(\mathrm{Quo}(X))$; in particular, $\mathrm{Quo}^{\mathrm{e}}_X(\mathbb{P})$ is a complete sublattice of $\mathrm{Quo}^{\mathrm{e}}(\mathbb{P})$ and this sublattice is isomorphic to $\mathrm{Quo}(X)$.

Proof of Claim 3.2. For a set X, $\Delta_X := \{(x, x) : x \in X\}$ denotes the smallest element of $\mathrm{Quo}(X)$. To prove Claim 3.2, consider the functions

$$\varphi'_X \colon \mathrm{Quo}(X) \to \mathrm{Quo}(P) \text{ defined by } \varphi'_X(\rho) := \rho \cup \Delta_P \text{ and}$$
$$\varphi^\flat_X \colon \varphi'_X(\mathrm{Quo}(X)) \to \mathrm{Quo}^{\mathrm{e}}(\mathbb{P}) \text{ defined by } \varphi^\flat_X(\rho) := \rho \vee \mu.$$

It follows from (3.1) and $\Delta_P \cup \mu = \mu$ that $\varphi_X = \varphi^\flat_X \circ \varphi'_X$. Thus, as φ'_X is clearly an embedding, it suffices to show that $\varphi^\flat_X$ is also one. As $\mathrm{rtr}(\mu \cup \rho) = \mu \vee \rho$ by (3.1), it is clear that $\varphi^\flat_X$ commutes with joins. In particular, $\varphi^\flat_X$ is isotone. For $\rho \in \mathrm{Quo}(P)$, let ρ^- denote the *irreflexive version* $\rho \setminus \Delta_P$ of ρ. For $\rho \in \varphi'_X(\mathrm{Quo}(X))$, we have that $\rho^- \subseteq X^2$. Assume that $\rho_j \in \varphi'_X(\mathrm{Quo}(X))$ for all j in an index set J and $(x, y) \in \bigwedge_{j \in J} \varphi^\flat_X(\rho_j) = \bigcap_{j \in J} \varphi^\flat_X(\rho_j)$; we need to show that $(x, y) \in \varphi^\flat_X\big(\bigcap_{j \in J} \rho_j\big)$. Since this is trivial if $(x, y) \in \mu$, we assume that $(x, y) \notin \mu$, that is, $x \nleq y$. As $(x, y) \in \bigcap_{j \in J} \varphi^\flat_X(\rho_j)$, we have that $(x, y) \in \varphi^\flat_X(\rho_j) = \rho_j \vee \mu$ for all $j \in J$. We claim that, for each $j \in J$, there are elements $z_0 = x$, z_1, z_2, and $z_3 = y$ such that

$$z_1, z_2 \in X,\ z_1 \ne z_2,\ (z_0, z_1), (z_2, z_3) \in \mu, \text{ and } (z_1, z_2) \in \rho_j^-; \tag{3.5}$$

at present, these elements might depend on j. To show the existence of these elements, note that (3.1) and $(x, y) \in (\rho_j \vee \mu) \setminus \mu$ provide us with a *minimal* integer $n \in \mathbb{N}^+$ and elements $z'_0, z'_1, \dots, z'_n \in P$ such that $z'_0 = x$, $z'_n = y$, and for each $i \in \{1, \dots, n\}$, the pair (z'_{i-1}, z'_i) is either in ρ_j^- or in $\mu \setminus \rho_j$. In particular, $z'_{i-1} \ne z'_i$. As $(x, y) \notin \mu$, there is an i such that $(z'_{i-1}, z'_i) \in \rho_j^-$; we assert that this i is uniquely determined. For the sake of contradiction, assume that there is an $s \in \{1, \dots, n\}$ such that $s \ne i$ and (z'_{s-1}, z'_s) is also in ρ_j^-; we can also assume that s is chosen so that $s - i > 0$ and $s - i$ is minimal. There are two cases. First, let $s = i + 1$. Then $(z'_{i-1}, z'_{i+1}) \in \rho_j$ since ρ_j is transitive. Hence, either $z'_{i-1} \ne z'_{i+1}$ and z'_i could be removed from the sequence or $z'_{i-1} = z'_{i+1}$ and both z'_i and z'_{i+1} could be removed; both possibilities contradict the minimality of n. Second, let $i + 1 < s$. Then the minimality of $s - i$ yields that the pairs (z'_i, z'_{i+1}), ..., (z'_{s-2}, z'_{s-1}) are in $\mu \setminus \rho_j \subseteq \mu^-$. By the transitivity of μ^-, $(z'_i, z'_{s-1}) \in \mu^-$, that is, $z'_i < z'_{s-1}$. Hence, z'_i and z'_{s-1} are in the same component $\mathbb{Y}$ of $\mathbb{P}$. Since (z'_{i-1}, z'_i) and (z'_{s-1}, z'_s) belong to ρ_j^-, both z'_i and z'_{s-1} are in X. Hence, $X \cap Y$ contains two distinct elements, z'_i and z'_{s-1}. This is a contradiction again, proving the uniqueness of i. By this uniqueness, $(z'_{t-1}, z'_t) \in \mu \setminus \rho_j \subseteq \mu$ for $t \in \{1, \dots, n\} \setminus \{i\}$, and the transitivity of μ implies that (z'_0, z'_i) and (z'_{i+1}, z'_n) are in μ. Thus, we conclude the validity of (3.5) by letting $(z_0, z_1, z_2, z_3) := (z'_0, z'_{i-1}, z'_i, z'_n)$.

With the elements occurring in (3.5), we proceed as follows. Let $\mathbb{Y}_0$ and $\mathbb{Y}_3$ be the components of $\mathbb{P}$ that contain $x = z_0$ and $y = z_3$, respectively. As (z_0, z_1) and (z_2, z_3) are in μ, we have that $z_1 \in Y_0$ and $z_2 \in Y_3$. Since $(z_1, z_2) \in \rho_j^-$, we also have that $z_1, z_2 \in X$. Hence, $z_1 \in X \cap Y_0$ and $z_2 \in X \cap Y_3$. Combining this with $|X \cap Y_0| \le 1$ and $|X \cap Y_3| \le 1$, we obtain that z_1 and z_2 are uniquely determined, and so they do not depend on j. Hence $(z_1, z_2) \in \bigcap_{j \in J} \rho_j$, and we obtain that $(x, y) \in \left(\bigcap_{j \in J} \rho_j\right) \vee \mu = \varphi^\flat_X\left(\bigcap_{j \in J} \rho_j\right)$. So $\bigcap_{j \in J} \varphi^\flat_X(\rho_j) \subseteq \varphi^\flat_X\left(\bigcap_{j \in J} \rho_j\right)$. The converse inclusion is obvious since $\varphi^\flat_X$ is isotone. Thus, $\varphi^\flat_X$ commutes with meets. So $\varphi^\flat_X$ is a complete lattice homomorphism. For $\rho \in \mathrm{Quo}(P)$, let $\rho\rceil_X$ denote the restriction of ρ to X. That is, $\rho\rceil_X = \rho \cap (X \times X)$. The assumptions of Claim 3.2 together with (3.1) imply easily that for $\rho \in \varphi'_X(\mathrm{Quo}(X))$, we have that $\rho = \varphi^\flat_X(\rho)\rceil_X \cup \Delta_P$. This yields the injectivity of $\varphi^\flat_X$ and we have proved Claim 3.2. □

