The ordered set of principal congruences of a countable lattice

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To the memory of András P. Huhn

Abstract. For a lattice L, let $\mathrm{Princ}(L)$ denote the ordered set of principal congruences of L. In a pioneering paper, G. Grätzer characterized the ordered set $\mathrm{Princ}(L)$ of a finite lattice L; here we do the same for a countable lattice. He also showed that every bounded ordered set H is isomorphic to $\mathrm{Princ}(L)$ of a bounded lattice L. We prove a related statement: if an ordered set H with a least element is the union of a chain of principal ideals (equivalently, if $0 \in H$ and H has a cofinal chain), then H is isomorphic to $\mathrm{Princ}(L)$ of some lattice L.

1. Introduction

1.1. Historical background. A classical theorem of Dilworth [1] states that every finite distributive lattice is isomorphic to the congruence lattice of a finite lattice. Since Dilworth's result, the *congruence lattice representation problem* has attracted many researchers, and dozens of papers belonging to this topic have been written. The progress is mile-stoned by Huhn [12] and Schmidt [14], reached its summit in Wehrung [15] and Růžička [13], and was summarized in Grätzer [6]; see also Czédli [3] and Grätzer [10] for some additional, recent references.

In [7], Grätzer started an analogous topic of Lattice Theory. Namely, for a lattice L, let $\mathrm{Princ}(L) = \langle \mathrm{Princ}(L), \subseteq \rangle$ denote the ordered set of principal congruences of L. A congruence is *principal* if it is generated by a pair $\langle a, b \rangle$ of elements. Ordered sets and lattices with 0 and 1 are called *bounded*. Clearly, if L is a bounded lattice, then $\mathrm{Princ}(L)$ is a bounded ordered set. The pioneering theorem in Grätzer [7] states the converse: each bounded ordered set P is isomorphic to $\mathrm{Princ}(L)$ for an appropriate bounded lattice L. Actually, the lattice he constructed is of length 5. Up to isomorphism, he also characterized finite bounded ordered sets as ordered sets $\mathrm{Princ}(L)$ of finite lattices L.

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1.2. Terminology. Unless otherwise stated, we follow the standard terminology and notation of Lattice Theory; see, for example, Grätzer [8]. Our terminology for weak perspectivity is the classical one taken from Grätzer [5]. *Ordered sets* are nonempty sets equipped with *orderings*, that is, with reflexive, transitive, antisymmetric relations. Note that an ordered set is often called a *partially ordered set*, or a *poset*, or an *order*.

1.3. Our result. Motivated by Grätzer's theorem mentioned above, our goal is to prove the following theorem. A set X is *countable* if it is finite or countably infinite, that is, if $|X| \leq \aleph_0$. An ordered set P is *directed* if each two-element subset of P has an upper bound in P. Nonempty down-sets of P and subsets $\downarrow c = \{x \in P : x \leq c\}$ are called *order ideals* and *principal* (*order*) *ideals*, respectively.

Theorem 1.1.

(i) *An ordered set* $P = \langle P; \leq \rangle$ *is isomorphic to* $\mathrm{Princ}(L)$ *for some* countable *lattice* L *if and only if* P *is a countable directed ordered set with zero.*

(ii) *If* P *is an ordered set with zero and it is the union of a chain of principal ideals, then there exists a lattice* L *such that* $P \cong \mathrm{Princ}(L)$ *and* $|L| \leq |P| + \aleph_0$.

An alternative way of formulating the condition in part (ii) is to say that $0 \in P$ and there is a cofinal chain in P; see the first paragraph of Section 5 for the definition of cofinality. For a pair $\langle a, b \rangle \in L^2$ of elements, the least congruence collapsing a and b is denoted by $\mathrm{con}(a, b)$ or $\mathrm{con}_L(a, b)$. As it was pointed out in Grätzer [7], the rule

$$\mathrm{con}(a_i, b_i) \subseteq \mathrm{con}(a_1 \wedge b_1 \wedge a_2 \wedge b_2, a_1 \vee b_1 \vee a_2 \vee b_2) \text{ for } i \in \{1, 2\} \tag{1.1}$$

implies that $\mathrm{Princ}(L)$ is always a directed ordered set with zero. Therefore, the first part of the theorem will easily be concluded from the second one. To compare part (ii) of our theorem to Grätzer's result, note that a bounded ordered set P is always a union of a (one-element) chain of principal ideals. Of course, no *bounded* lattice L can represent P by $P \cong \mathrm{Princ}(L)$ if P has no greatest element.

1.4. Basic idea. Let $\langle Q; \leq \rangle$ be the ordered set given in Step (d) of Figure 1. Choose a cofinal chain $\{c_0 < c_1 < \dots\}$ in Q. In our case, this chain is $\{c_0 < c_1\}$. The figure shows how to construct a lattice M in several steps (in our case, four steps) such that $\mathrm{Princ}(M) \cong \langle Q; \leq \rangle$. In the figure, each ordered set on the left is isomorphic to the ordered set of principal congruences of the corresponding lattice on the right. Note that the lattice obtained in Step (b) is an interval of M. Below, we have a closer look at our mysterious steps leading to M.

In general, Step (a) of Figure 1 is the following. If $\langle H; \nu \rangle$ is a modular lattice of length 2, then it is isomorphic to the ordered set $\mathrm{Princ}(L)$ for the

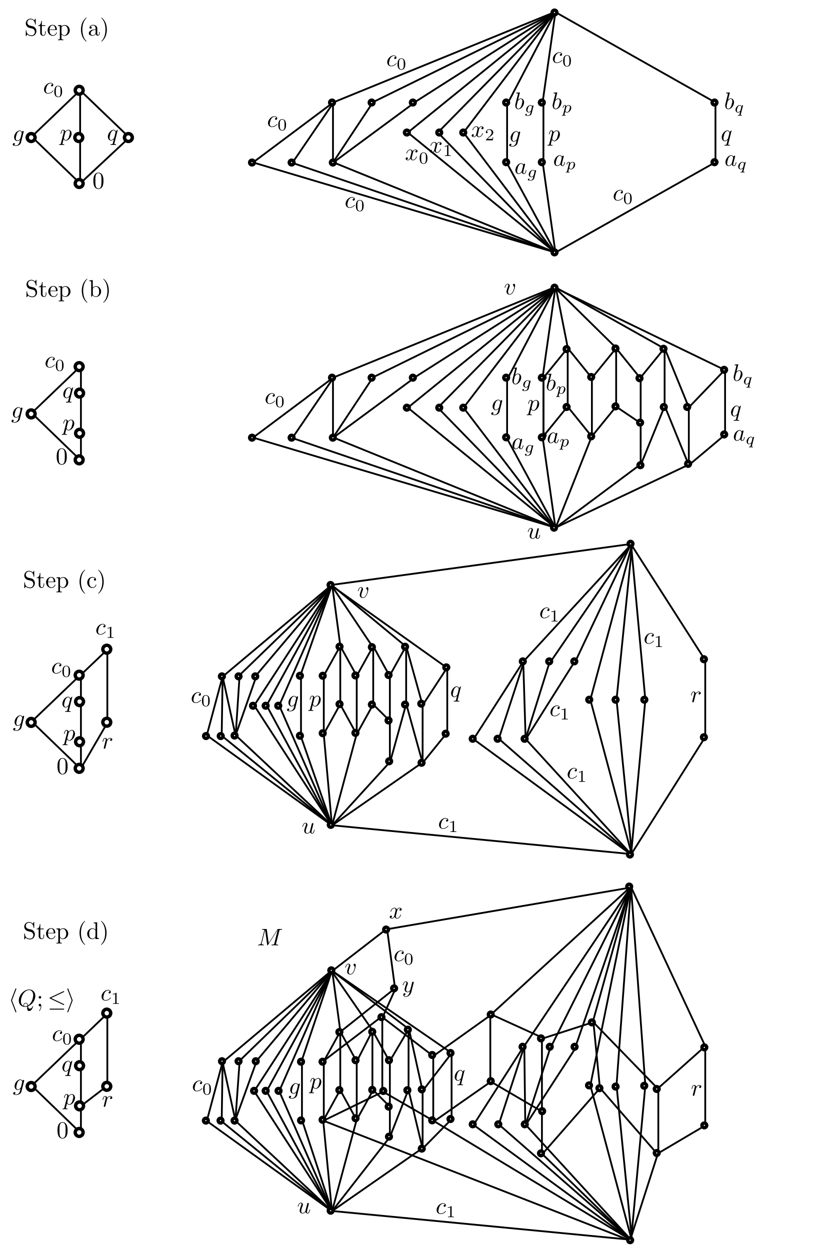

FIGURE 1. An example for our construction

bounded lattice L in Figure 5. The thin edges are labeled by the elements of $H \setminus \{0_H, 1_H\}$, while the thick edges of Figure 5 and the rest of the nontrivial intervals by 1_H. In this way, the labeling provides an isomorphism between

$\mathrm{Princ}(L)$ and $\langle H;\nu\rangle$. Note that most of the largest labels and previous labels are not indicated in Figure 1.

Next, assume that $\langle H;\nu\rangle$ is a bounded ordered set and that L is the bounded lattice constructed such that $\mathrm{Princ}(L)$ is isomorphic to $\langle H;\nu\rangle$. According to Lemma 5.3, if $\nu^{\blacktriangleright}$ is an ordering and $\nu \subseteq \nu^{\blacktriangleright} \subseteq H^2$, then we can extend L to a bounded lattice $L^{\blacktriangleright}$ such that $\langle H;\nu^{\blacktriangleright}\rangle \cong \mathrm{Princ}(L^{\blacktriangleright})$. This construction consists of repeated applications of single steps, which are called *horizontal extensions*; see Figures 7 and 8, or see Steps (b) and (d) in Figure 1. The transition from L to $L^{\blacktriangleright}$ is called a *multi-step horizontal extension*. The first multi-step horizontal extension is practically the same as that in Grätzer [7]. However, in general, we shall also perform vertical extensions; see a few lines below. Generally, after infinitely many vertical extensions, neither the lattice L, nor its interval $[b_p, 1]$ (in Figure 7) is of finite length. Thus, horizontal extensions become much more complicated than those in Grätzer [7]; the elements x and y in Figure 1 indicate why. In particular, M in Figure 1 without x and y would not work, and the lattice in Figure 8 would be inappropriate if d_1^{pq} were a coatom. The complexity of vertical extensions makes it necessary to introduce several auxiliary concepts.

Since the ordered set in Theorem 1.1 has no largest element in general, we also need vertical extensions. Assume that $\mathrm{Princ}(L) \cong \langle H;\nu\rangle$ and that $\langle H;\nu\rangle$ extends to a bounded ordered set $\langle H^{\blacktriangle};\nu^{\blacktriangle}\rangle$ such that H is an order ideal of $\langle H^{\blacktriangle};\nu^{\blacktriangle}\rangle$, $0_{H^{\blacktriangle}} = 0_H$, and, except for $1_{H^{\blacktriangle}}$, each new element is incomparable with any other element distinct from $0_{H^{\blacktriangle}}$ and $1_{H^{\blacktriangle}}$. (Note that L is not necessarily bounded and so H need not have a largest element.) It is not difficult to extend L to a larger lattice $L^{\blacktriangle}$ such that $\mathrm{Princ}(L^{\blacktriangle}) \cong \langle H^{\blacktriangle},\nu^{\blacktriangle}\rangle$, see Figure 6. We refer to this $L^{\blacktriangle}$ as a *vertical extension*. For example Step (c) in Figure 1 is a vertical extension. Note that neither our treatment for horizontal extensions, nor that for vertical ones uses the fact that the orderings in question are antisymmetric. Thus, without extra work, we deal with these extensions in a slightly more general setting.

Next, let $\{c_0 < c_1 < c_2 < \dots\} = \{c_\iota : \iota < \kappa\}$ be a cofinal chain in the ordered set $\langle P;\le\rangle$ of Theorem 1.1, and assume that we have already constructed a lattice L_ι such that $\mathrm{Princ}(L_\iota)$ is isomorphic to the principal ideal $\downarrow c_\iota$ of $\langle P;\le\rangle$. In order to extend L_ι to a lattice $L_{\iota+1}$ such that $\mathrm{Princ}(L_{\iota+1})$ is isomorphic to $\downarrow c_{\iota+1}$, we perform a vertical extension followed by a multi-step horizontal extension. Finally, at limit ordinals, we form directed unions.

1.5. Method. First of all, we need the key idea from Grätzer [7]. However, while [7] is based on an 11-element gadget lattice, we need a gadget consisting of more elements; see Figure 2.

Second, we feel that without the quasi-coloring technique developed in Czédli [3], the investigations leading to this paper would have not even begun. As opposed to colorings, the advantage of quasi-colorings is that we have joins (equivalently, the possibility of generation) in their range sets. This allows us to decompose our construction into a sequence of elementary steps. Each step is accompanied by a quasiordering. If several steps, possibly infinitely many steps, are carried out, then the join of the corresponding quasiorderings gives a satisfactory insight into the construction. Even if it is the "coloring versions" of some lemmas that we only use at the end, it is worth allowing their quasi-coloring versions since in this way the proofs will be simpler and the lemmas become more general.

Third, the idea of using appropriate auxiliary structures is taken from Czédli [2]. Their role is to accumulate all the assumptions our induction steps will need.

1.6. Outline. The rest of the paper is devoted to the proof of Theorem 1.1. Quasi-colored lattices, zigzags and auxiliary structures, which are the basic concepts we need in the proof, are introduced in Section 2. Vertical and horizontal extensions, which are our main constructive steps, are discussed in Sections 3 and in the longest section, Section 4, respectively. Finally, Section 5 completes the proof by transfinite induction.

2. Quasi-colorings and auxiliary structures

2.1. Quasi-colored lattices. A *quasiordered set* is a structure $\langle H;\nu\rangle$ where $H \neq \varnothing$ is a set and $\nu \subseteq H^2$ is a reflexive, transitive relation on H. Quasiordered sets are also called preordered ones. Instead of $\langle x,y\rangle \in \nu$, we usually write $x \leq_\nu y$. Also, we write $x <_\nu y$ and $x \parallel_\nu y$ for the conjunction of $x \leq_\nu y$ and $y \nleq_\nu x$, and that of $\langle x,y\rangle \notin \nu$ and $\langle y,x\rangle \notin \nu$, respectively. Similarly, $x =_\nu y$ will stand for the conjunction of $x \leq_\nu y$ and $y \leq_\nu x$. If $g \in H$ and $x \leq_\nu g$ for all $x \in H$, then g is a *greatest element* of H; *least elements* are defined dually. They are not necessarily unique; if they are, then they are denoted by 1_H and 0_H. If for all $x, y \in H$, there exists a $z \in H$ such that $x \leq_\nu z$ and $y \leq_\nu z$, then $\langle H;\nu\rangle$ is a *directed* quasiordered set. Given $H \neq \varnothing$, the set of all quasiorderings on H is denoted by $\mathrm{Quord}(H)$. It is a complete lattice with respect to set inclusion. For $X \subseteq H^2$, the least quasiorder on H that includes X is denoted by $\mathrm{quo}(X)$. We write $\mathrm{quo}(x,y)$ instead of $\mathrm{quo}(\{\langle x,y\rangle\})$.

Let L be an ordered set or a lattice. For $x, y \in L$, $\langle x,y\rangle$ is called an *ordered pair* of L if $x \leq y$. The set of ordered pairs of L is denoted by $\mathrm{Pairs}^{\leq}(L)$. If $X \subseteq L$, then $\mathrm{Pairs}^{\leq}(X)$ will stand for $X^2 \cap \mathrm{Pairs}^{\leq}(L)$. Note that we shall often use the fact that $\mathrm{Pairs}^{\leq}(S) \subseteq \mathrm{Pairs}^{\leq}(L)$ holds for subsets S of L; this explains why we work with ordered pairs rather than intervals. Note also that $\langle a,b\rangle$ is an ordered pair iff b/a is a quotient; however, the concept of ordered pairs fits better to previous work with quasi-colorings.

By a *quasi-colored lattice* we mean a structure $\mathcal{L} = \langle L;\gamma, H,\nu\rangle$ where L is a lattice, $\langle H;\nu\rangle$ is a quasiordered set, $\gamma\colon \mathrm{Pairs}^{\leq}(L) \to H$ is a surjective map, and for all $\langle u_1,v_1\rangle, \langle u_2,v_2\rangle \in \mathrm{Pairs}^{\leq}(L)$,

(C1) if $\gamma(\langle u_1,v_1\rangle) \leq_\nu \gamma(\langle u_2,v_2\rangle)$, then $\mathrm{con}(u_1,v_1) \leq \mathrm{con}(u_2,v_2)$;

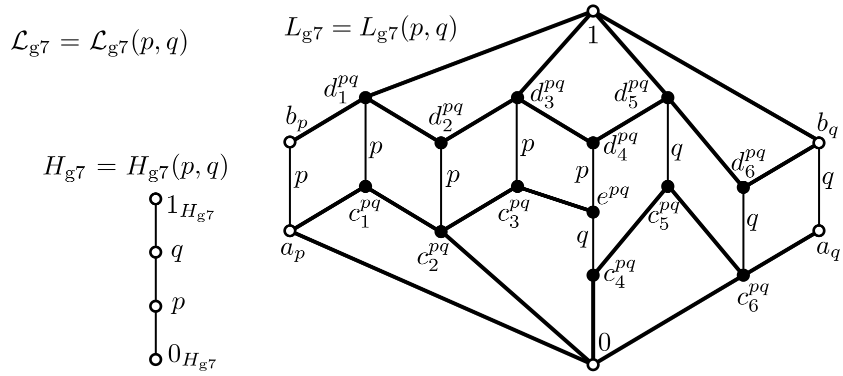

FIGURE 2. Our gadget, $\mathcal{L}_{g7}(p,q)$

(C2) if $\operatorname{con}(u_1, v_1) \le \operatorname{con}(u_2, v_2)$, then $\gamma(\langle u_1, v_1\rangle) \le_\nu \gamma(\langle u_2, v_2\rangle)$.

