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WHICH DISTRIBUTIVE LATTICES HAVE 2-DISTRIBUTIVE SUBLATTICE LATTICES?

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I. Introduction

The concept of *n-distributivity* was introduced by HUHN (cf. [4] and [5]). A lattice is said to be n-distributive $(n \geqq 1)$ if it satisfies the identity

$$x \wedge \bigvee_{i=0}^{n} y_i \leqq \bigvee_{j=0}^{n} \left(x \wedge \bigvee_{\substack{i=0 \\ i \neq j}}^{n} y_i \right). \tag{1}$$

The n-distributivity of subalgebra lattices (or congruence lattices) of universal algebras proved to be an important property in several cases. E.g., as it was proved by HUHN ([3] and [4]), the subgroup lattice of an abelian group A is n-distributive iff every finitely generated subgroup of A can be generated by at most n elements.

Sublattice lattices were investigated by FILIPPOV [1]. He gave necessary and sufficient conditions for having isomorphism between sublattice lattices of two given lattices. Lattices having modular and (upper) semi-modular sublattice lattices were characterized by KOH [6] and LAKSER [7], respectively.

Our aim is to characterize distributive lattices having n-distributive sublattice lattices in case $n \leqq 2$.

II. Preliminaries

For an *idempotent* algebra A let $\mathrm{Su}(A)$ denote the lattice of subalgebras of A. (It contains the empty set as a subalgebra.) Let us recall a non-published result of A. P. HUHN:

LEMMA 1. *For an arbitrary idempotent algebra A and $n \geqq 1$, $\mathrm{Su}(A)$ is n-distributive iff for any subset H of A we have*

$$[H] = \bigcup \{[G]\colon G \subseteqq H \quad \text{and} \quad |G| \leqq n\}. \tag{2}$$

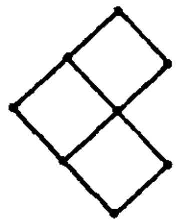
(Here, $[H]$ means the subalgebra generated by H and $\bigcup$ stands for the set-theoretical union.)

PROOF. Suppose $\mathrm{Su}(A)$ is n-distributive and $H = \{h_0, h_1, \ldots, h_m\} \subseteqq A$ for some $m \geqq n$. n-distributivity implies m-distributivity for all $m \geqq n$ (HUHN [5]), so for an arbitrary $a \in [H]$, we have

$$a \in \{a\} \wedge \bigvee_{i=0}^{m} \{h_i\} \subseteqq \bigvee_{j=0}^{m} \left(\{a\} \wedge \bigvee_{\substack{i=0 \\ i \neq j}}^{m} \{h_i\} \right).$$

Hence $a\in[\{h_0, \dots, h_{j-1}, h_{j+1}, \dots, h_m\}]$ for some j. This proves (2) for any *finite* H whence (2) holds for *any* subset H. Conversely, suppose that (2) holds. Then for any $Y_i\in \mathrm{Su}(A)$, we have $\bigvee_{i=0}^{n} Y_i = \bigcup_{j=0}^{n} \bigvee_{\substack{i=0\\ i\neq j}}^{n} Y_i$ whence the n-distributivity of $\mathrm{Su}(A)$ follows easily. Q.e.d.

Now we define the concept of the *special sum* of lattices. Let $(I, \leqq)$ be a chain and for every $i\in I$ let L_i be a lattice. Let $\sum_{i\in I} L_i$ denote the *ordinal sum* of lattices in the usual sense. (I.e., consider the disjoint union of the L_i-s and let $x\leqq y$ mean that $x\in L_i, y\in L_j$ and $i<j$, or $x, y\in L_i$ and $x\leqq y$.) For $a, b\in \sum_{i\in I} L_i$, let $a\vartheta b$ denote that "a is the greatest element of L_i, b is the least element of L_j and $i<j$, for some $i, j\in I$". Let Θ be the *equivalence* relation on $\sum_{i\in I} L_i$ generated by the binary relation ϑ. Then, as it can be seen easily, Θ is a congruence relation. Now, denoting by $\sum'_{i\in I} L_i$, the definition of the special sum is the following: $\sum'_{i\in I} L_i = \sum_{i\in I} L_i/\Theta$. Let us agree that we write $\sum'_{i\in I} L_i = \sum_{i\in I} L_i$ iff Θ is the equality relation. Denoting the lattice



by K we can state our main

THEOREM. *For any distributive lattice L the following three conditions are equivalent:*