Resuming the proof of Theorem 2.4, for a poset $\mathbb{X}$, we define

$$\begin{aligned} \mathrm{Edge}^{-1}(\mathbb{X}) &:= \{(x, y) : (y, x) \in \mathrm{Edge}(\mathbb{X})\} \\ &= \{(x, y) : x, y \in X \text{ and } y \prec x\}. \end{aligned}$$

With the help of this notation, we introduce two crucial concepts for the proof as follows. By a *strip* of $\mathbb{P}$, we mean a nonempty subset S of $\mathrm{Edge}^{-1}(\mathbb{P})$ such that for all $\mathbb{X} \in \mathrm{Comp}(\mathbb{P})$, we have that $|S \cap \mathrm{Edge}^{-1}(\mathbb{X})| \le 1$. An *upper thread* is a subset D of $\mathrm{Max}(\mathbb{P})$ such that for all $\mathbb{X} \in \mathrm{Comp}(\mathbb{P})$, we have that $|D \cap \mathrm{Max}(\mathbb{X})| = 1$. Similarly, if $D \subseteq \mathrm{Min}(\mathbb{P})$ such that $|D \cap \mathrm{Min}(\mathbb{X})| = 1$ for all $\mathbb{X} \in \mathrm{Comp}(\mathbb{P})$, then D is a *lower thread*. Upper threads and lower threads are called *threads*. Note the difference: "≤ 1" for strips but "$= 1$" for threads. Some strips, the S_i's, and are visualized in Figures 1 and 2; these figures present all threads, the D_i's. The stipulations on $\mathbb{P}$ allow us to choose a partition $\mathrm{Stp}(\mathbb{P})$ of $\mathrm{Edge}^{-1}(\mathbb{P})$ such that

$$\mathrm{Stp}(\mathbb{P}) \text{ consists of } \mathrm{ce}(\mathbb{P}) \text{ blocks and these blocks are strips of } \mathbb{P}. \tag{3.6}$$

E.g., we can choose $\mathrm{Stp}(\mathbb{P}) := \{S_1, \dots, S_5\}$ and $\mathrm{Stp}(\mathbb{P}) := \{S_1, \dots, S_4\}$ for $\mathbb{P}$ in Figure 1 and that in Figure 2, respectively.

Part 2. This part of the proof is needed only for Parts (B) and (C) of the theorem.

Claim 3.3. If $\mathbb{P}$ is a forest of finite component size, then for every $(v,u) \in \text{Edge}^{-1}(\mathbb{P})$, $\text{qu}_\mu(v,u) \not\leq \bigvee\{\text{qu}_\mu(y,x) : (y,x) \in \text{Edge}^{-1}(\mathbb{P}) \setminus \{(v,u)\}\}$.

Proof of Claim 3.3. Suppose the contrary. Then, combining (3.1) and (3.2) with $\mu = \text{rtr}(\text{Edge}(\mathbb{P}))$, we can find a *shortest* sequence $z_0 = v$, $z_1, \dots$, z_{n-1}, $z_n = u$ of elements of P such that for each $s \in \{1, \dots, n\}$, either $z_{s-1} \prec z_s$, or $z_{s-1} \succ z_s$ but $(z_{s-1}, z_s) \neq (v,u)$. Our sequence, being a shortest one, is repetition-free. Hence, in particular, none of the $z_1, \dots$, z_{n-1} is u or v. But then $z_0 = v$, $z_1, \dots$, z_{n-1}, $z_n = u$, $z_{n+1} = z_0$ is a non-singleton circle in the graph of $\mathbb{P}$, which is a contradiction proving Claim 3.3. □

For a strip $S \in \text{Stp}(\mathbb{P})$, see (3.6), S is a set of reversed edges and so a subset of P^2, whence $\text{qu}_\mu(S)$ makes sense. For $\mathbb{T} \in \text{Comp}(\mathbb{P})$, $|\text{Edge}^{-1}(\mathbb{T}) \cap S| \leq 1$. Thus, using a shortest sequence like in the previous paragraph and observing that this sequence cannot "jump" from one component to another one, we obtain easily that if $(y,x) \in S \in \text{Stp}(\mathbb{P})$ and $x, y \in \mathbb{T} \in \text{Comp}(\mathbb{P})$,

$$\text{then } \text{qu}_\mu(y,x) = \text{qu}_\mu(S) \cap \nabla_T^+. \tag{3.7}$$

Claim 3.4. If $\mathbb{P}$ is a forest of finite component size, then $\text{Quo}^{\text{e}}(\mathbb{P})$ contains an at most $\text{LASp}(\text{ce}(\mathbb{P}))$-element subset G such that for each $S \in \text{Stp}(\mathbb{P})$, $\text{qu}_\mu(S)$ is the meet of some members of G.

Proof of Claim 3.4. For a set H, we denote the *power set lattice* of H by $\text{Pow}(H)$; in this lattice, $\vee$, $\wedge$ and $\leq$ are $\cup$, $\cap$, and $\subseteq$, respectively. Since $|\text{Stp}(\mathbb{P})| = \text{ce}(\mathbb{P})$, we know from Czédli [9] that $\text{Pow}(\text{Stp}(\mathbb{P}))$ has a $\text{LASp}(\text{ce}(\mathbb{P}))$-element generating set G_0. The members of G_0 are subsets of $\text{Stp}(\mathbb{P})$. For each strip S, the singleton $\{S\} \in \text{Pow}(\text{Stp}(\mathbb{P}))$ belongs to the sublattice $[G_0]$ generated by G_0. Since $\text{Pow}(\text{Stp}(\mathbb{P}))$ is distributive, $\{S\}$ is obtained by applying a disjunctive normal form to appropriate elements of G_0. That is, $\{S\}$ is the union of intersections of elements of $\text{Pow}(\text{Stp}(\mathbb{P}))$. However, $\{S\}$ is join-irreducible (i.e., union-irreducible), whereby no union is needed and $\{S\}$ is the intersection of some members of G_0. So

$$\{S\} = \bigcap\{X : S \in X \in G_0\} \text{ for every } S \in \text{Stp}(\mathbb{P}). \tag{3.8}$$

For $X \in G_0$, we let

$$\text{qu}_\mu(X) := \bigvee\{\text{qu}_\mu(S) : S \in X\} = \text{qu}_\mu\big(\bigcup\{S : S \in X\}\big), \tag{3.9}$$

$$\text{and we define } G := \{\text{qu}_\mu(X) : X \in G_0\}. \tag{3.10}$$

We are going to show that G satisfies the requirements of Claim 3.4. As $|G_0| = \text{LASp}(\text{ce}(\mathbb{P}))$, we have that $|G| \leq \text{LASp}(\text{ce}(\mathbb{P}))$, as required.