This concept is taken from Czédli [3]. By the "antichain variant" of (Ci) we mean the condition obtained from (Ci) by substituting the equality sign for $\le_\nu$ and $\le$. Prior to [3], the name "coloring" was used for surjective maps satisfying the antichain variant of (C2) in Grätzer, Lakser, and Schmidt [11], and for surjective maps satisfying the antichain variant of (C1) in Grätzer [6, page 39]. However, in [3], [6], and [11], $\gamma(\langle u, v\rangle)$ was defined only for covering pairs $u \prec v$. To emphasize that $\operatorname{con}(u_1, v_1)$ and $\operatorname{con}(u_2, v_2)$ belong to the ordered set $\operatorname{Princ}(L)$, we usually write $\operatorname{con}(u_1, v_1) \le \operatorname{con}(u_2, v_2)$ rather than $\operatorname{con}(u_1, v_1) \subseteq \operatorname{con}(u_2, v_2)$. It follows easily from (C1), (C2), (1.1), and the surjectivity of γ that if $\langle L; \gamma, H, \nu\rangle$ is a quasi-colored lattice, then $\langle H; \nu\rangle$ is a directed quasiordered set with a least element; possibly with many least elements.

The quasi-colored lattice $\mathcal{L}_{g7} = \mathcal{L}_{g7}(p,q) = \langle L_{g7}; \gamma_{g7}, H_{g7}, \nu_{g7}\rangle$ depicted in Figure 2 is the basic gadget of the paper. In this notation, the subscript g comes from "gadget" while "7" comes from (A15′), see later. (Note that 7 is sufficiently large to have reasonably convenient proofs; smaller values might cause problems or at least inconvenience.) The gadget $\mathcal{L}_{g7}$ consists of a 19-element lattice L_{g7}, a quasiordered set $\langle H_{g7}; \nu_{g7}\rangle$, which is actually a chain, and γ_{g7} is defined by the figure as follows: for $\langle x, y\rangle \in \operatorname{Pairs}^{\le}(L_{g7})$,

$$\gamma_{g7}(\langle x, y\rangle) = \begin{cases} p, & \text{if } \langle x, y\rangle \text{ is a } p\text{-colored edge in the figure,} \\ q, & \text{if } \langle x, y\rangle \text{ is a } q\text{-colored edge or } \langle x, y\rangle = \langle c_4^{pq}, d_4^{pq}\rangle, \\ 0_{H_{g7}}, & \text{if } x = y, \\ 1_{H_{g7}}, & \text{otherwise (if the interval } [x, y] \text{ contains a thick edge).} \end{cases}$$

It is straightforward to see that $\mathcal{L}_{g7}$ is a quasi-colored lattice; this task is particularly easy if one uses the description of congruences given in Grätzer [9].

2.2. Horizontal distance and zigzags. For a lattice or ordered set L, the *interior ordered set* of L is $\mathcal{L}^{-01} = \langle L^{-01}; \leq\rangle$, where $\leq$ is inherited from L and $L^{-01} = L \setminus \{0_L, 1_L\}$ in the sense that $\{0_L, 1_L\}$ denotes the set, possibly the empty set, consisting of the least and greatest elements of L. In particular, $L^{-01} = L$ iff L has neither a least, nor a greatest element. The *interior comparability graph* or, in short, the *graph* of L is $\mathcal{G}^{\mathrm{icg}}(L) = \langle L^{-01}; \not\parallel\rangle$; its vertex set is L^{-01}, and $\langle x, y\rangle$ is an *edge* of this graph iff $x \leq y$ or $y \leq x$. For $X, Y \subseteq L^{-01}$, if

$$z_0 \in X,\ z_n \in Y,\ z_0 \not\parallel z_1 \not\parallel \cdots \not\parallel z_n, \text{ and } |\{z_0, \dots, z_n\}| = n + 1, \tag{2.1}$$

then $\langle z_0, \dots, z_n\rangle$ is a $\mathcal{G}^{\mathrm{icg}}(L)$*-path* from X to Y, or between X and Y, of *length* n. The (*horizontal*) *distance* $\boldsymbol{\delta}(X, Y) \in \mathbb{N}_0$ of $X, Y \subseteq L^{-01}$ is the minimum of lengths of $\mathcal{G}^{\mathrm{icg}}(L)$-paths between X and Y; it is ∞ if there is no path from X to Y. In the most important case of (2.1), $X = \{x\}$ and $Y = \{y\}$; then we write x, y, and $\boldsymbol{\delta}(x, y)$ instead of X, Y, and $\boldsymbol{\delta}(X, Y)$. Note that $\boldsymbol{\delta}\colon L^{-01} \times L^{-01} \to \mathbb{N}_0 = \{0, 1, 2, \dots\}$ is a distance function. That is, $\boldsymbol{\delta}(x, y) = 0 \iff x = y$, $\boldsymbol{\delta}(x, y) = \boldsymbol{\delta}(y, x)$, and $\boldsymbol{\delta}(x, y) + \boldsymbol{\delta}(y, z) \geq \boldsymbol{\delta}(x, z)$ hold for all $x, y, z \in L^{-01}$.

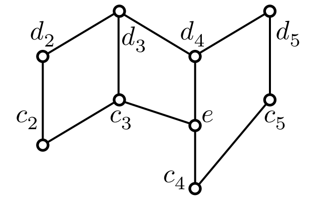

FIGURE 3. The ordered set Z_7

Next, consider a 9-tuple

$$Z_7 = \langle c_2, d_2; c_3, d_3; c_4, e, d_4; c_5, d_5\rangle \tag{2.2}$$

of elements belonging to L^{-01} . If the ordering of L^{-01} (equivalently, the ordering of L) restricted to $Z_7^{\mathrm{set}} := \{c_2, d_2, c_3, d_3, c_4, e, d_4, c_5, d_5\}$ is the one given by Figure 3, then we call Z_7 a *zigzag* of $\mathcal{G}^{\mathrm{icg}}(L)$. To explain the mysterious subscript 7, note that the labeling provides a canonical embedding of Z_7^{set} into $L_{\mathrm{g}7}$, see Figure 2. The subsets $\{c_2, d_2\}$ and $\{c_4, e, d_4\}$ are called the *lower fibers* of Z_7 given in (2.2) while $\{c_3, d_3\}$ and $\{c_5, d_5\}$ are its *upper fibers*. Note the terminological difference: although the fibers of a zigzag are chains, it has only four fibers but much more chains. If Z_7 is zigzag of $\mathcal{G}^{\mathrm{icg}}(L)$ such that its lower fibers are order ideals and its upper fibers are order filters in $\mathcal{L}^{-01}$, then Z_7 is called a *tight zigzag* of $\mathcal{G}^{\mathrm{icg}}(L)$. We say that a $\mathcal{G}^{\mathrm{icg}}(L)$-path (2.1) *goes through* the tight zigzag Z_7 if each fiber of Z_7 contains at least one of the elements $z_0, \dots, z_n$. For a subset X of L^{-01} and $n \in \mathbb{N} = \{1, 2, \dots\}$, the *neighborhood* with radius n of X is

$$\mathrm{Nbh}_n(X) = \{y \in L : \boldsymbol{\delta}(x, y) \leq n \text{ for some } x \in X\}.$$

Clearly, the inclusions $X \subseteq \mathrm{Nbh}_1(X) \subseteq \mathrm{Nbh}_2(X) \subseteq \dots$ hold.

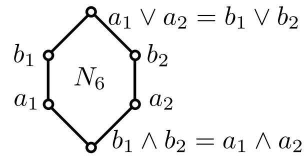

Figure 4. The lattice N_6

Next, let L be a (not necessarily bounded) lattice. We say that a quadruple $\langle a_1, b_1, a_2, b_2\rangle \in L^4$ is an N_6-*quadruple* of L if

$$\{b_1 \wedge b_2 = a_1 \wedge a_2,\ a_1 < b_1,\ a_2 < b_2,\ a_1 \vee a_2 = b_1 \vee b_2\}$$

is a six-element sublattice, see Figure 4. If, in addition, $b_1 \wedge b_2 = 0_L$ and $a_1 \vee a_2 = 1_L$, then we speak of a *spanning* N_6-*quadruple*. For a subset X of L^2, the least lattice congruence including X is denoted by $\mathrm{con}(X)$. In particular, $\mathrm{con}(\{\langle a, b\rangle\}) = \mathrm{con}(a, b)$. The least and the largest congruence of L are denoted by Δ_L and ∇_L, respectively.

2.3. Auxiliary structures and their substructures. Now, we are in the position to define the key concept we need. In the present paper, by an *auxiliary structure* we mean a structure

$$\mathcal{L} = \langle L; \gamma, H, \nu, \delta, \varepsilon, \mathcal{Z}\rangle \tag{2.3}$$

such that the following eight properties hold:

(A1) $\langle L; \gamma, H, \nu\rangle$ is a quasi-colored lattice and $|L| \geq 3$.

(A2) The quasiordered set $\langle H; \nu\rangle$ has exactly one least element, 0_H, at most one greatest element, and at least three elements.

(A3) δ and ε are $H \to L$ maps such that $\delta(0_H) = \varepsilon(0_H)$ and, for all $x \in H \setminus \{0_H\}$, $\delta(x) \prec \varepsilon(x)$; note that we usually write a_x and b_x instead of $\delta(x)$ and $\varepsilon(x)$, respectively.

(A4) For all $p \in H$, $\gamma(\langle \delta(p), \varepsilon(p)\rangle) = p$, that is, $\gamma(\langle a_p, b_p\rangle) = p$.

(A5) If p and q are distinct elements of $H \setminus \{0_H\}$, then $\langle \delta(p), \varepsilon(p), \delta(q), \varepsilon(q)\rangle$, also denoted by $\langle a_p, b_p, a_q, b_q\rangle$, is an N_6-quadruple of L.

(A6) For all $p \in H \setminus \{0_H\}$, the subsets $D_p := \{x \in L : 0_L \neq x \leq a_p\}$ and $U_p := \{x \in L : b_p \leq x \neq 1_L\}$ are sublattices. (The notation comes from "down" and "up". If L has no greatest element 1_L, then $x \neq 1_L$ means no restriction and $U_p = \uparrow b_p$, the principal filter generated by b_p. Similarly, $D_p = \downarrow a_p$ if L has no least element 0_L.)

(A7) For all $p \in H$ and $\langle x, y\rangle \in \mathrm{Pairs}^{\leq}(D_p) \cup \mathrm{Pairs}^{\leq}(U_p)$, if $x \neq y$, then $p \leq_\nu \gamma(\langle x, y\rangle)$.

(A8) $\mathcal{Z}$ is a set of tight zigzags of $\mathcal{G}^{\mathrm{icg}}(L)$. (Note that $\mathcal{Z}$ need not contain all tight zigzags of $\mathcal{G}^{\mathrm{icg}}(L)$. In particular, $\mathcal{Z}$ can be the empty set.)

We say that $\mathcal{L}$ in (2.3) is a *strong auxiliary structure* if it is an auxiliary structure and the following five additional properties hold.

(A9) H has a (unique) greatest element 1_H, and L is a bounded lattice.
(A10) The set $\{x \in L : 0_L \prec x \prec 1_L\}$ consists of at least three elements.
(A11) $\mathrm{con}\big(\{\langle a_r, b_r\rangle : r \in H \text{ and } r \neq 1_H\}\big) \neq \nabla_L$.
(A12) For all $p \in H^{-01}$ and $Z_7 \in \mathcal{Z}$, we have $\mathrm{Nbh}_1(\{a_p, b_p\}) \cap Z_7^{\mathrm{set}} = \varnothing$.
(A13) If $p, q \in H^{-01}$ such that $p \neq q$ and $\langle a_p, b_p, a_q, b_q\rangle$ is a spanning N_6-quadruple, then each $\mathcal{G}^{\mathrm{icg}}(L)$-path from $\{a_p, b_p\}$ to $\{a_q, b_q\}$ goes through at least one tight zigzag from $\mathcal{Z}$.

The conjunction of (A1), (A2), and (A10) imply (A9); we will not rely on this observation. Next, we mention three additional properties of strong auxiliary structures. The first one, (A14′), is a straightforward consequence of the fact that if x belongs to the set mentioned in (A10), then x is a complement of all elements in $L \setminus \{0_L, x, 1_L\}$. The next one follows from (A12) and (A13), and the third one from the second and $\mathrm{Nbh}_1(D_r \cup U_r) \subseteq \mathrm{Nbh}_2(\{a_r, b_r\})$.

(A14′) if Ψ is a congruence of L distinct from ∇_L, then $\{0_L\}$ and $\{1_L\}$ are singleton Ψ-blocks.
(A15′) For all $p, q \in H \setminus \{0_H\}$ such that $\langle a_p, b_p, a_q, b_q\rangle$ is a spanning N_6-quadruple, $\boldsymbol{\delta}(\{a_p, b_p\}, \{a_q, b_q\}) \geq 7$.
(A16′) For all $p, q \in H \setminus \{0_H\}$ such that $\langle a_p, b_p, a_q, b_q\rangle$ is a spanning N_6-quadruple, if $x \in \mathrm{Nbh}_1(D_p \cup U_p)$ and $y \in \mathrm{Nbh}_1(D_q \cup U_q)$, then the elements x and y are complementary, that is, $x \wedge y = 0_L$ and $x \vee y = 1_L$.

If $\langle H; \nu\rangle$ is a quasiordered set, then $\Theta_\nu = \nu \cap \nu^{-1}$ is known to be an equivalence relation, and the definition $[x]\Theta_\nu \leq [y]\Theta_\nu \iff x \leq_\nu y$ turns the quotient set H/Θ_ν into an ordered set $\langle H/\Theta_\nu; \leq\rangle$. The importance of our auxiliary structures is first shown by the following lemma.

Lemma 2.1. *If $\mathcal{L}$ in* (2.3) *is an auxiliary structure, then the ordered set* $\mathrm{Princ}(L)$ *is isomorphic to $\langle H/\Theta_\nu; \leq\rangle$. In particular, if ν is an ordering, then* $\mathrm{Princ}(L)$ *is isomorphic to the ordered set $\langle H; \nu\rangle$.*

Proof. Clearly, $\mathrm{Princ}(L) = \{\mathrm{con}(x, y) : \langle x, y\rangle \in \mathrm{Pairs}^{\leq}(L)\}$. Consider the map $\varphi\colon \mathrm{Princ}(L) \to H/\Theta_\nu$, defined by $\mathrm{con}(x, y) \mapsto [\gamma(\langle x, y\rangle)]\Theta_\nu$. If $\mathrm{con}(x_1, y_1) = \mathrm{con}(x_2, y_2)$, then $[\gamma(\langle x_1, y_1\rangle)]\Theta_\nu = [\gamma(\langle x_2, y_2\rangle)]\Theta_\nu$ follows from (C2). Hence, φ is a map. It is surjective since so is γ. Finally, it is bijective and an order isomorphism by (C1) and (C2). ☐

We say that an auxiliary structure $\mathcal{L} = \langle L; \gamma, H, \nu, \delta, \varepsilon, \mathcal{Z}\rangle$ is *countable* if $|L| \leq \aleph_0$. In this case, by the surjectivity of γ, $|H| \leq \aleph_0$ also holds. Next, we give an example.

Example 2.2. Let H be a set, finite or infinite, such that $0_H, 1_H \in H$ and $|H| \geq 3$. Let us define $\nu = \mathrm{quo}\big((\{0_H\} \times H) \cup (H \times \{1_H\})\big)$; note that $\langle H; \nu\rangle$ is an ordered set (actually, a modular lattice of length 2). Let L be the lattice depicted in Figure 5, where $\{h, g, p, q, \dots\}$ is the set $H^{-01} = H \setminus \{0_H, 1_H\}$.

For $x \prec y$, $\gamma(\langle x,y\rangle)$ is defined by the labeling of edges like in case of $\mathcal{L}_{\mathrm{g7}}$. In particular, $\gamma(\langle z,z\rangle) = 0_H$ for all $z \in L$, and $[x,y]$ includes a thick (unlabeled) edge iff $\gamma(\langle x,y\rangle) = 1_H$. Let $\delta(0_H) = \varepsilon(0_H) = x_0$ and $\mathcal{Z} = \varnothing$. For $s \in H\setminus\{0_H\}$, we define $\delta(s) = a_s$ and $\varepsilon(s) = b_s$. Now, obviously, $\mathcal{L} = \langle L;\gamma,H,\nu,\delta,\varepsilon,\mathcal{Z}\rangle$ is a strong auxiliary structure. If $|H| \le \aleph_0$, then $\mathcal{L}$ is countable. Note that the black-filled elements form a simple, selfdual sublattice, which is usually denoted by $M_{3,3}$. Hence, L is a selfdual lattice.

