The product of two von Neumann n-frames, its characteristic, and modular fractal lattices

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Abstract. Let L be a bounded lattice. If for each $a_1 < b_1 \in L$ and $a_2 < b_2 \in L$ there is a lattice embedding $\psi : [a_1, b_1] \to [a_2, b_2]$ with $\psi(a_1) = a_2$ and $\psi(b_1) = b_2$ then we say that L is a *quasifractal*. If ψ can always be chosen an isomorphism or, equivalently, if L is isomorphic to each of its nontrivial intervals then L will be called a *fractal lattice*. For a ring R with 1 let $\mathcal{V}(R)$ denote the lattice variety generated by the submodule lattices of R-modules. Varieties of this kind are completely described in [16]. The prime field of characteristic p will be denoted by F_p.

Let $\mathcal{U}$ be a lattice variety generated by a nondistributive modular quasifractal. The main theorem says that $\mathcal{U}$ is neither too small nor too large in the following sense: there is a unique $p = p(\mathcal{U})$, a prime number of zero, such that $\mathcal{V}(F_p) \subseteq \mathcal{U}$ and for any $n \geq 3$ and any nontrivial (normalized von Neumann) n-frame $(\vec{a}, \vec{c}) = (a_1, \dots, a_n, c_{12}, \dots, c_{1n})$ of any lattice in $\mathcal{U}$, $(\vec{a}, \vec{c})$ is of characteristic p. We do not know if $\mathcal{U} = \mathcal{V}(F_p)$ in general; however we point out that for any ring R with 1, $\mathcal{V}(R) \subseteq \mathcal{U}$ implies $\mathcal{V}(R) = \mathcal{V}(F_p)$. It will not be hard to show that $\mathcal{U}$ is Arguesian.

The main theorem does have a content, for it has been shown in [2] that each of the $\mathcal{V}(F_p)$ is generated by a single fractal lattice L_p; moreover we can stipulate either that L_p is a continuous geometry or that L_p is countable.

The proof of the main theorem is based on the following result of the present paper: if $(\vec{a}, \vec{c})$ is a nontrivial m-frame and $(\vec{u}, \vec{v})$ is an n-frame of a modular lattice L with $m, n \geq 3$ such that $u_1 \vee \cdots \vee u_n = a_1$ and $u_1 \wedge u_2 = a_1 \wedge a_2$ then these two frames have the same characteristic and, in addition, they determine a nontrivial mn-frame $(\vec{b}, \vec{d})$ of the same characteristic in a canonical way, which we call the *product frame*.

1. Introduction and definitions

Following [2], by a *fractal lattice* or, shortly, *fractal* we mean a bounded lattice L such that for each $a < b \in L$ the interval $[a, b]$ is isomorphic with L. For motivation and an application of this concept cf. [2], and cf. Giudici [11] for an overview of fractals coming from the theory of bisimple rings. If for any $a_1 < b_1 \in L$ and $a_2 < b_2 \in L$ there is a lattice embedding $[a_1, b_1] \to [a_2, b_2]$ with $a_1 \mapsto a_2$ and $b_1 \mapsto b_2$ then L is called a *quasifractal*. Fractals are clearly quasifractals but it is

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an open problem if every quasifractal is a fractal. From now on we assume that fractals and quasifractals consist of at least two elements. The two element lattice is a fractal, and so is the bounded chain of rational numbers between 0 and 1. Using the fact that the theory of countable atomless boolean algebras is $\aleph_0$-categorical, one should find it easy to verify that the (unique) countable atomless boolean lattice is a fractal. An additional distributive countable fractal, which is neither a chain nor a complemented lattice, is given in [2]. These distributive fractals and the L_p to be mentioned later in the Introduction are the only known countable fractal lattices at present.

For a ring R with 1 let $\mathcal{V}(R)$ denote the lattice variety generated by the submodule lattices of R-modules. Notice that $\mathcal{V}(R)$ is a congruence variety and congruence varieties of this kind are completely described in [16]. For $n \in \mathbf{N}$ let $\mathrm{Sub}(R^n) = \mathrm{Sub}({}_R R^n)$ denote the lattice of submodules of ${}_R R^n$. The residue class of integers modulo m will be denoted by $\mathbf{Z}_m$. Let P denote the set of prime numbers. For $p \in P \cup \{0\}$ let F_p stand for the prime field of characteristic p. I.e., $F_0 = \mathbf{Q}$ and $F_p = \mathbf{Z}_p$ for $p \in P$. The congruence varieties $\mathcal{V}(F_p)$, $p \in P \cup \{0\}$, are exactly the minimal modular nondistributive congruence varieties from Freese [4], cf. also Freese, Herrmann and Huhn [8].

It is shown in [2] that each of these $\mathcal{V}(F_p)$ is generated by a countable fractal lattice L_p. However, fractal lattices seem to be rare phenomena and [2] presents continuously many lattice varieties such that none of them is *fractal generated*, i.e., no fractal lattice generates it. The main theorem of the present paper gives a much stronger statement on *modular* fractal generated varieties, and even on modular quasifractal generated varieties. Namely, it asserts that for each modular nondistributive lattice variety $\mathcal{U}$ generated by a quasifractal there is a unique $p = p(\mathcal{U}) \in P \cup \{0\}$ such that (i) $\mathcal{V}(F_p)$ is included in $\mathcal{U}$ and (ii) for any $M \in \mathcal{U}$, $n \geq 3$ and any nontrivial (normalized von Neumann) n-frame $(\vec{a}, \vec{c})$ of M, $(\vec{a}, \vec{c})$ is of characteristic p. This says that $\mathcal{U}$ cannot be too large and, by giving the exact lower bound, also says that $\mathcal{U}$ cannot be too small. Of course we cannot say $n \geq 2$ instead of $n \geq 3$, for no meaningful characteristic of 2-frames can be defined. (Indeed, any nontrivial 2-frame generates the same M_3 up to isomorphism.) It remains an open problem if $\mathcal{U} = \mathcal{V}(F_p)$. As a corollary to the main theorem we show that $\mathcal{V}(F_p)$ is the only congruence variety of the form $\mathcal{V}(R)$ which is included in $\mathcal{U}$. By giving an infinite chain of lattice varieties between $\mathcal{V}(F_p)$ and $\mathcal{V}(\mathbf{Z}_{p^2})$, for a prime p, we also show that this corollary is weaker than the main theorem.