Next, we are going to show that for S occurring in (3.8), we have that

$$\text{qu}_\mu(S) = \bigwedge\{\text{qu}_\mu(X) : X \in G_0 \text{ and } S \in X\}. \tag{3.11}$$

The "$\subseteq$" part of (3.11) is clear since $S \in X$ implies, by (3.9), that $\mathrm{qu}_\mu(S) \le \mathrm{qu}_\mu(X)$. To verify the converse inclusion, assume that $p, q \in P$ such that

$$p \neq q \ \text{ and } \ (p,q) \in \mathrm{qu}_\mu(X) \text{ for all } X \in G_0 \text{ containing } S. \tag{3.12}$$

For an X such that $S \in X \in G_0$, as in (3.12), the containment $(p,q) \in \mathrm{qu}_\mu(X)$, $\mu = \mathrm{rtr}(\mathrm{Edge}(\mathbb{P}))$, and (3.1) yield a *shortest* sequence[4]

$$\vec{r}(X) = \big(r_0(X) = p,\, r_1(X),\, \dots,\, r_{m(X)-1}(X), r_{m(X)}(X) = q\big) \tag{3.13}$$

of elements of P such that for each $s \in \{1, \dots, m(X)\}$,

(c1) $r_{s-1}(X) \prec r_s(X)$ or
(c2) $(r_{s-1}(X), r_s(X)) \in \mathrm{qu}_\mu(S')$ for some $S' \in X$.

By the same reason, the containment in $\mathrm{qu}_\mu(S')$ in (c2) also has an expansion to a sequence of elements of P. Thus, assuming that this expansion has already been made, we can replace (c2) by the following condition:

(c2′) $(r_{s-1}(X), r_s(X)) \in S' \subseteq \mathrm{Edge}^{-1}(\mathbb{P})$ for some $S' \in X$.

So, for each $s \in \{1, \dots, m(X)\}$, either (c1) or (c2′) holds. In both cases, $r_{s-1}(X)$ and $r_s(X)$ are the two endpoints of an edge of the (undirected) graph of $\mathbb{P}$. Hence, $\vec{r}(X)$ is a path in the graph of $\mathbb{P}$. In a forest, the shortest path connecting two vertices is unique (as otherwise we would get a circle). Consequently, for $X \in G_0$ such that $S \in X$,

$$\text{the sequence } \vec{r}(X) \text{ does not depend on } X. \tag{3.14}$$

This allows us to write m, $\vec{r}$, and r_i instead of $m(X)$, $\vec{r}(X)$, and $r_i(X)$, respectively. As distinct strips are disjoint, S' in (c2′) is uniquely determined. Keeping in mind that X has contained S since (3.12), (3.14) and (c2′) yield that for each $s \in \{1, \dots, m\}$, either (c1), i.e. $r_{s-1} \prec r_s$, or

(c3) $(r_{s-1}, r_s) \in S' \subseteq \mathrm{Edge}^{-1}(\mathbb{P})$ for some S' belonging to all the $X \in G_0$ that contain S.

It follows from (3.8) that S' in (c3) is S. Hence, if (c3) holds for a subscript $s \in \{1, \dots, m\}$, then $(r_{s-1}, r_s) \in S \subseteq \mathrm{qu}_\mu(S)$. If (c1) holds for this s, then $(r_{s-1}, r_s) \in \mathrm{qu}_\mu(S)$ again since $\mu \subseteq \mathrm{qu}_\mu(S)$. Therefore, we obtain that $(p,q) = (r_0, r_m) \in \mathrm{qu}_\mu(S)$ by transitivity. We have obtained the converse inclusion for (3.11). Hence, we have proved (3.11) and, thus, Claim 3.4. □

Next, we present a "convexity property" of $\mathbb{P}$; it holds for any poset, not only for $\mathbb{P}$ in Theorem 2.4.

Claim 3.5. For $a < b \in \mathbb{P}$ and $\rho \in \mathrm{Quo}^{\mathrm{e}}(\mathbb{P})$, we have that $(b,a) \in \rho$ if and only if $(y,x) \in \rho$ for all edges $[x,y]$ such that $a \le x \prec y \le b$.

Proof of Claim 3.5. The "if part" of Claim 3.5 is clear by transitivity. To see the "only if part", assume that $(b,a) \in \rho$ and $[x,y]$ is an edge in the interval $[a,b]$. Then $(y,b) \in \mu \subseteq \rho$, $(b,a) \in \rho$, and $(a,x) \in \mu \subseteq \rho$ imply $(y,x) \in \rho$ by transitivity, proving Claim 3.5. □

[4]For convenience, we write this sequence as a vector.

Part 3. Now we are in the position to prove Part (C) of the theorem. So let $\mathbb{P}$ be a chain of length $\mathrm{ce}(\mathbb{P})$ for a while. By Kim, Kwon, and Lee [14], $\mathrm{Quo}^{\mathrm{e}}(\mathbb{P})$ is a Boolean lattice with $\mathrm{ce}(\mathbb{P})$ atoms. Alternatively, since all the necessary tools occur in the present paper, we can argue shortly as follows. For each member ρ of $\mathrm{Quo}^{\mathrm{e}}(\mathbb{P})$, Claim 3.5 gives that

$$\rho = \bigvee\{\mathrm{qu}_\mu(y,x) : (y,x) \in \mathrm{Edge}^{-1}(\mathbb{P}) \cap \rho\}.$$

By Claim 3.3, none of the joinands can be removed, and it follows that $\mathrm{Quo}^{\mathrm{e}}(\mathbb{P})$ is (order) isomorphic to the powerset lattice of $\mathrm{Edge}^{-1}(\mathbb{P})$, whereby it is isomorphic to the Boolean lattice with $\mathrm{ce}(\mathbb{P})$ atoms. Therefore, Part (C) of Theorem 2.4 follows from Czédli [9].

Part 4. Now we turn our attention to Parts (A) and (B) of the theorem. For an antichain, the theorem asserts the same as Lemma 1.1. Thus, in the rest of the proof, we assume that $\mathbb{P}$ is not an antichain. For Part (B), we choose G according to Claim 3.4. For Part (A), let $G := \mathrm{Stp}(\mathbb{P})$; see (3.6). The following fact is trivial for Part (A) and it follows from Claim 3.4 for Part (B).

Fact 3.6. $|G| = \mathrm{ce}(\mathbb{P})$ for Part (A), $|G| = \mathrm{LASp}(\mathrm{ce}(\mathbb{P}))$ for Part (B), and (for both of these two parts of the theorem) for each $S \in \mathrm{Stp}(\mathbb{P})$, $\mathrm{qu}_\mu(S)$ is the meet of some members of G.