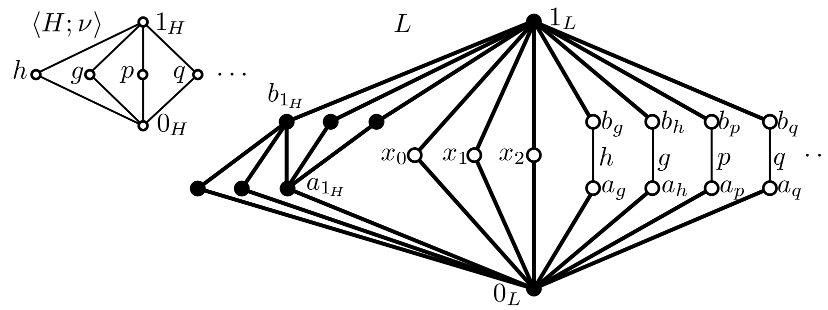

FIGURE 5. The auxiliary structure in Example 2.2

Substructures are defined in the natural way; note that $\nu = \nu' \cap H^2$ will not be required below. Namely,

Definition 2.3. Let

$$\mathcal{L} = \langle L;\gamma,H,\nu,\delta,\varepsilon,\mathcal{Z}\rangle \quad \text{and} \quad \mathcal{L}' = \langle L';\gamma',H',\nu',\delta',\varepsilon',\mathcal{Z}'\rangle$$

be auxiliary structures. We say that $\mathcal{L}$ is a *substructure* of $\mathcal{L}'$ if

(i) L is a sublattice of L', $H \subseteq H'$, $\nu \subseteq \nu'$, and $0_{H'} = 0_H$;

(ii) γ is the restriction of γ' to $\mathrm{Pairs}^{\le}(L)$, δ is the restriction of δ' to H, and ε is the restriction of ε' to H.

If, in addition,

(iii) $\mathcal{L}'$ is strong, $\mathcal{Z} \subseteq \mathcal{Z}'$, $0_L = 0_{L'}$, $1_L = 1_{L'}$, $1_{H'} \in H$,

(iv) for all $x \in L'$, if $0_{L'} \prec x \prec 1_{L'}$, then $x \in L$, and

(v) for each $Z_7 \in \mathcal{Z}'$, if $Z_7^{\mathrm{set}} \cap L \ne \varnothing$, then $Z_7 \in \mathcal{Z}$,

then $\mathcal{L}$ is a *tight substructure* of $\mathcal{L}'$

Assume that $\mathcal{L}$ is a tight substructure of $\mathcal{L}'$. Since $\nabla_L = \mathrm{con}_L(0_L,1_L) \in \mathrm{Princ}(L)$, $\langle H;\nu\rangle$ has a unique largest element by (C1), (C2), and (A2), and we have $1_H = 1_{H'}$. It is straightforward to see that $\mathcal{L}$ is strong; for example, (A11) follows by restricting the corresponding congruence of L' to L, while (A13) is a consequence of 2.3(v). Hence, we can emphasize that

$$\text{if } \mathcal{L} \text{ is a tight substructure of } \mathcal{L}', \text{ then } \mathcal{L},\mathcal{L}' \text{ are strong and } 1_H = 1_{H'}. \tag{2.4}$$

If $\mathcal{L}$ is a substructure (resp., tight substructure) of $\mathcal{L}'$, then we say $\mathcal{L}'$ is an *extension* (resp., *tight extension*) of $\mathcal{L}$. Clearly, if $\mathcal{L}$, $\mathcal{L}'$, and $\mathcal{L}''$ are auxiliary structures such that $\mathcal{L}$ is a substructure of $\mathcal{L}'$ and $\mathcal{L}'$ is a substructure of $\mathcal{L}''$, then $\mathcal{L}$ is a substructure of $\mathcal{L}''$; the same holds for strong auxiliary structures and their tight substructures. This transitivity will often be used in Section 5; sometimes implicitly. The next two sections indicate how easily and efficiently we can work with auxiliary structures.

3. Vertical extensions

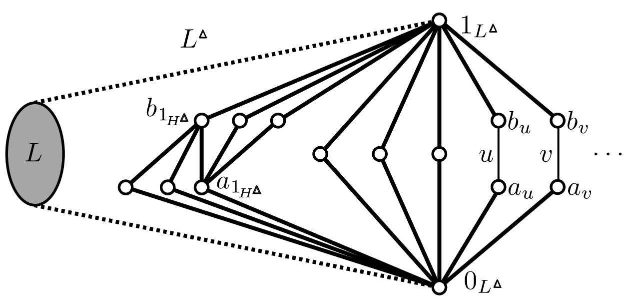

FIGURE 6. The auxiliary structure $\mathcal{L}^{\vartriangle}$

Generalizing the idea behind Example 2.2, this section captures, in terms of extensions of auxiliary structures, how to add an antichain and a new top element to the quasiordered set H of colors (even if H has no top element). For an auxiliary structure $\mathcal{L} = \langle L; \gamma, H, \nu, \delta, \varepsilon, \mathcal{Z}\rangle$ and an arbitrary (possibly empty) set K, we define the following objects. Let $H^{\vartriangle}$ be the disjoint union $H \cup K \cup \{1_{H^{\vartriangle}}\}$, let $\mathcal{Z}^{\vartriangle} = \varnothing$, and let $0_{H^{\vartriangle}} = 0_H$. Define $\nu^{\vartriangle} \in \mathrm{Quord}(H^{\vartriangle})$ by

$$\nu^{\vartriangle} = \mathrm{quo}\big(\nu \cup (\{0_{H^{\vartriangle}}\} \times H^{\vartriangle}) \cup (H^{\vartriangle} \times \{1_{H^{\vartriangle}}\})\big).$$

Consider the lattice $L^{\vartriangle}$ defined by Figure 6, where $u, v, \dots$ denote the elements of K. The thick dotted lines indicate $\le$ but not necessarily $\prec$; they are edges only if L is bounded. Note that all "new" lattice elements distinct from $0_{L^{\vartriangle}}$ and $1_{L^{\vartriangle}}$, that is, all elements of $L^{\vartriangle} \setminus (L \cup \{0_{L^{\vartriangle}}, 1_{L^{\vartriangle}}\})$, are complements of all "old" elements. Extend δ and ε to maps $\delta^{\vartriangle}, \varepsilon^{\vartriangle} \colon H^{\vartriangle} \to L^{\vartriangle}$ by letting $\delta^{\vartriangle}(w) = a_w$ and $\varepsilon^{\vartriangle}(w) = b_w$ for $w \in K \cup \{1_{H^{\vartriangle}}\}$. Define $\gamma^{\vartriangle} \colon \mathrm{Pairs}^{\le}(L^{\vartriangle}) \to H^{\vartriangle}$ by

$$\gamma^{\vartriangle}(\langle x, y\rangle) = \begin{cases} \gamma(\langle x, y\rangle), & \text{if } \langle x, y\rangle \in \mathrm{Pairs}^{\le}(L),\\ w, & \text{if } x = a_w,\ y = b_w, \text{ and } w \in K,\\ 0_{H^{\vartriangle}}, & \text{if } x = y,\\ 1_{H^{\vartriangle}}, & \text{otherwise.}\end{cases}$$

As usual, we use thick edges in Figure 6 instead of labeling them by $1_{H^{\vartriangle}}$. Finally, let $\mathcal{L}^{\vartriangle} = \langle L^{\vartriangle}; \gamma^{\vartriangle}, H^{\vartriangle}, \nu^{\vartriangle}, \delta^{\vartriangle}, \varepsilon^{\vartriangle}, \mathcal{Z}^{\vartriangle}\rangle$. The proof of the following lemma is based on $\mathrm{Nbh}_1(L) = L$; the straightforward details will be omitted.

Lemma 3.1. *If $\mathcal{L}$ is an auxiliary structure, then $\mathcal{L}^{\vartriangle}$ is a strong auxiliary structure. Furthermore, $\mathcal{L}$ is a substructure of $\mathcal{L}^{\vartriangle}$ and $|L^{\vartriangle}| \le |L| + |K| + \aleph_0$.*

Since a new bottom element and a new top element are added, we say that $\mathcal{L}^{\vartriangle}$ is obtained from $\mathcal{L}$ by a *vertical extension*; this motivates the triangle aiming upwards in its notation. Note that if L is a selfdual lattice, then so is $L^{\vartriangle}$.

4. Horizontal extensions of auxiliary structures

The purpose of this section is to capture, in terms of tight extensions of auxiliary structures, how to increase the quasiorder ν of $H = \langle H; \nu\rangle$ by a "single step" in case H has a largest element. If $x \in L_{\mathrm{g7}}^{-01}$ belongs to the boundary of the planar lattice L_{g7}, then we have $\boldsymbol{\delta}(e^{pq}, x) = \boldsymbol{\delta}(c_4^{pq}, x) \ge 3$ for $x \ne c_6^{pq}$ and $\boldsymbol{\delta}(e^{pq}, x) = \boldsymbol{\delta}(c_4^{pq}, x) = 2$ for $x = c_6^{pq}$, where $\boldsymbol{\delta}$ is understood in the graph $\langle L_{\mathrm{g7}}^{-01}; \nparallel\rangle$. This explains that although the planar lattice L_{g7} is not selfdual, the elements c_4^{pq} and e^{pq} behave similarly in most of our considerations. That is, we can often treat L_{g7} as if it were a selfdual lattice with $c_4^{pq} = e^{pq}$. When doing so, we will refer to "*quasi-duality*". Although the last two components below have not yet been defined, note that, for $i \in \{1, \dots, 6\}$, the two rows of the following matrix

$$\begin{pmatrix} a_p & b_p & a_q & b_q & c_i^{pq} & d_i^{pq} & D_p & U_p & U_p^q & D_p^q \\ b_q & a_q & b_p & a_p & d_{7-i}^{pq} & c_{7-i}^{pq} & U_q & D_q & D_p^q & U_p^q \end{pmatrix}$$

correspond to each other via quasi-duality.

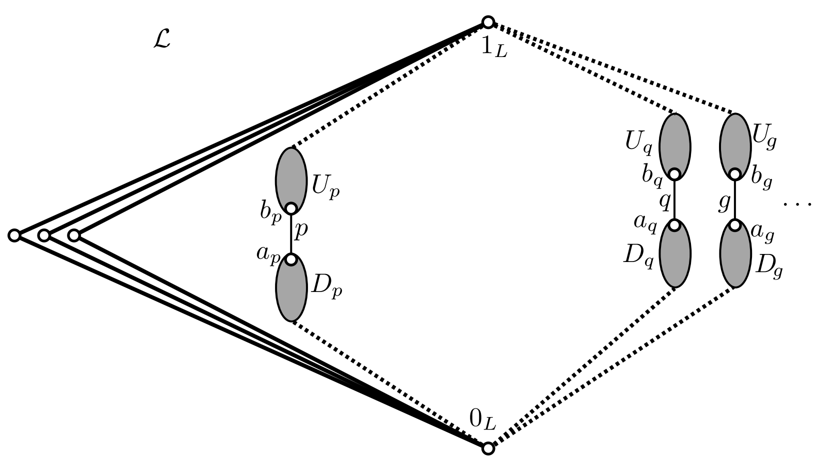

FIGURE 7. Starting from $\mathcal{L}$, ...

Assume that

$$\begin{gathered} \mathcal{L} = \langle L; \gamma, H, \nu, \delta, \varepsilon, \mathcal{Z}\rangle \text{ is a strong auxiliary structure,} \\ p, q \in H^{-01}, \text{ and } \langle a_p, b_p, a_q, b_q\rangle \text{ is a spanning } N_6\text{-quadruple.} \end{gathered} \tag{4.1}$$

We define a structure $\mathcal{L}^{\triangleright} = \mathcal{L}^{\triangleright}(p,q)$ as follows, and it will take a lot of work to prove that it is a strong auxiliary structure. We call $\mathcal{L}^{\triangleright}$ a *horizontal extension* of $\mathcal{L}$; this explains the horizontal triangle in the notation. The construction of $\mathcal{L}^{\triangleright}$ from $\mathcal{L}$ is illustrated in Figures 7 and 8. Note that our lattices can contain much more elements and in a more complicated way than depicted in these two figures. The convex subsets (actually, convex sublattices) of L defined in (A6) are indicated with light grey shapes. The solid lines represent the covering relation but the dotted lines, which still stand for the ordering, are not necessarily edges. For example, if $b_g \prec 1_L$, then $U_g = \{b_g\}$ and the respective dotted line denotes the covering $b_g \prec 1_L$; however, the dotted line is not a covering if U_g has no largest element. As usual, the thick (unlabeled) edges are colored by 1_H or $1_{H^{\triangleright}}$.

First, we change the N_6-sublattice $\{0_L, a_p, b_p, a_q, b_q, 1_L\}$ into an $L_{\mathrm{g}7}$, depicted in Figure 2, that is, we insert the black-filled circle-shaped elements into L. Next, to each $x \in U_p \setminus \{b_p\}$, indicated by an empty-filled little square in Figure 8, we add a new upper cover x^+ of x, which is indicated by a black-filled little square. The set of these new upper covers plus d_1^{pq} is denoted by U_p^q. The elements d_1^{pq} and c_1^{pq} will also be denoted by b_p^+ and a_p^+, respectively. For $x_1, x_2 \in U_p$, we let $x_1^+ \leq x_2^+$ iff $x_1 \leq x_2$. This means that the ordered subset $U_p \cup U_p^q$ of $L^{\triangleright}$ is isomorphic to the direct product of $U_p \times \mathsf{C}_2$, where C_2 is the 2-element chain. Finally (and similarly), to each $y \in D_q \setminus \{a_q\}$, we add a new lower cover y^- of y (indicated by a black-filled little square). We let $a_q^- = c_6^{pq}$ and $b_q^- = d_6^{pq}$. For $y_1, y_2 \in D_q$, $y_1^- \leq y_2^- \iff y_1 \leq y_2$. In this way, we have obtained an ordered set denoted by $L^{\triangleright}$; see also (4.9) later for more exact details. We will prove soon that $L^{\triangleright}$ is a lattice and L is a sublattice in it; then it will be clear that

$$\begin{aligned} x^+ &= x \vee c_1^{pq} \quad \text{for } x \in \{a_p\} \cup U_p \quad \text{and} \\ y^- &= y \wedge d_6^{pq} \quad \text{for } y \in \{b_q\} \cup D_q. \end{aligned} \tag{4.2}$$

Note that while Grätzer [7] constructed a lattice of length 5, here even the interval, say, $[b_p, 1_{L^{\triangleright}}]$ can be of infinite length.

Next, set $H^{\triangleright} = H$ and $Z_7^{pq} = \langle c_2^{pq}, d_2^{pq}; c_3^{pq}, d_3^{pq}; c_4^{pq}, e^{pq}, d_4^{pq}; c_5^{pq}, d_5^{pq}\rangle$. Let $\mathcal{Z}^{\triangleright} = \mathcal{Z} \cup \{Z_7^{pq}\}$. In $\mathrm{Quord}(H^{\triangleright})$, we define $\nu^{\triangleright} = \mathrm{quo}\big(\nu \cup \{\langle p,q\rangle\}\big)$. Since ν is reflexive and transitive, we have that

$$\langle r_1, r_2\rangle \in \nu^{\triangleright} \iff r_1 \leq_\nu p \text{ and } q \leq_\nu r_2, \text{ or } r_1 \leq_\nu r_2, \tag{4.3}$$

for arbitrary $r_1, r_2 \in H^{\triangleright}$. Hence, it follows easily from the validity of (A2) and (A9) in $\mathcal{L}$ that $\langle H^{\triangleright}; \nu^{\triangleright}\rangle$ has a unique largest element $1_{H^{\triangleright}}$, a unique least element $0_{H^{\triangleright}}$, and we have $1_{H^{\triangleright}} = 1_H$ and $0_{H^{\triangleright}} = 0_H$. We extend γ to a map

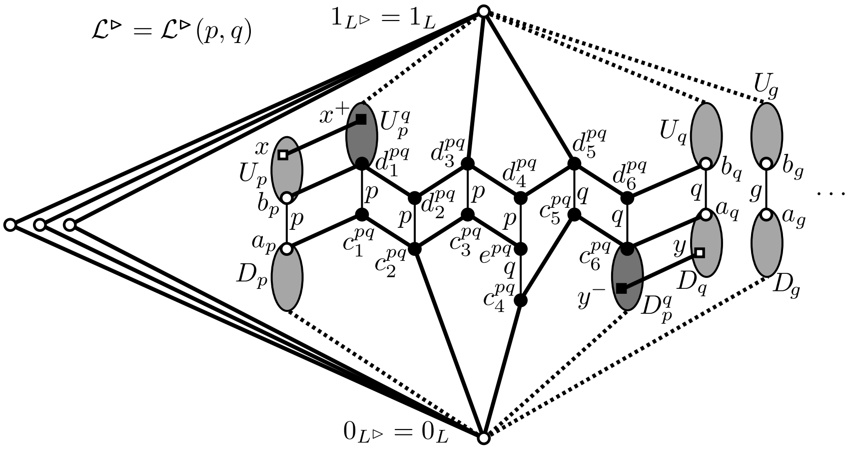

FIGURE 8. $\dots$, we obtain $\mathcal{L}^{\triangleright}$

$\gamma^{\triangleright}\colon \mathrm{Pairs}^{\leq}(L^{\triangleright}) \to H^{\triangleright}$ by

$$\gamma^{\triangleright}(\langle x,y\rangle) = \begin{cases} \gamma(\langle x,y\rangle), & \text{if } \langle x,y\rangle \in \mathrm{Pairs}^{\leq}(L),\\ \gamma(\langle z,t\rangle), & \text{if } \langle x,y\rangle = \langle z^+,t^+\rangle \in \mathrm{Pairs}^{\leq}(U_p^q \cup \{c_1^{pq}\}),\\ \gamma(\langle z,t\rangle), & \text{if } \langle x,y\rangle = \langle z^-,t^-\rangle \in \mathrm{Pairs}^{\leq}(D_p^q \cup \{d_6^{pq}\}),\\ p, & \text{if } \langle x,y\rangle \in \{\langle c_2^{pq},d_2^{pq}\rangle, \langle c_3^{pq},d_3^{pq}\rangle, \langle e^{pq},d_4^{pq}\rangle\},\\ q, & \text{if } \langle x,y\rangle \in \{\langle c_4^{pq},d_4^{pq}\rangle, \langle c_5^{pq},d_5^{pq}\rangle, \langle c_4^{pq},e^{pq}\rangle\},\\ 0_{H^{\triangleright}}, & \text{if } x=y,\\ 1_{H^{\triangleright}}, & \text{otherwise.} \end{cases}$$

