THE NUMBER OF RECTANGULAR ISLANDS BY MEANS OF DISTRIBUTIVE LATTICES

GÁBOR CZÉDLI

Dedicated to László Megyesi on his seventieth birthday (in 2009)

Abstract. If $A = (a_{ij})_{m\times n}$ is an $m \times n$ matrix of real numbers and $\alpha, \beta, \gamma, \delta$ are integers with $1 \le \alpha \le \beta \le m$ and $1 \le \gamma \le \delta \le n$ then the elements a_{ij} with $\alpha \le i \le \beta$ and $\gamma \le j \le \delta$ form a submatrix R which we call a *rectangle* of A. Let r be the least element (or one of the least elements) of R. If for every element a_{ij} of A which is neighbouring with R we have $a_{ij} < r$ then R is called a *rectangular island* of A. More precisely, R is called a rectangular island if whenever $(i, j) \in \big(\{1, \ldots, m\} \times \{1, \ldots, n\}\big) \setminus \big(\{\alpha, \ldots, \beta\} \times \{\gamma, \ldots, \delta\}\big)$, $(k, \ell) \in \{\alpha, \ldots, \beta\} \times \{\gamma, \ldots, \delta\}$, $|i - k| \le 1$ and $|j - \ell| \le 1$ then $a_{ij} < r$.

The first aim of the present paper is to determine the maximum of the number of rectangular islands of $m \times n$ matrices, for any fixed pair (m, n) of positive integers. The second aim is to point out that a purely lattice theoretic result on weak bases of distributive lattices in [1] is useful in combinatorics.

1. Motivations and definitions

For $n \in \mathbf{N}$ let $\mathbf{n} = [1, n] = \{1, \ldots, n\}$. For $m, n \in \mathbf{N}$ the set $\mathbf{m} \times \mathbf{n}$ will be called a *table* of size $m \times n$. In our figure and arguments we will consider $\mathbf{m} \times \mathbf{n}$ as a collection of *cells*; (i, j) will mean the j-th cell in the i-th row of cells. If (i, j) and (k, ℓ) are two cells of the table then their distance is $\sqrt{(i - k)^2 + (j - \ell)^2}$, the usual distance of their center points. Two cells with distance at most $\sqrt{2}$ are called *neighbouring cells*. For $1 \le \alpha \le \beta \le m$ and $1 \le \gamma \le \delta \le n$ the set $R = [\alpha, \beta] \times [\gamma, \delta] = \{\alpha, \ldots, \beta\} \times \{\gamma, \ldots, \delta\}$ is called a *rectangle* of $\mathbf{m} \times \mathbf{n}$. When $\alpha = \beta$ and $\gamma = \delta$ then R is called a *unit square*. So a unit square is a singleton consisting of a cell. Given a rectangle R, we will use the notations $\alpha(R)$, $\beta(R)$, ... to express that $R = [\alpha(R), \beta(R)] \times [\gamma(R), \delta(R)]$. Notice that rectangles are never empty and sometimes they will be treated as other tables.

By a (real) matrix of size $m \times n$ we mean a mapping $A\colon \mathbf{m} \times \mathbf{n} \to \mathbf{R}$, $(i, j) \mapsto a_{ij}$. Given A, for a rectangle R of the table $\mathbf{m} \times \mathbf{n}$ let $\min(A|_R)$ denote the minimum of $\{a_{ij} \colon (i, j) \in R\}$. We say that R is a *rectangular island* of the matrix A if $a_{ij} < \min(A|_R)$ holds for each $(i, j) \in (\mathbf{m} \times \mathbf{n}) \setminus R$ such that (i, j) is neighbouring with some cell of R. The set of rectangular islands of A will be denoted by $\mathcal{I}_{\mathrm{rect}}(A)$.

The primary motivation of the present paper comes from a recent result by Földes and Singhi [3] where "full segments" of vectors, which are just rectangular islands of $1 \times n$ tables in our terminology, are considered. According to Thm. 4 of

Date: June 11, 2007; revised: January 30, 2008.

Key words and phrases. Lattice, distributive lattice, weakly independent subset, weak basis, rectangular island, number of islands.

