Weak congruence semidistributivity laws and their conjugates

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Dedicated to the memory of Viktor Aleksandrovich Gorbunov

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Abstract. Lattice Horn sentences including Geyer's $SD(n,2)$ and their conjugates $C(n,2)$ are considered. $SD(2,2)$ is the meet semidistributivity law $SD_\wedge$. Both $SD(n,2)$ and $C(n,2)$ become strictly weaker when n grows. For varieties $\mathcal{V}$ the satisfaction of $SD(n,2)$ in $\{\mathrm{Con}(A) : A \in \mathcal{V}\}$ is characterized by a Mal'cev condition. Using this Mal'cev condition it is shown that $C(n,2) \models_{\mathrm{con}} SD(n,2)$, which means that, for every variety $\mathcal{V}$, whenever $C(n,2)$ holds in $\{\mathrm{Con}(A) : A \in \mathcal{V}\}$ then so does $SD(n,2)$. In particular, $C(2,2) \models_{\mathrm{con}} SD(2,2)$, which is a stronger statement than $SD_\vee \models_{\mathrm{con}} SD_\wedge$, the only previously known $\models_{\mathrm{con}}$ result between lattice Horn sentences "not below congruence modularity". Some other $\models_{\mathrm{con}}$ statements are also presented.

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I. Introduction and the main results

This paper is primarily concerned with Mal'cev conditions and the consequence relation $\models_{\mathrm{con}}$ between lattice Horn sentences in congruence (quasi)varieties.

Given a variety $\mathcal{V}$ of algebras, the class of congruence lattices of members of $\mathcal{V}$ will be denoted by

$$\mathrm{Con}(\mathcal{V}) = \{\mathrm{Con}(A) : A \in \mathcal{V}\}.$$

By a (universal lattice) Horn sentence we mean a first order sentence

$$(\forall x_0, \ldots, x_{t-1}) \big((p_1 = q_1 \;\&\; \ldots \;\&\; p_k = q_k) \Longrightarrow p = q\big) \tag{1}$$

where $p_1, \ldots, p_k, q_1, \ldots, q_k, p$ and q are lattice terms of the variables $x_0, \ldots, x_{t-1}$. Notice that using "$\leq$" instead of "$=$" in (1) would give the same notion modulo lattice theory. Lattice identities are special Horn sentences with $k = 0$ (or with $p_i = x_0$ and $q_i = x_0$ for all i). For convenience, lattice operations will be denoted by $+$ (join) and $\cdot$ (meet); $\bigwedge$ and & will denote conjunctions. The join semidistributivity law

$$SD_\vee : \qquad x+y=x+z \Longrightarrow x+y=x+yz$$

and the meet semidistributivity law

$$SD_\wedge : \qquad xy=xz \Longrightarrow xy=x(y+z)$$

are the most known Horn sentences that are not equivalent to lattice identities.

For a lattice H resp. class H of lattices and a Horn sentence λ let $H \models \lambda$ denote the fact that λ holds in H resp. in all members of H. The same symbol is used for the standard consequence relation between Horn sentences λ and μ: $\lambda \models \mu$ means that for every lattice L if $L \models \lambda$ then $L \models \mu$. If $\mathrm{Con}(\mathcal{V}) \models \lambda$ implies $\mathrm{Con}(V) \models \mu$ for every variety $\mathcal{V}$ then the notation

$$\lambda \models_{\mathrm{con}} \mu$$

is used. The statement $\lambda \models_{\mathrm{con}} \mu$ is said to be nontrivial if $\lambda \not\models \mu$. This fact, i.e. the conjunction of $\lambda \models_{\mathrm{con}} \mu$ and $\lambda \not\models \mu$, will be denoted by $\lambda \models^{\mathrm{nt}}_{\mathrm{con}} \mu$. Starting with Nation [22], there are many results of the form $\lambda \models^{\mathrm{nt}}_{\mathrm{con}} \mu$, cf., e.g., Day [6], [7], Day and Freese [8], Freese, Herrmann and [11], Jónsson [17], [18], Mederly [21], and [2], with various lattice identities. (As a related deep result, Freese [10] is also worth mentionig here.) These results are "below congruence modularity" in the sense that modularity $\models_{\mathrm{con}} \mu$. The only known $\lambda \models^{\mathrm{nt}}_{\mathrm{con}} \mu$ type result not below congruence modularity is

$$SD_\vee \models^{\mathrm{nt}}_{\mathrm{con}} SD_\wedge \tag{2}$$

from Hobby and McKenzie [14, p. 112]. One of our goals is to strengthen (2) and, by generalizing (2), to present infinitely many $\lambda \models^{\mathrm{nt}}_{\mathrm{con}} \mu$ results not below modularity.

Given a lattice identity λ, the class of varieties $\{\mathcal{V} : \mathrm{Con}(\mathcal{V}) \models \lambda\}$ is a weak Mal'cev class by Wille [26] and Pixley [24]. In other words, (the satisfaction of) λ (in congruence varieties) can be characterized by a weak Mal'cev condition. In many cases, all being covered by Chapter XIII in Freese and McKenzie [12], $\{\mathcal{V} : \mathrm{Con}(\mathcal{V}) \models \lambda\}$ is known to be a Mal'cev class. E.g., the distributivity resp. modularity are characterized by the famous Mal'cev conditions given by Jónsson [16] resp. Day [5].

Now let λ be a Horn sentence. Then $\{\mathcal{V} : \mathrm{Con}(\mathcal{V}) \models \lambda\}$ is known to be a weak Mal'cev class only in certain cases described in [3]; these cases include $SD_\wedge$ and $SD_\vee$. Using commutator theory, Lipparini [20] and Kearnes and Szendrei [19] have recently proved that $\{\mathcal{V} : \mathrm{Con}(\mathcal{V}) \models SD_\wedge\}$ is a Mal'cev class. For a direct approach (and also for an important application of the corresponding Mal'cev condition) cf. Willard [25], and cf. also Hobby and McKenzie [14] for the locally finite case. Using ideas from [1], [3] and [25] we present Mal'cev conditions for infinitely many Horn sentences. These Mal'cev conditions provide the key to our $\lambda \models^{\mathrm{nt}}_{\mathrm{con}} \mu$ type achievements.

For $n \geq 2$ put $\mathbf{n} = \{0, 1, \ldots, n-1\}$ and let $P_2(\mathbf{n})$ denote $\{S : S \subseteq \mathbf{n} \text{ and } |S| \geq 2\}$. For $\emptyset \neq H \subseteq P_2(\mathbf{n})$ we define the generalized meet semidistributivity law $SD(n, H)$ as follows:

$$\alpha\beta_0 = \alpha\beta_1 = \ldots = \alpha\beta_{n-1} \Longrightarrow \alpha \prod_{I \in H} \sum_{i \in I} \beta_i \leq \beta_0.$$

Equivalently, $SD(n,H)$ is

$$\alpha\beta_0 = \alpha\beta_1 = \ldots = \alpha\beta_{n-1} \Longrightarrow \alpha\beta_0 = \alpha \prod_{I\in H}\sum_{i\in I}\beta_i.$$

When $H = P_2(\mathbf{n})$, $SD(n,H)$ will be denoted by $SD(n,2)$. Notice that

$$SD(n,2):\qquad \alpha\beta_0 = \alpha\beta_1 = \ldots = \alpha\beta_{n-1} \Longrightarrow \alpha \prod_{0\le i<j<n}(\beta_i+\beta_j) \le \beta_0$$

has been studied by Geyer [13], and $SD(2,2)$ is exactly $SD_\wedge$.