(i) $\mathrm{Su}(L)$ *is 2-distributive*

(ii) L *contains neither a sublattice isomorphic to K nor a three-element antichain (antichain means a set of pairwise incomparable elements)*

(iii) L *is isomorphic to a special sum $\sum'_{i\in I} L_i$, where for each $i\in I$, L_i is a chain or $L_i = \underset{\sim}{2}\times C$ for some chain C. ($\underset{\sim}{2}$ denotes the two-element chain.)*

REMARK. As for 1-distributivity, which is the usual distributivity, it is very easy to show that an arbitrary lattice L has distributive sublattice lattice iff L is a chain.

In what follows lattices isomorphic to $\underset{\sim}{2}\times C$ for some chain C will be referred to as *ladders*. The following sections deal with the proof of the theorem, namely the implications (iii)→(i), (i)→(ii) and (ii)→(iii) are proved.

III. The first part of the proof

In this section the implications (iii)→(i) and (i)→(ii) will be verified.

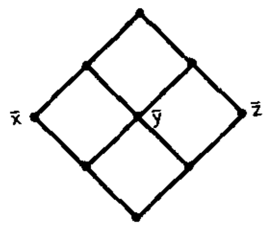
PROPOSITION 1. *If a lattice L is a chain or a ladder then* $\mathrm{Su}(L)$ *is 2-distributive. If L_2 is a homomorphic image of a lattice L_1 and* $\mathrm{Su}(L_1)$ *is 2-distributive then so is* $\mathrm{Su}(L_2)$.

The proof is straightforward by making use of Lemma 1, so it will be omitted. It can be easily shown that

$$\mathrm{Su}\left(\sum_{i\in I} L_i\right) \cong \prod_{i\in I} \mathrm{Su}\,(L_i) \tag{3}$$

for an arbitrary ordinal sum $\sum_{i\in I} L_i$ (cf. also FILIPPOV [1, Lemma 1.2]). Now Proposition 1 and (3) yield the proof of (iii)→(i).

To prove (i)→(ii) suppose L is a distributive lattice and $\mathrm{Su}(L)$ is 2-distributive. Since neither $\mathrm{Su}(K)$ nor $\mathrm{Su}(\underset{\sim}{2}^3)$ are 2-distributive by Lemma 1, L does not contain any sublattice isomorphic to K or $\underset{\sim}{2}^3$. Suppose $\{a, b, c\}$ is a three-element antichain in L. Then $\{a\vee b, a\vee c, b\vee c\}$ cannot be a three-element antichain since otherwise it would generate a sublattice isomorphic to $\underset{\sim}{2}^3$ (cf. GRÄTZER [2, p. 45]). Hence $a\vee b\vee c\in\{a\vee b, a\vee c, b\vee c\}$ and, by the lattice theoretical Duality Principle, $a\wedge b\wedge c\in\{a\wedge b, a\wedge c, b\wedge c\}$. If we had $a\vee b\vee c=a\vee b$ and $a\wedge b\wedge c=b\wedge c$ then $c=(a\vee b\vee c)\wedge c=(a\vee b)\wedge c\leqq(a\vee b)\wedge(a\vee c)=a\vee(b\wedge c)=a\vee(a\wedge b\wedge c)=a$ would contradict $a\| c$. So $a\vee c=a\vee b\vee c$ and $a\wedge c=a\wedge b\wedge c$ can be assumed. Now $[a, b, c]=$ $=[\{a, b, c\}]$ is a homomorphic image of $\mathrm{FD}(3)/\Theta$ where $\mathrm{FD}(3)$ denotes the free distributive lattice freely generated by $\{x, y, z\}$ and Θ denotes the smallest congruence for which $a\wedge c\,\Theta\, a\wedge b\wedge c$ and $a\vee c\,\Theta\, a\vee b\vee c$. Since the structure of $\mathrm{FD}(3)$ is well-known (c.f. GRÄTZER [2, p. 46]), it is easy to check that $\mathrm{FD}(3)/\Theta$ is the following lattice:



An arbitrary homomorphism $\varphi\colon \mathrm{FD}(3)/\Theta\to[a, b, c]$, $\bar{x}\mapsto a$, $\bar{y}\mapsto b$, $\bar{z}\mapsto c$ must be injective because a, b, c are pairwise incomparable. Hence $\mathrm{FD}(3)/\Theta$ can be embedded into L. So can K, which is a contradiction. Thus the proof of (i)→(ii) is complete.

IV. A decomposition of lattices

The (decomposability) lemma given below will be an important tool to prove (ii)→(iii). First, for a *partially ordered set* L, we set

$$C(L) = \{x\in L\colon\ x \not\parallel y \text{ for all } y\in L\}$$

and

$$C'(L) = \{x\in C(L)\colon\ x \neq 0_L \text{ and } x \neq 1_L\}.$$

(Note that L is not necessarily bounded and the above two sets may coincide.)

LEMMA 2. *An arbitrary lattice* L *is isomorphic to a special sum* $\sum_{i\in I}' L_i$ *where, for all* $i\in I$, L_i *is a chain or* $C'(L_i)=\emptyset$.

Before proving this lemma we need some preliminaries. Define the binary relation $\varrho=\varrho_L$ on L in the following way: set $a\varrho b$ iff one of the conditions

— $a\parallel b$

— $a \not\parallel b$ and $[a, b]\cap C(L) = \emptyset$

— $a \not\parallel b$ and $[a, b] \subseteq C(L)$

holds, where $[a, b]=\{x\in L\colon a\leqq x\leqq b$ or $b\leqq x\leqq a\}$.

PROPOSITION 2. *For an arbitrary lattice* L, $\varrho=\varrho_L$ *is an equivalence relation.*

PROOF. Suppose we have $a\varrho b$ and $b\varrho c$ for some $a, b, c\in L$ and let us show that $a\varrho c$. Evidently ϱ is reflexive and symmetric so, by the Duality Principle, we have to deal only with the following four cases.

Case 1. $a\parallel b$ and $b\parallel c$. If $a\not\parallel c$, say $a<c$, then $[a, c]\cap C(L)=\emptyset$ because $x\parallel b$ for all $x\in[a, c]$.

Case 2. $a\parallel b$ and $b<c$. Suppose $a\not\parallel c$, then $a<c$. Now $[a, c]\cap C(L)= =([a\vee b, c]\cap C(L))\cup(([a, a\vee b]\setminus\{a\vee b\})\cap C(L))$. But, for all $x\in[a, a\vee b]\setminus\{a\vee b\}$, $x\parallel b$ and $[a\vee b, c]\subseteq[b, c]$ so we have $[a, c]\cap C(L)=\emptyset$.

Case 3. $a<b$ and $b<c$. If $[a, b]\subseteq C(L)$ and $[b, c]\subseteq C(L)$ then $[a, c]= =[a, b]\cup[b, c]\subseteq C(L)$. If $[a, b]\cap C(L)=\emptyset$ then $[a, c]\cap C(L)=([a, b]\cap C(L))\cup \cup([b, c]\cap C(L))=\emptyset$.

Case 4. $a<b$ and $b>c$. If $a\not\parallel c$, say $a<c$, then $[a, c]\subseteq[a, b]$ and either $[a, c]\subseteq[a, b]\subseteq C(L)$ or $[a, c]\cap C(L)\subseteq[a, b]\cap C(L)=\emptyset$. This completes the proof of Proposition 2.