This research was partially supported by the NFSR of Hungary (OTKA), grant no. T 049433 and K 60148.

[3], $1 \times n$ tables A whose entries form the lexicographic length sequence of some binary maximal instantaneous code are characterized by $|\mathcal{I}_{\mathrm{rect}}(A)|$ many equations. This makes the maximum of $\{|\mathcal{I}_{\mathrm{rect}}(A)| \colon A \text{ is an } 1 \times n \text{ table}\}$ important in coding theory, and this is why determining

$$f(m,n) = \max\{|\mathcal{I}_{\mathrm{rect}}(A)| \colon A \text{ is an } m \times n \text{ matrix}\}$$

seems to be interesting.

The particular case of determining $f(1,n)$ has already been solved by Härtel [4]. It will be clear soon that determining $f(m,n)$ is much harder than determining $f(1,n)$. Perhaps the two dimensional problem, i.e. determining $f(m,n)$, may look rather easy at the first sight, and one might expect to generalize it to the analogous problem in higher dimensions easily. However, at the end of the paper we will point out where the present approach fails in the three dimensional space, and now the reader is invited to guess $f(100,100)$ before further reading.

Beside giving further motivations, the following example explains our terminology. Let us consider a rectangular lake whose bottom is divided into $(m+2) \times (n+2)$ cells. In other words, we identify the bottom of the lake with the table $\{0,1,\dots,m+1\} \times \{0,1,\dots,n+1\}$. The height of the bottom (above see level) is constant on each cell but definitely less than the height of the lake shore. Let a_{ij} be the height of the cell (i,j) for $(i,j) \in \mathbf{m} \times \mathbf{n}$ and let $\min(A|_{\mathbf{m}\times\mathbf{n}}) - 1$ be the height of (i,j) otherwise. Now a rectangle R of the table $\mathbf{m} \times \mathbf{n}$ is a rectangular island in our sense iff there is a possible water level such that R is an island of the lake in the usual sense. There are other examples requiring only $m \times n$ cells; for example, a_{ij} may mean a colour on a gray-scale (before we convert the picture to black and white), transparency (against X-rays), or melting temperature.

Only a very minimal knowledge of lattice theory is assumed: the notion of a distributive lattice, cf., e.g., Grätzer [5] or any textbook on universal algebra or lattice theory. Let $L = (L; \vee, \wedge)$ be a finite distributive lattice. Following [1], a subset H of L is called *weakly independent* if for any $k \in \mathbf{N}$ and $h, h_1, \dots, h_k \in H$ which satisfy $h \le h_1 \vee \dots \vee h_k$ there exists an $i \in \{1,\dots,k\}$ such that $h \le h_i$. Maximal weakly independent subsets are called *weak bases* of L. It is pointed out in [1] that the set $J_0(L)$ of join-irreducible elements and all maximal chains are weak bases of L. The main theorem of [1] asserts that

Lemma 1. *Any two weak bases of a finite distributive lattice have the same number of elements.*

Although this statement initiated some further research like [2], Lengvárszky [7] and some others mentioned in [7], all this happened within lattice theory. This is probably the first time when Lemma 1 is applied in another branch of mathematics.

2. Auxiliary statements

Let R and S be rectangles of $\mathbf{m} \times \mathbf{n}$. We say that R and S are *far from each other* if they are disjoint and no cell of R is neighbouring with some cell of S. In other words, if the distance of any cell in R from any cell in S is at least two. The set of rectangles of $\mathbf{m} \times \mathbf{n}$ will be denoted by $\mathcal{R}(\mathbf{m} \times \mathbf{n})$. Notice that, by definition, the empty set does not belong to $\mathcal{R}(\mathbf{m} \times \mathbf{n})$. Since the notion of rectangular islands does not make it comfortable to work with these islands, the following easy lemma will be important.