Now with $SD(n,H)$ we associate its conjugate Horn sentence $C(n,H)$ as follows. Let α and $\beta_{i,I}$ ($i \in I \in H$) be the variables of $C(n,H)$. Denoting $\{I \in H : j \in I\}$ by H_j, $C(n,H)$ is

$$\bigwedge_{I\in H}\Big((\alpha \le \sum_{i\in I}\beta_{i,I}) \ \&\ \bigwedge_{i\in I}(\beta_{i,I} \le \alpha + \sum_{j\in I\setminus\{i\}}\beta_{j,I})\Big) \Longrightarrow$$
$$\alpha \le \sum_{I\in H_0}\beta_{0,I} + \alpha(\sum_{I\in H_1}\beta_{1,I} + \alpha(\sum_{I\in H_2}\beta_{2,I} + \alpha(\ldots + \alpha\sum_{I\in H_{n-1}}\beta_{n-1,I})\ldots).$$

The conjugate of $SD(n,2) = SD(n,P_2(\mathbf{n}))$ is denoted by $C(n,2)$; it is the following Horn sentence:

$$\Big(\bigwedge_{i<j}^{0,n-1}(\alpha \le \beta_{ij}+\beta_{ji}) \ \&\ \bigwedge_{i\ne j}^{0,n-1}(\beta_{ij} \le \alpha + \beta_{ji})\Big) \Longrightarrow$$
$$\alpha \le \sum_{j\ne 0}^{0,n-1}\beta_{0j} + \alpha(\sum_{j\ne 1}^{0,n-1}\beta_{1j} + \alpha(\sum_{j\ne 2}^{0,n-1}\beta_{2j} + \alpha(\ldots\alpha\sum_{j\ne n-1}^{0,n-1}\beta_{n-1,j})\ldots).$$

For example, $C(2,2)$, the conjugate of $SD_\wedge$, is (clearly equivalent to):

$$C(2,2):\qquad x+y = x+z = y+z \Longrightarrow x+y = x+yz. \tag{3}$$

Our main results are as follows; the proofs will be given in the next chapter.

Theorem 1. *For every $n \ge 2$ and $\emptyset \ne H \subseteq P_2(\mathbf{n})$, $\{\mathcal{V} : \mathcal{V}$ is a variety and $\mathrm{Con}(\mathcal{V}) \models SD(n,H)\}$ is a Mal'cev class.*

A concrete Mal'cev condition will be given in Theorem 9.

Theorem 2. *For every $n \ge 2$ and $\emptyset \ne H \subseteq P_2(\mathbf{n})$, $C(n,H) \models_{\mathrm{con}} SD(n,H)$.*

Theorem 3. *For every $n \geq 2$ and $\emptyset \neq H \subseteq P_2(\mathbf{n})$, ($SD(n,H)$ and modularity) $\models_{\rm con}$ distributivity.*

To justify the notation used in Theorem 3 let us mention that the conjunction of two Horn sentences is equivalent to a single Horn sentence modulo lattice theory. While ($SD_\wedge$ and modularity) $\models$ distributivity, the five element nonmodular lattice M_3 witnesses that ($SD(n,2)$ and modularity) $\not\models$ distributivity for $n > 2$. Hence $\models_{\rm con}$ in Theorem 3 is nontrivial in many cases. The same is true for Theorem 2, as it is pointed out by the following

Corollary 4. *For every $n \geq 2$, $C(n,2) \models^{\rm nt}_{\rm con} SD(n,2)$.*

Notice that $C(2,2)$ is a weaker Horn sentence than $SD_\vee$. Indeed, $SD_\vee \models C(2,2)$ is trivial, and $C(2,2) \not\models SD_\vee$ is witnessed by Figure 1.

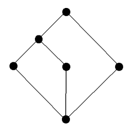

Figure 1

Hence Corollary 4 for $n = 2$ is a stronger result than (2), and it is worth separate formulating.

Corollary 5. $C(2,2) \models^{\rm nt}_{\rm con} SD_\wedge$.

Now we formulate a statement on the relations among the Horn sentences $C(n,H)$ and $SD(n,H)$.

Proposition 6. *Let $k > 2$, $m \geq 2$, $n \geq 2$, $\emptyset \neq H \subseteq P_2(\mathbf{n})$ and $\emptyset \neq K \subseteq P_2(\mathbf{m})$. Then*

(a) *$SD(k,2)$ is strictly weakening in k, i.e., $SD(k-1,2) \models SD(k,2)$ but $SD(k,2) \not\models SD(k-1,2)$;*

(b) *$C(k,2)$ is strictly weakening in k, i.e., $C(k-1,2) \models C(k,2)$ but $C(k,2) \not\models C(k-1,2)$;*

(c) *$SD(2,2) \models SD(n,H)$;*

(d) *$SD(m,K) \not\models C(n,H)$;*

(e) *$C(m,2) \not\models SD(n,H)$ and, moreover, $SD_\vee \not\models SD(n,H)$.*

Since Proposition 6 does not answer all questions, the remarks concluding the paper will add some further information. Part (d) of Proposition 6 can be strengthened to

Theorem 7. *For any $m, n \geq 2$, $\emptyset \neq K \subseteq P_2(\mathbf{m})$ and $\emptyset \neq H \subseteq P_2(\mathbf{n})$ we have $SD(m,K) \not\models_{\rm con} C(n,H)$.*

The Mal'cev conditions we are going to present in the following chapter are far from being simple. However, they are useful to prove Theorems 2 and 3. Interestingly enough, for all known $\lambda \models^{\mathrm{nt}}_{\mathrm{con}} \mu$ statement $\{\mathcal{V} : \mathrm{Con}(\mathcal{V}) \models \mu\}$ is known to be a Mal'cev class (even if $\lambda \models_{\mathrm{con}} \mu$ was proved or can be proved without Mal'cev conditions). The proof of Theorem 7 is also based on our Mal'cev condition, and resorting to Theorem 7 is, at present, the only way to prove (d) of Proposition 6. On the other hand, we could not solve the naturally arising problem if $SD(n, 2) \models_{\mathrm{con}} SD(n-1, 2)$ is true or not.

II. Proofs and technical statements

Like in some previous papers, e.g. in [1] and [3], our Mal'cev conditions will be given by certain graphs. This is not just an economic way to establish the appropriate Mal'cev conditions, it is also a possible way to work with them. For any lattice term $p(\alpha_0, \ldots, \alpha_{n-1})$ and integer $k \geq 2$ we define a graph $G_k(p)$ associated with p. The edges of $G_k(p)$ will be coloured by the variables $\alpha_0, \ldots, \alpha_{n-1}$, and two distinguished vertices, the so-called left and right *endpoints*, will have special roles. In figures, the endpoints will always be placed on the left-hand side and on the right-hand side, respectively. By $E(G_k(p))$ we denote the edge set of $G_k(p)$. An α-coloured edge connecting the vertices x and y will often be denoted by (x, α, y). Before defining $G_k(p)$ we introduce two kinds of operations for graphs. We obtain the *parallel connection* of graphs G_1 and G_2 by taking disjoint copies of G_1 and G_2 and identifying their left (right, resp.) endpoints, cf. Figure 2.