It is pointed out in [2] that the metric completion $\overline{L_p}$ of L_p is a continuous geometry and also a fractal lattice, and $\overline{L_p}$ still generates $\mathcal{V}(F_p)$. So if we consider fractalness as something good and expected then, from equational point of view, our result is close to saying that von Neumann [18] defined continuous geometries in an optimal and more or less the only possible way.

To prove the main result on fractal lattices some results on von Neumann's (normalized) n-frames are necessary, and these results also belong to the main achievements of the paper. The lattice operations join and meet will be denoted by $+$ and $\cdot$ (mostly juxtaposition) such that meets take precedence over joins. The indices we use will be *positive* integers so, for example, $i \le n$ is understood as $1 \le i \le n$. For definition, let $2 \le n$, let L be a nontrivial modular lattice with 0 and 1 and let $\vec{a} = (a_1, \dots, a_n) \in L^n$ and $\vec{c} = (c_{12}, \dots, c_{1n}) \in L^{n-1}$. We say that $(\vec{a}, \vec{c})$ is a *spanning n-frame of L* if the following equations hold for all $1 \le j \le n$ and $2 \le k \le n$:

$$\begin{gathered} \sum_{i \le n} a_i = 1, \quad a_j \sum_{i \ne j} a_i = 0, \\ a_1 + c_{1k} = a_k + c_{1k} = a_1 + a_k, \quad a_1 c_{1k} = a_k c_{1k} = 0. \end{gathered} \tag{1}$$

When referring to (1) we will often use without further notice that it implies $a_1 a_k = 0$ for $2 \le k \le n$. We almost always assume that the spanning n-frame is nontrivial, i.e., $0 \ne 1$. Notice that either the frame is trivial or $|\{a_1, \dots, a_n, c_{12}, \dots, c_{1n}\}| = 2n - 1$. The definition of a spanning n-frame means that the a_i are the atoms of a spanning Boolean lattice $\mathbf{2}^n$ and $\{a_1, c_{1k}, a_k\}$ generates an M_3 with bottom $0 = 0_L$ for $k \in \{2, \dots, n\}$. By the *order* of the frame we mean n. Notice that if we do not assume that 0 and 1 in formula (1) are the bottom and the top of L then we arrive at the notion of an n-frame in a modular lattice which is not necessarily a spanning one. However, we will mainly use spanning n-frames (of L or of an interval of L) only. Notice also that von Neumann [18], page 19, calls c_{1k} the axis of perspectivity between the intervals $[0, a_1]$ and $[0, a_k]$ and we will shortly call c_{1k} as the *axis of $\langle a_1, a_k \rangle$-perspectivity.*

For a ring R with 1 let $e_i = (0, \dots, 0, 1, 0, \dots, 0)$ (1 at the ith position, $1 \le i \le n$) form a canonical basis of R^n, and let $a_i = Re_i$ and $c_{1j} = R(e_1 - e_j)$. Then $(\vec{a}, \vec{c})$ is a spanning n-frame in the submodule lattice $\mathrm{Sub}(R^n)$; it is called the *canonical spanning n-frame of* $\mathrm{Sub}(R^n)$.

Given a spanning n-frame $(\vec{a}, \vec{c})$ we can define $c_{k1} = c_{1k}$ for $2 \le k \le n$, and for 1, j, k distinct let $c_{jk} = (c_{1j} + c_{1k})(a_j + a_k)$. Then, according to Lemma 5.3 in page 118 in von Neumann [18] (cf. also Freese [5]), for i, j, k distinct we have

$$\begin{gathered} c_{ik} = (c_{ij} + c_{jk})(a_i + a_k) \\ a_i + c_{ij} = a_j + c_{ij} = a_i + a_j, \quad a_i c_{ij} = a_j c_{ij} = a_i a_j = 0. \end{gathered} \tag{2}$$

This means that the index 1 has no special role if we consider the system $\vec{c}\,'$ of the c_{ij} $(i \ne j)$ rather then $\vec{c}$. To make a distinction between $(\vec{a}, \vec{c})$ and $(\vec{a}, \vec{c}\,')$ the latter will be called an *extended spanning n-frame.* When we do calculations with a frame we often pass to the elements of the extended frame. Sometimes, to unify some definitions or arguments, we allow the formal definition of a *trivial axis* $c_{ii} = 0$, $1 \le i \le n$. This notation harmonizes with the canonical spanning n-frame of $\mathrm{Sub}(R^n)$ and makes formula (1) valid also for $k = 1$. However, according to tradition, the trivial axes belong neither to frames nor to extended frames.

Now let $n \ge 3$ and consider the extended spanning frame. For $i, j, k \in \{1, \dots, n\}$ distinct let $1e_{ij} = c_{ij}$ and for $\ell \in \mathbf{N}$ let

$$(\ell + 1)e_{ij} = (a_i + a_j)\big((\ell e_{ij} + a_k)(c_{ik} + a_j) + c_{jk}\big).$$

For $n = 3$, (i, j) uniquely determines k while for $n \ge 4$ von Neumann [18] proved that ℓe_{ij} does not depend on k. The *characteristic of the spanning n-frame* $(\vec{a}, \vec{c})$, denoted by $\mathrm{char}(\vec{a},\vec{c})$, is the smallest $\ell \in \mathbf{N}$ such that $\ell e_{12} = a_1$ while $\mathrm{char}(\vec{a},\vec{c})$ is zero if no such ℓ exists. It follows from p. 284 in Freese [6] that $\ell e_{ij} = a_i$ implies that $\mathrm{char}(\vec{a},\vec{c})$ divides ℓ. Notice that the characteristic of $(\vec{a},\vec{c})$ is 1 if and only if $(\vec{a},\vec{c})$ is a trivial frame.

From now on the general assumption for the rest of this section is that either $n \geq 4$ or L is Arguesian and $n \geq 3$. Then, for $i \neq j \in \{1,\dots,n\}$,

$$R_{ij} = R(a_i,a_j) = \{x \in L : a_j + x = a_i + a_j,\ \ a_j x = 0\}$$

is a ring associated with the frame by von Neumann [18] ($n \geq 4$) and by Day and Pickering [3] ($n = 3$ and L being Arguesian); cf. also Freese [5] or Herrmann [12]. This ring is called the *auxiliary ring* of the frame; the terminology is justified by [18] and [3] where it is shown that, up to isomorphism, $R(a_i,a_j)$ does not depend on (i,j). The zero resp. unit of $R(a_i,a_j)$ is a_i resp. c_{ij}, and the ring addition will be denoted by $\oplus$. (The multiplication of the auxiliary ring will not be used in this paper.) It is known that ℓe_{ij} is $1_{R_{ij}} \oplus \cdots \oplus 1_{R_{ij}}$ and $a_i = 0_{R_{ij}}$ in $R(a_i,a_j)$. Therefore the characteristic of $(\vec{a},\vec{c})$ is that of the auxiliary ring $R(a_1,a_2)$. We will sometimes exploit that $\ell e_{ij} \in R(a_i,a_j)$ and therefore $(\ell e_{ij})a_j = 0$ and $(\ell e_{ij}) + a_j = a_i + a_j$.