Next, we are going to choose a set $\mathrm{Thd}(\mathbb{P})$ of threads of $\mathbb{P}$; it is not unique in general. Choose $\mathbb{M}_1, \mathbb{M}_2 \in \mathrm{Comp}(\mathbb{P})$ such that

$$\mathbb{M}_1 \text{ and } \mathbb{M}_2 \text{ witness the minimum in (2.6).} \tag{3.15}$$

Although $\mathbb{M}_1$ and $\mathbb{M}_2$ are not uniquely determined in general, we fix an $\mathbb{M}_1$ and an $\mathbb{M}_2$ satisfying (3.15). If $\mathrm{cmp}(\mathbb{P}) \geq 3$, then choose a set $T^{(\mathrm{up})}$ of upper threads of $\mathbb{P}^{(0)}_{\mathbb{M}_1,\mathbb{M}_2}$ such that

$$|T^{(\mathrm{up})}| = \max\{|\mathrm{Max}(\mathbb{X})| : \mathbb{X} \in \mathrm{Comp}(\mathbb{P}^{(0)}_{\mathbb{M}_1,\mathbb{M}_2})\}$$

and each $x \in \mathrm{Max}(\mathbb{P}^{(0)}_{\mathbb{M}_1,\mathbb{M}_2})$ belongs to a thread $D \in T^{(\mathrm{up})}$. Dually, if $\mathrm{cmp}(\mathbb{P}) \geq 3$, then we choose a set $T^{(\mathrm{lo})}$ of lower threads of $\mathbb{P}^{(0)}_{\mathbb{M}_1,\mathbb{M}_2}$ such that

$$|T^{(\mathrm{lo})}| = \max\{|\mathrm{Min}(\mathbb{X})| : \mathbb{X} \in \mathrm{Comp}(\mathbb{P}^{(0)}_{\mathbb{M}_1,\mathbb{M}_2})\}$$

and $\mathrm{Min}(\mathbb{P}^{(0)}_{\mathbb{M}_1,\mathbb{M}_2}) = \bigcup\{D : D \in T^{(\mathrm{lo})}\}$. For $\mathbb{P}$ in Figure 1, $T^{(\mathrm{up})} = \{D_1, D_2\}$ and $T^{(\mathrm{lo})} = \{D_3, D_4\}$; in Figure 2, $T^{(\mathrm{up})} = \{D_1\}$ and $T^{(\mathrm{lo})} = \{D_2\}$. If $\mathrm{cmp}(\mathbb{P}) = 2$, then we consider $\emptyset$ a thread of the "empty poset" $\mathbb{P}^{(0)}_{\mathbb{M}_1,\mathbb{M}_2}$ and we let $T^{(\mathrm{up})} := \{\emptyset\}$ and $T^{(\mathrm{lo})} := \{\emptyset\}$. By adding an element of $\mathrm{Max}(\mathbb{M}_1)$ and that of $\mathrm{Max}(\mathbb{M}_2)$ to a thread $D \in T^{(\mathrm{up})}$, we can extend D to an upper thread of $\mathbb{P}$; note that D has $|\mathrm{Max}(\mathbb{M}_1)|\cdot|\mathrm{Max}(\mathbb{M}_2)|$ such extensions. Taking all $D \in T^{(\mathrm{up})}$ into account, these extensions form a set that we denote by $T^{(\mathrm{up})}_{\mathrm{ext}}$. We define $T^{(\mathrm{lo})}_{\mathrm{ext}}$ dually, and we let $\mathrm{Thd}(\mathbb{P}) := T^{(\mathrm{up})}_{\mathrm{ext}} \cup T^{(\mathrm{lo})}_{\mathrm{ext}}$. As $\mathbb{P}$

is not an antichain, $T^{(\mathrm{up})}_{\mathrm{ext}} \cap T^{(\mathrm{lo})}_{\mathrm{ext}} = \emptyset$. By (2.4) and (2.5), $|\mathrm{Thd}(\mathbb{P})| = \mathrm{thn}(\mathbb{P}, \mathbb{M}_1, \mathbb{M}_2)$.

Clearly, for all $\mathbb{T} \in \mathrm{Comp}(\mathbb{P})$, $i \in \{1, 2\}$, $x_* \in \mathrm{Min}(\mathbb{T})$, $y_* \in \mathrm{Min}(\mathbb{M}_i)$, $x^* \in \mathrm{Max}(\mathbb{T})$, and $y^* \in \mathrm{Max}(\mathbb{M}_i)$, there exist $D', D'' \in \mathrm{Thd}(\mathbb{P})$ such that

$$\{x_*, y_*\} \subseteq D' \text{ and } \{x^*, y^*\} \subseteq D''. \tag{3.16}$$

For $D \in \mathrm{Thd}(\mathbb{P})$, $|D| = \mathrm{cmp}(\mathbb{P})$ by definitions. So Lemma 1.1 and Definition 2.2 allow us to pick an $f_4(\mathrm{cmp}(\mathbb{P}))$-element generating set $H^{(0)}_D$ of $\mathrm{Quo}(D)$. By Claim 3.2, $\mathrm{Quo}^{\mathrm{e}}_D(\mathbb{P})$, which is a complete sublattice of $\mathrm{Quo}^{\mathrm{e}}(\mathbb{P})$, also has an $f_4(\mathrm{cmp}(\mathbb{P}))$-element generating set $H_D := \varphi_D(H^{(0)}_D)$. Let

$$H := \bigcup_{D \in \mathrm{Thd}(\mathbb{P})} H_D;\text{ note that } |H| = \mathrm{thn}(\mathbb{P}, \mathbb{M}_1, \mathbb{M}_2) \cdot f_4(\mathrm{cmp}(\mathbb{P})). \tag{3.17}$$

Next, let $i \in \{1, 2\}$ and keep (3.15) in mind. If $\mathrm{spp}(\mathbb{M}_i) = 0$ (that is, if $|M_i| = 1$), then $F^{(0)}(\mathbb{M}_i) := \emptyset$. If $\mathbb{M}_i$ is a finite chain with more than one element, then let $F^{(0)}(\mathbb{M}_i) := \{\nabla_{M_i}\}$. If $\mathbb{M}_i$ is not a chain, then let $F^{(0)}(\mathbb{M}_i)$ be the union of $\{\nabla_{M_i}\}$ and the set $Y(\mathbb{M}_i)$; see (2.2). In all cases, $F^{(0)}(\mathbb{M}_i) \subseteq \mathrm{Quo}^{\mathrm{e}}(\mathbb{M}_i)$. Note that

$$\text{for } i \in \{1, 2\}, \quad |F^{(0)}(\mathbb{M}_i)| = \mathrm{spp}(\mathbb{M}_i). \tag{3.18}$$

Now we pass from $\mathbb{M}_i$ to $\mathbb{P}$. For $i \in \{1, 2\}$, let ψ_i stand for $\psi_{\mathbb{M}_i}$; see (3.4). Let $M^{\mathrm{sl}}_i := \psi_i\big(\mathrm{Quo}^{\mathrm{e}}(\mathbb{M}_i)\big)$ and $F(M^{\mathrm{sl}}_i) := \psi_i(F^{(0)}(\mathbb{M}_i))$; the superscript of M^{sl}_i is just a reminder to "**s**ub**l**attice". Note the difference: $\mathbb{M}_i$ is in $\mathrm{Comp}(\mathbb{P})$ and so it consists of some elements of P while M^{sl}_i is a sublattice of $\mathrm{Quo}^{\mathrm{e}}(\mathbb{P})$ and consists of some quasiorders of P. As a subset of M^{sl}_i, $F(M^{\mathrm{sl}}_i)$ is also a subset of $\mathrm{Quo}^{\mathrm{e}}(\mathbb{P})$. It is clear by (2.2) and (3.18) that for $i \in \{1, 2\}$

$$F(M^{\mathrm{sl}}_i) \cup \mathrm{Edge}^{-1}(\mathbb{M}_i) \text{ generates } M^{\mathrm{sl}}_i, \quad \nabla^+_{M_i} \in F(M^{\mathrm{sl}}_i), \tag{3.19}$$

$$\text{and } \quad |F(M^{\mathrm{sl}}_i)| = \mathrm{spp}(\mathbb{M}_i). \tag{3.20}$$