PROPOSITION 3. *Let* L *be a lattice and* M *a* ϱ_L*-class in* L. *Then either* $M\subseteq C(L)$ *or* $M\cap C(L)=\emptyset$. *If* $M\subseteq C(L)$ *then* M *is a chain. If* $M\cap C(L)=\emptyset$ *then* M *has neither greatest element nor least element, and exactly one of the following four possibilities*

$$[M] = M,\quad [M] = M\cup\{1_M\},\quad [M] = M\cup\{0_M\},\quad [M] = M\cup\{0_M, 1_M\}$$

holds where $0_M, 1_M\in C(L)\setminus M$ *and they are the zero and unit of* $[M]$, *respectively.*

PROOF. It is sufficient to prove the last statement. Let $M\cap C(L)=\emptyset$. Then $C(M)=\emptyset$ whence M has neither greatest nor least element. Suppose M is not a join sub-semilattice of L, say $a, b\in M$ but $a\vee b\notin M$, and let us show that $a\vee b\in C(L)$ and $M\cup\{a\vee b\}$ is a join-semilattice. Since $a\bar{\varrho}a\vee b$ (i.e., $a\varrho a\vee b$ does not hold), there is an $x\in[a, a\vee b]\cap C(L)$. Now $a\|b$ implies $x\not\leq b$ and $b\in M$ implies $x\not\geq b$ and so $a\vee b\leq x\in[a, a\vee b]$ implies $a\vee b=x\in C(L)$. Now x is the unit of $M\cup\{x\}$ because otherwise $x<c$ for some $c\in M$ and $x\in[a, c]\cap C(L)=\emptyset$, a contradiction. If, for $d, e\in M$, $y=d\vee e\notin M$, then $y\leq x$. But the role of x and y can be interchanged so $x\leq y$ as well. I.e., $M\cup\{a\vee b\}$ is a join-semilattice indeed and the proof is complete by the Duality Principle.

PROOF OF LEMMA 2. Let $\varrho=\varrho_L$ and for $D_1, D_2\in L/\varrho$ we define $D_1\leq D_2$ by the formula $(\exists X_1\in D_1)(\exists X_2\in D_2)(X_1\leq X_2)$. An easy calculation shows that $(L/\varrho, \leq)$ is a chain. We assert $L\cong\sum'_{D\in L/\varrho}[D]$. Let $\varphi\colon \sum_{D\in L/\varrho}[D]\to L$ be the map for which $\varphi|_{[D]}$ is the natural embedding of D into L. It follows easily that φ is a homomorphism onto L. It is also seen that if $D, E\in L/\varrho$, $D<E$ and $x\in[E]\cap[D]$ then $x=1_D=0_E$, and $D\prec E$ or $D\prec\{x\}\prec E$. Therefore $L\cong\sum_{D\in L/\varrho}[D]/\mathrm{Ker}\,\varphi=\sum'_{D\in L/\varrho}[D]$, indeed. If $D\in L/\varrho$ is not a chain then $C'([D])=\emptyset$ follows from Proposition 3. Q.e.d.

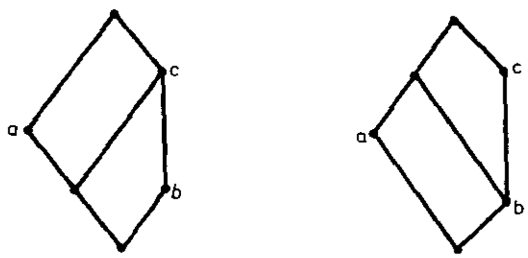
V. The second part of the proof

Having Lemma 2, the proof of (ii)→(iii) will be complete if we prove the following

LEMMA 3. *Let L be a distributive lattice which satisfies* (ii), $C'(L)=\emptyset$ *and* $|L|\geq 3$. *Then L is of the form $\sum_{i\in I}L_i=\sum'_{i\in I}L_i$ where $(I, \leq)$ is a chain and for all $i\in I$, L_i is a ladder or consists of a single element.*

The proof of this lemma requires several preliminary statements.