It is well-known by von Neumann [18] that the auxiliary ring of the canonical n-frame of the submodule lattice $\mathrm{Sub}(R^n)$ is isomorphic with R, cf. also Herrmann [12]. Finally let us emphasize the terminology: spanning frames have two numerical attributes: the order (which was denoted by n) and the characteristic.

To help the reader to understand our calculations in modular lattices more quickly and also to shorten some of these calculations the following notations will be in effect. We use $=^i$ resp. $=^{\mathrm{f}}$ to indicate that formula (i) resp. some basic property of frames is used. In many cases, $=^{\mathrm{f}}$ means the same as $=^1$. When an application of the modular law uses the relation $x \leq z$ then, beside using $=^{\mathrm{m}}$, x resp. z will be underlined resp. doubly underlined. For example, $(\underline{x} + y)(\underline{\underline{x+z}}) =^{\mathrm{m}} x + y(x+z)$. The use of the shearing identity (cf. Thm. IV.1.1 in Grätzer [10]) is indicated by $=^{\mathrm{s}}$ and underlining the subterm "sheared": $x(\underline{y} + z) =^{\mathrm{s}} x\big(y(x+z) + z\big)$ or even $x(\underline{y} + z) =^{\mathrm{s}} xz$ when $y(x+z) = 0$. When $x_1 \geq x_2 \dots x_k$ for some easy reason then we write $\overline{x_1} x_2 \dots x_k$ to indicate that this expression is considered as $x_2 \dots x_k$. In other words, we overline meetands that can be omitted. We can even iterate or combine our notations; e.g., $\overline{(\overline{x_1} x_2 + x_3)} x_4$ gives x_4 (and indicates $x_2 \leq x_1$ and $x_4 \leq x_2 + x_4$) while $=^{m5}$ refers to modularity and formula (5).

2. The product frame and its characteristic

Let L be a bounded modular lattice for the whole section. For $m,n \in \mathbf{N} \setminus \{1\}$ let $(\vec{a},\vec{c})$ be a spanning m-frame of L and let $(\vec{u},\vec{v})$ be a spanning n-frame of the interval $[0,a_1]$. We say that $(\vec{a},\vec{c})$ is the *outer frame* while $(\vec{u},\vec{v})$ will be called the *inner frame.* We define a spanning mn-frame of L as follows. For $1 \leq i \leq n$ and $1 \leq j \leq m$ let

$$b_i^j = (u_i + c_{1j})a_j \quad \text{and } d_{1i}^{1j} = \big(v_{1i} + c_{1j}(u_i + a_j)\big)(u_1 + b_i^j). \tag{3}$$

Let $\vec{b}$ denote the vector of the all the b_i^j such that b_1^1 is the first component. Let $\vec{d}$ denote the vector of all the d_{1i}^{1j}, $(i,j) \neq (1,1)$. With d_{1i}^{1j} playing the role of the axis of $\langle b_1^1, b_i^j \rangle$-perspectivity, $(\vec{b}, \vec{d})$ is a candidate for a spanning mn-frame of L. It will be called the *product frame of the outer and inner frames.*

For convenient later use we reformulate (3) without relying on trivial axes and providing simpler expressions for some particular values of indices:

$$\begin{gathered} b_i^1 = u_i \quad \text{for } 1 \le i \le n, \\ b_i^j = (u_i + c_{1j})a_j \quad \text{for } 2 \le j \le m \text{ and } 1 \le i \le n, \\ d_{11}^{1j} = (u_1 + a_j)c_{1j} \quad \text{for } 2 \le j \le m, \\ d_{1i}^{11} = v_{1i} \quad \text{for } 2 \le i \le n, \\ d_{1i}^{1j} = \big(v_{1i} + c_{1j}(u_i + a_j)\big)(u_1 + b_i^j) \quad \text{for } 2 \le j \le m \text{ and } 2 \le i \le n. \end{gathered} \tag{4}$$

It is easy to see that (4) leads to the same $(\vec{b}, \vec{d})$ as formula (3). Consequently, the product frame $(\vec{b}, \vec{d})$ clearly determines the inner frame $(\vec{u}, \vec{v})$. The notations above are fixed in this section.

Theorem 1. (A) *$(\vec{b}, \vec{d})$ is indeed a spanning mn-frame of L.*

(B) *Let $m \ge 3$ and $n \ge 3$. Then the outer frame $(\vec{a}, \vec{c})$, the inner frame $(\vec{u}, \vec{v})$ and the product frame $(\vec{b}, \vec{d})$ have the same characteristic.*

Proof. Formula (1) together with the isomorphism theorem of modular lattices (cf., e.g., Thm. IV.1.2 in Grätzer [10]) yield that the map $\varphi_j : [0, a_1] \to [0, a_j]$, $x \mapsto (x + c_{1j})a_j$ is an isomorphism. (Clearly, φ_1 is the identical map.) This gives

$$a_j = \sum_{i \le n} b_i^j \tag{5}$$

and we conclude $\sum_{j \le m} \sum_{i \le n} b_i^j = \sum_{j \le m} a_j =^{\mathrm{f}} 1$. Further, for $i \le n$ and $j \le m$,

$$b_i^j \sum_{(r,s) \neq (i,j)} b_r^s = b_i^j \Big(\sum_{s \neq j} \sum_{r \le n} b_r^s + \sum_{r \neq i} b_r^j \Big) =^5 b_i^j \Big(\underline{\sum_{s \neq j}} a_s + \sum_{r \neq i} b_r^j \Big) =^{\mathrm{sf}}$$

$$b_i^j \sum_{r \neq i} b_r^j = \varphi_j \Big(b_i^1 \sum_{r \neq i} b_r^1 \Big) =^{\mathrm{f}} \varphi_j(0) = 0.$$

Since the rest of the required equations are trivial for $j = 1$, we assume $2 \le j \le m$ in the sequel.