Finally, after letting $\delta^{\triangleright} = \delta$, and $\varepsilon^{\triangleright} = \varepsilon$, we define

$$\mathcal{L}^{\triangleright}(p,q) = \mathcal{L}^{\triangleright} \quad \text{as} \quad \langle L^{\triangleright}; \gamma^{\triangleright}, H^{\triangleright}, \nu^{\triangleright}, \delta^{\triangleright}, \varepsilon^{\triangleright}, \mathcal{Z}^{\triangleright}\rangle. \tag{4.4}$$

Lemma 4.1. *If $\mathcal{L}$ satisfies* (4.1)*, then $L^{\triangleright}$ is a bounded lattice. Furthermore, S^{pq}, given by* (4.5) *below, and L are $\{0,1\}$-sublattices of $L^{\triangleright}$.*

Proof. First, we describe the ordering of $L^{\triangleright}$ more precisely; this description is the real definition of $L^{\triangleright}$. Consider the following subsets of $L^{\triangleright}$:

$$\begin{aligned}
&N^{pq} = \{c_1^{pq},\dots,c_6^{pq}, d_1^{pq},\dots,d_6^{pq}, e^{pq}\} \cup U_p^q \cup D_p^q \quad \text{(new elements)},\\
&B_\ell^{pq} = \{0_L = 0_{L^{\triangleright}}, a_p, b_p, 1_L\} \cup D_p \cup U_p \quad \text{(left boundary)},\\
&B_r^{pq} = \{0_L, a_q, b_q, 1_L = 1_{L^{\triangleright}}\} \cup D_q \cup U_q \quad \text{(right boundary)},\\
&B^{pq} = B_\ell^{pq} \cup B_r^{pq} \quad \text{(boundary)},\\
&R_\ell^{pq} = D_p \cup U_p \cup U_p^q \cup \{c_1^{pq}\} \quad \text{(left region)},\\
&R_r^{pq} = D_q \cup U_q \cup D_p^q \cup \{d_6^{pq}\} \quad \text{(right region)},\\
&S^{pq} = N^{pq} \cup B^{pq}, \quad \text{and}\\
&L_{\mathrm{g7}}^{pq} = \{0_L, a_p, b_p, a_q, b_q, c_1^{pq},\dots,c_6^{pq}, e^{pq}, d_1^{pq},\dots,d_6^{pq}, 1_{L^{\triangleright}}\}.
\end{aligned} \tag{4.5}$$

In case of four sets above, we call these sets and their elements "left" or "right" simply because of their positions in our figures. The definitions of some of these sets above are redundant; for example, $d_1^{pq} \in U_p^q$. The ordering within B^{pq}, which is a subset of L, is inherited from L. By definition, $L_{\mathrm{g}7}^{pq} \cong L_{\mathrm{g}7} = L_{\mathrm{g}7}(p,q)$. The ordering within $U_p \cup \{a_p\} \cup U_p^q \cup \{c_1^{pq}\}$ and that within $D_q \cup \{b_q\} \cup D_p^q \cup \{d_6^{pq}\}$ are already clear; for example, if $x \in U_p \cup \{a_p\}$ and $y^+ \in U_p^q \cup \{c_1^{pq}\}$, then $x \leq y^+$ iff $x \leq_L y$. Thus, since a_p is the top element of D_p and b_q is the bottom of U_q, the ordering within R_ℓ^{pq} and that within R_r^{pq} are defined. (A6) implies that

$$\langle R_\ell^{pq}; \leq\rangle \text{ and } \langle R_r^{pq}; \leq\rangle \text{ are lattices.} \tag{4.6}$$

The above facts, together with $L_{\mathrm{g}7}^{pq} \cong L_{\mathrm{g}7}$ and even $L_{\mathrm{g}7}^{pq} = L_{\mathrm{g}7}(p,q)$, define the ordering within S^{pq}. A routine argument verifies that $S^{pq} = \langle S^{pq}; \leq_{S^{pq}}\rangle$ is a lattice; the details are omitted. Observe that $1_{L^\triangleright} \notin R_\ell^{pq} \cup R_r^{pq}$ and

$$\text{for all } x \in R_\ell^{pq} \text{ and } y \in R_r^{pq}, \quad x \wedge_{S^{pq}} y = 0_{L^\triangleright}. \tag{4.7}$$

Therefore, for $x \in N^{pq}$, there is a unique least element x^* of B^{pq} such that $x \leq_{S^{pq}} x^*$. Similarly, for $x \in N^{pq}$, there is a unique largest element x_* of B^{pq} such that $x_* \leq_{S^{pq}} x$. If $x \in L$, then we let $x^* = x_* = x$. In this way, $x \mapsto x^*$ and $x \mapsto x_*$ are maps from $L^\triangleright$ to L. Note that

$$\begin{aligned} &(x^-)^* = x \text{ and } (y^*)^- = y \text{ for } x \in D_q \cup \{b_q\} \text{ and } y \in D_p^q \cup \{d_6^{pq}\};\\ &(x^+)_* = x \text{ and } (y_*)^+ = y \text{ for } x \in U_p \cup \{a_p\} \text{ and } y \in U_p^q \cup \{c_1^{pq}\}. \end{aligned} \tag{4.8}$$

Using these maps, the exact definition of the ordering in $L^\triangleright$ is described as follows: for $x, y \in L^\triangleright$,

$$x \leq_{L^\triangleright} y \iff \begin{cases} x \leq_L y, & \text{if } x, y \in L, \text{ or}\\ x \leq_{S^{pq}} y, & \text{if } x, y \in S^{pq}, \text{ or}\\ x \leq_L y_*, & \text{if } x \in L \setminus S^{pq} \text{ and } y \in N^{pq}, \text{ or}\\ x^* \leq_L y, & \text{if } x \in N^{pq} \text{ and } y \in L \setminus S^{pq}. \end{cases} \tag{4.9}$$

Observe that for $u_1, u_3 \in B^{pq}$ and $u_2 \in N^{pq}$, the conjunction of $u_1 \leq_{S^{pq}} u_2$ and $u_2 \leq_{S^{pq}} u_3$ implies $\{0_{L^\triangleright}, 1_{L^\triangleright}\} \cap \{u_1, u_3\} \neq \varnothing$. Hence, it is straightforward to see that $\leq_{L^\triangleright}$ is an ordering and $\leq_L$ is the restriction of $\leq_{L^\triangleright}$ to L. Note that, for $x \in L^\triangleright$,

$$x_* = 1_{L \cap \downarrow x} \quad \text{and} \quad x^* = 0_{L \cap \uparrow x}; \tag{4.10}$$

that is, x_* is the greatest element of $L \cap \downarrow x$, and dually for x^*. Unless otherwise specified, $\leq$, $\|$, $\vee$, $\downarrow e_{pq}$, etc. will be understood in $L^\triangleright$.

Next, we define a mapping $u \mapsto \widehat{u}$ from L to S^{pq}. For $u \in L \setminus \{0_L\}$, either $u \in D_p$, or $u \vee b_p \in U_p \cup \{1_L\}$ is the smallest element u^p of $B_\ell^{pq} \cap \uparrow u$. Similarly, $B_r^{pq} \cap \uparrow u$ has a smallest element u^q. If $u \neq 0_{L^\triangleright}$, then it follows from (A16′) that $1_{L^\triangleright} \in \{u^p, u^q\}$. Hence, for each $u \in L$, $B^{pq} \cap \uparrow u$ has a smallest element;

we denote it by $\widehat{u}$. For $u \in N^{pq}$, we let $\widehat{u} = u$. Note that,

$$\begin{aligned} &\text{for every } u \in L^{\triangleright},\ \widehat{u} \text{ is the smallest element of } S^{pq} \cap {\uparrow}u, \\ &\qquad\qquad \text{and } u \in {\downarrow}U_p \setminus {\downarrow}a_p \text{ implies } \widehat{u} = u \vee_L b_p. \end{aligned} \tag{4.11}$$

For later reference, we quasi-dualize (4.11). For $u \in L \setminus \{1_L\}$, either $u \in U_q$, or $u \wedge a_q \in D_q \cup \{0_L\}$ is the largest element u_q of $B_r^{pq} \cap {\downarrow}u$. Similarly, $B_\ell^{pq} \cap {\downarrow}u$ has a largest element u_p. Hence, for each $u \in L$, $B^{pq} \cap {\downarrow}u$ has a largest element; we denote it by $\breve{u}$. For $u \in N^{pq}$, we let $\breve{u} = u$. Note that,

$$\begin{aligned} &\text{for every } u \in L^{\triangleright},\ \breve{u} \text{ is the largest element of } S^{pq} \cap {\downarrow}u, \\ &\qquad\qquad \text{and } u \in {\uparrow}D_q \setminus {\uparrow}b_q \text{ implies } \breve{u} = u \wedge_L a_q. \end{aligned} \tag{4.12}$$

Next, for $x \parallel y \in L^{\triangleright}$, we want to show that x and y has a join in $L^{\triangleright}$. There are several cases to consider. The order ideal generated by U_p^q will be denoted by ${\downarrow}U_p^q$. Since U_p^q is directed, ${\downarrow}U_p^q$ will turn out to be a lattice ideal of $L^{\triangleright}$.

Case 4.2. We claim that if $\{x, y\} \subseteq L$, then L, $x \vee_L y$ is the join of x and y in $L^{\triangleright}$. To prove this, we can assume that $\{x, y\} \subseteq {\downarrow}U_p^q$, since otherwise $\{x, y\}$ has no upper bound outside L. Let $z \in N^{pq}$ be an upper bound of $\{x, y\}$. We obtain from (4.10) that $x \le_L z_*$ (even if $x \in S^{pq}$) and $y \le_L z_*$. Hence, $x \vee_L y \le_L z_* \le z$, proving $x \vee_{L^{\triangleright}} y = x \vee_L y$.

Case 4.3. We claim that if $\{x, y\} \subseteq S^{pq}$, then $x \vee_{S^{pq}} y$ is the join of x and y in $L^{\triangleright}$. (We have already mentioned that $\langle S^{pq}; \le\rangle$ is a lattice.) For $\{x, y\} \subseteq R_\ell^{pq}$ or $\{x, y\} \subseteq R_r^{pq}$, this follows from (4.8) and (4.10) in a straightforward way by considering several cases. Next, assume that $x \in R_\ell^{pq}$ and $y \in R_r^{pq}$, and suppose, for a contradiction, that $z \in L \setminus \{1_{L^{\triangleright}}\}$ is an upper bound of $\{x, y\}$. We have $x^* \le_L z$ and $y^* \le_L z$ by (4.10). Since $x \le x^* \le z < 1_L$, the element $x \in R_\ell^{pq}$ has a nontrivial upper bound in L. Thus, $x^* = x \in D_p \cup U_p \subseteq \mathrm{Nbh}_1(D_p \cup U_p)$. Since $y \in R_r^{pq}$, we have $y^* \in D_q \cup U_q \subseteq \mathrm{Nbh}_1(D_q \cup U_q)$. Hence, the validity of (A16′) in $\mathcal{L}$ yields that $1_L = x^* \vee_L y^* \le_L z <_L 1_L$, a contradiction. Thus, we conclude the validity of $x \vee_{S^{pq}} y = x \vee_{L^{\triangleright}} y$ for the case $\{x, y\} \subseteq R_\ell^{pq} \cup R_r^{pq}$. The same holds in the remaining case $\{x, y\} \not\subseteq R_\ell^{pq} \cup R_r^{pq}$, because then all upper bounds of $\{x, y\}$ belong to S^{pq}.

Case 4.4. For $x \in L \setminus B^{pq} = L \setminus S^{pq}$ and $y \in N^{pq} = S^{pq} \setminus L$, we claim that

$$x \vee_{L^{\triangleright}} y = \begin{cases} \widehat{x} \vee_{S^{pq}} y, & \text{if } x \in {\downarrow}U_p^q \ \ \text{(equivalently, if } x \in {\downarrow}U_p\text{)}, \\ x \vee_L y^*, & \text{if } x \notin {\downarrow}U_p^q \ \ \text{(equivalently, if } x \notin {\downarrow}U_p\text{)}. \end{cases} \tag{4.13}$$

To prove this, first we assume that $x \in {\downarrow}U_p^q$. We conclude from (A16′) that $\widehat{x} \in D_p \cup U_p$ and $x \in \mathrm{Nbh}_1(D_p \cup U_p)$. Suppose, for a contradiction, that $\{x, y\}$ has an upper bound z in $L \setminus \{1_L\}$. We have $y^* \le_L z$ by (4.10). Since $y^* \in D_q \cup U_q \subseteq \mathrm{Nbh}_1(D_q \cup U_q)$, (A16′) yields $z = 1$, contradicting $z \in L \setminus \{1_L\}$. Hence, all upper bounds of $\{x, y\}$ belong to S^{pq}. This proves the first half of (4.13). The second half is obvious, because $x \in L \setminus {\downarrow}U_p^q$ has no upper bound outside L.

Cases 4.2–4.4 prove that $L^{\triangleright}$ is a join-semilattice. By quasi-duality, it is a lattice. Cases 4.2, 4.3, and their quasi-duals also prove that L and S^{pq} are $\{0,1\}$-sublattices of $L^{\triangleright}$. This completes the proof of Lemma 4.1. $\square$

We need a lemma from Dilworth [4], see also Grätzer [5, Theorem III.1.2].

Lemma 4.5. *If L is a lattice and $\langle u_1, v_1\rangle, \langle u_2, v_2\rangle \in \mathrm{Pairs}^{\le}(L)$, then the following three conditions are equivalent.*

(i) $\mathrm{con}(u_1, v_1) \le \mathrm{con}(u_2, v_2)$*;*

(ii) $\langle u_1, v_1\rangle \in \mathrm{con}(u_2, v_2)$*;*

(iii) *there exists an $n \in \mathbb{N}$ and there are $x_i \in L$ for $i \in \{0, \dots, n\}$ and $\langle y_{ij}, z_{ij}\rangle \in \mathrm{Pairs}^{\le}(L)$ for $\langle i, j\rangle \in \{1, \dots, n\} \times \{0, \dots, n\}$ such that the following equalities and inequalities hold:*

$$\begin{aligned} &u_1 = x_0 \le x_1 \le \cdots \le x_{n-1} \le x_n = v_1\\ &y_{i0} = x_{i-1},\ y_{in} = u_2,\ z_{i0} = x_i, \text{ and } z_{in} = v_2 \text{ for } 1 \le i \le n,\\ &y_{i,j-1} = z_{i,j-1} \wedge y_{ij} \text{ and } z_{i,j-1} \le z_{ij} \text{ for } j \text{ odd, } i, j \in \{1, \dots, n\},\\ &z_{i,j-1} = y_{i,j-1} \vee z_{ij} \text{ and } y_{i,j-1} \ge y_{ij} \text{ for } j \text{ even, } i, j \in \{1, \dots, n\}. \end{aligned} \tag{4.14}$$

The situation of Lemma 4.5 is outlined in Figure 9; note that the elements depicted do not form a sublattice in general and they are not necessarily distinct. The second half of (4.14) says that, in terms of Grätzer [5], $\langle y_{i,j-1}, z_{i,j-1}\rangle$ is *weakly* up or down perspective into $\langle y_{ij}, z_{ij}\rangle$; up for j odd and down for j even. Besides weak perspectivity, we recall that $\langle x_1, y_1\rangle$ is *perspective* to $\langle x_2, y_2\rangle$ if there are $i, j \in \{1, 2\}$ such that $i \ne j$, $x_i = x_j \wedge y_i$, and $y_j = x_j \vee y_i$. *Projectivity* is the reflexive transitive closure of perspectivity.

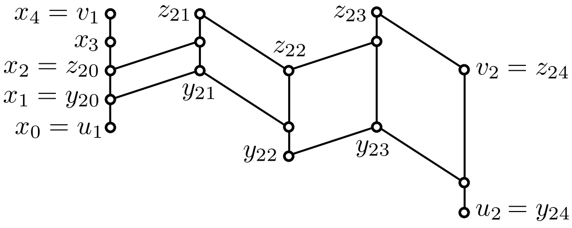

FIGURE 9. Illustrating Lemma 4.5 for $n = 4$

For a quasiordered set $\langle H; \nu\rangle$, we say that $p \in H$ is a *join* of the elements $q_1, \dots, q_n \in H$, in notation, $p =_\nu \bigvee_{i=1}^{n} q_i$, if $q_i \le_\nu p$ for all i and, for every $r \in H$, the conjunction of $q_i \le_\nu r$ for $i = 1, \dots, n$ implies $p \le_\nu r$. Even if a join exists, it need not be unique in the usual sense, but it is unique modulo $=_\nu$.