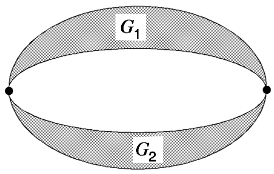

Figure 2

By taking disjoint graphs $H_1, \ldots, H_k$ ($k \geq 2$) such that $H_i \cong G_1$ for i odd and $H_i \cong G_2$ for i even, and identifying the right endpoint of H_i and the left endpoint of H_{i+1} for $i = 1, 2 \ldots, k-1$ we obtain the *serial connection* of length k of G_1 and G_2. (The left endpoint of H_1 and the right one of H_k are the endpoints of the serial connection, cf. Figure 3.)

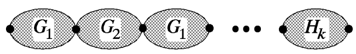

Figure 3

Now, if p is a variable then, for any $k \geq 2$, let $G_k(p)$ be the graph depicted in Figure 4,

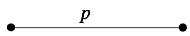

Figure 4

which consists of a single edge coloured by p. Let $G_k(p_1+p_2)$ resp. $G_k(p_1p_2)$ be the serial connection of length k resp. the parallel connection of graphs $G_k(p_1)$ and $G_k(p_2)$. Now we have defined $G_k(p)$ for lattice terms p with *binary* operations. However, p is often given by means of $\sum$ and $\prod$ as well. Then we always assume a fixed binary representation of p. Although each fixed binary form makes the rest of the paper work and the corresponding $G_2(p)$ does not depend too much on this form, we note that $G_k(p)$ ($k \geq 3$) heavily depends on the binary representation chosen. E.g., $G_3\big((\beta_0+\beta_1)+\beta_2\big)$ has eight vertices while $G_3\big(\beta_1+(\beta_2+\beta_0)\big)$ has only six.

For an algebra A, a lattice term $p = p(\alpha_0, \ldots, \alpha_{n-1})$, congruences $\hat\alpha_0, \ldots, \hat\alpha_{n-1} \in \mathrm{Con}(A)$, $a_0, a_1 \in A$ and $k \geq 2$ we say that a_0 and a_1 *can be connected by* $G_k(p)$ *in the algebra* A if there is a map φ (referred to as the connecting map) from the vertex set of $G_k(p)$ into A such that a_0 and a_1 are the images of the left and right endpoints, respectively, and for every edge $(x, \alpha_i, y) \in E(G_k(p))$ we have $\big(\varphi(x), \varphi(y)\big) \in \hat\alpha_i$. If it is necessary, we can emphasize that the colour α_i is represented by the congruence $\hat\alpha_i$. The following statement from [3] was proved by an easy induction.

Lemma 8. *With the above notations,* $(a_0, a_1) \in p(\hat\alpha_0, \ldots, \hat\alpha_{n-1})$ *iff* a_0 *and* a_1 *can be connected by* $G_k(p)$ *in* A *for some* $k \geq 2$ *iff there is a* $k_0 \geq 2$ *such that* a_0 *and* a_1 *can be connected by* $G_k(p)$ *in* A *for all* $k \geq k_0$.

Now with any pair of (finite coloured) graphs G' and G'' we associate a strong Mal'cev condition $U(G' \leq G'')$ in the following way, cf. [3]. Let $\alpha_0, \ldots, \alpha_{n-1}$ be the colours occurring on edges of G' and G'', and let $X = \{x_0, x_1, \ldots, x_{t-1}\}$ and $F = \{f_0, f_1, \ldots\}$ be the vertex sets of G' and G'', respectively, with x_0, x_1, f_0, f_1 being the endpoints. For $0 \leq j \leq t-1$ and $0 \leq i \leq n-1$ let $\alpha_i(j)$ be the smallest s such that there is an α_i-coloured path in G' connecting x_j and x_s. (By convention, the empty path connecting x_j with itself is α_i-coloured.) Now $U(G' \leq G'')$ is defined to be the following (strong Mal'cev) condition:

"There exist t-ary terms $f(x_0, \ldots, x_{t-1})$ ($f \in F$) which satisfy (1) the *endpoint identities* $f_0(x_0, \ldots, x_{t-1}) = x_0$ and $f_1(x_0, \ldots, x_{t-1}) = x_1$, and (2) for every edge $(f, \alpha_i, g) \in E(G'')$ the corresponding identity $f(x_{\alpha_i(0)}, x_{\alpha_i(1)}, \ldots, x_{\alpha_i(t-1)}) = g(x_{\alpha_i(0)}, x_{\alpha_i(1)}, \ldots, x_{\alpha_i(t-1)})$."

The identity associated with the edge (f, α_i, g) above will often be denoted by $I(f, \alpha_i, g)$.

Now let $n \geq 2$ be fixed, and define lattice terms $\beta_i^{(k)} = \beta_i^{(k)}(\alpha, \beta_0, \ldots, \beta_{n-1})$, $0 \leq i < n$, $0 \leq k$, via induction as follows. Let $\beta_i^{(0)} = \beta_i$, and let $\beta_i^{(j+1)} = \beta_i + \alpha\beta_{i+1}^{(j)}$. Here the subscript $i+1$ is understood modulo n, and the same convention applies for subscripts of β in the sequel. Theorem 1 is an easy consequence of the following theorem.

Theorem 9. *Let $n \geq 2$ and $\emptyset \neq H \subseteq P_2(\mathbf{n})$. Then, for an arbitrary variety $\mathcal{V}$, the following three conditions are equivalent.*

(i) $\mathrm{Con}(\mathcal{V}) \models SD(n, H)$.

(ii) *The Mal'cev condition "there is a $k \geq 2$ such that*

$$U_k := U\big(G_2(\alpha \prod_{I\in H} \sum_{i\in I} \beta_i) \leq G_k(\beta_0^{(k)})\big) \text{ "}$$

holds in $\mathcal{V}$.

(iii) $(x_0, x_1) \in \beta_0^{(k)}(\hat\alpha, \hat\beta_0, \ldots, \hat\beta_{n-1})$ *for some k where X is the vertex set of $G_2 = G_2(\alpha \prod_{I\in H} \sum_{i\in I} \beta_i)$, x_0 and x_1 are the endpoints, $\hat\alpha$ resp. $\hat\beta_i$ denote the congruence generated by $\{(x, y) \in X^2 : (x, \alpha, y) \in E(G_2)\}$ resp. $\{(x, y) \in X^2 : (x, \beta_i, y) \in E(G_2)\}$ in the free algebra $F_{\mathcal{V}}(X)$.*

Proof. (i) $\Longrightarrow$ (iii): Let $A = F_{\mathcal{V}}(X)$. With the notation $\hat\beta_i^{(k)} = \beta_i^{(k)}(\hat\alpha, \hat\beta_0, \ldots, \hat\beta_{n-1})$, an evident induction gives $\hat\beta_i^{(0)} \subseteq \hat\beta_i^{(1)} \subseteq \hat\beta_i^{(2)} \subseteq \ldots$ for $0 \leq i < n$. Hence $\hat\beta_i^{(\omega)} := \bigcup_{k=0}^{\infty} \hat\beta_i^{(k)} \in \mathrm{Con}(A)$. Suppose $(a, b) \in \hat\alpha \cap \hat\beta_i^{(\omega)}$. Then $(a, b) \in \hat\alpha \cap \hat\beta_i^{(k)}$ for some k, which gives $(a, b) \in \hat\alpha \cap \hat\beta_{i-1}^{(k+1)} \subseteq \hat\alpha \cap \hat\beta_{i-1}^{(\omega)}$ for all i, i.e.,

$$\hat\alpha \cap \hat\beta_0^{(\omega)} \supseteq \hat\alpha \cap \hat\beta_1^{(\omega)} \supseteq \ldots \supseteq \hat\alpha \cap \hat\beta_{n-1}^{(\omega)} \supseteq \hat\alpha \cap \hat\beta_0^{(\omega)}.$$

Hence all the $\hat\alpha \cap \hat\beta_i^{(\omega)}$ are equal, and (i) gives $\hat\alpha \prod_{I\in H} \sum_{i\in I} \hat\beta_i^{(\omega)} \leq \hat\beta_0^{(\omega)}$. Using Lemma 8 we conclude

$$(x_0, x_1) \in \hat\alpha \prod_{I\in H} \sum_{i\in I} \hat\beta_i \subseteq \hat\alpha \prod_{I\in H} \sum_{i\in I} \hat\beta_i^{(\omega)} \subseteq \hat\beta_0^{(\omega)}.$$

Hence $(x_0, x_1) \in \hat\beta_0^{(k)} = \beta_0^{(k)}(\hat\alpha,\ \hat\beta_0, \ldots, \hat\beta_{n-1})$ for some k, i.e., (iii) holds.