Now Fact 3.6, (3.17), and (3.19) allow us to define

$$E := F(M^{\mathrm{sl}}_1) \cup F(M^{\mathrm{sl}}_2) \cup G \cup H. \tag{3.21}$$

It follows from Fact 3.6, (3.17), and (3.20) that, depending on whether we deal with Part (A) or Part (B) of the theorem, $|E|$ satisfies (2.7) or (2.8), respectively. Thus, it suffices to show that E generates $\mathrm{Quo}^{\mathrm{e}}(\mathbb{P})$. Let K be a complete sublattice of $\mathrm{Quo}^{\mathrm{e}}(\mathbb{P})$ such that $E \subseteq K$. To complete the proof, we need to show that $\mathrm{Quo}^{\mathrm{e}}(\mathbb{P}) \subseteq K$. It follows from $H_D \subseteq H \subseteq E$ that

$$\text{for all } D \in \mathrm{Thd}(\mathbb{P}), \quad \mathrm{Quo}^{\mathrm{e}}_D(\mathbb{P}) \subseteq K. \tag{3.22}$$

Next, we claim that

$$\text{for any } i \in \{1, 2\} \text{ and } (y, x) \in \mathrm{Edge}^{-1}(\mathbb{M}_i), \ \mathrm{qu}_\mu(y, x) \in K. \tag{3.23}$$

To see this, take the unique $S \in \mathrm{Stp}(\mathbb{P})$ that contains the reversed edge (y, x) of $\mathbb{M}_i$. Since $G \subseteq E$, Fact 3.6 yields that $\mathrm{qu}_\mu(S)$ is in K. So is $\nabla^+_{M_i}$ by

$F(M_i^{\rm sl})\subseteq E$ and (3.19). Hence (3.7) gives that $\mathrm{qu}_\mu(y,x)=\mathrm{qu}_\mu(S)\wedge\nabla^+_{M_i}\in K$, proving (3.23). Now it follows from (3.23), $F(M_i^{\rm sl})\subseteq E$, and (3.19) that for all $i\in\{1,2\}$,

$$M_i^{\rm sl}\subseteq K,\text{ that is, for any } x,y\in M_i,\ \ \mathrm{qu}_\mu(x,y)\in K. \tag{3.24}$$

Claim 3.7. For any $x_*\in\mathrm{Min}(\mathbb{P})$ and $y^*\in\mathrm{Max}(\mathbb{P})$, if $x_*\leq y^*$, then $\mathrm{qu}_\mu(y^*,x_*)\in K$.

Proof of Claim 3.7. Let $\mathbb{T}$ be the unique component of $\mathbb{P}$ that contains x_* and y^*. We can assume that $x_*<y^*$. By (3.24), we can also assume that $\mathbb{T}\notin\{\mathbb{M}_1,\mathbb{M}_2\}$. For $i\in\{1,2\}$, pick an element $x_{*i}\in\mathrm{Min}(\mathbb{M}_i)$ and an element $y_i^*\in\mathrm{Max}(\mathbb{M}_i)$ such that $x_{*i}\leq y_i^*$. In virtue of (3.16), we can select $D_i',D_i''\in\mathrm{Thd}(\mathbb{P})$ such that $\{x_*,x_{*i}\}\subseteq D_i'$ and $\{x^*,x_i^*\}\subseteq D_i''$. By the choice of H_D (with $D=D_i'$ or $D=D_i''$), Claim 3.2, and (3.17),

$$\mathrm{qu}_\mu(y^*,y_i^*)\in K\text{ and }\mathrm{qu}_\mu(x_{*i},x_*)\in K\text{ for }i\in\{1,2\}. \tag{3.25}$$

We know from (3.24) that

$$\text{for } i\in\{1,2\},\ \ \mathrm{qu}_\mu(y_i^*,x_{*i})\in K. \tag{3.26}$$

We assert that

$$h(y^*,x_*):=\bigwedge_{i=1}^{2}\bigl(\mathrm{qu}_\mu(y^*,y_i^*)\vee\mathrm{qu}_\mu(y_i^*,x_{*i})\vee\mathrm{qu}_\mu(x_{*i},x_*)\bigr)\in K \tag{3.27}$$

and

$$\mathrm{qu}_\mu(y^*,x_*)\leq h(y^*,x_*)\leq\nabla^+_T. \tag{3.28}$$

Observe that (3.27) is clear by (3.25) and (3.26). The first inequality in (3.28) is trivial by transitivity. To show the second inequality in (3.28), assume that $(u,v)\in P^2$ belongs to the meet in (3.27), that is, it belongs to $h(y^*,x_*)$. We also assume that $u\not\leq v$ since otherwise $(u,v)\in\nabla^+_T$ is trivial. Let $i\in\{1,2\}$. Using the description of joins, see (3.1), (3.2), and (3.3), we can take a *shortest* sequence $\vec{w}: w_0=u,w_1,w_2,\dots,w_{t-1},w_t=v$ of pairwise different elements of P such that for each $j\in\{1,\dots,t\}$, one of the following four alternatives holds:

(d1) $w_{j-1}<w_j$ (and so w_{j-1} and w_j belong to the same component of $\mathbb{P}$);
(d2) $(w_{j-1},w_j)=(y^*,y_i^*)$;
(d3) $(w_{j-1},w_j)=(y_i^*,x_{*i})$;
(d4) $(w_{j-1},w_j)=(x_{*i},x_*)$.

Since $u\not\leq v$, at least one of (d2), (d3), and (d4) holds some $j\in\{1,\dots,t\}$. Hence, at least one of the elements w_j belongs to $T\cup M_i$. This fact and the parenthesized comment in (d1) yield that all the w_j belong to $T\cup M_i$. In particular, $\{u,v\}=\{w_0,w_t\}\subseteq T\cup M_i$. As opposed to the elements w_j for $j\in\{1,\dots,t-1\}$, we know that $\{u,v\}$ does not depend on $i\in\{1,2\}$. Hence, $\{u,v\}\subseteq(T\cup M_i)\cap(T\cup M_{3-i})=T\cup(M_i\cap M_{3-i})=T\cup\emptyset=T$. Thus, $(u,v)\in\nabla^+_T$, proving the second inequality of (3.28) and (3.28) itself.