Lemma 2. *Let $\mathcal{H}$ be a subset of $\mathcal{R}(\mathbf{m}\times\mathbf{n})$. Then the following two conditions are equivalent.*

(i) There exists a matrix $A\colon \mathbf{m}\times\mathbf{n}\to\mathbf{R}$ such that $\mathcal{H}=\mathcal{I}_{rect}(A)$;

(ii) $\mathbf{m}\times\mathbf{n}\in\mathcal{H}$, and for any $R,S\in\mathcal{H}$ either $R\subseteq S$ or $S\subseteq R$ or R and S are far from each other.

In what follows, subsets $\mathcal{H}$ of $\mathcal{R}(\mathbf{m}\times\mathbf{n})$ satisfying the (equivalent) conditions of Lemma 2 will be called *systems of rectangular islands*. We will of course work with (ii) rather than (i). Our task is to determine the maximum of $|\mathcal{H}|$ for these systems.

Proof. Suppose $\mathcal{H}=\mathcal{I}_{\mathrm{rect}}(A)$. Then $\mathbf{m}\times\mathbf{n}\in\mathcal{H}$ is evident. If the rest of (ii) fails then one can easily find a cell $(i,j)\in R\setminus S$ which is neighbouring with some cell of S. Since S is an island, we have $a_{ij}<\min(A|_S)$. Similarly, there is a cell $(k,\ell)\in S\setminus R$ which is neighbouring with some cell of R and so $a_{k\ell}<\min(A|_R)$. Hence $a_{ij}<\min(A|_S)\le a_{k\ell}<\min(A|_R)$ contradicts $(i,j)\in R$. This proves (i) $\Rightarrow$ (ii).

The converse implication will be proved via induction on mn. For $mn=1$ or $|\mathcal{H}|=1$ everything is clear. Suppose $mn>1$, $|\mathcal{H}|>1$ and (ii) holds for $\mathcal{H}$. Let $R_1,\dots,R_k$ be the maximal elements of $\mathcal{H}\setminus\{\mathbf{m}\times\mathbf{n}\}$. Clearly, $\mathcal{H}_i=\{S\in\mathcal{H}\colon S\subseteq R_i\}$ satisfies (ii) for the table R_i, $1\le i\le k$. Hence, by the induction hypothesis, there is a matrix $A_i\colon R_i\to\mathbf{R}$ such that $\mathcal{I}_{\mathrm{rect}}(A_i)=\mathcal{H}_i$ for each $i\in\{1,\dots,k\}$. Now choose an $r\in\mathbf{R}$ such that r is strictly less than the minimum of the elements of A_i for all $i\in\{1,\dots,k\}$. Then the union of the A_i, $1\le i\le k$, and the constant mapping $\mathbf{m}\times\mathbf{n}\setminus(R_1\cup\dots\cup R_k)\to\{r\}$ is an $\mathbf{m}\times\mathbf{n}\to\mathbf{R}$ matrix A. Since the R_i are pairwise far from each other, we conclude $\mathcal{H}\subseteq\mathcal{I}_{\mathrm{rect}}(A)$. Then using the (i) $\Rightarrow$ (ii) direction we obtain $\mathcal{H}=\mathcal{I}_{\mathrm{rect}}(A)$. $\square$

Let $\max(\mathcal{H})$ denote the set of maximal elements of $\mathcal{H}\setminus\{\mathbf{m}\times\mathbf{n}\}$ with respect to set inclusion. Since the one element system is not a problem, in what follows we usually assume that the system $\mathcal{H}$ of rectangular islands is not a singleton. Notice that $\max(\mathcal{H})=\emptyset$ iff $|\mathcal{H}|=1$. For $R\in\mathcal{H}$, let $\mathcal{H}|_R$ denote $\{S\in\mathcal{H}\colon S\subseteq R\}$. Then $\mathcal{H}|_R$ is clearly a system of rectangular islands of the table R. Since the elements of $\max(\mathcal{H})$ are far from each other, we have

$$\mathcal{H}=\{\mathbf{m}\times\mathbf{n}\}\cup\dot{\bigcup}_{R\in\max(\mathcal{H})}\mathcal{H}|_R\ ,$$

where the dot in the formula indicates that the $\mathcal{H}|_R$, $R\in\max(\mathcal{H})$, are pairwise disjoint. The lattice of all subsets of $\mathbf{m}\times\mathbf{n}$ will be denoted by

$$\mathcal{P}(\mathbf{m}\times\mathbf{n})=\big(\mathcal{P}(\mathbf{m}\times\mathbf{n});\cup,\cap\big).$$

It is a finite distributive lattice. Notice that $\big(\mathcal{R}(\mathbf{m}\times\mathbf{n})\cup\{\emptyset\};\subseteq\big)$ is also a lattice but it is not distributive in general.