PROPOSITION 4. *Let a, b, c be elements of a distributive lattice satisfying* (ii). *If $a\|b, a\|c$ and $b<c$ then $[a, b, c]$ is isomorphic to one of the following two lattices*



PROOF. It is an immediate consequence of the well-known fact (cf. GRÄTZER [2, p. 14]) that the free distributive lattice generated by the partially ordered set $(\{a, b, c\}, b\leq c)$ is isomorphic to K.

Let us agree that any use of Proposition 4 in case of $[x, y, z]$ will also mean $x\|y, x\|z$ and $y\leq z$.

An element x in L is called *join-reducible* if $x=a\vee b\notin\{a,b\}$ for some $a,b\in L$. An element x is said to be *reducible (doubly reducible)* if it is either (both) join-reducible or (and) meet-reducible.

PROPOSITION 5. *The lattice L from Lemma* 3 (*i.e., L distributive,* (ii) *and* $C'(L)=\varnothing$) *does not contain any doubly reducible element.*

PROOF. Suppose $x\in L$ is doubly reducible. Then $x=e\vee b=f\wedge c$, $e\|b$, $f\|c$ and $a\|x$ for some $a,b,c,e,f\in L$. Now $\{a,b,e\}$ and $\{a,c,f\}$ are not antichains so $a\|c$, $a\prec f$, $a\|b$, $a\succ e$ can be assumed. Consider $[a,b,c]$. Then we have $a\vee b\leqq f$, $a\vee c\not\leqq f$, $a\wedge c\geqq e$ and $a\wedge b\not\geqq e$, which contradicts Proposition 4. Q.e.d.

For a lattice L define the binary relation ψ_L on L as follows:

Set $a\psi_L b$ iff for any $e,f\in[a\wedge b,a\vee b]$, where e is join-reducible (in L) and f is meet-reducible (in L), $e\not\leqq f$ holds.

LEMMA 4. *Let L be a distributive lattice satisfying* (ii) *and* $C'(L)=\varnothing$. *Then $\psi=\psi_L$ is a congruence, L/ψ is a chain and $a\|b$ implies $a\psi b$ for any $a,b\in L$.*

PROOF. Throughout the proof, let e and f stand for join-reducible and meet-reducible elements of L, respectively. Suppose $a\|b$ but $a\psi b$ does not hold. So $e,f\in[a\wedge b,a\vee b]$ and $e\leqq f$ for some e,f. First, exactly one of $a\|e$ and $b\|e$ holds because otherwise $e=a\wedge b$ or $e=f=a\vee b$ would contradict Proposition 5. Suppose $a\|e$ and $b\nparallel e$. If we had $e\prec b$, then e would be doubly reducible by Proposition 4 (considering $[a,e,b]$). If $b\prec e$, then $a\|f$ (otherwise $f=a\vee b$ is doubly reducible) and, considering $[a,e,f]$, e or f is doubly reducible. Thus $e=b$. Similarly, $f\in\{a,b\}$. Since $f\leqq e$, $f=b=e$ which contradicts Proposition 5. Now we have shown:

$$a\|b \quad \text{implies} \quad a\psi b. \tag{4}$$

Suppose we have $a\psi b$ and $b\psi c$. Since ψ is reflexive by Proposition 5 and symmetric, to prove transitivity only the following four cases have to be considered.

Case 1. $a\prec b$ and $b\prec c$ but $a\bar{\psi}c$. Then there are $e,f\in[a,c]$, $e\prec f$. Both $b\|e$ and $b\|f$ do not hold by Propositions 4 and 5. Say $b\nparallel f$, and so, from $\{e,f\}\nsubseteq[a,b]$, $b\prec f$. Then $e\prec b$ because otherwise $\{b\vee e,f\}\subseteq[b,c]$. Choose an $x\in L$ such that $x\|b$. Both $e\leqq x$ and $x\leqq f$ lead to a contradiction: $a\leqq e\leqq x\wedge b\leqq b$ or $b\leqq x\vee b\leqq f\leqq c$, respectively. Therefore $x\|e$ and $x\|f$. But regarding $[x,e,f]$ Probositions 4 and 5 give a contradiction.