First we consider the case $i = 1$. We have to show that $\{b_1^1, d_{11}^{1j}, b_1^j\} =^4 \{u_1, (u_1 + a_j)c_{1j}, (u_1 + c_{1j})a_j\}$ generates an M_3. Indeed, we have

$$\begin{gathered} \underline{u_1} + (\underline{\underline{u_1 + a_j}})c_{1j} =^{\mathrm{m}} (u_1 + a_j)(u_1 + c_{1j}), \\ \underline{u_1} + (\underline{\underline{u_1 + c_{1j}}})a_j =^{\mathrm{m}} (u_1 + a_j)(u_1 + c_{1j}), \\ \underline{(u_1 + a_j)c_{1j}} + (\underline{\underline{u_1 + c_{1j}}})a_j =^{\mathrm{m}} (u_1 + c_{1j})\big(\underline{a_j} + (\underline{\underline{u_1 + a_j}})c_{1j}\big) =^{\mathrm{m}} \\ (u_1 + c_{1j})(u_1 + a_j)(a_j + c_{1j}) =^{\mathrm{f}} (u_1 + c_{1j})(u_1 + a_j)\overline{(a_1 + a_j)}, \end{gathered}$$

while the meet of any two is 0 since $u_1 c_{1j} \le a_1 c_{1j} =^{\mathrm{f}} 0$, $u_1 a_j \le a_1 a_j =^{\mathrm{f}} 0$ and $c_{1j} a_j =^{\mathrm{f}} 0$.

From now on let $2 \le i \le n$ and $2 \le j \le m$. We have to show that $\{b_1^1,\ d_{1i}^{1j},\ b_i^j\} =^4 \{u_1,\ \big(v_{1i}+c_{1j}(u_i+a_j)\big)(u_1+b_i^j),\ (u_i+c_{1j})a_j\}$ generates an M_3. The meets are obtained easily:

$$\begin{gathered}
u_1 b_i^j \le^4 a_1 a_j = 0,\\
u_1 d_{1i}^{1j} =^4 u_1\big(v_{1i} + \underline{c_{1j}(u_i+a_j)}\big)\overline{(u_1+b_i^j)} =^{\mathrm{sf}} u_1 v_{1i} =^{\mathrm{f}} 0,\\
d_{1i}^{1j} b_i^j =^4 \big(v_{1i}+c_{1j}(u_i+a_j)\big)\overline{(u_1+b_i^j)}b_i^j =^4 \big(v_{1i}+\underline{c_{1j}(u_i+a_j)}\big)(\underline{\underline{u_i+c_{1j}}})a_j =^{\mathrm{m}}\\
\big(v_{1i}(u_i+\underline{c_{1j}})+c_{1j}(u_i+a_j)\big)a_j =^{\mathrm{sf}} c_{1j}(\overline{u_i+a_j})a_j =^{\mathrm{f}} 0.
\end{gathered}$$

The next task is to show that each of the three elements is below the join of the other two. Clearly, $d_{1i}^{1j} \le^4 u_1 + b_i^j = b_1^1 + b_i^j$. Further,

$$\begin{gathered}
d_{1i}^{1j} + b_i^j =^4 \big(v_{1i}+c_{1j}(u_i+a_j)\big)(\underline{\underline{u_1+b_i^j}}) + \underline{b_i^j} =^{\mathrm{m}}\\
(u_1+b_i^j)\big(v_{1i}+b_i^j+c_{1j}(u_i+a_j)\big) =^4\\
(u_1+b_i^j)\big(v_{1i}+\underline{(u_i+c_{1j})a_j}+c_{1j}(\underline{\underline{u_i+a_j}})\big) =^{\mathrm{m}}\\
(u_1+b_i^j)\Big(v_{1i}+(u_i+a_j)\big((\underline{\underline{u_i+c_{1j}}})a_j+\underline{c_{1j}}\big)\Big) =^{\mathrm{m}}\\
(u_1+b_i^j)\big(v_{1i}+(u_i+a_j)(u_i+c_{1j})(a_j+c_{1j})\big) =^{\mathrm{f}}\\
(u_1+b_i^j)\big(v_{1i}+(u_i+a_j)(u_i+c_{1j})(\overline{a_1+a_j})\big) \ge (u_1+b_i^j)(v_{1i}+u_i) \ge^f u_1 = b_1^1,
\end{gathered}$$

and finally

$$\begin{gathered}
b_1^1 + d_{1i}^{1j} =^4 \underline{u_1} + \big(v_{1i}+c_{1j}(u_i+a_j)\big)(\underline{\underline{u_1+b_i^j}}) =^{\mathrm{m}}\\
(u_1+b_i^j)\big(u_1+v_{1i}+c_{1j}(u_i+a_j)\big) =^{\mathrm{f}} (u_1+b_i^j)\big(u_1+\underline{u_i}+c_{1j}(\underline{\underline{u_i+a_j}})\big) =^{\mathrm{m}}\\
(u_1+b_i^j)\big(u_1+(u_i+c_{1j})(u_i+a_j)\big) \ge\\
b_i^j(u_i+c_{1j})(u_i+a_j) =^4 b_i^j.
\end{gathered}$$

This proves part (A) of the theorem.

Now, to prove part (B), we define

$$d_{i1}^{j1} = d_{1i}^{1j} \quad \text{and} \quad d_{ii}^{jk} = (d_{1i}^{1j}+d_{1i}^{1k})(b_i^j+b_i^k)$$

for $i \in \{1,\dots,n\}$ and $j,k \in \{1,\dots,m\}$ with $(1,1) \ne (i,j) \ne (i,k) \ne (1,1)$. Then (2) (for the product frame) gives that d_{ii}^{jk} is the axis of $\langle b_i^j, b_i^k\rangle$-perspectivity in the extended frame. We claim that for $2 \le j \le m$

$$c_{1j} = \sum_{i \le n} d_{ii}^{1j}. \tag{6}$$

First we show that

$$\big(v_{1i}+c_{1j}(u_i+a_j)\big)(u_i+a_j) \le c_{1j}. \tag{7}$$

Indeed, $\big(v_{1i}+\underline{c_{1j}(u_i+a_j)}\big)\big(\underline{\underline{u_i+a_j}}\big) =^{\mathrm{m}} v_{1i}(u_i+\underline{a_j})+c_{1j}(u_i+a_j) =^{\mathrm{s}} c_{1j}(u_i+a_j)$, which gives formula (7). For $2\leq i\leq n$ we have

$$\begin{gathered}
d_{ii}^{1j} = (d_{1i}^{11}+d_{1i}^{1j})(b_i^1+b_i^j) =^{4} \Big(\underline{v_{1i}}+\big(\underline{\underline{v_{1i}+c_{1j}(u_i+a_j)}}\big)(u_1+b_i^j)\Big)(u_i+b_i^j) =^{\mathrm{m}}\\
\big(v_{1i}+c_{1j}(u_i+a_j)\big)(v_{1i}+u_1+b_i^j)(u_i+b_i^j) =^{\mathrm{f}}\\
\big(v_{1i}+c_{1j}(u_i+a_j)\big)\overline{(u_i+u_1+b_i^j)}(u_i+b_i^j) =^{4}\\
\big(v_{1i}+c_{1j}(u_i+a_j)\big)\big(\underline{u_i}+(\underline{\underline{u_i+c_{1j}}})a_j\big) =^{\mathrm{m}}\\
\big(v_{1i}+c_{1j}(u_i+a_j)\big)(u_i+c_{1j})(u_i+a_j).
\end{gathered}$$