Lemma 3. *Let $\mathcal{H}$ be a system of rectangular islands of $\mathbf{m}\times\mathbf{n}$. Then $\mathcal{H}$ is a weakly independent subset of $\mathcal{P}(\mathbf{m}\times\mathbf{n})$. Consequently, $|\mathcal{H}|\le mn$.*

Proof. Suppose

$$R\subseteq R_1\cup\dots\cup R_k \tag{1}$$

where $R,R_1,\dots,R_k\in\mathcal{H}$. We can assume that (1) is irredundant in the sense that there is no $i\in\{1,\dots,k\}$ with $R\subseteq R_1\cup\dots\cup R_{i-1}\cup R_{i+1}\cup\dots\cup R_k$. Then no R_i is disjoint from R, and the R_i are pairwise incomparable. If $R\subseteq R_i$ for some

i then we are done. In the opposite case $k \geq 2$ and $R_i \subset R$ for all $i \in \{1, \ldots, k\}$. Then there is a cell c of R which is neighbouring with R_1. (Indeed, take a cell c_0 in $R \setminus R_1$ and a cell $c_1 \in R_1$, and walk from c_1 to c_0 within R, stepping from cell to neighbouring cell.) Since the R_j for $1 < j$ are far from R_1, c does not belong to $R_1 \cup \cdots \cup R_k$. This contradicts (1).

Finally, to derive the last sentence of the lemma, extend $\mathcal{H}$ to a weak basis $\mathcal{H}'$ of $\mathcal{P}(\mathbf{m}\times\mathbf{n})$, and consider a maximal chain $\mathcal{C}$ in $\mathcal{P}(\mathbf{m}\times\mathbf{n})$. Then $\mathcal{C}$ has $mn+1$ elements and it is also a weak basis. Hence we conclude from Lemma 1 that $|\mathcal{H}'| = mn + 1$. On the other hand, the empty set belongs to every weak basis but not to $\mathcal{H}$, so $|\mathcal{H}| \leq |\mathcal{H}'| - 1 = mn$. $\square$

Notice that Lemma 3 together with the first two sentences in the proof of Lemma 6 clearly imply the result of Härtel [4] on $f(1, n)$.

The proof of Lemma 3 does not pay any attention how we extend $\mathcal{H}$ to a weak basis. However, we will need a particular extension in what follows. Let $\mathcal{H}$ be a system of rectangular islands of $\mathbf{m} \times \mathbf{n}$, and let $\mathcal{C}$ be a set of unit squares of $\mathbf{m} \times \mathbf{n}$. We say that $\mathcal{C}$ is a *companion set of* $\mathcal{H}$ if $\mathcal{C} \cap \mathcal{H} = \emptyset$ and $\mathcal{H} \cup \mathcal{C} \cup \{\emptyset\}$ is a weak basis of $\mathcal{P}(\mathbf{m} \times \mathbf{n})$. Notice that in this case $|\mathcal{H} \cup \mathcal{C}| = mn$ by Lemma 1. A cell $(i, j) \in \mathbf{m} \times \mathbf{n}$ is called an *outer cell of* $\mathcal{H}$ if (i, j) is not in $\bigcup_{R \in \max(\mathcal{H})} R$. In particular, if $|\mathcal{H}| = 1$ then all cells of $\mathbf{m} \times \mathbf{n}$ are outer cells of $\mathcal{H}$. The set of outer cells of $\mathcal{H}$ will be denoted by $\operatorname{out}(\mathcal{H})$. Notice that $\operatorname{out}(\mathcal{H})$ is never empty, for the members of $\max(\mathcal{H})$ are pairwise far from each other.