Case 2. $a\|b$ and $b\prec c$. If $a\|c$ then $a\psi c$ by (4). Otherwise $a\prec c$. Set $b'=a\vee b$, then from $[a,b']\subseteq[a\wedge b,a\vee b]$ and $[b',c]\subseteq[b,c]$ we get $a\psi b'$ and $b'\psi c$ whence, by Case 1, $a\psi c$ follows.

Case 3. $a\|b$, $b\|c$ and $a\prec c$. Now, by Proposition 4, we have either $[a,c]\subseteq$ $\subseteq[a\wedge b,a\vee b]$ or $[a,c]\subseteq[b\wedge c,b\vee c]$, and so $a\psi c$.

Case 4. $a\prec b$ and $c\prec b$ and $a\prec c$. Then $a\psi c$ follows from $[a,c]\subseteq[a,b]$.

Now we have that ψ is an equivalence relation. Let us have $a,b,c\in L$, $a\psi b$. If $a\psi c$ then $a\wedge c\psi a\psi b\psi b\wedge c$, while in case of $a\bar{\psi}c$ $a\wedge c\psi b\wedge c$ follows from (4). Therefore, by the Duality Principle, ψ is a congruence. Finally, (4) implies that L/ψ is a chain. Q.e.d.

COROLLARY 1. *Let* L *be a distributive lattice satisfying* (ii) *and* $C'(L)=\emptyset$. *Then* $L=\sum'_{D\in L/\psi} D=\sum_{D\in L/\psi} D$, *where* ψ *denotes* ψ_L, *for all* $D\in L/\psi$, $C'(D)=\emptyset$ *and* $\psi_D=D^2$.

LEMMA 5. *Let* L *be a distributive lattice for which* $\psi_L=L^2$, $C'(L)=\emptyset$, $|L|\geqq 5$ *and* (ii) *hold. Suppose* L *contains neither a join-irreducible unit nor a meet-irreducible zero. Then* L *is a ladder.*

PROOF. The proof consists of several steps. Let E and F denote the set of join-reducible and meet-reducible elements of L, respectively. Then $e\nleqq f$ for any $e\in E$ and $f\in F$ and therefore $E\cap F=\emptyset$.

STEP 1. F is an ideal and E is a dual ideal of L and both are chains.

PROOF. First we show that F is a chain. Suppose $a, b\in F$, $a\| b$. Let $a=x\wedge y$ and $b=u\wedge v$ where $x\| y$ and $u\| v$. Since $\{a, u, v\}$ is not an antichain, $a<u$ and $a\| v$ can be assumed. Similarly, $b<y$ and $b\| x$ can be also assumed. So $a=x\wedge(y\wedge u)$ and $b=v\wedge(y\wedge u)$. Hence $x\| y\wedge u$ and $v\| y\wedge u$ and $x\| v$ (from $a\| b$), which is a contradiction. I.e., F is a chain and so is E. Now let $f\in F$, $x\in L\setminus F$ and $x<f$. Then x is not the zero of L and so $x\| y$ for some $y\in L$. Here $y\| f$ since otherwise $f\geqq x\vee y\in E$. Considering $[y, x, f]$, Proposition 4 leads to a contradiction because of $f\notin E$. I.e., F is an ideal and E is a dual ideal by the Duality Principle. Q.e.d.

STEP 2. Both E and F have at least two elements.

PROOF. Suppose $|F|<2$ and let a, b be incomparable elements in L. Then, by Proposition 4, $x\| a$ implies $x=b$ and $x\| b$ implies $x=a$ for any $x\in L$. Thus $L=\{a, b\}\cup(a\wedge b]\cup[a\vee b)$, which is a contradiction. The proof is complete by the Duality Principle.

STEP 3. If $a, b\in L\setminus(E\cup F)$, $f\in F$, $a\leqq b$ and $f<a$ then $a=b$.

