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TWO NOTES ON n-LATTICES

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Abstract. By an n-lattice we mean a non-empty set with $n \geq 2$ semilattice operations such that the roundabout absorption laws are satisfied. This notion was introduced by K. Horiuchi, who studied mainly the case $n = 3$ and called 3-lattices as trices. Two examples for n-lattices are presented; one of them is due to Horiuchi for $n \leq 4$.

By an n-fold semilattice we mean an algebra $L = (L; \sqcup_1, \dots, \sqcup_n)$ such that every $(L; \sqcup_i)$, $1 \leq i \leq n$, is a semilattice. If an n-fold semilattice satisfies the roundabout absorption law

$$(\dots((x \sqcup_{1\sigma} y) \sqcup_{2\sigma} y) \sqcup_{3\sigma} \dots) \sqcup_{n\sigma} y = y \tag{1}$$

for all permutations $\sigma \in S_n$ then it will be called an *n-lattice*. 2-lattices are the usual lattices, and 3-lattices are also called *trices*. The notions above are due to K. Horiuchi [2, 3]. To point out that n-lattices have reasonable algebraic properties we define some terms via induction. Let $t_1(x_1, x_2) = x_1 \sqcup_1 x_2$, and let $t_{i+1}(x_1, \dots, x_{2^{i+2}}) = t_i(x_1, \dots, x_{2^{i+1}}) \sqcup_{i+1} t_i(x_{2^{i+1}+1}, \dots, x_{2^{i+2}})$. Clearly, t_n is a near unanimity term in the variety of n-lattices, and even in the variety of n-fold semilattices satisfying (1) for the identical permutation only. Hence n-lattices are congruence distributive, cf. Mitschke [4]. To make n-lattices more attractive, this short note is devoted to two examples and their properties.

We say that an n-lattice is *fully absorptive* if it satisfies the identity

$$(\dots((x \sqcup_{i_1} y) \sqcup_{i_2} y) \sqcup_{i_3} \dots) \sqcup_{i_k} y = y \tag{2}$$

for all sequences $i_1, \dots, i_k$ such that $\{i_1, \dots, i_k\} = \{1, \dots, n\}$. Here $k \geq n$. It is easy to see that lattices ($n = 2$) and trices are fully absorptive, but we conjecture that n-lattices for $n \geq 4$ are not always fully absorptive.

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From a fully absorptive k-lattice we can easily obtain n-lattices for all $n > k$ by repeating some of the basic operations. However, *nontrivial* n-lattices $L = (L; \sqcup_1, \dots, \sqcup_n)$; i.e., n-lattices such that none of the $(L; \sqcup_1, \dots, \sqcup_{i-1}, \sqcup_{i+1}, \dots, \sqcup_n)$, $1 \le i \le n$, is an $(n-1)$-lattice, cannot be produced this way.

Now, following Horiuchi [3], let $n \ge 2$ and let C be a chain with $|C| \ge n+1$. Denote $\{X \subseteq C : |X| = n\}$ by $P_n(C)$. For $X \subseteq C$ and $i \le |X|$ the i-element order ideal resp. order filter of the chain X will be denoted by $X_{\downarrow i}$ resp. $X^{\uparrow i}$. If $i = 0$ then $X_{\downarrow i} = X^{\uparrow i} = \emptyset$. For $A, B \in P_n(C)$ let $A \sqcup_i B = (A \cup B)_{\downarrow i} \cup (A \cup B)^{\uparrow n-i}$. The following statement settles a conjecture of Horiuchi.

Proposition 1 (Horiuchi for $n \le 4$). *If C is a chain with $|C| \ge n+1$ then $L = (P_n(C); \sqcup_0, \dots, \sqcup_n)$ is a fully absorptive $(n+1)$-lattice.*

Proof. Clearly, L is an $(n+1)$-fold semilattice. Now let $i_1, \dots, i_k$ be a sequence with $\{i_1, \dots, i_k\} = \{0, \dots, n\}$, and let $A, B \in P_n(C)$. Let $D_0 = A$, $D_j = D_{j-1} \sqcup_{i_j} B$, $1 \le j \le k$, and $E_j = D_j \setminus B$, $1 \le j \le k$. Since $D_j = (E_{j-1} \cup B)_{\downarrow j} \cup (E_{j-1} \cup B)^{\uparrow n-j}$, we conclude that $E_1 \supseteq E_2 \supseteq \cdots \supseteq E_k$. We claim that $E_k = \emptyset$. Suppose the contrary and let $a \in E_k$. Then $0 \le |\{b \in B : b < a\}| \le |B| = n$, so there is a smallest t, $1 \le t \le k$, such that $i_t = |\{b \in B : b < a\}|$. Since $a \notin B$, $|\{b \in B : a < b\}| = n - i_t$. Hence $a \notin (D_{t-1} \cup B)_{\downarrow i_t}$, $a \notin (D_{t-1} \cup B)^{\uparrow n-i_t}$, and therefore $a \notin D_{t-1} \sqcup_{i_t} B = D_t$. This contradicts $a \in E_k \subseteq E_t \subseteq D_t$, so we have $E_k = \emptyset$. Hence $D_k \subseteq B$, and $|D_k| = n = |B|$ gives $D_k = B$, which simply means that (2) holds for A and B. ☐