Hence

$$\begin{gathered}
\sum_{i\leq n} d_{ii}^{1j} = \sum_{2\leq i\leq n}(d_{11}^{1j}+d_{ii}^{1j}) =^{4}\\
\sum_{2\leq i\leq n}\Big(\underline{(u_1+a_j)c_{1j}}+\big(v_{1i}+c_{1j}(u_i+a_j)\big)(\underline{\underline{u_i+c_{1j}}})(u_i+a_j)\Big) =^{\mathrm{m}}\\
\sum_{2\leq i\leq n}(u_i+c_{1j})\Big((u_1+a_j)\underline{\underline{c_{1j}}}+\underline{\big(v_{1i}+c_{1j}(u_i+a_j)\big)(u_i+a_j)}\Big) =^{m7}\\
\sum_{2\leq i\leq n}(\overline{u_i+c_{1j}})c_{1j}\Big(u_1+\underline{a_j}+\big(v_{1i}+c_{1j}(u_i+a_j)\big)(\underline{\underline{u_i+a_j}})\Big) =^{\mathrm{m}}\\
\sum_{2\leq i\leq n} c_{1j}\Big(u_1+(u_i+a_j)\big(\underline{a_j}+v_{1i}+c_{1j}(\underline{\underline{u_i+a_j}})\big)\Big) =^{\mathrm{m}}\\
\sum_{2\leq i\leq n} c_{1j}\Big(u_1+(u_i+a_j)\big(v_{1i}+(a_j+c_{1j})(u_i+a_j)\big)\Big) =^{\mathrm{f}}\\
\sum_{2\leq i\leq n} c_{1j}\Big(u_1+(u_i+a_j)\overline{\big(v_{1i}+(\overline{a_1+a_j})(u_i+a_j)\big)}\Big) =\\
\sum_{2\leq i\leq n} c_{1j}(u_1+u_i+a_j) = \underline{\underline{c_{1j}}}(u_1+u_2+a_j)+\underline{\sum_{3\leq i\leq n} c_{1j}(u_1+u_i+a_j)} =^{\mathrm{m}}\\
c_{1j}\Big(u_1+u_2+a_j+\sum_{3\leq i\leq n} c_{1j}(u_1+u_i+a_j)\Big) =\\
c_{1j}\Big(u_1+u_2+\sum_{3\leq i\leq n}\big(\underline{a_j}+c_{1j}(\underline{\underline{u_1+u_i+a_j}})\big)\Big) =^{\mathrm{m}}\\
c_{1j}\Big(u_1+u_2+\sum_{3\leq i\leq n}(a_j+c_{1j})(u_1+u_i+a_j)\Big) =^{\mathrm{f}}\\
c_{1j}\Big(u_1+u_2+\sum_{3\leq i\leq n}(\overline{a_1+a_j})(u_1+u_i+a_j)\Big) =\\
c_{1j}(u_1+u_2+\cdots+u_n+a_j) =^{5} c_{1j}(a_1+a_j) =^{\mathrm{f}} c_{1j}.
\end{gathered}$$

This proves (6).

Now, armed with (6), for $1, j, k \in \{1, \dots, m\}$ distinct and $i \le n$ we have

$$d_{ii}^{jk} =^2 (d_{ii}^{1j} + d_{ii}^{1k})(b_i^j + b_i^k) \le \Big(\sum_{t\le n} d_{tt}^{1j} + \sum_{t\le n} d_{tt}^{1k}\Big)\Big(\sum_{t\le n} b_t^j + \sum_{t\le n} b_t^k\Big) =^{5,6}$$
$$(c_{1j} + c_{1k})(a_j + a_k) =^2 c_{jk}.$$

This and (6) give that for any $j \ne k \in \{1, \dots, m\}$ and $i \le n$ we have

$$d_{ii}^{jk} \le c_{jk}. \tag{8}$$

Working in the auxiliary ring $R(a_1, a_2)$ of $(\vec{a}, \vec{c})$ we define

$$1e = c_{12}, \quad (\ell+1)e = (a_1 + a_2)\big((\ell e + a_3)(c_{13} + a_2) + c_{23}\big)$$

while in the auxiliary ring $R(b_i^1, b_i^2)$ we have

$$1e_i = d_{ii}^{12}, \quad (\ell+1)e_i = (b_i^1 + b_i^2)\big((\ell e_i + b_i^3)(d_{ii}^{13} + b_i^2) + d_{ii}^{23}\big).$$

Notice that ℓ is the characteristic of $(\vec{a}, \vec{c})$ resp. $(\vec{b}, \vec{d})$ iff ℓ is the smallest positive integer with $\ell e = a_1$ resp. $\ell e_i = b_i^1$. The choice of i is irrelevant in the previous sentence, for the order of the extended frame is at least four (in fact, at least nine) whence the auxiliary rings $R(b_i^1, b_i^2)$, $i \in \{1, \dots, n\}$, are isomorphic. We obtain from (5) and (8) via induction on ℓ that $\ell e_i \le \ell e$ for $i \le n$, which yields $\sum_{i\le n} \ell e_i \le \ell e$. Therefore, denoting the use of $\ell e_i \in R(b_i^1, b_i^2)$ and $\ell e \in R(a_1, a_2)$ by $=^R$, we obtain

$$\begin{aligned} \ell e =^R \ell e(a_1 + a_2) =^5 \ell e\Big(\sum_{i\le n}(b_i^1 + b_i^2)\Big) =^R \\ \ell e\Big(\sum_{i\le n}(b_i^2 + \ell e_i)\Big) =^5 \underline{\underline{\ell e}}\Big(a_2 + \underline{\sum_{i\le n} \ell e_i}\Big) =^{\mathrm{m},R} \sum_{i\le n} \ell e_i. \end{aligned} \tag{9}$$

Now we are ready to show that, for any $\ell \in \mathbf{N}$, $\ell e = a_1$ iff $\ell e_1 = b_1^1$; this will clearly give part (B) of Theorem 1, for $\mathrm{char}(\vec{u}, \vec{v}) = \mathrm{char}(\vec{b}, \vec{d})$ is evident.