(iii) $\Longrightarrow$ (ii): Suppose (iii). By Lemma 8, x_0 and x_1 can be connected by $G_t(\beta_0^{(k)})$ in $F_{\mathcal{V}}(X)$ for some $t \geq 2$. Since $\beta_0^{(k)} \leq \beta_0^{(k+1)}$ in all lattices, it is not hard to see that both k and t can be enlarged, and therefore $t = k$ can be assumed†. Now the routine technique of deriving strong Mal'cev conditions, cf. e.g. Wille [26], Pixley [24] and [3], yields that U_k holds in $\mathcal{V}$.

(ii) $\Longrightarrow$ (i): Suppose $k \geq 2$, U_k holds in $\mathcal{V}$, $A \in \mathcal{V}$, $\hat\alpha, \hat\beta_0, \ldots, \hat\beta_{n-1} \in \mathrm{Con}(A)$ and $\hat\alpha\hat\beta_0 = \ldots = \hat\alpha\hat\beta_{n-1}$. Let (a_0, a_1) belong to $\hat\alpha \prod_{I\in H} \sum_{i\in I} \hat\beta_i$; we have to show that $(a_0, a_1) \in \hat\beta_0$. By Lemma 8, there is an $s \geq 2$ such that a_0 and a_1 can be connected by $G_s(\alpha \prod_{I\in H} \sum_{i\in I} \beta_i)$ in A. Hence there are finitely many elements $c_{I,0} = a_0$, $c_{I,1}$, …, $c_{I,m_I} = a_1$ for each $I \in H$ such that $(c_{I,j}, c_{I,j+1}) \in \bigcup_{i\in I} \hat\beta_i$ for $0 \leq j < m_I$.

Now $G_2(\alpha \prod_{I\in H} \sum_{i\in I} \beta_i)$ is depicted in Figure 5

† Essentially by the same reason, $U_k \models U_{k+1}$, i.e., "$(\exists k)(U_k)$" is a Mal'cev condition, indeed.

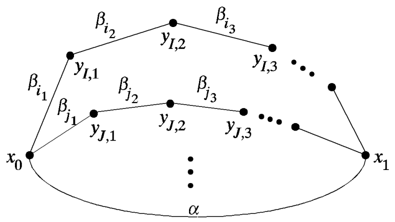

Figure 5

where $I, J \ldots \in H$, $I = \{i_1 < i_2 < i_3 < \ldots\}$ and $J = \{j_1 < j_2 < j_3 < \ldots\}$. The inner (i.e., not endpoint) vertices of this graph are denoted by $y_{I,1}$, $y_{I,2}, \ldots$ $(I \in H)$; the corresponding variables in the Mal'cev condition U_k are called *inner variables*.

Now we define some subgraphs, referred to as *permitted subgraphs*, of $G_k(\beta_0^{(k)})$. The only permitted subgraph of height k is $G_k(\beta_0^{(k)})$ itself. By definition, $G_k(\beta_0^{(k)})$ is a serial connection of length k of $G_k(\alpha\beta_1^{(k-1)})$ and the single edge graph $G_k(\beta_0)$; the copies of $G_k(\alpha\beta_1^{(k-1)})$ in the serial connection are the permitted subgraphs of height $k-1$. Each copy of $G_k(\beta_1^{(k-1)})$, i.e. each permitted subgraph of height $k-1$ without its α-edge connecting its endpoints, is a serial connection of length k of $G_k(\beta_1)$ and $G_k(\alpha\beta_2^{(k-2)})$; the copies of $G_k(\alpha\beta_2^{(k-2)})$ are the permitted subgraphs of height $k-2$. And so on, for $0 \le j < k$, the permitted subgraphs of height j are isomorphic to $G_k(\alpha\beta_{k-j}^{(j)})$, and each of them is a subgraph of a permitted subgraph of height $j+1$. (Of course, according to our general agreement, the subscript $k-j$ is understood modulo n.) In particular, the permitted subgraphs of height 0 are isomorphic to $G_k(\alpha\beta_k^{(0)}) = G_2(\alpha\beta_k)$. For $k = 4$ the situation is outlined in Figure 6.

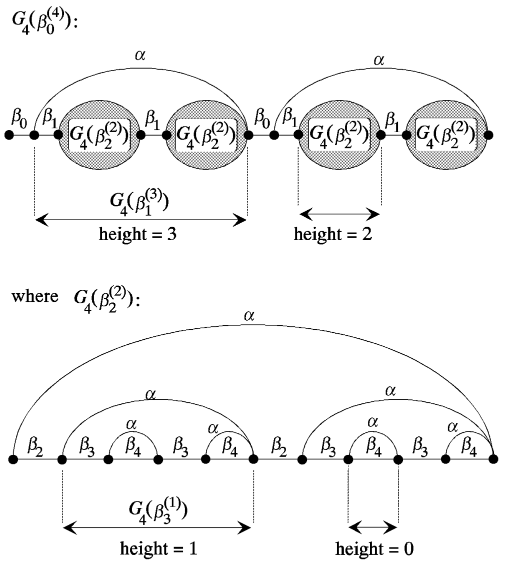

Figure 6

The expression "permitted subgraph" will mean a permitted subgraph of $G_k(\beta_0^{(k)})$ of height j for some $0 \le j \le k$.

The term symbols in the strong Mal'cev condition U_k are vertices in $G_k(\beta_0^{(k)})$, so they are endpoints of permitted subgraphs; this fact will be utilized in the sequel. Let $m = 2 + \sum_{I \in H}(|I| - 1)$, the number of vertices in $G_2(\alpha \prod_{I \in H} \sum_{i \in I} \beta_i)$.

Claim 10. Let f and g be the endpoints of a permitted subgraph and let

$$\vec{u} = (a_0, a_1, d_2, \ldots, d_{m-1}) \in \{a_0\} \times \{a_1\} \times A^{m-2}$$

be arbitrary. Then $f(\vec{u})\ \hat{\alpha}\ g(\vec{u})$.

Since (f, α, g) is an edge of the permitted subgraph in question, using the identity $I(f, \alpha, g)$ associated with this edge we obtain

$$f(\vec{u})\ \hat{\alpha}\ f(a_0, a_0, d_2, \ldots, d_{m-1}) = g(a_0, a_0, d_2, \ldots, d_{m-1})\ \hat{\alpha}\ g(\vec{u}),$$

proving Claim 10.