PROOF. Suppose $a<b$, then $a\| c$ for some $c\in L$. In case $c\| b$, by Proposition 4, $\{a, b\}\cap(E\cup F)\neq\emptyset$. Thus $c<b$. Hence $b\geqq a\vee c\in E$, which contradicts Step 1.

STEP 4. If $f\in F$, $a, b\in L\setminus(E\cup F)$, $a>f$, $b>f$ and f is not the zero element of L, then $a=b$.

PROOF. If $a\neq b$ then $a\| b$ by Step 3. Choose an element q in L so that $f\| q$. Now $q\notin F$ by Step 1. $\{a, b, q\}$ is not an antichain, so $q\in E$ contradicts Step 1 and $q\notin E$ contradicts Step 3.

STEP 5. If $f\in F$, f is not the zero of L and $[f)\cap(L\setminus(E\cup F))=\emptyset$, then F is a prime ideal.

PROOF. Suppose $a\wedge b\in F$ but $a\notin F$ and $b\notin F$. $\{a, b\}\nsubseteq E$ by Step 1 so, e.g., $a\notin E$. Consequently, by Step 1 and Proposition 4, we have $a\| f$, $f<b$ and $a\vee f\| b$. Hence, by Step 1, $b\notin E$, which is a contradiction. Q.e.d.

Now F' is defined as follows: If F is a prime ideal then set $F'=F$. Otherwise set $F'=F\cup\{f'\}$ where $f'\in L\setminus(E\cup F)$ and, for some $f\in F$, $0_L\neq f<f'$. E' is defined dually. The previous steps yield the correctness of this definition.

STEP 6. F' is a prime ideal and a chain, and in case $F' \neq F, f'$ is the greatest element of F'. The dual statement is valid for E'.

PROOF. It is enough to consider the case $F' \neq F$. First we show that $g < f'$ for any $g \in F$. If $g \nless f'$ for some $g \in F$, then $g \| f'$ by Step 1. Since $g \wedge f' \neq 0_L$ by Step 1, $b \| f' \wedge g$ for some $b \in L \setminus F$. We have $b \| g$ by Step 1 and, $\{f', g, b\}$ being not an antichain, $b < f'$. Applying Proposition 4 to $[g, b, f']$ we get $f' \in E$ which is a contradiction. Therefore f' is the greatest element of F and, by Step 1, F' is a chain. Suppose $x \in L$ and $x < f'$ but $x \notin F$. For *any* $f \in F$ $x \| f$ by Steps 1 and 3, however $x \nparallel x \wedge f \in F$ is a contradiction. Therefore F' is an ideal. Suppose $a, b \notin F'$, but $a \wedge b \in F'$. Since $a \| b, b \notin E$ can be assumed by Step 1. From Step 3 we have $f' \| b$. Since $\{b, f', a\}$ is not an antichain, $f' < a$. Considering $[b, f', a]$, Proposition 4 yields $f' \in F$ or $a \wedge b \| f'$, which is a contradiction. Q.E.D.

STEP 7. $E' \cap F' = \emptyset$.

PROOF. Suppose $E' \cap F' \neq \emptyset$. Since $E \cap F = \emptyset$, we have $e' = f'$. Consider the set $H = \{x \in L : x \| f'\}$. $H \neq \emptyset$. Suppose H consists of a single element x. Let $f_1, f_2 \in F, f_1 < f_2$. Since $f_2 \notin C'(L), f_2 \| x$ and, considering $[x, f_2, f']$, Proposition 4 yields a contradiction. Suppose $x, y \in H$, $x \nleqq y$. Then $x > y$ and considering $[f', y, x]$, Proposition 4 yields a contradiction again. Q.e.d.

STEP 8. $E' \cup F' = L$.

PROOF. Suppose $x \in L \setminus (E \cup F)$. Let $0_L \neq f \in F$ and $1_L \neq e \in E$. Since $x \nleqq f$ and $e \nleqq x$ by Step 1, we have $f < x$ or $x < e$ by Step 1 and Proposition 4. Therefore Step 4 (or its dual statement) implies $x = f'$ or $x = e'$. Q.e.d.