Suppose $\ell e_1 = b_1^1$. Since all the auxiliary rings $R(b_i^1, b_i^2)$ are isomorphic, $\ell e_i = b_i^1$ for all $i \le n$. Hence $\ell e =^9 \sum_{i\le n} \ell e_i = \sum_{i\le n} b_i^1 =^5 a_1$. Conversely, let us assume that $\ell e = a_1$. Then

$$\ell e_1 \le^{R9} \ell e(b_1^1 + b_1^2) = \underline{\underline{a_1}}(\underline{b_1^1} + b_1^2) =^{\mathrm{m}5} a_1 b_1^2 + b_1^1 =^{\mathrm{f}5} b_1^1,$$

whence

$$b_1^1 =^R \underline{\underline{b_1^1}}(\underline{\ell e_1} + b_1^2) =^{\mathrm{m}} \ell e_1 + b_1^1 b_1^2 =^{\mathrm{f}} \ell e_1,$$

completing the proof of Theorem 1. $\square$

In the light of Herrmann [12], two remarks on the above proof are relevant. Defining $(\vec{b}, \vec{d})$ via d_{1i}^{jj} and d_{11}^{1j} would lead to an easier proof of part (A) via Lemma 2.1 of [12] but then we should derive (4) for part (B). After verifying (6) the whole situation is described by a single spanning frame, the product frame of order at least nine. Hence [12], which gives a solution for the word problem of modular lattices generated by a *single* frame, offers an alternative argument for the last part of the proof. However, the present approach within the theory of modular lattices (motivated by [12] of course) is more elegant (and shorter) than coordinate-wise computations in sublattices of direct powers of modules.

3. Quasifractal generated modular lattice varieties

With the notations from Section 1 the main theorem of the paper is the following.

Theorem 2. *Let $\mathcal{U}$ be a nondistributive modular lattice variety generated by a quasifractal. Then*

(i) *there exists a unique $p = p(\mathcal{U})$, a prime number or zero, such that $\mathcal{V}(F_p) \subseteq \mathcal{U}$;*

(ii) *for any $n \geq 3$ and any nontrivial n-frame $(\vec{a}, \vec{c})$ of an arbitrary lattice in $\mathcal{U}$, $\mathrm{char}(\vec{a}, \vec{c}) = p(\mathcal{U})$;*

(iii) *$\mathcal{U}$ is Arguesian.*

Proof. Let L be a quasifractal that generates $\mathcal{U}$. Since L is not distributive, it has a nontrivial 2-frame (a_1, a_2, c_{12}) (i.e., (a_1, a_2, c_{12}) generates an M_3 sublattice). The interval $[a_1a_2, a_1]$ is Arguesian by Theorem 1 of Freese [7], where this theorem is attributed to B. Jónsson. Since L can be embedded in $[0, a_1]$, L is Arguesian as well. This yields part (iii) of Theorem 2. The rest of the proof is based on the following two observations.

($*$) Any two spanning frames of L with orders at least 3 have the same characteristic, and for each $n \geq 3$ there is a spanning n-frame in L.

($**$) If $n \geq 3$, $h > 1$ and some lattice in $\mathcal{U}$ has an n-frame of characteristic h then there exists a prime divisor p of h such that L has a spanning n-frame of characteristic p.

In order to prove ($*$), let $(\vec{a}, \vec{c})$ and $(\vec{u}, \vec{v})$ be spanning frames of orders at least 3. Then an appropriate lattice embedding $L \to [0, a_1]$ sends $(\vec{u}, \vec{v})$ to a spanning frame of $[0, a_1]$. This embedding preserves the characteristic of $(\vec{u}, \vec{v})$. Hence it follows from Theorem 1 that $\mathrm{char}(\vec{a}, \vec{c}) = \mathrm{char}(\vec{u}, \vec{v})$, proving the first half of ($*$).

We have already seen that L has a nontrivial 2-frame, say, (x_1, x_2, y_{12}). Since L is a quasifractal, we can assume that (x_1, x_2, y_{12}) is a spanning 2-frame of L and there is a spanning 2-frame of $[0, x_1]$. Then the product of these two 2-frames is a spanning 4-frame of L by Theorem 1. Repeating this argument with a 4-frame instead of a 2-frame we obtain a spanning 2^{2^2}-frame of L. And so one, we see that for each $k \in \mathbf{N}$, L has a spanning 2^{2^k}-frame, say $(\vec{a}, \vec{c})$. If $3 \leq n \leq 2^{2^k}$ then the first n of the a_i together with the c_{1i}, $2 \leq i \leq n$, constitute a nontrivial n-frame. Since L is a quasifractal, it has a spanning n-frame, too. This proves ($*$).

Now, to verify ($**$), let $n \geq 3$ and let us consider an n-frame of characteristic $h > 1$ in an arbitrary lattice of $\mathcal{U}$. Let S denote the sublattice generated by this n-frame. Corollary 4.3 of Herrmann [12] says that S is a subdirect product of certain submodule lattices $\mathrm{Sub}(\mathbf{Z}^n_{p^k})$ with p^k dividing h and p being a prime. Therefore some $\mathrm{Sub}(p^{k-1}\mathbf{Z}^n_{p^k})$, an ideal of $\mathrm{Sub}(\mathbf{Z}^n_{p^k})$, belongs to $\mathcal{U}$. Hence $\mathrm{Sub}(\mathbf{Z}^n_p) \cong \mathrm{Sub}(p^{k-1}\mathbf{Z}^n_{p^k}) \in \mathcal{U}$; the isomorphism follows by an easy argument (submodules are the same as subgroups and $p^{k-1}\mathbf{Z}_{p^k}^n \cong \mathbf{Z}_p^n$ as groups since the $(0,\dots,0,p^{k-1},0,\dots,0)$ form a basis in $p^{k-1}\mathbf{Z}_{p^k}^n$), or this follows from Corollary 3.4 (and the sentence preceding Lemma 3.3) of Herrmann [12]. It belongs to the folklore of lattice theory that $\mathrm{Sub}(\mathbf{Z}_p^n)$ is a subdirectly irreducible lattice, cf. e.g. Freese [5] or page 240 in Grätzer [10] or Herrmann [12]. Hence the famous lemma of Jónsson [17] yields that L has an ultrapower L' and L' has a sublattice K such that $\mathrm{Sub}(\mathbf{Z}_p^n)$ is a homomorphic image of K. Now we will use Theorem 1.6 of Freese [6]; note that we could use Freese [5] instead. Freese's theorem says that for any $g > 1$, n-frames of characteristic g are projective configurations in modular lattices. Hence the canonical n-frame of $\mathrm{Sub}(\mathbf{Z}_p^n)$ has a preimage in $K \subseteq L'$ which is an n-frame of characteristic p. Having an n-frame of characteristic p is a first-order property, so Łoś' Theorem applies and we conclude that L has an n-frame of characteristic p. Since L is a quasifractal, it has a spanning n-frame of characteristic p, too. This proves $(**)$.