Based on (3.28) and (3.27), now we show that

$$\text{for any } u, v \in P, \text{ if } u < v, \text{ then } \mathrm{qu}_\mu(v,u) \in K. \tag{3.29}$$

First, we deal with the particular case where $(v,u) \in \mathrm{Edge}^{-1}(\mathbb{P})$. Let $S \in \mathrm{Stp}(\mathbb{P})$ be the unique strip that contains (v,u). Since $G \subseteq E$, we know from Fact 3.6 that $\mathrm{qu}_\mu(S)$ is in K. Pick $u_* \in \mathrm{Min}(\mathbb{P})$ and $v^* \in \mathrm{Max}(\mathbb{P})$ such that $u_* \leq u \prec v \leq v^*$ and let $\mathbb{T}$ be the unique component of $\mathbb{P}$ that contains these four elements. Combining (3.7), Claim 3.5, and (3.28), we have that

$$\begin{aligned}\mathrm{qu}_\mu(v,u) = \mathrm{qu}_\mu(S) \wedge \mathrm{qu}_\mu(v,u) &\overset{\text{Claim 3.5}}{\leq} \mathrm{qu}_\mu(S) \wedge \mathrm{qu}_\mu(v^*,u_*)\\ \overset{(3.28)}{\leq} \mathrm{qu}_\mu(S) \wedge h(v^*,u_*) &\overset{(3.28)}{\leq} \mathrm{qu}_\mu(S) \wedge \nabla_T^+ \overset{(3.7)}{=} \mathrm{qu}_\mu(v,u).\end{aligned}$$

These inequalities and (3.27) yield that $\mathrm{qu}_\mu(v,u) = \mathrm{qu}_\mu(S) \wedge h(v^*,u_*) \in K$. That is, the particular case $u \prec v$ of (3.29) holds. It follows from transitivity and Claim 3.5 that

$$\text{for any } a < b \in \mathbb{P}, \quad \mathrm{qu}_\mu(b,a) = \bigvee\{\mathrm{qu}_\mu(y,x) : a \leq x \prec y \leq b\}. \tag{3.30}$$

Thus, the particular case implies the general case, and we have shown (3.29). So we have also proved Claim 3.7 since it is a particular case of (3.29). ☐

Now we are ready to show that

$$\text{for each } \mathbb{T} = (T; \mu_T) \in \mathrm{Comp}(\mathbb{P}), \quad \nabla_T^+ \in K. \tag{3.31}$$

Here, of course, μ_T stands for the restriction $\mu\rceil_T$ of μ to T. We can assume that $|T| > 1$. In virtue of (3.29), it suffices to show that $\nabla_T^+ = \bigvee\{\mathrm{qu}_\mu(y,x) : (y,x) \in \mathrm{Edge}^{-1}(T)\}$, that is, $\nabla_T^+ = \mathrm{qu}_\mu(\mathrm{Edge}^{-1}(T))$. To see the "$\leq$" part, assume that $(x,y) \in \nabla_T^+$. As $\mathbb{T}$ is a *connected* component, there are elements $z_0 = x, z_1, \dots, z_{t-1}, z_t = y$ in T such $(z_{j-1}, z_j) \in \mathrm{Edge}(\mathbb{T}) \cup \mathrm{Edge}^{-1}(\mathbb{T})$ for every $j \in \{1, \dots, t\}$. Regardless whether (z_{j-1}, z_j) is in $\mathrm{Edge}(\mathbb{T})$ or it is in $\mathrm{Edge}^{-1}(\mathbb{T})$, we have that $(z_{j-1}, z_j) \in \mathrm{qu}_\mu(z_{j-1}, z_j) \subseteq \mathrm{qu}_\mu(\mathrm{Edge}^{-1}(\mathbb{T}))$. Thus the "$\leq$" part of the required equality follows by transitivity. As the converse inequality is trivial by Claim 3.1, we conclude (3.31).

The following easy assertion will be useful.

$$\text{For any } a, b \in P, \text{ we have that } \mathrm{qu}_\mu(a,b) = \mu \cup (\mathrm{idl}(a) \times \mathrm{fil}(b)). \tag{3.32}$$

The "$\geq$" part is clear by transitivity. To show the converse inequality, assume that $(p,q) \in \mathrm{qu}_\mu(a,b) \setminus \mu$. By (3.1) and (3.2), there is a sequence $x_0 = p, x_1, \dots, x_{t-1}, x_t = q$ of elements in P such that, for each $i \in \{0, \dots, t-1\}$, $(x_i, x_{i+1}) \in \mu$ or $(x_i, x_{i+1}) = (a,b)$. As $p \not\leq q$, there is at least one i such that $(x_i, x_{i+1}) = (a,b)$. The least such i shows that $p = x_0 \leq x_1 \cdots \leq x_i = a$, whence $p \in \mathrm{idl}(a)$, while the largest such i yields that $b = x_{i+1} \leq \cdots \leq x_t = q$, whereby $q \in \mathrm{fil}(b)$. Hence, $(p,q) \in \mathrm{idl}(a) \times \mathrm{fil}(b) \subseteq \mu \cup (\mathrm{idl}(a) \times \mathrm{fil}(b))$, proving (3.32).

Claim 3.8. For any $\mathbb{T}_1, \mathbb{T}_2 \in \text{Comp}(\mathbb{P})$, if $\mathbb{T}_1 \neq \mathbb{T}_2$, $\{\mathbb{M}_1, \mathbb{M}_2\} \cap \{\mathbb{T}_1, \mathbb{T}_2\} \neq \emptyset$, $a \in T_1$ and $b \in T_2$, then $\text{qu}_\mu(a, b) \in K$.

Proof of Claim 3.8. To show this, pick $a_* \in \text{Min}(\mathbb{T}_1)$, $a^* \in \text{Max}(\mathbb{T}_1)$, $b_* \in \text{Min}(\mathbb{T}_2)$, and $b^* \in \text{Max}(\mathbb{T}_2)$ such that $a_* \leq a \leq a^*$ and $b_* \leq b \leq b^*$. Let

$$\beta^{\text{up}} := \text{qu}_\mu(a^*, b^*) \vee \text{qu}_\mu(b^*, b) \text{ and } \gamma_{\text{dn}} := \text{qu}_\mu(a, a_*) \vee \text{qu}_\mu(a_*, b_*).$$

We claim that

$$\text{qu}_\mu(a, b) = \beta^{\text{up}} \wedge \gamma_{\text{dn}} \in K. \tag{3.33}$$

Using the assumption $\{\mathbb{M}_1, \mathbb{M}_2\} \cap \{\mathbb{T}_1, \mathbb{T}_2\} \neq \emptyset$, (3.16) allows us to select $D', D'' \in \text{Thd}(\mathbb{P})$ such that $\{a_*, b_*\} \subseteq D'$ and $\{a^*, b^*\} \subseteq D''$. These two inclusions together with $H \subseteq E$ imply that $\text{qu}_\mu(a_*, b_*)$ and $\text{qu}_\mu(a^*, b^*)$ are in K. Combining this with (3.29), we obtain that $\beta^{\text{up}} \wedge \gamma_{\text{dn}} \in K$.