Now we define a map $\tau : E' \to F'$ as follows:

$$e\tau = \begin{cases} f \wedge e, & \text{if } f \| e \quad \text{for some } f \in F' \\ f', & \text{if } f < e \quad \text{for all } f \in F'. \end{cases}$$

STEP 9. The definition of τ is correct.

PROOF. Suppose $f_1 < f_2, f_1 \| e$ and $f_2 \| e$ for some $f_1, f_2 \in F', e \in E'$. Considering $[e, f_1, f_2]$, Proposition 4 and Step 6 imply $e \wedge f_1 = e \wedge f_2$. Suppose we have an $e \in E'$ such that $e \nparallel f$ for all $f \in F'$. Steps 6, 8 and $C'(L) = \emptyset$ imply $e = 1 = 1_L$.

We have $a, b \in L$ such that $a \| b$ and $a \vee b = 1$. By Steps 6 and 8 $a \in F'$ and $b \in E'$ can be assumed. If $a \in F$ then, for some $c, d \in L$, $a = c \wedge d, c \| d$ and, from $1 = c \vee b = d \vee b$, $\{b, c, d\}$ is an antichain. Therefore $a = f' \notin F$ and $e\tau = 1\tau$ is defined. Q.e.d.

STEP 10. The map τ is a bijective lattice homomorphism.

PROOF. First we show that $e_1 < e_2$ $(e_1, e_2 \in E')$ implies $e_1\tau < e_2\tau$. As we have already seen in the proof of Step 9, $e_2\tau = f'$ implies $e_2 = 1$. Therefore $e_2 \neq 1$ can be assumed. Let $e_i\tau = e_i \wedge f_i$ $(i = 1, 2, f_i \in F')$. Then $f_2 \| e_1, e_i\tau = f_2 \wedge e_i$ $(i = 1, 2)$ and $f_2 \vee e_1 \nparallel e_2$. Considering $[f_2, e_1, e_2]$, Proposition 4 implies $e_1\tau < e_2\tau$. Now, if $f' \in F' \setminus F$ exists then $a \wedge b \in F$ and $a \| b$ for some $a, b \in L \setminus F$. Hence, by Step 6, $f' \in \{a, b\}$ and $f' = (a \vee b)\tau$. If $f \in F$ then, by Step 6, $f = c \wedge d$ and $c \| d$ for some $c \in F'$ and $d \in E'$. So $f = d\tau$. I.e., τ is surjective. Q.e.d.

Now let $\underset{\sim}{2}=\{0, 1\}$ be the two-element chain and let us define a map $\eta: \underset{\sim}{2}\times E' \to L$ by $(1, e)\mapsto e$ and $(0, e)\mapsto e\tau$. Our previous steps imply that η is a (required) isomorphism between $\underset{\sim}{2}\times E'$ and L. The proof of Lemma 5 is complete.

Finally, Lemmas 4 and 5 and Corollary 1 imply Lemma 3.

References

[1] N. D. Filippov, Projectivity of lattices, *Mat. Sbornik*, **70** (1966), 36—54 (Russian).
[2] G. Grätzer, *Lattice theory*, W. H. Freeman and Co. (San Francisco, 1971).
[3] A. P. Huhn, Scwach distributive Verbände, *Acta F.R.N. Univ. Comen. — Mathematica*, Nim. cislo (1971), 51—56. (Summary of the author's lecture on the conference Letná Skola 1970, Ceslovensko.)
[4] A. P. Huhn, Scwach distributive Verbände. I, *Acta Sci. Math. Szeged*, **33** (1972), 297—305.
[5] A. P. Huhn, n-distributivity and some questions on the equational theory of lattices, *Colloq. M. Soc. J. Bolyai*, **17**. Contributions to the universal algebra (Szeged, 1975), 167—178.
[6] K. M. Koh, On sublattices of a lattice, *Nanta Math.*, **6** (1973), 68—79.
[7] H. Lakser, A note on the lattice of sublattices of a finite lattice, *Nanta Math.*, **6** (1973), 55—57.

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