Claim 11. Let f and g be the endpoints of a permitted subgraph. If there exists a $\vec{u} \in \{a_0\} \times \{a_1\} \times \{a_0, a_1\}^{m-2}$ with $f(\vec{u})\ \hat{\alpha}\hat{\beta}_0 \dots \hat{\beta}_{n-1}\ g(\vec{u})$ then $f(\vec{v})\ \hat{\alpha}\hat{\beta}_0 \dots \hat{\beta}_{n-1}\ g(\vec{v})$ holds for all $\vec{v} \in \{a_0\} \times \{a_1\} \times \{a_0, a_1\}^{m-2}$.

It suffices to show that if $2 \le i < m$ and the i-th component of $\vec{u} = (a_0, a_1, u_2, \dots, u_{m-1})$ is $u_i = a_0$ then $f(\vec{v})\ \hat{\alpha}\hat{\beta}_0 \dots \hat{\beta}_{n-1}\ g(\vec{v})$ holds for $\vec{v} = (a_0, a_1, u_2, \dots, u_{i-1}, a_1, u_{i+1}, \dots, u_{m-1})$. Fix an $I \in H$ and consider the m-tuples $\vec{w}^{(j)} = (a_0, a_1, u_2, \dots, u_{i-1}, c_{I,j}, u_{i+1}, \dots, u_{m-1})$, $j = 0, 1, \dots, m_I$. Then $\vec{w}^{(0)} = \vec{u}$ and $\vec{w}^{(m_I)} = \vec{v}$, so it suffices to show via induction that for all $j \le m_I$

$$f(\vec{w}^{(j)})\ \hat{\alpha}\hat{\beta}_0 \dots \hat{\beta}_{n-1}\ g(\vec{w}^{(j)}). \tag{4}$$

When $j = 0$, (4) states what we have assumed. Now suppose (4) for some $j < m_I$. Since $(c_{I,j}, c_{I,j+1}) \in \bigcup_{\ell \in I} \hat{\beta}_\ell$, there is an $\ell \in I$ with $(c_{I,j}, c_{I,j+1}) \in \hat{\beta}_\ell$, and we have $f(\vec{w}^{(j)})\ \hat{\beta}_\ell\ f(\vec{w}^{(j+1)})$ and $g(\vec{w}^{(j)})\ \hat{\beta}_\ell\ g(\vec{w}^{(j+1)})$. Using (4) for j and transitivity we infer $f(\vec{w}^{(j+1)})\ \hat{\beta}_\ell\ g(\vec{w}^{(j+1)})$. By Claim 10, $f(\vec{w}^{(j+1)})\ \hat{\alpha}\ g(\vec{w}^{(j+1)})$. Since $\hat{\alpha}\hat{\beta}_0 = \dots = \hat{\alpha}\hat{\beta}_{m-1}$, we conclude (4) for $j+1$. We have shown that a_0 can be changed to a_1 at the ith component; the transition from a_1 to a_0 follows similarly. This proves Claim 11.

Claim 12. Let f and g be the endpoints of a permitted subgraph S. Then for all $\vec{u} \in \{a_0\} \times \{a_1\} \times \{a_0, a_1\}^{m-2}$ we have $f(\vec{u})\ \hat{\alpha}\hat{\beta}_0 \dots \hat{\beta}_{n-1}\ g(\vec{u})$.

We prove this claim via induction on the height of S. Suppose S is of height 0, i.e., $S = G_k(\alpha\beta_k)$. We define $\vec{u} = (u_0, \dots, u_{m-1}) \in \{a_0\} \times \{a_1\} \times \{a_0, a_1\}^{m-2}$ as follows. Let $u_0 = a_0$, and for all edge $(x_0, \beta_k, y_{I,1}) \in E\big(G_2(\alpha \prod_{I \in H} \sum_{i \in I} \beta_i)\big)$, cf. Figure 5, let the component of $\vec{u}$ corresponding to $y_{I,1}$ be a_0. Let the rest of the components be defined as a_1. Since $2 \le |I|$ for all $I \in H$, for each β_k-coloured edge of $G_2(\alpha \prod_{I \in H} \sum_{i \in I} \beta_i)$ the components of $\vec{u}$ corresponding to the endpoints of this edge are equal. Hence the identity $I(f, \beta_k, g)$ applies and we obtain $f(\vec{u}) = g(\vec{u})$. This gives $f(\vec{u})\ \hat{\alpha}\hat{\beta}_0 \dots \hat{\beta}_{n-1}\ g(\vec{u})$ for one $\vec{u}$, whence it holds for all $\vec{u}$ in virtue of Claim 11.

Now let S be of height $k - j$, $0 \le j < k$. Then S is a serial connection of length k of graphs $G_k(\beta_j)$ and $S' = G_k(\alpha\beta_{j+1}^{(k-j-1)})$. Let $h_0 = f$, h_1, $\dots$, $h_k = g$ be the endpoints of copies of $G_k(\beta_j)$ and S' in this serial connection, cf. Figure 7.

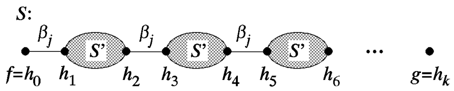

Figure 7

As previously, we can choose a $\vec{u} \in \{a_0\} \times \{a_1\} \times \{a_0, a_1\}^{m-2}$ such that, applying the identity associated with $(h_t, \beta_j, h_{t+1}) \in E(S)$, we obtain $h_t(\vec{u}) = h_{t+1}(\vec{u})$ for t even, $0 \le t < k$. Since each copy of S' in Figure (7) is a permitted subgraph of height $k - j - 1$, the induction hypothesis yields $h_t(\vec{u})\ \hat{\alpha}\hat{\beta}_0 \dots \hat{\beta}_{m-1}\ h_{t+1}(\vec{u})$ for $0 < t < k$, t odd. By transitivity, $\big(f(\vec{u}), g(\vec{u})\big) = \big(h_0(\vec{u}), h_k(\vec{u})\big) \in \hat{\alpha}\hat{\beta}_0 \dots \hat{\beta}_{n-1}$. This holds for one carefully chosen $\vec{u}$, whence for all $\vec{u} \in \{a_0\} \times \{a_1\} \times \{a_0, a_1\}^{m-2}$ in virtue of Claim 11. Claim 12 has been shown.

Now let us apply Claim 12 for the whole graph $G_k(\hat{\beta}_0^{(k)})$ with endpoints f_0 and f_1; we obtain $(a_0, a_1) = \big(f_0(\vec{u}), f_1(\vec{u})\big) \in \hat{\alpha}\hat{\beta}_0 \ldots \hat{\beta}_{m-1} \subseteq \hat{\beta}_0$ for arbitrary $\vec{u} \in \{a_0\} \times \{a_1\} \times \{a_0, a_1\}^{m-2}$. This proves (ii) $\Longrightarrow$ (i) and Theorem 9. $\diamond$