To avoid syntactical errors in subsequent formulas we often have to convert cells to unit squares and vice versa. For a cell $c = (i, j)$ let $c^+ = \{c\}$. For a set S of cells let $S^+ = \{c^+ \colon c \in S\}$. (If the reader prefers not to make a notational distinction between cells and unit squares, then he can simply disregard $^+$ in what follows; probably this is the best strategy at first reading.)

Lemma 4. *Let $\mathcal{H}$ be a system of rectangular islands of $\mathbf{m} \times \mathbf{n}$. Then $\mathcal{H}$ has a companion set $\mathcal{C}$. Moreover, for each companion set $\mathcal{C}$ of $\mathcal{H}$ there is a unique outer cell c of $\mathcal{H}$ such that c^+ does not belong to $\mathcal{C}$.*

Proof. We prove the lemma via induction on $|\mathcal{H}|$. For $|\mathcal{H}| = 1$ we can choose any cell $c = (i, j)$, and let $\mathcal{C} = \big((\mathbf{m} \times \mathbf{n}) \setminus \{c\}\big)^+$. Then Lemma 1 and $|\mathcal{H} \cup \mathcal{C} \cup \{\emptyset\}| = mn + 1$ easily give that $\mathcal{H} \cup \mathcal{C} \cup \{\emptyset\}$ is a weak basis, whence $\mathcal{C}$ is a companion set of $\mathcal{H}$.

Suppose now that $|\mathcal{H}| > 1$, $\max(\mathcal{H}) = \{R_1, \ldots, R_k\}$ and, for $1 \leq i \leq k$, $\mathcal{C}_i$ is a companion set of $\mathcal{H}|_{R_i}$ in the table R_i. The induction hypothesis gives $\Big|\mathcal{H}|_{R_i} \cup \mathcal{C}_i\Big| = |R_i|$. Fix a cell $c \in \operatorname{out}(\mathcal{H})$ arbitrarily. We claim that

$$\mathcal{C} = \mathcal{C}_1 \cup \cdots \cup \mathcal{C}_k \cup \operatorname{out}(\mathcal{H})^+ \setminus \{c^+\} \text{ is a companion set of } \mathcal{H}. \tag{2}$$

It is straightforward to see that $\mathcal{H} \cup \mathcal{C} \cup \{\emptyset\}$ is weakly independent. Since

$$|\mathcal{H} \cup \mathcal{C} \cup \{\emptyset\}| = \Big|\dot{\bigcup}_{1 \leq i \leq k} (\mathcal{H}|_{R_i} \cup \mathcal{C}_i) \dot{\cup} \big(\operatorname{out}(\mathcal{H})^+ \setminus \{c^+\}\big) \dot{\cup} \{\mathbf{m} \times \mathbf{n}, \emptyset\}\Big| =$$
$$1 + |\operatorname{out}(\mathcal{H})^+| + \sum_{1 \leq i \leq k} \Big|(\mathcal{H}|_{R_i} \cup \mathcal{C}_i)\Big| = 1 + |\operatorname{out}(\mathcal{H})| + \sum_{1 \leq i \leq k} |R_i| =$$
$$1 + \Big|\operatorname{out}(\mathcal{H}) \,\dot{\cup}\, \dot{\bigcup}_{1 \leq i \leq k} R_i\Big| = 1 + |\mathbf{m} \times \mathbf{n}| = 1 + mn,$$

$\mathcal{H} \cup \mathcal{C} \cup \{\emptyset\}$ is a weak basis by Lemma 1. This proves (2).

Finally, to show the last sentence of the lemma, let $\mathcal{C}$ be an arbitrary companion set of $\mathcal{H}$. We can assume that $(m,n) \neq (1,1)$, for otherwise $\mathcal{C}$ is necessarily empty and the unique unit square of the table does the job. If $\mathrm{out}(\mathcal{H})^+ \subseteq \mathcal{C}$ then $\mathbf{m} \times \mathbf{n} \subseteq \bigcup_{c \in \mathrm{out}(\mathcal{H})} c^+ \cup \bigcup_{R \in \max(\mathcal{H})} R$ would contradict the weak independence of $\mathcal{H} \cup \mathcal{C}$. If $c, d \in \mathrm{out}(\mathcal{H})^+ \setminus \mathcal{C}$ with $c \neq d$ then $\mathcal{H} \cup \mathcal{C} \cup \{\emptyset, d\}$ would still be weakly independent, contradicting the maximality of $\mathcal{H} \cup \mathcal{C} \cup \{\emptyset\}$. □