Lemma 4.6 ("Chain Lemma" for quasi-colored lattices)**.** *If $\langle L; \gamma, H, \nu\rangle$ is a quasi-colored lattice and $\{u_0 \le u_1 \le \cdots \le u_n\}$ is a finite chain in L, then*

$$\gamma(\langle u_0, u_n\rangle) =_\nu \bigvee_{i=1}^{n} \gamma(\langle u_{i-1}, u_i\rangle) \quad \text{holds in } \langle H; \nu\rangle. \tag{4.15}$$

Proof. Let $p = \gamma(\langle u_0, u_n\rangle)$ and $q_i = \gamma(\langle u_{i-1}, u_i\rangle)$. Since $\mathrm{con}(u_{i-1}, u_i) \leq \mathrm{con}(u_0, u_n)$, (C2) yields $q_i \leq_\nu p$ for all i. Next, assume that $r \in H$ such that $q_i \leq_\nu r$ for all i. By the surjectivity of γ, there exists a $\langle v, w\rangle \in \mathrm{Pairs}^{\leq}(L)$ such that $\gamma(\langle v, w\rangle) = r$. It follows by (C1) that $\langle u_{i-1}, u_i\rangle \in \mathrm{con}(u_{i-1}, u_i) \leq \mathrm{con}(v, w)$. Since $\mathrm{con}(v, w)$ is transitive and collapses the pairs $\langle u_{i-1}, u_i\rangle$, it collapses $\langle u_0, u_n\rangle$. Hence, $\mathrm{con}(u_0, u_n) \leq \mathrm{con}(v, w)$, and (C2) implies $p \leq_\nu r$. □

Now, we are in the position to prove the main lemma of the paper.

Lemma 4.7. *The structure $\mathcal{L}^\triangleright = \mathcal{L}^\triangleright(p, q)$, which is defined in* (4.4) *with assumption* (4.1)*, is a strong auxiliary structure, and $\mathcal{L}$ is a tight substructure of $\mathcal{L}^\triangleright$. Furthermore, $|L^\triangleright| \leq \aleph_0 + |L|$.*

Proof. Since we work both in $\mathcal{L}$ and $\mathcal{L}^\triangleright$, relations, operations and maps are often subscripted by the relevant structure; in the absence of subscripts, we are in $\mathcal{L}^\triangleright$. By Lemma 4.1, $L^\triangleright$ is a bounded lattice. We obtain from (4.3) that (A2) holds for $\mathcal{L}^\triangleright$. It follows trivially from the construction and Lemma 4.1 that $\mathcal{L}^\triangleright$ satisfies (A3), (A4), (A5), (A9), and (A10).

Next, we deal with (A6). Let $r \in H \setminus \{0_H\}$ and $\{x, y\} \subseteq U_r$; we have to prove that $x \vee y \in U_r$. Equivalently, we have to prove that $x \vee y \neq 1_{L^\triangleright}$. For $\{x, y\} \subseteq L$, this follows from the validity of (A6) in $\mathcal{L}$. If $\{x, y\} \subseteq N^{pq}$, then both x and y belong to $\{c_1^{pq}\} \cup U_p^q$, since $b_r \in {\downarrow}x \cap {\downarrow}y$ and $b_r \in L$. Hence, $x \vee y \neq 1_{L^\triangleright}$, because R_ℓ^{pq} is sublattice by (4.6). Therefore, we can assume that $x \in L \setminus S^{pq}$ and $y \in N^{pq}$. We obtain from (4.10) that $b_r \leq y_*$ and $y \in \{c_1^{pq}\} \cup U_p^q$. If we had $y_* = a_p$, then $b_r \leq a_p$ would contradict either (A5), if $r \neq p$, or (A3), if $r = p$. Hence, $y_* \in U_p$, and for later reference, we note that

$$\text{for all } r \in H, \quad b_r \nleq a_p. \tag{4.16}$$

Since $\mathcal{L}$ satisfies (A6) and $\{x, y_*\} \subseteq U_r$, we obtain $x \vee y_* \neq 1_L$. Clearly, $x \vee y_* \in U_p$. Using that R_ℓ^{pq} is a sublattice of $L^\triangleright$, it follows that $(x \vee y_*) \vee y \in R_\ell^{pq}$. Thus, $x \vee y = x \vee (y_* \vee y) = (x \vee y_*) \vee y \neq 1_{L^\triangleright}$. Consequently, U_r is a sublattice of $\mathcal{L}$, and $\mathcal{L}^\triangleright$ satisfies (A6) by quasi-duality.

The members of $\mathrm{Pairs}^{\leq}(L^\triangleright) \setminus \big(\mathrm{Pairs}^{\leq}(L) \cup \mathrm{Pairs}^{\leq}(S^{pq})\big)$ will be called *mixed pairs.* In other words, a pair is called mixed if exactly one of its components belongs to L. By the definition of $\gamma^\triangleright$,

$$\text{if } \langle x, y\rangle \text{ is a mixed pair, then } \gamma^\triangleright(\langle x, y\rangle) = 1_{H^\triangleright}. \tag{4.17}$$

In order to verify (A7), assume that $r \in H$, $\langle x, y\rangle \in \mathrm{Pairs}^{\leq}(U_r)$, and $x \neq y$. If $\langle x, y\rangle \in \mathrm{Pairs}^{\leq}(L)$, in other words, $\langle x, y\rangle$ is an old pair, or $\langle x, y\rangle$ is a mixed pair, or $\gamma^\triangleright(\langle x, y\rangle) = 1_{H^\triangleright}$, then $r \leq_{\nu^\triangleright} \gamma^\triangleright(\langle x, y\rangle)$ follows from $\nu \subseteq \nu^\triangleright$, (4.17), and the validity of (A7) in $\mathcal{L}$. Hence, we can assume that $\langle x, y\rangle \in \mathrm{Pairs}^{\leq}(N^{pq})$. Since $0_L \neq b_r \leq x$ and $b_r \in L$, we obtain that $x \in U_p^q \cup \{c_1^{pq}\}$. Actually, we have $x \in U_p^q$, since $x = c_1^{pq}$, combined with (4.10) and $(c_1^{pq})_* = a_p$, would contradict (4.16). Similarly, $y \in U_p^q$. Clearly, $x_*, y_* \in U_p \subseteq L$. Hence, using the definitions, the validity of (A7) in $\mathcal{L}$, and (4.8), we obtain that

$r \leq_\nu \gamma(\langle x_*, y_*\rangle) = \gamma^\rhd(\langle (x_*)^+, (y_*)^+\rangle) = \gamma^\rhd(\langle x, y\rangle)$. Thus, by quasi-duality, $\mathcal{L}^\rhd$ satisfies (A7).

Next, we prove that, for $\langle x, y\rangle \in \mathrm{Pairs}^{\leq}(L^\rhd)$,

$$\gamma^\rhd(\langle x, y\rangle) = 1_{H^\rhd} \Longrightarrow \mathrm{con}_{L^\rhd}(x, y) = \nabla_{L^\rhd}. \tag{4.18}$$

First, assume that $\langle x, y\rangle \in \mathrm{Pairs}^{\leq}(L)$ and $\gamma^\rhd(\langle x, y\rangle) = 1_{H^\rhd}$, that is, $\gamma(\langle x, y\rangle) = 1_H$. Since (C1) holds in $\mathcal{L}$ and $\gamma(\langle 0_L, 1_L\rangle) \leq_\nu 1_H = \gamma(\langle x, y\rangle)$, we obtain $\nabla_L = \mathrm{con}_L(0_L, 1_L) \leq \mathrm{con}_L(x, y)$. Hence, $\langle 0_L, 1_L\rangle \in \mathrm{con}_L(x, y)$. Using Lemma 4.5 and the fact that L is a sublattice of $L^\rhd$ by Lemma 4.1, we obtain $\langle 0_{L^\rhd}, 1_{L^\rhd}\rangle \in \mathrm{con}_{L^\rhd}(x, y)$, and (4.18) holds in this case. Second, assume that $\langle x, y\rangle \in \mathrm{Pairs}^{\leq}(L^\rhd)$ is a mixed pair in the sense of (4.17), and keep in mind that (A14′), which is a consequence of (A10), holds in $\mathcal{L}^\rhd$. Figure 8 shows (and it is straightforward to prove) that there exist $x_1, y_1 \in B^{pq}$ such that $x \leq x_1 \prec y_1 \leq y$ and $\langle x_1, y_1\rangle$ is perspective to $\langle 0_{L^\rhd}, u\rangle$ or $\langle u, 1_{L^\rhd}\rangle$ for some $u \in (L^\rhd)^{-01}$. Hence, we conclude $\mathrm{con}_{L^\rhd}(x, y) = 1_{H^\rhd}$ from (A14′). Third, we are left with the case $\langle x, y\rangle \in \mathrm{Pairs}^{\leq}(N^{pq})$ and $\gamma^\rhd(\langle x, y\rangle) = 1_{H^\rhd}$. If $[x, y]$ contains a thick edge $x_1 \prec y_1$ of Figure 8, then the previous case applies. Otherwise, either $\langle x, y\rangle \in \mathrm{Pairs}^{\leq}(U_p^q \cup \{c_1^{pq}\})$, or $\langle x, y\rangle \in \mathrm{Pairs}^{\leq}(D_p^q \cup \{d_6^{pq}\})$. By quasi-duality, we can assume the first alternative. Using (4.8), we have that $\langle x, y\rangle = \langle (x_*)^+, (y_*)^+\rangle$, which is perspective to $\langle x_*, y_*\rangle$. Thus, $\mathrm{con}_{L^\rhd}(x, y) = \mathrm{con}_{L^\rhd}(x_*, y_*)$, $\gamma^\rhd(\langle x, y\rangle) = \gamma^\rhd(\langle x_*, y_*\rangle)$, and the first case applies since $\langle x_*, y_*\rangle$ belongs to $\mathrm{Pairs}^{\leq}(L)$. This proves (4.18).

Let Θ denote the congruence of L described in (A11). Let

$$\begin{aligned}
\Psi &= \{\langle x^+, y^+\rangle : x, y \in U_p \cup \{a_p\} \text{ and } \langle x, y\rangle \in \Theta\},\\
\Gamma &= \{\langle x^-, y^-\rangle : x, y \in D_q \cup \{b_q\} \text{ and } \langle x, y\rangle \in \Theta\} \quad \text{and}\\
\Phi &= \{e^{pq}, c_4^{pq}, d_4^{pq}\}^2 \cup \{c_2^{pq}, d_2^{pq}\}^2 \cup \{c_3^{pq}, d_3^{pq}\}^2 \cup \{c_5^{pq}, d_5^{pq}\}^2.
\end{aligned}$$

Here, Ψ and Γ are equivalence relations on the sets $\{c_1^{pq}\} \cup U_p^q$ and $\{d_6^{pq}\} \cup D_p^q$, respectively, and Φ is the equivalence on $\{e^{pq}, c_2^{pq}, \dots, c_5^{pq}, d_2^{pq}, \dots, d_5^{pq}\}$ whose blocks are the fibers of Z_7^{pq}. Let $\Theta^\rhd = \Theta \cup \Psi \cup \Gamma \cup \Phi$; its blocks are the Θ-blocks, the Ψ-blocks, the Γ-blocks, and the Φ-blocks. The restriction of Θ to subset $X \subseteq L$ will be denoted by $\Theta\rceil_X$. Since the Θ-blocks, the $\Theta\rceil_{\{a_p\}\cup U_p}$-blocks, and the $\Theta\rceil_{\{b_q\}\cup D_q}$-blocks are convex sublattices, so are the $\Theta^\rhd$-blocks.

To prove that $\Theta^\rhd$ is a congruence, assume that $\langle x, y\rangle \in \Theta^\rhd \cap \mathrm{Pairs}^{\leq}(L^\rhd)$, $x \neq y$, and $z \in L^\rhd \setminus \{0_L, 1_L\}$ such that $x \leq z$; we claim that

$$\langle x \vee z, y \vee z\rangle \in \Theta^\rhd. \tag{4.19}$$

Since Θ is taken from (A11), (A14′) gives that $\{x, y\} \cap \{0_L, 1_L\} = \varnothing$. By the convexity of $\Theta^\rhd$-blocks, we can assume that $y \parallel z$. Based on Grätzer [8, Lemma 11], a tedious but straightforward argument shows that $\Theta^\rhd\rceil_{S^{pq}}$ is a congruence. Since so is $\Theta = \Theta^\rhd\rceil_L$, we can assume that $\{x, y, z\} \nsubseteq S^{pq}$ and $\{x, y, z\} \nsubseteq L$. Since both L and N^{pq} are unions of $\Theta^\rhd$-blocks, there are two cases to consider.

First, assume that $\langle x, y\rangle \in \mathrm{Pairs}^{\leq}(L)$ and $z \in N^{pq} = L^{\triangleright} \setminus L$. It follows from $0_{L^{\triangleright}} < x < z$ that $c_1^{pq} \leq z$. If $z \in U_p^q$, then $\langle x \vee b_p, y \vee b_p\rangle \in \Theta \cap U_p^2 = \Theta^{\triangleright}\rceil_{S^{pq}} \cap U_p^2$, $\langle x \vee z, y \vee z\rangle = \langle x \vee b_p \vee z, y \vee b_p \vee z\rangle$, and the fact that $\Theta^{\triangleright}\rceil_{S^{pq}}$ is a congruence give (4.19). Hence, we consider $z = c_1^{pq}$. If we have $y \leq b_p$, then $y \parallel z = c_1^{pq}$, $y \leq d_1^{pq}$, and $c_1^{pq} \prec d_1^{pq}$ yield $\langle x \vee z, y \vee z\rangle = \langle c_1^{pq}, d_1^{pq}\rangle \in \Theta^{\triangleright}$. Thus, we assume that $y \nleq b_p$, that is, $b_p < b_p \vee y$. Since $x \leq c_1^{pq}$, we have $x \leq (c_1^{pq})_* = a_p \leq b_p$ by (4.10). Hence, $\langle b_p, b_p \vee y\rangle = \langle b_p \vee_L x, b_p \vee_L y\rangle \in \Theta \neq \nabla_L$. Thus, (A14′) yields $b_p \vee y \neq 1_L = 1_{L^{\triangleright}}$, implying $y \in {\downarrow}U_p$. Using (4.11) and $y \nleq a_p$, we have that $\widehat{y} = y \vee b_p$. Therefore, $y \vee z = \widehat{y} \vee_{S^{pq}} c_1^{pq}$; either by (4.13), if $y \in L \setminus S^{pq}$, or because $y = \widehat{y} \in S^{pq}$. Since $\widehat{y} = y \vee b_p$, we obtain that $\langle b_p, \widehat{y}\rangle = \langle x \vee b_p, y \vee b_p\rangle \in \Theta \subseteq \Theta^{\triangleright}$. Joining this with $z = c_1^{pq}$, we have $\langle d_1^{pq}, y \vee z\rangle \in \Psi$, which implies (4.19) since $\langle x \vee z, d_1^{pq}\rangle = \langle c_1^{pq}, d_1^{pq}\rangle \in \Psi$.

Second, assume that $\langle x, y\rangle \in \mathrm{Pairs}^{\leq}(N^{pq})$ and $z \in L \setminus S^{pq}$. We still have $\{x, y\} \cap \{0_{L^{\triangleright}}, 1_{L^{\triangleright}}\} = \varnothing$, by the definition of $\Theta^{\triangleright}$. Since $x < z < 1_L$ implies $x \in D_p^q \cup \{d_6^{pq}\}$, the definition of $\Theta^{\triangleright}$ gives $\{x, y\} \subseteq D_p^q \cup \{d_6^{pq}\}$. By (4.8) and the definition of $\Theta^{\triangleright}$, we obtain $\langle x^*, y^*\rangle \in \Theta$. We have $x^* \leq z$ since $x \leq z$. If we had $z \in {\downarrow}U_p$, then $z \in \mathrm{Nbh}_1(D_p \cup U_p)$, $x^* \in D_q \cup U_q \subseteq \mathrm{Nbh}_1(D_q \cup U_q)$, and the validity of (A16′) in L would imply $x^* = z \wedge_L x^* = 0_L$, whence $x = 0_L$, contradicting $\{x, y\} \cap \{0_{L^{\triangleright}}, 1_{L^{\triangleright}}\} = \varnothing$. Hence, $z \notin {\downarrow}U_p$, and (4.13) yields that $\langle x \vee z, y \vee z\rangle = \langle x^* \vee_L z, y^* \vee_L z\rangle \in \Theta \subseteq \Theta^{\triangleright}$. This proves (4.19). Since $\Theta^{\triangleright}$ is an equivalence relation, (4.19) and its quasi-dual imply that $\Theta^{\triangleright}$ is a congruence on $L^{\triangleright}$. Since it is distinct from $\nabla_{L^{\triangleright}}$, $\mathcal{L}^{\triangleright}$ satisfies (A11).