Proof of Theorem 2. Let $\mathcal{V}$ be a variety with $\mathrm{Con}(\mathcal{V}) \models C(n, H)$, and let us consider the graph $G_2(\alpha \prod_{I \in H} \sum_{i \in I} \beta_i)$, cf. Figure 5. The vertex set of this graph is denoted by X. For $i \in I \in H$, the path x_0, $y_{I,1}$, $y_{I,2}$, $\ldots$, x_1 contains a unique β_i-coloured edge; let $\hat{\beta}_{i,I}$ be the smallest congruence of the free algebra $F_{\mathcal{V}}(X)$ that collapses the endpoints of this edge. The congruence generated by (x_0, x_1) is denoted by $\hat{\alpha}$. Clearly, $\hat{\alpha}$ and the $\hat{\beta}_{i,I}$ ($i \in I$, $I \in H$) satisfy the premise of $C(n, H)$. Since $C(n, H)$ holds in $\mathrm{Con}(F_{\mathcal{V}}(X))$,

$$(x_0, x_1) \in \hat{\alpha} \le \hat{\beta}_0 + \hat{\alpha}(\hat{\beta}_1 + \hat{\alpha}(\hat{\beta}_2 + \ldots + \hat{\alpha}\hat{\beta}_{n-1}) \ldots) \tag{5}$$

where $\hat{\beta}_i := \sum_{I \in H_i} \hat{\beta}_{i,I}$ ($0 \le i < n$, $H_i = \{I \in H : i \in I\}$). Notice that the right-hand side of (5) is just $\beta_0^{(n-1)}(\hat{\alpha}, \hat{\beta}_0, \ldots, \hat{\beta}_{n-1})$, and $\hat{\alpha}$, $\hat{\beta}_0$, $\ldots$, $\hat{\beta}_{n-1}$ are exactly the congruences occurring in (iii) of Theorem 9. Hence $\mathrm{Con}(\mathcal{V}) \models SD(n, H)$ by Theorem 9. The proof is complete. $\diamond$

Proof of Theorem 3. Suppose, to obtain a contradiction, that $\mathcal{V}$ is a congruence modular but not congruence distributive variety such that $\mathrm{Con}(\mathcal{V}) \models SD(n, H)$. Let $k := 1 + \sum_{I \in H} |I - 1| = |X| - 1$ where X is the vertex set of $G_2 := G_2(\alpha \prod_{I \in H} \sum_{i \in I} \beta_i)$, cf. Figure 5. Since $|I| \ge 2$ for $I \in H$, $k \ge 2$. If $k = 2$ then $SD(n, H)$ is equivalent to $SD_\wedge$ modulo lattice theory, and the theorem follows from ($SD_\wedge$ and modularity) $\models$ distributivity. Thus we can assume that $k \ge 3$.

Now, recalling Huhn's lattice identity

$$\mathrm{dist}_k: \qquad x \sum_{i=0}^{k} y_i = \sum_{j=0}^{k} (x \sum_{i \ne j}^{0,k} y_i),$$

it is known that $\mathrm{dist}_k \models_{\mathrm{con}}$ distributivity, cf. Nation [22]. Therefore $\mathrm{Con}(\mathcal{V}) \not\models \mathrm{dist}_k$, so we can take an algebra $A \in \mathcal{V}$ with $\mathrm{Con}(A) \not\models \mathrm{dist}_k$. We conclude from Huhn [15, Thm. 1.1(C)] that there is a prime field K such that $L(PG_k(K))$, the subspace lattice of the k-dimensional projective geometry over K, is a sublattice of $\mathrm{Con}(A)$. Let M be the vector space over K freely generated by X. Then $L(PG_k(K))$ is isomorphic to $L(M)$, the subspace lattice of M, so we conclude that $SD(n, H)$ holds in $L(M)$.

Now the desired contradiction proving Theorem 3 is supplied by the following statement.

Claim 13. $SD(n, H)$ fails in the subspace lattice $L(M)$ defined above.

Indeed, for $0 \le i < n$, let $\hat\beta_i \in L(M)$ be the subspace spanned by $\{u - v : (u, \beta_i, v) \in E(G_2)\}$, and let $\hat\alpha := K(x_1 - x_0)$, the (cyclic) subspace spanned by $\{u - v : (u, \alpha, v) \in E(G_2)\} = \{x_0 - x_1\}$. Since for each edge (u, β_i, v) either u or v is an endpoint of no other β_i-coloured edge, and $\{u, v\} \ne \{x_0, x_1\}$, it is easy to conclude that $x_1 - x_0 \notin \hat\beta_i$. Hence $\hat\alpha\hat\beta_0 = \ldots = \hat\alpha\hat\beta_{n-1} = 0$. By the construction, $x_1 - x_0 \in \hat\alpha \prod_{I\in H} \sum_{i\in I} \hat\beta_i$ but $x_1 - x_0 \notin \hat\beta_0$. So $SD(n, H)$ fails in $L(M)$. This proves Claim 13 and Theorem 3. ◇

Proof of Proposition 6. (a) $SD(k-1, 2) \models SD(k, 2)$ is evident. It is easy to see that $SD(k, 2)$ holds for any $k+1$ elements in a lattice that do not form an antichain. Let M_k denote the $k+2$ element lattice with a k element antichain, then $SD(k, 2)$ holds but $SD(k-1, 2)$ fails in M_k. Hence $SD(k, 2) \not\models SD(k-1, 2)$.

(b) $C(k-1, 2) \models C(k, 2)$ is easy, so we do not detail it. For $t > 1$ let L_t be the lattice depicted in Figure 8.

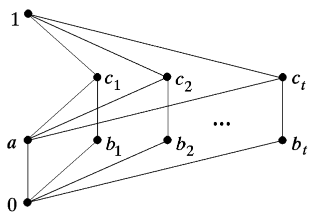

Figure 8

The substitution $\alpha = b_k$, $\beta_{ij} = b_{i+1}$ $(i \ne j,\ 0 \le i < k-1,\ 0 \le j < k-1)$ shows that $C(k-1, 2)$ fails in L_k. Now we show that $C(k, 2)$ holds in L_k. Suppose the contrary and fix $\alpha, \beta_{ij} \in L_k$ $(i < k,\ j < k,\ i \ne j)$ satisfying the premise of $C(k, 2)$ such that, with the notation $\beta_i := \sum_{j\ne i} \beta_{ij}$,

$$\alpha \not\le \beta_0 + \alpha(\beta_1 + \alpha(\beta_2 + \ldots + \alpha\beta_{k-1})\ldots). \tag{6}$$

Then $\alpha \not\le \beta_{ij}$, for otherwise $\alpha \le \beta_i$ would contradict (6). Hence $\beta_{ij} \ne 0$, for otherwise $\alpha \le \beta_{ij} + \beta_{ji} = \beta_{ji}$, which we have already excluded.

Case 1: $\alpha = 1$. Then $\beta_{ij} = a$ would lead to $1 = \alpha = a + \beta_{ji} \Longrightarrow \beta_{ji} = 1 \ge \alpha$, a contradiction. Hence $\{\beta_{ij} : i \ne j\} \subseteq \{b_1, \ldots, b_k, c_1, \ldots, c_k\}$. For a given i, the β_{ij} must belong to the same $\{b_{\varphi(i)}, c_{\varphi(i)}\}$, for otherwise $\beta_i = 1 \ge \alpha$. Since $\beta_{ij} + \beta_{ji} \ge \alpha = 1$, $\varphi : \{0, \ldots, k-1\} \to \{1, \ldots, k\}$ is injective, and therefore surjective. Hence the right-hand side of (6) is $\sum_{i\ne j} \beta_{ij} \ge b_1 + \ldots + b_k = 1$, a contradiction.