Given a system $\mathcal{H}$ of rectangular islands, a set $\mathcal{D}$ of unit squares is called an *extended companion set* of $\mathcal{H}$ if $\mathrm{out}(\mathcal{H})^+ \subseteq \mathcal{D}$ and there is a unit square $d \in \mathrm{out}(\mathcal{H})^+$ such that $\mathcal{D} \setminus \{d\}$ is a companion set of $\mathcal{H}$. It is clear from Lemma 4 that $\mathcal{H}$ has an extended companion set. Moreover, if $\mathcal{D}$ is an extended companion set then for each $d \in \mathrm{out}(\mathcal{H})^+$, $\mathcal{D} \setminus \{d\}$ is a companion set of $\mathcal{H}$; this comes from the fact that c right before (2) was chosen arbitrarily. Now Lemma 1 implies

Lemma 5. *If $\mathcal{D}$ is an extended companion set and $\mathcal{C}$ is a companion set of the system $\mathcal{H}$ of rectangular islands then*

$$|\mathcal{H}| + |\mathcal{C}| = |\mathcal{H}| + |\mathcal{D}| - 1 = mn\,.$$

3. The main result and the rest of its proof

Given a real number x, $\lfloor x \rfloor$ resp. $\lceil x \rceil$ will denote the greatest resp. least integer such that $\lfloor x \rfloor \le x$ resp. $x \le \lceil x \rceil$. The usual calculation rules $\lfloor x \rfloor + \lfloor y \rfloor \le \lfloor x+y \rfloor$ and $\lceil x \rceil + \lceil y \rceil \ge \lceil x+y \rceil$ will be used without further notice.

Theorem 1. *Given a table of size $m \times n$, the maximum number of rectangular islands, i.e.* $\max\{|\mathcal{I}_{rect}(A)| \colon A$ *is an $m \times n$ matrix*$\}$ $= \max\{|\mathcal{H}| \colon \mathcal{H}$ *is a system of rectangular islands of* $\mathbf{m} \times \mathbf{n}\}$, *is*

$$f(m,n) = \lfloor (mn + m + n - 1)/2 \rfloor.$$

As a particular case, we obtain Härtel's result which, in our terminology, says that $f(1,n) = n$. The proof of the theorem consists of the following two lemmas.

Lemma 6. $f(m,n) \ge \lfloor (mn + m + n - 1)/2 \rfloor$.

Proof. When $m = 1$ then $\mathcal{H} = \big\{\{1\} \times [1,i] \colon i \in \mathbf{n}\big\}$ is a system of rectangular islands. This shows that $f(1,n) \ge |\mathcal{H}| = n = \lfloor (1 \cdot n + 1 + n - 1)/2 \rfloor$. By the commutativity of f the lemma holds for the case $1 \in \{m,n\}$, and it clearly holds for $m = n = 2$.

Now we show that

$$f(m_1 + 1 + m_2, n) \ge f(m_1, n) + 1 + f(m_2, n). \tag{3}$$

Indeed, in the table $T = [1, m_1 + 1 + m_2] \times \mathbf{n}$, let $R_1 = [1, m_1] \times \mathbf{n}$ and $R_2 = [m_1 + 2, m_1 + 1 + m_2] \times \mathbf{n}$. Then, for $i \in \{1,2\}$, there is a system $\mathcal{H}_i$ of rectangular islands in the subtable R_i with $|\mathcal{H}_i| = f(m_i, n)$. Clearly, $\mathcal{H} = \mathcal{H}_1 \cup \{T\} \cup \mathcal{H}_2$ is a system of rectangular islands of T and $|\mathcal{H}| = f(m_1,n) + 1 + f(m_2,n)$ shows (3).