Next, we prove the converse of (4.18). Assume that $\langle x, y\rangle \in \mathrm{Pairs}^{\leq}(L^{\triangleright})$ such that $\gamma^{\triangleright}(\langle x, y\rangle) \neq 1_{H^{\triangleright}}$; we want to show that $\mathrm{con}_{L^{\triangleright}}(x, y) \neq \nabla_{L^{\triangleright}}$. Since this is clear if $x = y$, we assume $x \neq y$. First, if $x, y \in L$, then let $r = \gamma(\langle x, y\rangle)$. Applying (C1) to γ and (A4) to $\mathcal{L}$, we obtain $\mathrm{con}_L(x, y) = \mathrm{con}_L(a_r, b_r)$. Hence $\Theta^{\triangleright}$, which we used in the previous paragraphs, collapses $\langle x, y\rangle$, and $\mathrm{con}_{L^{\triangleright}}(x, y) \subseteq \Theta^{\triangleright} \subset \nabla_{L^{\triangleright}}$. Second, if $\{x, y\} \cap L = \varnothing$, then there exists a pair $\langle x', y'\rangle \in \mathrm{Pairs}^{\leq}(\{a_p\} \cup U_p) \cup \mathrm{Pairs}^{\leq}(\{b_q\} \cup D_q) \subseteq \mathrm{Pairs}^{\leq}(L)$ such that $\gamma^{\triangleright}(\langle x, y\rangle) = \gamma^{\triangleright}(\langle x', y'\rangle)$ and $\langle x, y\rangle$ is projective to $\langle x', y'\rangle$. For example, if $\langle x, y\rangle \in \mathrm{Pairs}^{\leq}(\{c_1^{pq}\} \cup U_p^q)$, then we let $\langle x', y'\rangle = \langle x_*, y_*\rangle$. Thus, since projective pairs generate the same congruence, this case reduces to the first case. Finally, $|L \cap \{x, y\}| = 1$ is excluded by (4.17). Now, after verifying the converse of (4.18), we have proved that, for all $\langle x, y\rangle \in \mathrm{Pairs}^{\leq}(L^{\triangleright})$,

$$\gamma^{\triangleright}(\langle x, y\rangle) = 1_{H^{\triangleright}} \iff \mathrm{con}_{L^{\triangleright}}(x, y) = \nabla_{L^{\triangleright}}. \tag{4.20}$$

Observe that $\gamma^{\triangleright}$ is isotone in the sense that

$$\text{if } w_1 \leq w_2 \leq w_3 \leq w_4, \text{ then } \gamma^{\triangleright}(\langle w_2, w_3\rangle) \leq_{\nu^{\triangleright}} \gamma^{\triangleright}(\langle w_1, w_4\rangle). \tag{4.21}$$

This follows from (4.17), from the definition of $\gamma^{\triangleright}$, and from the fact that γ is isotone by (C1) and (C2) .

Next, to prove that $\gamma^{\triangleright}$ satisfies (C1), assume that $\langle u_1, v_1\rangle$ and $\langle u_2, v_2\rangle$ belong to $\mathrm{Pairs}^{\leq}(L^{\triangleright})$ such that $\gamma^{\triangleright}(\langle u_1, v_1\rangle) \leq_{\nu^{\triangleright}} \gamma^{\triangleright}(\langle u_2, v_2\rangle)$. Let $r_i = \gamma^{\triangleright}(\langle u_i, v_i\rangle)$, for $i \in \{1, 2\}$. We have to show $\mathrm{con}_{L^{\triangleright}}(u_1, v_1) \leq \mathrm{con}_{L^{\triangleright}}(u_2, v_2)$.

By (4.20), we can assume that $r_2 \neq 1_{H^\triangleright}$. We also have that $r_1 \neq 1_{H^\triangleright}$, since otherwise $1_{H^\triangleright} = r_1 \leq_{\nu^\triangleright} r_2$ and the satisfaction of (A2) for $\mathcal{L}^\triangleright$ would give $r_2 = r_1 = 1_H$. Similarly, we can assume that r_1 and, consequently, also r_2 differ from $0_{H^\triangleright}$, since otherwise $u_1 = v_1$, and so $\operatorname{con}_{L^\triangleright}(u_1, v_1) = \Delta_{L^\triangleright}$ would clearly imply $\operatorname{con}_{L^\triangleright}(u_1, v_1) \leq \operatorname{con}_{L^\triangleright}(u_2, v_2)$. Thus, $r_1, r_2 \in H^{-01} = (H^\triangleright)^{-01}$. By the construction of $L^\triangleright$, $\langle u_i, v_i\rangle$ is projective to some $\langle u_i', v_i'\rangle \in \operatorname{Pairs}^{\leq}(L)$ such that $\gamma^\triangleright(\langle u_i, v_i\rangle) = \gamma^\triangleright(\langle u_i', v_i'\rangle)$. Projectivity implies $\operatorname{con}_{L^\triangleright}(u_i, v_i) = \operatorname{con}_{L^\triangleright}(u_i', v_i')$. Therefore, we can assume that $\langle u_1, v_1\rangle, \langle u_2, v_2\rangle \in \operatorname{Pairs}^{\leq}(L)$, because otherwise we could work with $\langle u_1', v_1'\rangle$ and $\langle u_2', v_2'\rangle$.

According to (4.3), we distinguish two cases. First, assume that $r_1 \leq_\nu r_2$. Since $\gamma^\triangleright$ extends γ, we have that

$$\gamma(\langle u_1, v_1\rangle) = \gamma^\triangleright(\langle u_1, v_1\rangle) = r_1 \leq_\nu r_2 = \gamma^\triangleright(\langle u_2, v_2\rangle) = \gamma(\langle u_2, v_2\rangle).$$

Applying (C1) to γ, we obtain $\langle u_1, v_1\rangle \in \operatorname{con}_L(u_1, v_1) \leq \operatorname{con}_L(u_2, v_2)$. Using Lemma 4.5 (i) $\Rightarrow$ (iii) in L and then Lemma 4.5 (iii) $\Rightarrow$ (i) in $L^\triangleright$, we obtain that $\operatorname{con}_{L^\triangleright}(u_1, v_1) \leq \operatorname{con}_{L^\triangleright}(u_2, v_2)$.

Second, assume that $r_1 \leq_\nu p$ and $q \leq_\nu r_2$. Since $\gamma^\triangleright(\langle a_p, b_p\rangle) = \gamma(\langle a_p, b_p\rangle) = p$ and $\gamma^\triangleright(\langle a_q, b_q\rangle) = q$ by (A4), the argument of the previous paragraph yields $\operatorname{con}_{L^\triangleright}(u_1, v_1) \leq \operatorname{con}_{L^\triangleright}(a_p, b_p)$ and $\operatorname{con}_{L^\triangleright}(a_q, b_q) \leq \operatorname{con}_{L^\triangleright}(u_2, v_2)$. Clearly (or applying Lemma 4.5 within S^{pq}), we have $\operatorname{con}_{L^\triangleright}(a_p, b_p) \leq \operatorname{con}_{L^\triangleright}(a_q, b_q)$. Hence, transitivity yields $\operatorname{con}_{L^\triangleright}(u_1, v_1) \leq \operatorname{con}_{L^\triangleright}(u_2, v_2)$. Consequently, $\gamma^\triangleright$ satisfies (C1).

Next, to prove that $\gamma^\triangleright$ satisfies (C2), we assume that $\langle u_1, v_1\rangle, \langle u_2, v_2\rangle \in \operatorname{Pairs}^{\leq}(L^\triangleright)$ such that $\operatorname{con}_{L^\triangleright}(u_1, v_1) \leq \operatorname{con}_{L^\triangleright}(u_2, v_2)$. Our purpose is to show that $\gamma^\triangleright(\langle u_1, v_1\rangle) \leq_{\nu^\triangleright} \gamma^\triangleright(\langle u_2, v_2\rangle)$. We can assume that $u_1 \neq v_1$ and, by (4.20), that $\operatorname{con}_{L^\triangleright}(u_2, v_2) \neq \nabla_{L^\triangleright}$. That is, $\{\operatorname{con}_{L^\triangleright}(u_1, v_1), \operatorname{con}_{L^\triangleright}(u_2, v_2)\}$ is disjoint from $\{\Delta_{L^\triangleright}, \nabla_{L^\triangleright}\}$. We obtain from (4.17) that none of $\langle u_1, v_1\rangle$ and $\langle u_2, v_2\rangle$ is a mixed pair. If $\langle u_i, v_i\rangle$ is a new pair, that is, if $\{u_i, v_i\} \in \operatorname{Pairs}^{\leq}(N^{pq})$, then we can consider an old pair $\langle u_i', v_i'\rangle$ such that $\gamma^\triangleright(\langle u_i', v_i'\rangle) = \gamma^\triangleright(\langle u_i, v_i\rangle)$ and so, since $\gamma^\triangleright$ satisfies (C1), $\operatorname{con}_{L^\triangleright}(u_i', v_i') = \operatorname{con}_{L^\triangleright}(u_i, v_i)$. Hence, we can assume that $\langle u_1, v_1\rangle$ and $\langle u_2, v_2\rangle$ are old pairs, that is, they belong to $\operatorname{Pairs}^{\leq}(L)$.

The starting assumption $\operatorname{con}_{L^\triangleright}(u_1, v_1) \leq \operatorname{con}_{L^\triangleright}(u_2, v_2)$ is witnessed by Lemma 4.5. Let $x_j, y_{ij}, z_{ij} \in L^\triangleright$, for $i \in \{1, \dots, n\}$ and $j \in \{0, \dots, n\}$, be elements that satisfy (4.14); see also Figure 9. To ease our terminology, the ordered pairs $\langle y_{ij}, z_{ij}\rangle$ will be called *witness pairs*. Since $\operatorname{con}_{L^\triangleright}(u_2, v_2) \neq \nabla_{L^\triangleright}$, none of the witness pairs generate $\nabla_{L^\triangleright}$. Thus, by (4.17) and (4.20),

$$\text{none of the witness pairs is mixed or } 1_{H^\triangleright}\text{-colored.} \tag{4.22}$$

Take two consecutive witness pairs, $\langle y_{i,j-1}, z_{i,j-1}\rangle$ and $\langle y_{ij}, z_{ij}\rangle$. Here $i, j \in \{1, \dots, n\}$, and (4.14) says that $\langle y_{i,j-1}, z_{i,j-1}\rangle$ is weakly perspective into $\langle y_{ij}, z_{ij}\rangle$. We want to show that

$$\gamma^\triangleright(\langle y_{i,j-1}, z_{i,j-1}\rangle) \leq_{\nu^\triangleright} \gamma^\triangleright(\langle y_{ij}, z_{ij}\rangle). \tag{4.23}$$

To make the notation easier, we let $y_0 = y_{i,j-1}$, $z_0 = z_{i,j-1}$, $\mathfrak{p}_0 = \langle y_0, z_0\rangle$, $y_1 = y_{ij}$, $z_1 = z_{ij}$, and $\mathfrak{p}_1 = \langle y_1, z_1\rangle$. With this notation, (4.23) turns into

$$\gamma^{\triangleright}(\mathfrak{p}_0) \leq_{\nu^{\triangleright}} \gamma^{\triangleright}(\mathfrak{p}_1); \tag{4.24}$$

this is what we have to prove now. We assume $y_0 < z_0$ since (4.24) trivially holds otherwise. Hence, $y_1 < z_1$ also holds. By (4.21), we can also assume perspectivity rather than weak perspectivity. That is, as depicted in Figure 10,

$$\text{either} \quad z_0 \parallel y_1, \quad z_0 \wedge y_1 = y_0 \quad \text{and} \quad z_0 \vee y_1 = z_1, \tag{4.25}$$

$$\text{or} \quad y_0 \parallel z_1, \quad y_0 \wedge z_1 = y_1 \quad \text{and} \quad y_0 \vee z_1 = z_0; \tag{4.26}$$

By (4.22), there are three cases to consider.

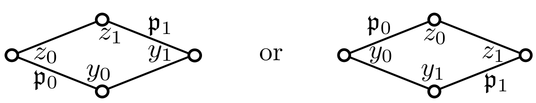

FIGURE 10. Perspectivities (4.25) and (4.26)

Case 4.8. If both $\mathfrak{p}_0$ and $\mathfrak{p}_1$ are old, then $\mathrm{con}_L(\mathfrak{p}_0) = \mathrm{con}_L(\mathfrak{p}_1)$. Applying (C2) for $\mathcal{L}$, we conclude the relation $\gamma(\mathfrak{p}_0) \leq_\nu \gamma(\mathfrak{p}_1)$. Thus, since $\gamma^{\triangleright}$ extends γ, (4.24) holds for old witness pairs.

Case 4.9. If $\mathfrak{p}_0, \mathfrak{p}_1 \in \mathrm{Pairs}^{\leq}(N^{pq})$, that is, if both $\mathfrak{p}_0$ and $\mathfrak{p}_1$ are new, then there are only few cases, and (4.24) follows in a straightforward way from our assumptions and the definition of $\gamma^{\triangleright}$. For example, let $\mathfrak{p}_0, \mathfrak{p}_1 \in \mathrm{Pairs}^{\leq}(U_p^q)$, and consider the old pairs $(\mathfrak{p}_0)_* := \langle (y_0)_*, (z_0)_*\rangle$ and $(\mathfrak{p}_1)_* := \langle (y_1)_*, (z_1)_*\rangle$. Since the maps in the second row of (4.8) are reciprocal lattice isomorphism between $\{a_p\} \cup U_p$ and $\{c_1^{pq}\} \cup U_p^q$, it follows that $(\mathfrak{p}_0)_*$ and $(\mathfrak{p}_1)_*$ are perspective. Hence, $\mathrm{con}_L((\mathfrak{p}_0)_*) = \mathrm{con}_L((\mathfrak{p}_1)_*)$, and we obtain from (C2), applied for $\mathcal{L}$, that $\gamma((\mathfrak{p}_0)_*) =_\nu \gamma((\mathfrak{p}_1)_*)$. On the other hand, we have $\gamma^{\triangleright}(\mathfrak{p}_0) = \gamma((\mathfrak{p}_0)_*)$ and $\gamma^{\triangleright}(\mathfrak{p}_1) = \gamma((\mathfrak{p}_1)_*)$ by (4.8) and the definition of $\gamma^{\triangleright}$. Hence, (4.24) follows by transitivity.

Case 4.10. Assume that one of $\mathfrak{p}_0$ and $\mathfrak{p}_1$ is old and one is new. We claim

$$\gamma^{\triangleright}(\mathfrak{p}_0) =_{\nu^{\triangleright}} \gamma^{\triangleright}(\mathfrak{p}_1), \tag{4.27}$$

which is a stronger statement than (4.24). Since the role of $\mathfrak{p}_0$ and $\mathfrak{p}_1$ in (4.27) is symmetric, we can assume that $\mathfrak{p}_0$ is old and $\mathfrak{p}_1$ is new. By quasi-duality, we also assume (4.25). Since $0_{L^{\triangleright}} < y_0 < y_1$, y_0 is an old element, and y_1 is a new one, we have $y_1, z_1 \in \{c_1^{pq}\} \cup U_p^q$ and $z_0 \in {\downarrow} U_p$.

First, assume that $y_1 = c_1^{pq}$; see Figure 11. We conclude from (4.9) that $y_0 \leq a_p$. Since $z_0 \nleq c_1^{pq}$, we have $z_0 \nleq a_p$. Hence, $\widehat{z}_0 = z_0 \vee b_p \leq z_1$ by (4.11). Let $u = z_0 \wedge b_p$; we have $y_0 \leq u$. Denote $\gamma(\langle b_p, \widehat{z}_0\rangle)$ and $\gamma(\langle u, z_0\rangle)$ by r and r', respectively. Since $\langle b_p, \widehat{z}_0\rangle$ and $\langle u, z_0\rangle$ are perspective in L, (C2) yields

$$r' =_\nu r. \tag{4.28}$$

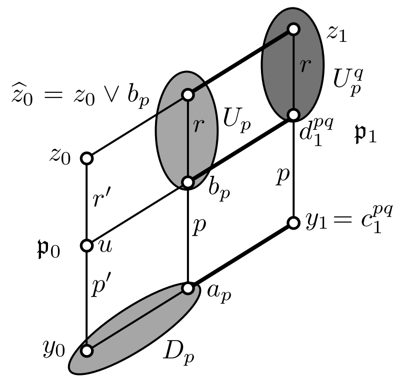

FIGURE 11. Case 4.10 with $y_1 = c_1^{pq}$

Since $y_0 \leq a_p < b_p$ and, by assumption (4.25), $y_0 = z_0 \wedge c_1^{pq}$, it follows that $y_0 = b_p \wedge y_0 \wedge b_p = b_p \wedge z_0 \wedge c_1^{pq} \wedge b_p = u \wedge a_p$. Denote $\gamma(\langle y_0, u\rangle)$ by p'. If $u \nleq a_p$, then $\langle y_0, u\rangle$ is perspective to $\langle a_p, b_p\rangle$ in L since $a_p \prec b_p$ by (A3), so (C2) yields $p' =_\nu p$. Otherwise, if $u \leq a_p$, then we obtain from $y_0 \leq u \leq z_0 \wedge a_p \leq z_0 \wedge c_1^{pq} = y_0$, (A1), and (A2) that $p' = \gamma(\langle u, u\rangle) = 0_H \leq_\nu p$. Hence,

$$p' \leq_\nu p, \text{ and even } p' =_\nu p \text{ if } u \nleq a_p. \tag{4.29}$$

As a subcase, assume that $\widehat{z}_0 = b_p$. (We would obtain this situation from Figure 11 by collapsing each of the r'-colored and r-colored intervals to its bottom.) We have that $y_0 < z_0 = z_0 \wedge b_p = u$. This excludes $u \leq a_p$, because $z_0 \nleq y_1 = c_1^{pq}$. Hence, $p' =_\nu p$ by (4.29). Since $z_1 = z_0 \vee y_1 = \widehat{z}_0 \vee c_1^{pq} = b_p \vee c_1^{pq} = d_1^{pq}$, we conclude

$$\gamma^{\triangleright}(\mathfrak{p}_0) = \gamma(\langle y_0, u\rangle) = p' =_\nu p = \gamma(\langle a_p, b_p\rangle) = \gamma^{\triangleright}(\langle c_1^{pq}, d_1^{pq}\rangle) = \gamma^{\triangleright}(\mathfrak{p}_1);$$

that is, (4.27) holds.