Hence, to prove (3.33) and, consequently, Claim 3.8, it suffices to prove the equality in (3.33). Actually, it suffices to show that $\text{qu}_\mu(a, b) \geq \beta^{\text{up}} \wedge \gamma_{\text{dn}}$, as the converse inequality is trivial. To do so, assume that $(p, q) \in \beta^{\text{up}} \wedge \gamma_{\text{dn}}$. Pick a shortest sequence $\vec{x} : x_0 = p, x_1, \ldots, x_h = q$ and a shortest sequence $\vec{y} : y_0 = p, y_1, \ldots, y_k = q$ of elements of P witnessing that $(p, q) \in \beta^{\text{up}} = \mu \vee \text{quo}(a^*, b^*) \vee \text{quo}(b^*, b)$ and $(p, q) \in \gamma_{\text{dn}} = \mu \vee \text{quo}(a, a_*) \vee \text{quo}(a_*, b_*)$, respectively, according (3.1). For the sequence $\vec{x}$, this means that for every $j \in \{1, \ldots, h\}$, exactly one of $x_{j-1} < x_j$, $(x_{j-1}, x_j) = (a^*, b^*)$, and $(x_{j-1}, x_j) = (b^*, b)$ holds; for $\vec{y}$ the meaning is analogous. First, assume that $q \in T_1$. If an element of the sequence $\vec{x}$ is in T_2, then all the *subsequent* elements of $\vec{x}$ are in T_2, contradicting that $q \in T_1$. Hence, no element of $\vec{x}$ is in T_2. So, for all $j \in \{1, \ldots, h\}$, (x_{j-1}, x_j) is neither (a^*, b^*) nor (b^*, b). Hence, $x_{j-1} < x_j$ for all $j \in \{1, \ldots, h\}$, and so $(p, q) = (x_0, x_h) \in \mu \subseteq \text{qu}_\mu(a, b)$. Let us summarize:

$$\text{if } q \in T_1, \text{ then } (p, q) \in \beta^{\text{up}} \text{ implies that } (p, q) \in \mu \subseteq \text{qu}_\mu(a, b). \tag{3.34}$$

Second, observe that if an element of $\vec{y}$ belongs to T_1, then all the *preceding* elements of $\vec{y}$ do so. Then, similarly to the argument for (3.34), we obtain that

$$\text{if } p \in T_2, \text{ then } (p, q) \in \gamma_{\text{dn}} \text{ implies that } (p, q) \in \mu \subseteq \text{qu}_\mu(a, b). \tag{3.35}$$

For $p \in T_2$ and $q \in T_1$, the elements p and q are incomparable, since $\mathbb{T}_1 \neq \mathbb{T}_2$. Hence, the "$\in \mu$" part of (3.34) or (3.35) implies that

$$(p, q) \in T_2 \times T_1 \text{ is impossible.} \tag{3.36}$$

We can assume that $p \not\leq q$ as otherwise $(p, q) \in \text{qu}_\mu(a, b)$ is obvious. But then each of the sequences $\vec{x}$ and $\vec{y}$ implies easily that $\{p, q\} \subseteq T_1 \cup T_2$. Thus, it follows from (3.34), (3.35), and (3.36) that there is only one case left, namely, $p \in T_1$ and $q \in T_2$. To settle this case, the rest of the proof assumes that $p \in T_1$ and $q \in T_2$. In the sequence $\vec{x}$, which begins in T_1 and terminates in T_2, let x_u be the first element that is in T_2. As we mentioned in the argument for (3.34), the subsequent elements $x_{u+1}, x_{u+2}, \ldots, x_h$ are all in T_2. Clearly, $x_u = b^* \geq b$. Using that, for $j \in \{u, \ldots, h-1\}$, either $(x_j, x_{j+1}) = (b^*, b)$ (and so $x_{j+1} = b \geq b$) or $(x_j, x_{j+1}) \in \mu$ (and so

$x_j \geq b \Rightarrow x_{j+1} \geq b$), it follows by induction on j that $q = x_h \geq b$, that is, $q \in \mathrm{fil}(b)$. Similarly, letting y_v denote the last member of $\vec{y}$ that belongs to T_1, the elements $y_v, y_{v-1}, \dots, y_0 = p$ are all in T_1. Then $y_v = a_* \in \mathrm{idl}(a)$. Using that each of the pairs (y_{j-1}, y_j), $j \in \{v, v-1, \dots, 1\}$, is in μ or is of the form (a, a_*), a trivial induction on j "downwards" gives that $p = y_0 \in \mathrm{idl}(a)$. Hence $(p,q) \in \mathrm{idl}(a) \times \mathrm{fil}(b)$. So $(p,q) \in \mathrm{qu}_\mu(a,b)$ by (3.32). We have proved (3.33) and Claim 3.8. □

Next, we assert that

$$\text{for all } a, b \in P, \text{ we have that } \mathrm{qu}_\mu(a,b) \in K. \tag{3.37}$$

To prove (3.37), note that if $a \leq b$, then $\mathrm{qu}_\mu(a,b) = \mu \in K$ is clear by, say, (3.24). If $b < a$, then $\mathrm{qu}_\mu(a,b) \in K$ follows from (3.29). If $\{a,b\} \subseteq M_i$ for some $i \in \{1,2\}$, then (3.19), (3.21), and (3.29) yield that $\mathrm{qu}_\mu(a,b) \in K$. If there is an $i \in \{1,2\}$ such that $|\{a,b\} \cap M_i| = 1$, then Claim 3.8 implies that $\mathrm{qu}_\mu(a,b) \in K$. After these considerations, we can assume that $a \parallel b$ and $\{a,b\} \subseteq P^{(0)}_{\mathbb{M}_1,\mathbb{M}_2} = P \setminus (M_1 \cup M_2)$. For $i \in \{1,2\}$, pick an element $c_i \in M_i$. It suffices to show that

$$\mathrm{qu}_\mu(a,b) = \big(\mathrm{qu}_\mu(a,c_1) \vee \mathrm{qu}_\mu(c_1,b)\big) \wedge \big(\mathrm{qu}_\mu(a,c_2) \vee \mathrm{qu}_\mu(c_2,b)\big), \tag{3.38}$$

as the term on the right is in K by Claim 3.8. The "$\leq$" in place of the equality sign in (3.38) is clear by transitivity. For $i \in \{1,2\}$, (3.32) yields that

$$\mathrm{qu}_\mu(a,c_i) \subseteq \mu \cup \big(\mathrm{idl}(a) \times M_i\big) \text{ and } \mathrm{qu}_\mu(c_i,b) \subseteq \mu \cup \big(M_i \times \mathrm{fil}(b)\big). \tag{3.39}$$

Letting

$$\eta_i := (M_i \times M_i) \cup (P^{(0)}_{\mathbb{M}_1,\mathbb{M}_2} \times M_i) \cup (M_i \times P^{(0)}_{\mathbb{M}_1,\mathbb{M}_2}),$$

(3.1) and (3.39) imply that

$$\mathrm{qu}_\mu(a,c_i) \vee \mathrm{qu}_\mu(c_i,b) \subseteq \mu \cup (\mathrm{idl}(a) \times \mathrm{fil}(b)) \cup \eta_i. \tag{3.40}$$

As $P^{(0)}_{\mathbb{M}_1,\mathbb{M}_2}$, M_1, and M_2 are pairwise disjoint, (3.40) implies that

$$\begin{aligned}
&\big(\mathrm{qu}_\mu(a,c_1) \vee \mathrm{qu}_\mu(c_1,b)\big) \wedge \big(\mathrm{qu}_\mu(a,c_2) \vee \mathrm{qu}_\mu(c_2,b)\big)\\
&\quad\subseteq \big(\mu \cup (\mathrm{idl}(a) \times \mathrm{fil}(b)) \cup \eta_1\big) \cap \big(\mu \cup (\mathrm{idl}(a) \times \mathrm{fil}(b)) \cup \eta_2\big)\\
&\quad= \mu \cup (\mathrm{idl}(a) \times \mathrm{fil}(b)) \cup (\eta_1 \cap \eta_2)\\
&\quad= \mu \cup (\mathrm{idl}(a) \times \mathrm{fil}(b)) \cup \emptyset \overset{(3.32)}{=} \mathrm{qu}_\mu(a,b).
\end{aligned}$$

This shows the "$\geq$" inequality and the whole (3.38). Thus, (3.37) holds.