Now let $h \in \mathbf{N}_0$ denote the common characteristic of nontrivial spanning frames of L with order at least three. Then h is well-defined by $(*)$. We distinguish two cases depending on h.

If $h > 1$ then $(**)$ gives a prime divisor p of h and an $n \geq 3$ such that L has a spanning n-frame of characteristic p. It follows from $(*)$ and quasifractalness that all nontrivial frames of L are of characteristic p and, furthermore, L has a spanning n-frame of characteristic p for each $3 \leq n \in \mathbf{N}$. Applying Corollary 4.3 of Herrmann [12] to the sublattice generated by any nontrivial n-frame of L we obtain that $\mathrm{Sub}(\mathbf{Z}_p^n) = \mathrm{Sub}(F_p^n) \in \mathcal{U}$, for all $n \geq 3$. It is clear from the theory of Mal'cev conditions (or cf. e.g. [16]) that these lattices generate $\mathcal{V}(F_p)$, so $\mathcal{V}(F_p) \subseteq \mathcal{U}$. This is the existence part of (i) when $h > 1$.

Now we need the fact that n-frames are projective configurations in the variety of modular lattices. This is the conjunction of Satz 1 in Huhn [15] with page 104 of Herrmann and Huhn [14], cf. also Lemma 3 in Freese, Herrmann and Huhn [8]. Hence any lattice Horn sentence of the form "for any n-frame $(\vec{x},\vec{y})$, $t_1(\vec{x},\vec{y}) = t_2(\vec{x},\vec{y})$", where t_1 and t_2 are lattice terms, is a equivalent with a lattice identity. (This idea is used in chapter XIII of Freese and McKenzie [9].) Therefore for each n and p there is a lattice identity $\mu_{n,p}$ such that $\mu_{n,p}$ holds in an arbitrary modular lattice M iff the characteristic of any n-frame of M divides p iff every *nontrivial* n-frame of M is of characteristic p. Therefore this property of L implies (ii) and, by excluding $\mathcal{V}(F_q) \subseteq \mathcal{U}$ for $q \neq p$, gives the uniqueness part of (i).

Now let us consider the case $h = 0$. Notice that $\mathcal{V}(F_p)$ with p prime contains a lattice with a nontrivial a 3-frame of characteristic p, namely $\mathrm{Sub}(\mathbf{Z}_p^3)$ with its canonical spanning 3-frame. Now it follows from $(*)$ and $(**)$ that, for all $n \geq 3$, any n-frame of an arbitrary lattice in $\mathcal{U}$ is of characteristic 0 and $\mathcal{V}(F_p) \not\subseteq \mathcal{U}$ for p prime. This makes (i) and (ii) clear as soon as we show that $\mathcal{V}(F_0) = \mathcal{V}(\mathbf{Q})$ is included in $\mathcal{U}$. To verify this inclusion, let χ be an arbitrary lattice identity satisfied by $\mathcal{U}$; we have to show that χ holds in $\mathcal{V}(\mathbf{Q})$. Let $(m_\chi, n_\chi) \in \mathbf{N}_0 \times \mathbf{N}$ be the pair of integers associated with χ in [16] and let k_χ be a sufficiently large integer (sufficient for χ and also for the dual of χ, cf. m in Theorem 1 of [16]). It is shown in [16] that for any ring R with 1, χ holds in $\mathcal{V}(R)$ iff χ holds in $\mathrm{Sub}(R^\ell)$ for some $k_\chi \le \ell \in \mathbf{N}$ iff χ holds in $\mathrm{Sub}(R^\ell)$ for all $k_\chi \le \ell \in \mathbf{N}$ iff $m_\chi x = n_\chi 1_R$ for some $x \in R$, i.e., m_χ divides n_χ in R.

Suppose by way of contradiction that $m_\chi = 0$ and let $\ell \in \mathbf{N}$ such that $\ell \ge k_\chi$ and $\ell \ge 3$. We know from $(*)$ that L has a spanning ℓ-frame of characteristic 0. Decompose the sublattice generated by this ℓ-frame into a subdirect product of subdirectly irreducible factors. According to Theorem 1.1 of Herrmann [12], each of these factors belongs to the following list of lattices:

$\mathrm{Sub}(\mathbf{Z}^\ell_{p^k})$ where p is a prime and $k \in \mathbf{N}$,
$\mathrm{Sub}(\mathbf{Q}^\ell)$,
$\mathrm{Sub}(\mathbf{Q}^\ell_p)$ where p is a prime and $\mathbf{Q}_p = \{i/j \in \mathbf{Q} : i, j \in \mathbf{Z},\ \ p \not| j\}$,
$\mathrm{Sub}(\mathbf{Q}^\ell_p)^{(d)}$, the dual of $\mathrm{Sub}(\mathbf{Q}^\ell_p)$, p is a prime.