We are left with the subcase $\widehat{z}_0 > b_p$. We obtain $p \leq_\nu r$ from (A7). Hence, using Lemma 4.6, we obtain $\gamma(\langle a_p, \widehat{z}_0\rangle) =_\nu r$. Since $z_1 = z_0 \vee c_1^{pq} = \widehat{z}_0 \vee c_1^{pq} = (\widehat{z}_0)^+$ by (4.13) and (4.2), it follows that $\gamma^{\triangleright}(\mathfrak{p}_1) = \gamma(\langle a_p, \widehat{z}_0\rangle)$, and we obtain $\gamma^{\triangleright}(\mathfrak{p}_1) =_\nu r$. Since (4.29), $p \leq_\nu r$, and (4.28) imply $p' \leq_\nu r'$ by transitivity, Lemma 4.6 yields $\gamma^{\triangleright}(\mathfrak{p}_0) =_\nu r'$. Therefore, (4.27) and (4.24) follow from (4.28).

Second, assume that $y_1 \in U_p^q$; see Figure 12. Let $r' = \gamma^{\triangleright}(\mathfrak{p}_0) = \gamma(\mathfrak{p}_0)$, $r = \gamma^{\triangleright}(\mathfrak{p}_1)$, and let $u = \widehat{z}_0 \wedge y_1 \in U_p$. Since $z_1 = z_0 \vee y_1 = \widehat{z}_0 \vee y_1$ by (4.13), $\langle u, \widehat{z}_0\rangle$ is up-perspective to $\mathfrak{p}_1$. Since $z_0 \leq \widehat{z}_0$ and $u \leq \widehat{z}_0$, we have $z_0 \vee u \leq \widehat{z}_0$. This, together with (4.11), $z_0 \leq z_0 \vee u$ and $z_0 \vee u \in S^{pq}$, implies $z_0 \vee u = \widehat{z}_0$. Hence, $\mathfrak{p}_0$ is perspective to $\langle u, \widehat{z}_0\rangle$. Thus, denoting $\gamma^{\triangleright}(\langle u, \widehat{z}_0\rangle) = \gamma(\langle u, \widehat{z}_0\rangle)$ by r'', the validity of (C2) in $\mathcal{L}$ gives $r' =_\nu r''$. On the other hand, the sublattice $U_p \cup U_p^q$ is isomorphic to $U_p \times \mathsf{C}_2$ by definitions, see also (4.8). Therefore, since $\langle u, \widehat{z}_0\rangle$ is up-perspective to $\mathfrak{p}_1$, it is straightforward to see that $\langle u, \widehat{z}_0\rangle$ is perspective to $\langle (y_1)_*, (z_1)_*\rangle$. By the definition of $\gamma^{\triangleright}$ and

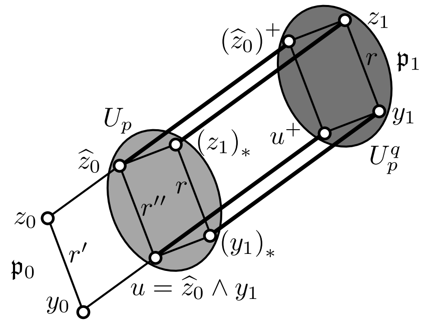

FIGURE 12. Case 4.10 with $y_1 \in U_p^q$

(4.8), $\gamma^{\rhd}(\langle (y_1)_*, (z_1)_*\rangle) = \gamma^{\rhd}(\mathfrak{p}_1) = r$. Applying (C2) in $\mathcal{L}$ to the above-mentioned perspective pairs, we obtain that $r'' =_\nu r$. Thus, we conclude $r' =_\nu r$ by transitivity, which implies (4.27). Its consequences, (4.24) and (4.23), also hold.

Cases 4.8–4.10 prove (4.23). Observe that (4.23) for $j = 1, \dots, n$ and transitivity yield $\gamma^{\rhd}(\langle x_{i-1}, x_i\rangle) = \gamma^{\rhd}(\langle y_{i0}, z_{i0}\rangle) \leq_{\nu^{\rhd}} \gamma^{\rhd}(\langle y_{in}, z_{in}\rangle) = \gamma^{\rhd}(\langle u_2, v_2\rangle)$. Hence, Lemma 4.6 implies $\gamma^{\rhd}(\langle u_1, v_1\rangle) \leq_{\nu^{\rhd}} \gamma^{\rhd}(\langle u_2, v_2\rangle)$. Therefore, $\mathcal{L}^{\rhd}$ satisfies (C2), and (A1) holds for $\mathcal{L}^{\rhd}$.

Next, to prove (A8), let $Z_7 \in \mathcal{Z}^{\rhd}$. Clearly, Z_7 is a zigzag of $\mathcal{G}^{\mathrm{icg}}(L^{\rhd})$; we only have to show that it is tight. Since the tightness of Z_7^{pq} is obvious by (4.9), we can assume that Z_7 is an old zigzag, that is, it belongs to $\mathcal{Z}$. By quasiduality, it suffices to deal with its upper fibers. So let F be an upper fiber of Z_7. Suppose, for a contradiction, that there exists an element $y \in L^{\rhd} \setminus L = N^{pq}$ such that $y \in {\uparrow}F$. Since there is an $f \in F$, which is a nonzero old element, such that $f \leq y$, we have that $y \in \{c_1^{pq}\} \cup U_p^q$. By (4.10), $f \leq y_*$. Since Z_7 is tight in $\mathcal{G}^{\mathrm{icg}}(L)$, F is a filter of L^{-01}. Hence, $y_* \in F \subseteq Z_7^{\mathrm{set}}$. On the other hand, $y_* \in \{a_p\} \cup U_p \subseteq \mathrm{Nbh}_1(\{a_p, b_p\})$. This yields $\mathrm{Nbh}_1(\{a_p, b_p\}) \cap Z_7^{\mathrm{set}} \neq \varnothing$, which contradicts the validity of (A12) in $\mathcal{L}$. This shows that all elements of $L^{-01} \cap {\uparrow}F$ are old. Hence, $L^{-01} \cap {\uparrow}F \subseteq F$ since Z_7 is a tight zigzag of $\mathcal{G}^{\mathrm{icg}}(L)$. Thus, Z_7 is a tight zigzag of $\mathcal{G}^{\mathrm{icg}}(L^{\rhd})$. Consequently, (A8) holds in $\mathcal{L}^{\rhd}$.

Next, to prove (A12), assume that $r \in H^{-01}$ and $Z_7 \in \mathcal{Z}^{\rhd} = \mathcal{Z} \cup \{Z_7^{pq}\}$. If $Z_7 \in \mathcal{Z}$, then $\mathrm{Nbh}_1(\{a_r, b_r\}) \cap Z_7^{\mathrm{set}} = \varnothing$ follows from the validity of (A12) for $\mathcal{L}$. Otherwise, if $Z_7 = Z_7^{pq}$, then we obtain $\mathrm{Nbh}_1(\{a_r, b_r\}) \cap Z_7^{\mathrm{set}} = \varnothing$ from $L^{-01} \cap {\downarrow}((Z_7^{pq})^{\mathrm{set}}) = L^{-01} \cap {\uparrow}((Z_7^{pq})^{\mathrm{set}}) = \varnothing$. Thus, $L^{\rhd}$ satisfies (A12).

Finally, to prove (A13), assume that $r, s \in H^{-01}$ such that $\langle a_r, b_r, a_s, b_s\rangle$ is a spanning N_6-quadruple and $\vec{z} = \langle z_0, \dots, z_n\rangle$ in (2.1) is a $\mathcal{G}^{\mathrm{icg}}(L^{\rhd})$-path from $\{a_r, b_r\}$ to $\{a_s, b_s\}$. We can assume that $\{z_0, \dots, z_n\} \not\subseteq L$, since otherwise the validity of (A13) in $\mathcal{L}$ implies that $\vec{z}$ goes through a tight zigzag belonging to $\mathcal{Z} \subseteq \mathcal{Z}^{\rhd}$. Suppose, for a contradiction, that $\vec{z}$ does not go through any tight zigzag from $\mathcal{Z}^{\rhd}$. A pair $\langle i, j\rangle$ of subscripts is called a *critical pair* (associated with $\vec{z}$) if $z_{i-1} \in L$, $\{z_i, \dots, z_j\} \subseteq N^{pq}$, and $z_{j+1} \in L$. The set of critical pairs is denoted by $\mathrm{CrP}(\vec{z})$. This set is nonempty by the indirect assumption. Clearly, $z_i, z_j \in (\{c_1^{pq}\} \cup U_p^q) \cup (\{d_6^{pq}\} \cup D_p^q)$ holds for every critical pair $\langle i, j\rangle$. If $z_i \in \{c_1^{pq}\} \cup U_p^q$ and $z_j \in \{d_6^{pq}\} \cup D_p^q$, or $z_j \in \{c_1^{pq}\} \cup U_p^q$ and $z_i \in \{d_6^{pq}\} \cup D_p^q$, then $\langle i, j\rangle \in \mathrm{CrP}(\vec{z})$ is called a *wide critical pair*; otherwise it is a *narrow critical pair.* If $\langle i_1, j_1\rangle \in \mathrm{CrP}(\vec{z})$ such that $j_2 \le j_1$ holds for all $\langle i_2, j_2\rangle \in \mathrm{CrP}(\vec{z})$, then $\langle i_1, j_1\rangle$ is called the *last critical pair* (associated with $\vec{z}$). The *last wide critical pair* and the *last narrow critical pair* are defined analogously. (We do not claim that both of them exist.) If $\langle i_1, j_1\rangle$ is the last wide critical pair, then $\langle z_1, \dots, z_{j_1}\rangle$ will be called the *essential part* of $\vec{z}$; if there is no wide critical pair, then the essential part is empty. We claim that there exists a wide critical pair. We prove this by induction on $|\mathrm{CrP}(\vec{z})|$. Let $\langle i, j\rangle$ be the last critical pair. We can assume that $\langle i, j\rangle \in \mathrm{CrP}(\vec{z})$ is narrow, since otherwise there is nothing to prove. By quasi-duality, we can also assume that $z_i, z_j \in \{c_1^{pq}\} \cup U_p^q$. Since $L^{-01} \cap \uparrow(\{c_1^{pq}\} \cup U_p^q) = \varnothing$, we have $z_{i-1} < z_i$ and $z_{j+1} < z_j$. By (4.10) and (4.6), we have that $z_{i-1} \le (z_i)_*$, $z_{j+1} \le (z_j)_*$, and $(z_i)_* \vee (z_j)_* \in L^{-01}$. Actually, $(z_i)_* \vee (z_j)_* \in \{a_p\} \cup U_p \subseteq \mathrm{Nbh}_1(\{a_p, b_p\})$, understood in L. By replacing the segment $\langle z_{i-1}, z_i, \dots, z_j, z_{j+1}\rangle$ in $\vec{z}$ by $\langle z_{i-1}, (z_i)_* \vee (z_j)_*, z_{j+1}\rangle$, we obtain a new $\mathcal{G}^{\mathrm{icg}}(L^{\triangleright})$-path $\vec{z}'$ from $\{a_r, b_r\}$ to $\{a_s, b_s\}$. It follows from (A12) that the element $(z_i)_* \vee (z_j)_*$, which is the only component of $\vec{z}'$ that need not occur in $\vec{z}$, cannot belong to an old tight zigzag. Obviously, $(z_i)_* \vee (z_j)_*$ does not belong to the new tight zigzag, Z_7^{pq}. Thus, $\vec{z}'$ does not go through any tight zigzag, because neither does $\vec{z}$. Therefore, since $\mathrm{CrP}(\vec{z}') = \mathrm{CrP}(\vec{z}) \setminus \{\langle i, j\rangle\}$, the induction hypothesis applies, and $\vec{z}'$ has a wide critical pair. So does $\vec{z}$, since $\vec{z}'$ and $\vec{z}$ have the same essential parts. This completes the induction, and we have shown that $\vec{z}$ has a wide critical pair $\langle i_0, j_0\rangle$. Thus, $\vec{z}$ goes trough the new tight zigzag $Z_7^{pq} \in \mathcal{Z}^{\triangleright}$, because so does the segment $\langle z_{i_0}, \dots, z_{j_0}\rangle$. This is a contradiction, which proves that (A13) holds in $L^{\triangleright}$. The proof of Lemma 4.7 is complete. $\square$

5. Approaching infinity

For an ordered set $P = \langle P; \le\rangle$ and a subset C of P, the restriction of the ordering of P to C will be denoted by $\le\rceil_C$. If each element of P has an upper bound in C, then C is a *cofinal subset* of P. The following two lemmas belong to the folklore; having no reference at hand, we will outline their easy proofs.

Lemma 5.1. *A countable ordered set is directed if and only if it has a cofinal chain.*

Proof. Let $P = \langle P; \le\rangle$ be a countable ordered set. Obviously, if P has a cofinal chain, then it is directed, no matter how large its cardinality is.

Conversely, assume that P is directed and countable. Denoting the least infinite ordinal by ω, there is an ordinal $\kappa \le \omega$ such that $P = \{p_i : i < \kappa\}$. Note that $\{i : i < \kappa\}$ is a subset of $\mathbb{N}_0 = \{0, 1, 2, \dots\}$. For $i, j < \kappa$, there exists a smallest k such that $p_i \le p_k$ and $p_j \le p_k$; we let $p_i * p_j = p_k$. This defines a binary operation $*$ on P. Let $q_0 = p_0$. For $0 < i < \kappa$, let $q_i = q_{i-1} * p_i$. A trivial induction shows that q_i is an upper bound of $\{p_0, p_1, \dots, p_i\}$, for all $i < \kappa$. Hence, $\{q_i : i < \kappa\}$ is a cofinal chain in P. □

Lemma 5.2. *If an ordered set $P = \langle P; \leq\rangle$ is the union of a chain of principal ideals, then it has a cofinal subset C such that $\langle C; \leq\rceil_C\rangle$ is a well-ordered set.*

Proof. The top elements of these principal ideals form a cofinal chain D in P. Let $\mathcal{H}(D) = \{X : X \subseteq D$ and $\langle X; \leq\rceil_X\rangle$ is a well-ordered set$\}$. For $X, Y \in \mathcal{H}(D)$, let $X \sqsubseteq Y$ mean that X is an order ideal of $\langle Y; \leq\rceil_Y\rangle$. Zorn's Lemma yields a maximal member C in $\langle \mathcal{H}(D), \sqsubseteq\rangle$. Clearly, $C = \langle C; \leq\rceil_C\rangle$ is a well-ordered set and it is a cofinal subset in D and also in P. □

Next, we prove a "multi-step" variant of the "1-step" Lemma 4.7. Its proof and a forthcoming part in the proof of Theorem 1.1 need transfinite inductions. Generally, but mostly only implicitly, some sort of uniqueness is desired at inductive definitions; this is easy to achieve in our case by fixing a large well-ordered set and choosing the first unused member of this set whenever we have to add a new element or a new color.

Lemma 5.3. *Assume that $\langle H; \nu\rangle$ and $\langle H^{\blacktriangleright}; \nu^{\blacktriangleright}\rangle$ are quasiordered sets with a unique least element $0_H = 0_{H^{\blacktriangleright}}$ and a unique largest element $1_H = 1_{H^{\blacktriangleright}}$ such that $H^{\blacktriangleright} = H$. If $\nu \subseteq \nu^{\blacktriangleright}$ and $\mathcal{L} = \langle L; \gamma, H, \nu, \delta, \varepsilon, \mathcal{Z}\rangle$ is a strong auxiliary structure such that $\langle a_p, b_p, a_q, b_q\rangle$ is a spanning N_6-quadruple for every pair $\langle p, q\rangle \in \nu^{\blacktriangleright} \setminus \nu$, then $\mathcal{L}$ has a tight extension $\mathcal{L}^{\blacktriangleright} = \langle L^{\blacktriangleright}; \gamma^{\blacktriangleright}, H^{\blacktriangleright}, \nu^{\blacktriangleright}, \delta^{\blacktriangleright}, \varepsilon^{\blacktriangleright}, \mathcal{Z}^{\blacktriangleright}\rangle$ such that $|L^{\blacktriangleright}| \leq \aleph_0 + |L|$. In particular, if $\mathcal{L}$ is countable, then so is $\mathcal{L}^{\blacktriangleright}$.*

Proof. We can assume $\nu \neq \nu^{\blacktriangleright}$ since otherwise $\mathcal{L}^{\blacktriangleright} = \mathcal{L}$ would do. Since every set can be well-ordered, we can write $\nu^{\blacktriangleright} \setminus \nu = \{\langle p_\iota, q_\iota\rangle : \iota < \kappa\}$, where κ is an ordinal number. In $\mathrm{Quord}(H)$, we define $\nu_\lambda = \mathrm{quo}\big(\nu \cup \{\langle p_\iota, q_\iota\rangle : \iota < \lambda\}\big)$ for $\lambda \leq \kappa$. Note that $\nu_0 = \nu$ and $\nu_\kappa = \nu^{\blacktriangleright}$. Let $\mathcal{L}_0 = \mathcal{L}$. For each $0 < \lambda \leq \kappa$, we want to define a strong auxiliary structure $\mathcal{L}_\lambda = \langle L_\lambda; \gamma_\lambda, H_\lambda, \nu_\lambda, \delta_\lambda, \varepsilon_\lambda, \mathcal{Z}_\lambda\rangle$ such that

$$\mathcal{L}_\mu \text{ is a tight substructure of } \mathcal{L}_\lambda \text{ for all } \mu \leq \lambda. \tag{5.1}$$

We define these $\mathcal{L}_\lambda$ by (transfinite) induction as follows. The strong auxiliary structure $\mathcal{L}_0 = \mathcal{L}$ has already been defined.