If $c_0 < c_1 < \cdots < c_n \in C$ and $0 \le i \le n$ then $B = \{c_0, \dots, c_n\} \setminus \{c_i\}$ and an arbitrary $A \in P_n(C)$ with $\{c_i\} \subseteq A \subseteq \{c_0, \dots, c_n\}$ witness that $(P_n(C); \sqcup_0, \dots, \sqcup_{i-1}, \sqcup_{i+1}, \dots, \sqcup_n)$ does not satisfy the roundabout absorption laws. Thus we have arrived at Horiuchi's remark (cf. [3]): L in Proposition 1 is a nontrivial $(n+1)$-lattice.

Now we present another example. Let $n \ge 3$ and let $H_1, \dots, H_n$ be lattices such that $|H_i| \ge 2$ for $i = 1, \dots, n$. Denote the Cartesian product $H_1 \times \cdots \times H_n$ by H. For $1 \le i \le n$ we define an operation $\sqcup_i$ on H by $(a_1, \dots, a_n) \sqcup_i (b_1, \dots, b_n) = (c_1, \dots, c_n)$ iff $c_i = a_i \wedge b_i$ and $c_j = a_j \vee b_j$ for $j \in \{1, \dots, n\} \setminus \{i\}$. Let $(H; \sqcup_1, \dots, \sqcup_n)$ be denoted by G.

Proposition 2. *G is a fully absorptive n-lattice. It is nontrivial. Moreover, if K is an $(n-1)$-fold semilattice reduct of G then the roundabout absorption law in K holds for no $\sigma \in S_{n-1}$.*

Proof. We prove the statement by way of contradiction. Suppose that $*_1, \dots, *_{n-1}$ are binary term functions of G such that $K = (H; *_1, \dots, *_{n-1})$ is an $(n-1)$-fold semilattice satisfying at least one of the roundabout absorption laws. Since subalgebras of G are subalgebras of K, we can assume that all the

H_i are two element, say $H_i = \{0, 1\}$. The subalgebra $S = \{(x_1, \dots, x_n) \in H : |\{i : x_i = 0\}| \le 1\}$ of G is also a subalgebra of K and it will have a special role later.

Let $B = (H; \vee, \wedge)$ be the direct product of $H_1, \dots, H_n$ as Boolean lattices. Let the kernel of the natural projection $B \to H_i$ be denoted by Θ_i, $1 \le i \le n$. Since lattices have no skew congruences, cf. e.g. Fraser and Horn [1], we easily obtain that these Θ_i, $1 \le i \le n$, generate the congruence lattice $\mathrm{Con}(B)$. But these Θ_i also belong to $\mathrm{Con}(G)$, so we obtain $\mathrm{Con}(B) \subseteq \mathrm{Con}(G) \subseteq \mathrm{Con}(K)$. Therefore the Θ_i, $1 \le i \le n$, guarantee that K is a direct product of the $T_i := K/\Theta_i$, $1 \le i \le n$. Since $|T_i| = 2$, we can assume that $T_i = (H_i; *_1, \dots, *_{n-1})$, $1 \le i \le n$. These T_i, being homomorphic images of K, also satisfy one of the roundabout absorption laws.

For $1 \le i \le n-1$, let $M_i = \{j : 1 \le j \le n \text{ and } 1 *_i 0 = 0 \text{ in } T_j\}$ and $J_i = \{j : 1 \le j \le n \text{ and } 1 *_i 0 = 1 \text{ in } T_j\}$. If $|M_i| \ge 2$ for some i, say $p \ne q \in M_i$, then $e_p = (1, \dots, 1, 0, 1, \dots, 1)$, the pth component being 0, and $e_q = (1, \dots, 1, 0, 1, \dots, 1)$, the qth component being 0, belong to S but $e_p *_i e_q$ does not. Hence $|M_i| \le 1$ for all i, $1 \le i \le n-1$. Therefore $\{1, \dots, n\} \setminus (M_1 \cup \dots \cup M_{n-1})$ is nonempty; let r be one of its elements. This means that all the $*_i$, $1 \le i \le n-1$, act like join on $H_r = \{0, 1\}$. Thus all the roundabout absorption laws fail in T_r, a contradiction. □

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