Case 2: α is a coatom, say $\alpha = c_1$. If we had $\beta_{ij} \in \{a, b_1\}$ for some pair (i, j), $i \ne j$, then $\alpha \not\le \beta_{ji} \le \alpha + \beta_{ij} = \alpha$ and $\alpha \le \beta_{ij} + \beta_{ji}$ would yield $\{\beta_{ij}, \beta_{ji}\} = \{a, b_1\}$, say $(\beta_{ij}, \beta_{ji}) = (a, b_1)$, and $\beta_i \ge a$ and $\beta_j \ge b_1$ would easily contradict (6). Hence $\{\beta_{ij} : i \ne j\} \subseteq \{b_2, \ldots, b_k, c_2, \ldots, c_k\}$, whence the previous φ cannot be injective, a contradiction.

Case 3: $\alpha = a$. Then $\{\beta_{ij} : i \neq j\} \subseteq \{b_1, \ldots, b_k\}$, φ is a bijection, and $\alpha + \beta_{ij} = \alpha + b_{\varphi(i)} = c_{\varphi(i)} \not\geq b_{\varphi(j)} = \beta_{ji}$ is a contradiction.

Case 4: α is another atom, say $\alpha = b_1$. Then $\{\beta_{ij} : i \neq j\} \subseteq \{a, b_2, \ldots, b_k, c_2, \ldots, c_k\}$. If $\beta_{ij} \neq a$ for all $i \neq j$ then φ cannot be a bijection. Hence $\beta_{ij} = a$ for some $i \neq j$, and $b_1 = \alpha \leq \beta_{ij} + \beta_{ji} = a + \beta_{ji}$ implies $\alpha \leq \beta_{ji}$, a contradiction. We have seen that $L_k \models C(k, 2)$. Hence $C(k, 2) \not\models C(k-1, 2)$, proving (b).

(c) To show $SD(2, 2) \models SD(n, H)$, firstly we assume that $|H| = 1$, say $H = \{\{0, 1, \ldots, t-1\}\}$. Then the statement follows via induction; indeed, after deriving $\alpha(\beta_1 + \ldots + \beta_{t-1}) = \alpha\beta_1 = \alpha\beta_0$ from the induction hypothesis, we can apply $SD_\wedge$ for the elements α, β_0 and $\beta_1 + \ldots + \beta_{t-1}$. From the $|H| = 1$ case the general case is evident.

(d) is a consequence of Theorem 7.

In order to show (e), let L be the set of convex polytopes in the $(n-1)$-dimensional Euclidean space E_{n-1}. By a polytope we mean the convex hull of finitely many points. Since polytopes can also be defined as bounded intersections of finitely many half spaces, cf., e.g., Ziegler [27], L is a lattice with intersection as meet and convex hull of union as join. First we show that $L \models SD_\vee$. Let $P, Q_1, Q_2 \in L$ such that $P + Q_1 = P + Q_2$. Let $R = P + Q_1 + Q_2 = P + Q_1 = P + Q_2$, and denote by V the vertex set of R. Then $\mathrm{conv}(V)$, the convex hull of V, is R but $\mathrm{conv}(R \setminus \{v\}) \neq R$ for all $v \in V$. We claim that

$$V \subseteq P \cup (Q_1 \cap Q_2). \tag{7}$$

Suppose $a \in V \setminus \big(P \cup (Q_1 \cap Q_2)\big) = \big(V \setminus (P \cup Q_1)\big) \cup \big(V \setminus (P \cup Q_2)\big)$, then $P + Q_i \subseteq \mathrm{conv}(R \setminus \{a\}) \subset R = P + Q_i$ for $i = 1$ or $i = 2$, a contradiction. This shows (7). Armed with (7) we conclude $P + Q_1 = R = \mathrm{conv}(V) \subseteq \mathrm{conv}\big(P \cup (Q_1 \cap Q_2)\big) = \mathrm{conv}(P) + \mathrm{conv}(Q_1 \cap Q_2) = P + Q_1Q_2$. Hence $L \models SD_\vee$; therefore $L \models C(2, 2)$ and, by (b), $L \models C(m, 2)$.

Now let $b_0, b_1, \ldots, b_{n-1} \in E_{n-1}$ be points in general position, i.e., they do not belong to a hyperplane. Then $S = \mathrm{conv}(\{b_0, \ldots, b_{n-1}\})$ is a symplex. For $i = 0, \ldots, n-1$ let $\beta_i := \mathrm{conv}(\{b_0, \ldots, b_{i-1}, b_{i+1}, \ldots, b_{n-1}\})$, a facet of the symplex. Choose an inner point a of the symplex, i.e., $a \in S \setminus \{\beta_0 \cup \beta_1 \cup \ldots \cup \beta_{n-1}\}$. Set $\alpha = \{a\}$. Since $\alpha\beta_i = \{a\} \cap \beta_i = \emptyset$, the polytopes $\alpha, \beta_0, \ldots, \beta_{n-1}$ easily witness that $SD(n, H)$ fails in L. This yields (e). Proposition 6 is proved. ◇

Proof of Theorem 7. Let $\mathcal{V}$ be the variety of (meet) semilattices. By Papert [23] $\mathrm{Con}(\mathcal{V}) \models SD(2, 2)$, so $\mathrm{Con}(\mathcal{V}) \models SD(m, K)$ by Proposition 6(c). We intend to show that $\mathrm{Con}(\mathcal{V}) \not\models C(n, H)$; suppose the contrary. The graph $G_2(\alpha \prod_{I \in H} \sum_{i \in I} \beta_i)$ will be denoted by G_2. With the notations of the proof of Theorem 2 we have

$$(x_0, x_1) \in \beta_0^{(n-1)}(\hat{\alpha}, \hat{\beta}_0, \ldots, \hat{\beta}_{n-1}). \tag{8}$$

For semilattice terms g_0 and g_1 over the vertex set $X = \{x_0, x_1, \ldots\}$ of G_2 and for a permitted subgraph S (cf. the proof of Theorem 9) of $G_k(\beta_0^{(n-1)})$ with vertex set F_S and endpoints f_{0S} and f_{1S} we define the following condition:

> "there exist semilattice terms $h(x_0, x_1, \ldots)$, $h \in F_S$, which satisfy the identities $f_{0S}(x_0, x_1, \ldots) = g_0(x_0, x_1, \ldots)$, $f_{1S}(x_0, x_1, \ldots) = g_1(x_0, x_1, \ldots)$ and for each $(h_1, \gamma, h_2) \in E(S)$ the identity $I(h_1, \gamma, h_2)$."

This condition will be denoted by $U^*(G_2 \le S; f_{0S} = g_0, f_{1S} = g_1)$. For example, $U^*(G_2 \le S; f_{0S} = x_0, f_{1S} = x_1)$ is the same as "$U(G_2 \le S)$ holds in $\mathcal{V}$".

From (8) we obtain $(x_1, x_0) \in \beta_0^{(n-1)}(\hat\alpha, \hat\beta_0, \dots, \hat\beta_{n-1})$, whence, similarly to the proof of (iii) $\Longrightarrow$ (ii) in Theorem 9, we conclude that there is a $k \ge 2$ such that

$$U^*(G_2 \le G_k(\beta_0^{(n-1)}); f_{0S} = x_1, f_{1S} = x_0) \text{ holds.} \tag{9}$$

(Interchanging x_0 and x_1 serves technical purposes.) We will use the fact that each semilattice term is, modulo semilattice theory, the meet of all variables occurring in it.