Finally, each element of $\mathrm{Quo^e}(\mathbb{P})$ is the join of (possibly infinitely many) elements of the form $\mathrm{qu}_\mu(a,b)$ where $a, b \in P$. Therefore, (3.37) implies that $\mathrm{Quo^e}(\mathbb{P}) \subseteq K$, completing the proof of Theorem 2.4.

□

Proof of Remark 2.5. We are going to use some parts of the previous proof. As $\mathrm{cmp}(\mathbb{P}) = 1$, each strip is a singleton and so we can consider it a reversed edge. Apart from this insignificant change, define G in the same way as we did right before Fact 3.6. Let $\mathbb{M}_1 = \mathbb{P}$ and take $Y(\mathbb{M}_1)$ from Definition 2.3(F). Changing (3.21) to $E := F(M_1^{\mathrm{sl}}) \cup G$, it follows from (2.2) and Fact 3.6 that E generates $\mathrm{Quo}^{\mathrm{e}}(\mathbb{P})$. As $|E|$ is what Remark 2.5 requires, the proof is complete. ☐

4. Some notes on Theorem 2.4

Remark 4.1. If $\mathbb{P}$ is an antichain such that $\mathrm{cmp}(\mathbb{P}) \geq 5$ is an accessible cardinal, then Theorem 2.4(B) says that $\mathrm{Quo}^{\mathrm{e}}(\mathbb{P})$ is 4-generated.

This means that Theorem 2.4 implies the first half Lemma 1.1 for sets with at least five elements. However, note that the proof of Theorem 2.4 uses Lemma 1.1.

Let $\{\mathbb{P}_i = (P_i; \leq_i) = (P_i; \mu_i) : \ i \in I\}$ be a set of posets that are assumed to be pairwise disjoint. The *cardinal sum* $\sum_{i\in I}^{\mathrm{card}} \mathbb{P}_i$ of these posets is $(\bigcup_{i\in I} P_i; \bigcup_{i\in I} \mu_i)$. E.g., for $\mathbb{P}$ in Theorem 2.4, $\mathbb{P} = \sum_{\mathbb{T}\in\mathrm{Comp}(\mathbb{P})}^{\mathrm{card}} \mathbb{T}$. If $I = \{1, 2\}$, then we prefer to write $\mathbb{P}_1 +^{\mathrm{card}} \mathbb{P}_2$. The following statement is a straightforward consequence of Theorem 2.4.

Corollary 4.2. *Assume that $\kappa \geq 5$ is an accessible cardinal. Let $\mathbb{A}_2$ denote the 2-element antichain. If $\mathbb{P}$ is the cardinal sum of κ many chains and the supremum of the lengths of these chains is* $\mathrm{ce}(\mathbb{P}) \in \mathbb{N}^+$, *then* $\mathrm{Quo}^{\mathrm{e}}(\mathbb{P})$ *and* $\mathrm{Quo}^{\mathrm{e}}(\mathbb{P} +^{\mathrm{card}} \mathbb{A}_2)$, *as complete lattices, are* $\big(10 + \mathrm{LASp}(\mathrm{ce}(\mathbb{P}))\big)$*-generated and* $\big(8 + \mathrm{LASp}(\mathrm{ce}(\mathbb{P}))\big)$*-generated, respectively.*

The *Y-poset* is the 4-element poset $\mathbb{Y} = (\{0, a, b, c\}, \leq)$ such that $0 \prec a \prec b$, $a \prec c$, but b and c are incomparable. It is straightforward but a bit tedious to check that $\mathrm{spp}(\mathbb{Y}) = 2$; this equality is witnessed by $Y(\mathbb{Y}) := \{\mathrm{qu}_\mu(b, c) \vee \mathrm{qu}_\mu(c, b)\}$, see Definition 2.3(F). Based on Theorem 2.4 and Corollary 4.2, we present some examples.

Example 4.3. Let $\kappa \geq 5$, let $\mathbb{P}$ be the cardinal sum of κ many chains of length 10^{100} each, and let $\mathbb{X}$ be the cardinal sum of κ many Y-posets. Then, as complete lattices, $\mathrm{Quo}^{\mathrm{e}}(\mathbb{P})$ is 347-generated, $\mathrm{Quo}^{\mathrm{e}}(\mathbb{P} +^{\mathrm{card}} \mathbb{A}_2)$ is 345-generated, $\mathrm{Quo}^{\mathrm{e}}(\mathbb{X})$ is $\big(4\cdot(2\cdot2\cdot2+1\cdot1\cdot1)+2+2\big)+3 = 43$-generated, and $\mathrm{Quo}^{\mathrm{e}}(\mathbb{X} +^{\mathrm{card}} \mathbb{A}_2)$ is $\big(4\cdot(2\cdot1\cdot1+1\cdot1\cdot1)+0+0\big)+3 = 15$-generated.

Remark 4.4. The proof of our theorem heavily uses Lemma 1.1. For infinite sets, this lemma is taken from Czédli and Kulin [10]. We could only deal with accessible cardinalities in [10] and in the earlier paper Czédli [4], on which [10] relies. This explains that Theorem 2.4 is also restricted to accessible cardinalities.

5. More about the dedication

The reader may wonder what happens if we drop the assumption "$|P|$ is an accessible cardinal" from Theorem 2.4. Because of this natural question, this is the right place to make some comments on the dedication the paper begins with.

Remark 5.1. There are several sources, including https://en.wikipedia.org/wiki/Vilmos_Totik and https://mta.hu/koztestuleti_tagok?PersonId=19540 where one can read about Professor Vilmos Totik. As a witness, here I add three stories; the third story sheds some light on the question mentioned above. First, Vilmos was one of my roommates in Loránd Eötvös Students' Hostel for (almost) three academic years from 1974 to 1977; his excellence was clear for all of us, fellow students, even at that time. Second, he received the Researcher of the Year award from the University of Szeged in the field of physical sciences on November 9, 2019; this was the first occasion when this new award was delivered; see https://u-szeged.hu/news-and-events/2019/honorary-titles-and . As this link shows, that day was splendid for us also for another reason: Doctor Honoris Causa title was awarded to Professor Ralph McKenzie, who hardly needs any introduction to the targeted readership. I still remember the dinner we organized for the evening of that splendid day, where the two celebrated professors of quite different fields of mathematics discussed, among other things, their fishing experiences. Third, when (around 1999) I mentioned the basic idea of my proofs in [4] and [7] to Professor Totik, it took him less than a minute to begin (and few minutes to complete) a proof showing that my method cannot work for all infinite cardinals. Later, when I forgot his proof, I could not find it again and I needed a professor of set theory to find the proof again. (In spite of many changes in subsequent proofs, see Czédli [8], Czédli and Kulin [10], and Czédli and Oluoch [12], the basic idea of dealing with sets of large cardinalities is, unfortunately, still the same and we seem to be far from strongly inaccessible cardinals.)

To conclude the stories, I wish Professor Vilmos Totik a happy birthday, further successes, and all the best.

Statements and declarations

Data availability statement

Data sharing is not applicable to this article as no datasets were generated or analyzed during the current study.

Competing interests

Not applicable as there are no interests to report.

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