Finally, we obtain the lemma from (3) via induction on mn as follows:

$$\begin{gathered} f(n, m+2)) = f(m+2, n) \ge f(m,n) + 1 + f(1,n) \ge \\ \lfloor (mn + m + n - 1)/2 \rfloor + 1 + n = \\ \lfloor (mn + m + n - 1)/2 + 1 + n \rfloor = \big\lfloor \big((m+2)n + (m+2) + n - 1\big)/2 \big\rfloor. \end{gathered}$$

□

Now, for a system $\mathcal{H}$ of rectangular islands of $\mathbf{m} \times \mathbf{n}$, we define the *deficiency* of $\mathcal{H}$ as $d(\mathcal{H}) = mn - |\mathcal{H}|$. Notice that Lemma 5 yields

$$d(\mathcal{H}) = |\mathcal{C}| = |\mathcal{D}| - 1 \tag{4}$$

for any companion set $\mathcal{C}$ and any extended companion set $\mathcal{D}$ of $\mathcal{H}$. Since $mn - \lceil (mn + 1 - m - n)/2 \rceil = \lfloor (mn + m + n - 1)/2 \rfloor$, the following lemma will complete the proof of Theorem 1.

Lemma 7. *For any system $\mathcal{H}$ of rectangular islands of $\mathbf{m} \times \mathbf{n}$,*

$$d(\mathcal{H}) \geq \lceil (mn + 1 - m - n)/2 \rceil.$$

Proof. We prove Lemma 7 via induction on mn. For $1 \in \{m, n\}$ Lemma 7 follows from Lemma 3, and for $m = n = 2$ it follows trivially via inspecting all cases. (Up to symmetry, there are only three possibilities for $\max(\mathcal{H})$.)

So we assume that $m \geq 2$, $n \geq 2$, $mn \geq 5$ and Lemma 7 is valid whenever the "area" of the table is less than mn. As the previous sentence indicates, subsets (mainly rectangles or the whole table) of $\mathbf{m} \times \mathbf{n}$ will often be treated as subsets of the Euclidian plane in the obvious way. This will cause no confusion; when we speak of the number $|S|$ of elements of some set S then S is understood as a subset of $\mathbf{m} \times \mathbf{n}$, and when we mention the area $\mu(S)$ of S or we want to magnify S then S is regarded as the corresponding subset of the plain. Notice that cells and unit squares are the same when they are considered as planar sets. The *area* $\mu(S)$ of S is defined in the usual way. In particular, $\mu(R) = (\beta(R) - \alpha(R) + 1)(\delta(R) - \gamma(R) + 1)$ for $R \in \mathcal{H}$. For simplicity, if S_1 and S_2 are subsets of the plane with $\mu(S_1 \cap S_2) = 0$ then $S_1 \cup S_2$ will be denoted by $S_1 \dot\cup S_2$ and we say that S_1 and S_2 do not overlap. When $\mu\big((S_1 \setminus S_2) \cup (S_2 \setminus S_1)\big) = 0$, then we will simply say that $S_1 = S_2$ modulo μ.