Multiplying (i.e., meeting) all terms by x_1, we infer from (9) that

$$U^*(G_2 \le G_k(\beta_0^{(n-1)}); f_0 = x_1, f_1 = x_0x_1) \quad \text{holds.} \tag{10}$$

We intend to show that for all permitted subgraphs S of $G_k(\beta_0^{(n-1)})$

$$U^*(G_2 \le S; f_{0S} = x_1, f_{1S} = x_0x_1) \quad \text{holds.} \tag{11}$$

This will be done via a downward induction on the height of S. If S is of height $n-1$ then (11) coincides with (10).

Now suppose that S is of height $n-1-t > 0$, i.e., $S = G_k(\beta_t^{(n-1-t)})$, and $U^*(G_2 \le S; f_{0S} = x_1, f_{1S} = x_0x_1)$ holds. We want to show the same for $T = G_k(\beta_{t+1}^{(n-2-t)})$. Let $g_0 = f_{0S}$, g_1, g_2, …, $g_k = f_{1S}$ be the endpoints needed to form S from $G_k(\beta_t)$ and T via serial connection, cf. Figure 9,

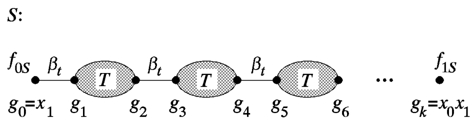

Figure 9

and suppose that all terms are chosen in $\mathcal{V}$ such that they witness $U^*(G_2 \le S; f_{0S} = x_1, f_{1S} = x_0)$. Let $A_t := \{u \in X : (u, \beta_t, x_1) \in E(G_2)\}$. Our argument uses the general convention that the colours on the arcs of G_2 (cf. Figure 5) occur from left to right order. This means that if $(x_0, \beta_{i_1}, y_{I,1})$, $(y_{I,1}, \beta_{i_2}, y_{I,2})$, $(y_{I,2}, \beta_{i_3}, y_{I,3})$, …, $(y_{I,\ell-1}, \beta_{i_\ell}, x_1)$ are adjacent consecutive edges from the left to the right then $i_1 < i_2 < i_3 \ldots < i_\ell$. Let $\breve\beta_i$ denote the smallest equivalence on X that includes $\{(u,v) \in X^2 : (u, \beta_i, v) \in E(G_2)\}$. It follows from the above-mentioned convention that

$$\text{for } \ u \in A_t \ \text{ and } \ j > t, \quad |[u]\breve\beta_j| = 1, \tag{12}$$

i.e., the $\breve\beta_j$-class of u is a singleton.

Suppose first that one of the g_i $(0 < i < k)$ contains some $u \in A_t$. Let d be the smallest integer such that g_d contains u, and let m be the largest integer such that g_d, $g_{d+1}, \dots, g_m$ all contain u. Since any two vertices of T are connected by a path containing the colours $\beta_{t+1}, \beta_{t+2}, \dots, \beta_{n-1}$ only, we conclude from (12) that if one of the endpoints of (a copy of) T contains u then all vertices (inner and endpoint vertices) of T contain u. Therefore d is odd and m is even, for otherwise g_{d-1} and g_d or g_m and g_{m+1} would be the endpoints of a copy of T.

Now we can change u to x_1 in all terms (vertices) between g_d and g_m (including g_d, g_m, and the inner vertices of the corresponding copies of T). We claim that the new terms obtained this way still witness that $U^*(G_2 \le S; f_{0S} = x_1, f_{1S} = x_0x_1)$ holds. Since (12) and $|[u]\breve{\alpha}| = 1$, u "was not used" within T, whence for every copy of T between g_d and g_m the identities associated with the edges of T hold. Since $(u, x_1) \in \breve{\beta}_t$, the identities $I(g_i, \beta_t, g_{i+1})$ remain valid for $d < i < m$, i even, and also for $i = d - 1$ and $i = m$. Hence the new terms do the job.

We have seen how to reduce the occurrences of elements of A_t. After doing this reduction in a finite number of steps we can get rid of all elements of A_t. Hence we can assume that

$$\text{no } \ u \in A_t \ \text{ occurs in our terms.} \tag{13}$$

From now on let m be the smallest number such that x_0 occurs in g_m. We claim that

$$g_j = x_1 \ \text{ for } \ 0 \le j < m. \tag{14}$$

This is true for $g_0 = f_{0S}$. If $g_{j-1} = x_1$, $j < m$ and $j-1$ is even then (13) and $I(g_{j-1}, \beta_t, g_j)$ yield $g_j = x_1$. If $g_{j-1} = x_1$, $j < m$ and $j-1$ is odd then the identity $I(g_{j-1}, \alpha, g_j)$ associated with $(g_{j-1}, \alpha, g_j) \in E(T)$ and the lack of x_0 in g_j give $g_j = x_1$. This induction shows (14).

If $m - 1$ is even then $I(g_{m-1}, \beta_t, g_m)$ cannot hold, for $g_{m-1} = x_1$, $(x_0, x_1) \notin \breve{\beta}_t$ but x_0 occurs in g_m. Consequently, $m - 1$ is odd and $(g_{m-1}, \alpha, g_m) \in E(S)$. Since $g_{m-1} = x_1$ and x_0 occurs in g_m, the identity $I(g_{m-1}, \alpha, g_m)$ can hold only if $g_m = x_0$ or $g_m = x_0x_1$. Hence either

$$U^*(G_2 \le T; f_{0T} = x_1, f_{1T} = x_0) \tag{15}$$

or

$$U^*(G_2 \le T; f_{0T} = x_1, f_{1T} = x_0x_1) \tag{16}$$

holds. Notice that (15) implies (16), for all terms h occurring in (15) can be replaced by hx_1. This completes the induction proving (11).

Applying (11) to the subgraphs of height 0, it follows that $U^*(G_2 \le G_k(\alpha\beta_{n-1}); f_0 = x_1, f_1 = x_0x_1)$ holds, which contradicts $(x_0, x_1) \notin \breve{\beta}_{n-1}$. This proves Theorem 7. $\diamond$

We conclude the paper with some remarks on Proposition 6. The five element nonmodular lattice N_5 witnesses that $SD_\vee \not\models C(3, \{\{0, 1, 2\}\})$ and so $C(2, 2) \not\models C(3, \{\{0, 1, 2\}\})$. This explains why Proposition 6 does not include a "conjugate" counterpart of (c).

We do not know if (e) holds with $C(m, K)$ instead of $C(2, 2)$ but the present proof of (e) is not appropriate to decide this. Indeed, if K is the center and $B_0, \dots, B_4$ are consecutive vertices of a (planar) regular pentagon then $\alpha = \mathrm{conv}(\{B_0, B_1, K\})$, $\beta_0 = \mathrm{conv}(\{B_1, B_2\})$, $\beta_1 = \mathrm{conv}(\{B_0, B_3, B_4\})$ and $\beta_2 = \mathrm{conv}(\{B_2, B_3, B_4\})$ witness that $C(3, \{\{0, 1, 2\}\})$ fails in L.

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Added on June 19, 1998. As an affirmative answer to the problem raised at the end of the first section, an anonymous referee has proved that $SD(n,H) \models_{\text{con}} SD_\wedge$ for every $n \geq 2$ and $\emptyset \neq H \subseteq P_2(\mathbf{n})$. The proof is based on Kearnes and Szendrei [19], Lipparini [20], and Theorem 3. Now Theorem 1 becomes a consequence of Proposition 6(c) and Willard [25], and the referee's method together with [3] gives a shorter proof of Theorem 2. However, the present approach to Theorems 1 and 2 can still be justified. Not only by its role in *finding* the results but also in the proofs of Theorem 7 and (the purely lattice theoretic) Proposition 6(d).

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