Now χ holds in the actual subdirectly irreducible factors, for they belong to $\mathcal{U}$. This excludes $\mathrm{Sub}(\mathbf{Q}^\ell_p)$ and $\mathrm{Sub}(\mathbf{Q}^\ell)$, for $m_\chi = 0$ divides $n_\chi > 0$ neither in $\mathbf{Q}_p$ nor in $\mathbf{Q}$. Hutchinson's duality theorem, cf. Theorem 7 of [16], yields that (m_χ, n_χ) is the same as $(m_{\chi^{(d)}}, n_{\chi^{(d)}})$ for the dual identity. This excludes the dual of $\mathrm{Sub}(\mathbf{Q}^\ell_p)$. We have excluded all but $\mathrm{Sub}(\mathbf{Z}^\ell_{p^k})$ from the list, whence at least one of the $\mathrm{Sub}(\mathbf{Z}^\ell_{p^k})$ belongs to $\mathcal{U}$. Hence there is an ℓ-frame, the canonical ℓ-frame of $\mathrm{Sub}(\mathbf{Z}^\ell_{p^k})$, in $\mathcal{U}$ whose characteristic is p^k rather than 0. This contradiction shows that $m_\chi \ne 0$. Therefore $m_\chi \mid n_\chi$ in $\mathbf{Q}$ and χ holds in $\mathcal{V}(\mathbf{Q})$. This completes the case $h = 0$ and the proof of Theorem 2. □

If L is a nondistributive modular quasifractal then, in virtue of Theorem 2, we can uniquely define the *characteristic of* L as $p(\mathcal{U})$ where $\mathcal{U}$ is the variety generated by L. Since the auxiliary ring of the canonical spanning n-frame of $\mathrm{Sub}(R^n)$ is isomorphic to R, cf. von Neumann [18] or Herrmann [12], we conclude the following corollary as an evident consequence of Theorem 2.

Corollary 1. *Let* $\mathcal{U}$ *and* $p = p(\mathcal{U})$ *be as in Theorem 2. If* R *is a ring with 1,* $n \ge 3$ *and* $\mathrm{Sub}(R^n) \in \mathcal{U}$ *then* $\mathrm{char}(R) = p(\mathcal{U})$.

Corollary 2. *Let* $\mathcal{U}$ *and* $p = p(\mathcal{U})$ *be as in Theorem 2. For any ring* R *with 1 if* $\mathcal{V}(R) \subseteq \mathcal{U}$ *then* $\mathcal{V}(R) = \mathcal{V}(F_p)$.

Proof. If $\mathcal{V}(R)$ is one of the $\mathcal{V}(F_p)$ then the statement is evident by Theorem 2. Now suppose that $\mathcal{V}(R)$ is distinct from all the $\mathcal{V}(F_p)$, p prime or zero, and $\mathcal{V}(R) \subseteq \mathcal{U}$. Then $\mathcal{V}(R)$ is one of the congruence varieties occurring in parts (1) and (2) of Theorem 5 in [16]. This theorem of [16] clearly implies the existence of a prime p such that $\mathcal{V}(\mathbf{Z}_p) \subseteq \mathcal{V}(R)$ but p is distinct from the characteristic of R. Using the canonical 3-frame of $\mathrm{Sub}(\mathbf{Z}^3_p)$ and that of $\mathrm{Sub}(R^3)$ we obtain two 3-frames with different characteristic in $\mathcal{U}$, which contradicts Theorem 2. □

Both part $(**)$ of Theorem 2 and Corollary 2 say that $\mathcal{U}$ cannot be too large. Although the latter may look more impressive, the former is a much stronger statement. Indeed, it follows easily from the following example and the description of the congruence varieties $\mathcal{V}(R)$ given in [16] that part $(**)$ of Theorem 2 applies but Corollary 2 does not apply for the varieties $\mathcal{U}_k$ defined below.

Example 1. Let p be a prime number, $3 \le k \in \mathbf{N}$ and let $\mathcal{U}_k$ be the lattice variety generated by $\{\mathrm{Sub}(\mathbf{Z}_{p^2}^k)\} \cup \mathcal{V}(F_p)$. Then $\mathcal{V}(F_p) \subset \mathcal{U}_k \subset \mathcal{U}_{k+1} \subset \mathcal{V}(\mathbf{Z}_{p^2})$.

Proof. It is evident by [16] that $\mathcal{V}(F_p) \subset \mathcal{V}(\mathbf{Z}_{p^2})$. Moreover, $\mathrm{Sub}(\mathbf{Z}_{p^2}^k)$ is an ideal in $\mathrm{Sub}(\mathbf{Z}_{p^2}^{k+1}) \in \mathcal{V}(\mathbf{Z}_{p^2})$. Hence the $\subseteq$ inclusions in the statement are clear and only the inequalities have to be verified. The identity χ_p of Herrmann and Huhn [13] (or the identity $\Delta(0,p)$ of [16]) holds in $\mathcal{V}(F_p)$ but fails in $\mathrm{Sub}(\mathbf{Z}_{p^2}^k)$, whence $\mathcal{V}(F_p) \neq \mathcal{U}_k$. For the next inequality it suffices to show that $\mathrm{Sub}(\mathbf{Z}_{p^2}^{k+1}) \notin \mathcal{U}_k$.

Suppose the opposite. We already know (from the argument with χ_p) that $\mathrm{Sub}(\mathbf{Z}_{p^2}^{k+1}) \notin \mathcal{V}(F_p)$. Since $\mathrm{Sub}(\mathbf{Z}_{p^2}^{k+1})$ is subdirectly irreducible, cf. e.g. Herrmann [12], the Jónsson lemma yields an ultraproduct $M = \prod_{i\in I} K_i/\mathcal{F}$ such that $\mathcal{F}$ is an ultrafilter over I, each K_i is either in $\mathcal{V}(F_p)$ or equals $\mathrm{Sub}(\mathbf{Z}_{p^2}^k)$, and $\mathrm{Sub}(\mathbf{Z}_{p^2}^{k+1})$ is a homomorphic image of a sublattice of M. Clearly, $M \notin \mathcal{V}(F_p)$. We will use Łoś' theorem without further warning. Since $\mathrm{Sub}(\mathbf{Z}_{p^2}^{k+1})$ is finite, there is a first-order lattice formula Φ which holds in a lattice iff the lattice is isomorphic with $\mathrm{Sub}(\mathbf{Z}_{p^2}^{k+1})$. If Φ did not hold in M then $\mathcal{F}$ would contain $\{i \in I : \Phi$ fails in $K_i\} = \{i \in I : K_i \in \mathcal{V}(F_p)\}$, whence $M \in \mathcal{V}(F_p)$ would be a contradiction. Hence Φ does hold in M, i.e., $M \cong \mathrm{Sub}(\mathbf{Z}_{p^2}^k)$. This is the desired contradiction, for $|M| = |\mathrm{Sub}(\mathbf{Z}_{p^2}^k)| < |\mathrm{Sub}(\mathbf{Z}_{p^2}^{k+1})|$ excludes that $\mathrm{Sub}(\mathbf{Z}_{p^2}^{k+1})$ is a homomorphic image of a sublattice of M. Finally, the third inequality follows from the second one if we substitute $k+1$ for k. $\square$

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