Successor step. Assume that λ is a successor ordinal, that is, $\lambda = \eta + 1$, and the strong auxiliary structures $\mathcal{L}_\iota = \langle L_\iota; \gamma_\iota, H_\iota, \nu_\iota, \delta_\iota, \varepsilon_\iota, \mathcal{Z}_\iota\rangle$ are already defined for all $\iota \leq \eta$ and (5.1) is satisfied up to η. There are two cases. First, if $p_\eta \leq_{\nu_\eta} q_\eta$, then $\nu_\lambda = \nu_\eta$ and we let $\mathcal{L}_\lambda = \mathcal{L}_\eta$. Second, if $p_\eta \nleq_{\nu_\eta} q_\eta$, then $\langle p_\eta, q_\eta\rangle \in \nu^{\blacktriangleright} \setminus \nu$ and $\langle a_p, b_p, a_q, b_q\rangle$ is a spanning N_6-quadruple by the assumptions of Lemma 5.3. Hence, Lemma 4.7 allows us to let $\mathcal{L}_\lambda = \mathcal{L}^{\triangleright}(p_\eta, q_\eta)$, which is a tight extension of $\mathcal{L}_\eta$. By transitivity and reflexivity, $\mathcal{L}_\mu$ is a tight substructure of $\mathcal{L}_\lambda$ for all $\mu \leq \lambda$. Thus, (5.1) is inherited by $\mathcal{L}_\lambda$ from $\mathcal{L}_\eta$.

Limit step. Assume that λ is a limit ordinal. As one would expect, we let $L_\lambda = \bigcup_{\eta<\lambda} L_\eta$, $\gamma_\lambda = \bigcup_{\eta<\lambda} \gamma_\eta$, $\delta_\lambda = \bigcup_{\eta<\lambda} \delta_\eta = \delta$, $\varepsilon_\lambda = \bigcup_{\eta<\lambda} \varepsilon_\eta = \varepsilon$, and $\mathcal{Z}_\lambda = \bigcup_{\eta<\lambda} \mathcal{Z}_\eta$. Note that $H_\lambda = H^\blacktriangleright = H$ and $\nu_\lambda = \bigcup_{\eta<\lambda} \nu_\eta$. We assert that

$$\mathcal{L}_\lambda = \langle L_\lambda; \gamma_\lambda, H_\lambda, \nu_\lambda, \delta_\lambda, \varepsilon_\lambda, \mathcal{Z}_\lambda\rangle$$

is a strong auxiliary structure satisfying (5.1). Since all the unions defining $\mathcal{L}_\lambda$ are directed unions, L_λ is a lattice and $\langle H_\lambda; \nu_\lambda\rangle$ is a quasiordered set. By the same reason, γ_λ, δ_λ, and ε_λ are maps. (Actually, $\delta_\lambda = \delta_0 = \delta$ and $\varepsilon_\lambda = \varepsilon$.) It is straightforward to check that all of (A1),…,(A13) hold for $\mathcal{L}_\lambda$; we only do this for (A1), that is, we verify (C1) and (C2), and also for (A11).

Assume $\gamma_\lambda(\langle u_1, v_1\rangle) \leq_{\nu_\lambda} \gamma_\lambda(\langle u_2, v_2\rangle)$. Since the unions are directed, there exists an $\eta < \lambda$ such that $u_1, v_1, u_2, v_2 \in L_\nu$ and we have $\gamma_\eta(\langle u_1, v_1\rangle) \leq_{\nu_\eta} \gamma_\eta(\langle u_2, v_2\rangle)$. Using that the auxiliary structure $\mathcal{L}_\eta$ satisfies (C1), we obtain $\mathrm{con}_{L_\eta}(u_1, v_1) \leq \mathrm{con}_{L_\eta}(u_2, v_2)$, that is, $\langle u_1, v_1\rangle \in \mathrm{con}_{L_\eta}(u_2, v_2)$. Using Lemma 4.5, we conclude $\langle u_1, v_1\rangle \in \mathrm{con}_{L_\lambda}(u_2, v_2)$ in the usual way. This implies $\mathrm{con}_{L_\lambda}(u_1, v_1) \leq \mathrm{con}_{L_\lambda}(u_2, v_2)$. Therefore, $\mathcal{L}_\lambda$ satisfies (C1).

Similarly, if $\mathrm{con}_{L_\lambda}(u_1, v_1) \leq \mathrm{con}_{L_\lambda}(u_2, v_2)$, then Lemma 4.5 easily implies the existence of an $\eta < \lambda$ such that $\langle u_1, v_1\rangle \in \mathrm{con}_{L_\eta}(u_2, v_2)$, that is, $\mathrm{con}_{L_\eta}(u_1, v_1) \leq \mathrm{con}_{L_\eta}(u_2, v_2)$. Thus, (C2) for $\mathcal{L}_\eta$ yields $\gamma_\eta(\langle u_1, v_1\rangle) \leq_{\nu_\eta} \gamma_\eta(\langle u_2, v_2\rangle)$ and we conclude $\gamma_\lambda(\langle u_1, v_1\rangle) \leq_{\nu_\lambda} \gamma_\lambda(\langle u_2, v_2\rangle)$. Hence, $\mathcal{L}_\lambda$ satisfies (C2) and (A1).

Next, for the sake of contradiction, suppose that (A11) fails in $\mathcal{L}_\lambda$. This implies that $\langle 0_{L_\lambda}, 1_{L_\lambda}\rangle$ belongs to $\bigvee\{\mathrm{con}_{L_\lambda}(a_p, b_p) : p \in H_\lambda^{-01}\}$, where the join is taken in the congruence lattice of L_λ. Since principal congruences are compact, there exists a finite subset $T \subseteq H_\lambda^{-01}$ such that $\langle 0_{L_\lambda}, 1_{L_\lambda}\rangle$ belongs to $\bigvee\{\mathrm{con}_{L_\lambda}(a_p, b_p) : p \in T\}$. Thus, there exists a finite sequence $0_{L_\lambda} = c_0, c_1, \dots, c_k = 1_{L_\lambda}$ of elements of L_λ such that, for $i = 1, \dots, k$, $\langle c_{i-1}, c_i\rangle \in \bigcup\{\mathrm{con}_{L_\lambda}(a_p, b_p) : p \in T\}$. Each of these memberships are witnessed by finitely many "witness" elements according to (4.14); see Lemma 4.5. Taking all these memberships into account, there are only finitely many witness elements all together. Hence, there exists an $\eta < \lambda$ such that L_η contains all these elements. Applying Lemma 4.5 in the converse direction, we obtain that $\langle 0_{L_\eta}, 1_{L_\eta}\rangle = \langle 0_{L_\lambda}, 1_{L_\lambda}\rangle$ belongs to $\bigvee\{\mathrm{con}_{L_\eta}(a_p, b_p) : p \in T\}$. This is a contradiction, because $\mathcal{L}_\eta$ satisfies (A11). Thus, $\mathcal{L}_\lambda$ is a strong auxiliary structure. The satisfaction of (5.1) for $\mathcal{L}_\lambda$ is evident.

We have seen that $\mathcal{L}_\nu$ is an auxiliary structure for all $\lambda \leq \kappa$ such that (5.1) holds. Letting λ equal κ and taking (2.4), (5.1), $\mathcal{L} = \mathcal{L}_0$ and $\langle H^\blacktriangleright; \nu^\blacktriangleright\rangle = \langle H_\kappa; \nu_\kappa\rangle$ into account, we obtain the existence part of the lemma. Finally, $|L^\blacktriangleright| \leq \aleph_0 + |L|$ and the last sentence of the lemma follow from the construction and basic cardinal arithmetics. $\square$

A non-empty subset X of a quasiordered set $\langle Y; \leq_Y\rangle$ is called a (*quasiorder*) *ideal* if for all $x \in X$ and $y \in Y$, $y \leq_Y x$ implies $y \in X$.

Lemma 5.4. *Let H be a quasiorder ideal of a quasiordered set $\langle H^{\blacktriangle}; \nu^{\blacktriangle}\rangle$. If $\langle H^{\blacktriangle}; \nu^{\blacktriangle}\rangle$ has a unique smallest element $0_{H^{\blacktriangle}}$ and a unique largest element $1_{H^{\blacktriangle}}$, then each auxiliary structure $\mathcal{L} = \langle L; \gamma, H, \nu^{\blacktriangle}\rceil_H, \delta, \varepsilon, \mathcal{Z}\rangle$ has an extension $\mathcal{L}^{\blacktriangle} = \langle L^{\blacktriangle}; \gamma^{\blacktriangle}, H^{\blacktriangle}, \nu^{\blacktriangle}, \delta^{\blacktriangle}, \varepsilon^{\blacktriangle}, \mathcal{Z}^{\blacktriangle}\rangle$ such that $|L^{\blacktriangle}| \leq |L| + |H^{\blacktriangle}| + \aleph_0$ and, furthermore, if $H \neq H^{\blacktriangle}$, then $\mathcal{L}^{\blacktriangle}$ is a strong auxiliary structure.*

Proof. We can assume that $H \neq H^{\blacktriangle}$, since otherwise we can let $\mathcal{L}^{\blacktriangle} = \mathcal{L}$. Let $\nu = \nu^{\blacktriangle}\rceil_H$. With $K = H^{\blacktriangle} \setminus (H \cup \{1_{H^{\blacktriangle}}\})$, let $\mathcal{L}^{\vartriangle} = \langle L^{\vartriangle}; \gamma^{\vartriangle}, H^{\vartriangle}, \nu^{\vartriangle}, \delta^{\vartriangle}, \varepsilon^{\vartriangle}, \mathcal{Z}^{\vartriangle}\rangle$ be the vertical extension of $\mathcal{L}$ from Lemma 3.1. It is a strong auxiliary structure. Clearly, $\nu^{\vartriangle} \subseteq \nu^{\blacktriangle}$. If we had a pair $\langle p, q\rangle \in \nu^{\blacktriangle} \setminus \nu^{\vartriangle}$ such that $p, q \in H$, then $\langle p, q\rangle \in \nu^{\blacktriangle}\rceil_H = \nu \subseteq \nu^{\vartriangle}$ would be a contradiction. This implies easily that, for every pair $\langle p, q\rangle \in \nu^{\blacktriangle} \setminus \nu^{\vartriangle}$, $\langle a_p, b_p, a_q, b_q\rangle$ is a spanning N_6-quadruple. Therefore, with $\langle H^{\vartriangle}, \nu^{\vartriangle}, H^{\blacktriangle}, \nu^{\blacktriangle}\rangle$ playing the role of $\langle H, \nu, H^{\blacktriangleright}, \nu^{\blacktriangleright}\rangle$, Lemma 5.3 yields a tight extension $\mathcal{L}^{\blacktriangle} := \mathcal{L}^{\blacktriangleright}$ of $\mathcal{L}^{\vartriangle}$. We have that

$$|L^{\blacktriangle}| \leq |L^{\vartriangle}| + \aleph_0 \leq |L| + |K| + \aleph_0 \leq |L| + |H^{\blacktriangle}| + \aleph_0$$

by the construction and basic cardinal arithmetics, and $\mathcal{L}^{\blacktriangle}$ is an extension of $\mathcal{L}$ by transitivity. Applying (2.4) to $\mathcal{L}^{\vartriangle}$ and $\mathcal{L}^{\blacktriangle}$, we conclude Lemma 5.4. ☐

We are now in the position to complete the proof of our theorem.

Proof of Theorem 1.1. In order to prove part (ii) of the theorem, assume that $P = \langle P; \nu_P\rangle$ is an ordered set with zero, and it is the union of a chain of principal ideals. We also assume that $|P| \geq 3$, since otherwise we can let $L = P$. By Lemma 5.2, there exist an ordinal number κ and a cofinal chain $C = \{c_\iota : \iota < \kappa\}$ in P such that $|{\downarrow} c_0| \geq 3$ and, for $\iota, \mu < \kappa$, we have $\iota < \mu \iff c_\iota < c_\mu$. The cofinality of C means that P is the union of the principal ideals ${\downarrow} c_\iota$, $\iota < \kappa$. For $1 \leq \lambda \leq \kappa$, let $H_\lambda = \bigcup_{\iota<\lambda} {\downarrow} c_\iota$, and let ν_λ be the restriction $\nu_P\rceil_{H_\lambda}$ of ν_P to H_λ. Note that H_λ is always an order ideal of P, but it is not a principal ideal in general. Our aim is to define, for each $1 \leq \lambda \leq \kappa$, an auxiliary structure

$$\mathcal{L}_\lambda = \langle L_\lambda; \gamma_\lambda, H_\lambda, \nu_\lambda, \delta_\lambda, \varepsilon_\lambda, \mathcal{Z}_\lambda\rangle$$

such that

$$\mathcal{L}_\mu \text{ is a substructure of } \mathcal{L}_\lambda \text{ for every } \mu \text{ with } 1 \leq \mu \leq \lambda. \tag{5.2}$$

To define $\mathcal{L}_1$, let $H' = {\downarrow} c_0 = H_1$ and $\nu' = (\{0_P\} \times H') \cup (H' \times \{c_0\})$. We define a strong auxiliary structure $\mathcal{L}' = \langle L'; \gamma', H', \nu', \delta', \varepsilon', \mathcal{Z}'\rangle$ from $\langle H'; \nu'\rangle$ exactly the same way as we defined $\mathcal{L}$ from $\langle H; \nu\rangle$ in Example 2.2, see also Figure 5. Note that $\mathcal{L}'$ is a strong auxiliary structure, all of its N_6-quadruples are spanning N_6-quadruples, and $\nu' \subseteq \nu_1$. This allows us to let $\mathcal{L}_1$ be the tight extension $(\mathcal{L}')^{\blacktriangle}$ of $\mathcal{L}'$ given by Lemma 5.3.

If $\lambda = \eta + 1$ is a successor ordinal, then c_η is the greatest element of H_λ and H_η is an order ideal of H_λ. Thus, we can let $\mathcal{L}_\lambda = (\mathcal{L}_\eta)^{\blacktriangle}$ by Lemma 5.4, and (5.2) follows from this lemma by transitivity. If λ is a limit ordinal, then we define $\mathcal{L}_\lambda$ almost as the (directed) union of the $\mathcal{L}_\eta$, $1 \leq \eta < \lambda$, in the following way. We let $\gamma_\lambda = \bigcup_{1\leq\eta<\lambda}\gamma_\eta$, $\delta_\lambda = \bigcup_{1\leq\eta<\lambda}\delta_\eta$, $\varepsilon_\lambda = \bigcup_{1\leq\eta<\lambda}\varepsilon_\eta$; we already know that $H_\lambda = \bigcup_{1\leq\eta<\lambda} H_\eta$ and $\nu_\lambda = \bigcup_{1\leq\eta<\lambda}\nu_\eta$; however, we let $\mathcal{Z}_\lambda = \varnothing$. The fact that $\mathcal{L}_\lambda$ is an auxiliary structure and the validity of (5.2) follow by a straightforward argument similar to the one we used in the limit step of the proof of Lemma 5.3.

We have defined $\mathcal{L}_\lambda$ for all $\lambda \leq \kappa$ such that (5.2) holds. Hence, in particular,

$$\mathcal{L}_\kappa = \langle L_\kappa; \gamma_\kappa, H_\kappa, \nu_\kappa, \delta_\kappa, \varepsilon_\kappa, \mathcal{Z}_\kappa\rangle = \langle L_\kappa; \gamma_\kappa, P, \nu_P, \delta_\kappa, \varepsilon_\kappa, \mathcal{Z}_\kappa\rangle$$

is an auxiliary structure. Thus, letting $L = L_\kappa$, Lemma 2.1 implies that $\mathrm{Princ}(L)$ is isomorphic to $\langle P; \nu_P\rangle$. This proves part (ii) of Theorem 1.1, since $|L| \leq |P| + \aleph_0$ follows from the construction by basic cardinal arithmetics.

In order to prove part (i), assume that L is a countable lattice. Obviously, we have $|\mathrm{Princ}(L)| \leq |\mathrm{Pairs}^{\leq}(L)| \leq \aleph_0$, and we mentioned at (1.1) that $\mathrm{Princ}(L)$ is always a directed ordered set with 0, no matter what the size $|L|$ of L is. The converse follows from part (ii) and Lemma 5.1. $\square$

Remark 5.5. Clearly, if $P = \langle P; \leq\rangle$ is a *bounded* ordered set (resp., a *finite* bounded ordered set), then we can choose a singleton cofinal chain, and our construction yields a bounded lattice (resp., a finite lattice) L with the property $\mathrm{Princ}(L) \cong P$. In this way, we obtain a new proof for the main result of Grätzer [7], except that while [7] constructs a lattice of length 5, our L is of larger (or even infinite) length in general.

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