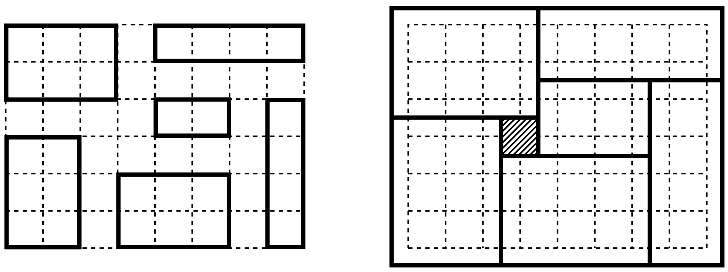

Figure 1

The key idea of the proof is that we can *magnify* the members of $\max(\mathcal{H})$ by half in all the four directions, and they still will not overlap after this magnification. For $R \in \max(\mathcal{H})$ let R^* denote what we obtain from R after this magnification. The magnification process is visualized in Figure 1, where $m = 6$, $n = 8$, and the table is depicted by dotted lines. The elements of $\max(\mathcal{H})$ are indicated by thick solid lines on the left. Similarly, the members of $\{R^* \colon R \in \max(\mathcal{H})\}$ are given by thick solid lines on the right. The shaded area is disjoint from each magnified maximal rectangular island. We can also magnify the table $T = \mathbf{m} \times \mathbf{n}$ by half in all the four directions; the *magnified table* will be denoted by T^* and its area is $\mu(T^*) = (m{+}1)(n{+}1)$. Notice that $R^* \subseteq T^*$ for all $R \in \max(\mathcal{H})$ and $\mu(R^* \cap S^*) = 0$ for distinct $R, S \in \max(\mathcal{H})$.

Let $\max(\mathcal{H}) = \{R_1, \dots, R_k\}$. In Figure 1, $k = 6$. Choose a companion set $\mathcal{C}_i$ of $\mathcal{H}|_{R_i}$ in the table R_i for each $i \in \{1, \dots, k\}$, and let

$$\mathcal{D} = \mathcal{C}_1 \mathbin{\dot\cup} \cdots \mathbin{\dot\cup} \mathcal{C}_k \mathbin{\dot\cup} \operatorname{out}(\mathcal{H})^+ . \tag{5}$$

Notice that the members of the union in (5) are pairwise disjoint and, by (2), $\mathcal{D}$ is an extended companion set of $\mathcal{H}$. Let $\mathcal{G}$ denote $(T^* \setminus T) \cup \operatorname{out}(\mathcal{H}) = T^* \setminus (R_1 \cup \cdots \cup R_k)$ in the plane. This equation is, and some of the following ones will automatically be, understood modulo μ. Then

$$\mu(\mathcal{G}) = m + n + 1 + |\operatorname{out}(\mathcal{H})|. \tag{6}$$

Moreover, in the plane we have

$$\mathcal{G} = \mathcal{E} \mathbin{\dot\cup} (R_1^* \setminus R_1) \mathbin{\dot\cup} \cdots \mathbin{\dot\cup} (R_k^* \setminus R_k) \tag{7}$$

where

$$\mathcal{E} = \mathcal{G} \setminus \big((R_1^* \setminus R_1) \cup \cdots \cup (R_k^* \setminus R_k)\big) = \mathcal{T}^* \setminus (R_1^* \mathbin{\dot\cup} \cdots \mathbin{\dot\cup} R_k^*).$$

In Figure 1, $\mathcal{E}$ is the shaded area. Notice that $\mu(\mathcal{E})$ is an integer, since so are $\mu(\mathcal{G})$ and the $\mu(R_i^* \setminus R_i) = \big(\beta(R_i) - \alpha(R_i) + 1\big) + \big(\delta(R_i) - \gamma(R_i) + 1\big) + 1$ for $i \in \{1, \dots, k\}$. It follows from the definition of $\mathcal{E}$ that

$$\mu(\mathcal{E}) + \sum_{i=1}^{k} \mu(R_i^*) = \mu(T^*) = (m+1)(n+1). \tag{8}$$

Now suppose that R_i is of size $u \times v$, i.e., $u = \beta(R_i) - \alpha(R_i) + 1$ and $v = \delta(R_i) - \gamma(R_i) + 1$. Then by the induction hypothesis we have

$$\begin{gathered} |\mathcal{C}_i| + \mu(R_i^* \setminus R_i) = |\mathcal{C}_i| + u + v + 1 = d(\mathcal{H}|_{R_i}) + u + v + 1 \geq \\ \lceil (uv + 1 - u - v)/2 \rceil + u + v + 1 = \lceil (uv + 1 - u - v)/2 + u + v + 1 \rceil = \\ \lceil (u+1)(v+1)/2 + 1 \rceil, \end{gathered}$$

whence

$$|\mathcal{C}_i| + \mu(R_i^* \setminus R_i) \geq \lceil \mu(R_i^*)/2 + 1 \rceil. \tag{9}$$

Now, applying the previous formulas and indicating their use like $=^{(4)}$, $\geq^{(9)}$, etc., let us compute:

$$\begin{gathered} d(\mathcal{H}) =^{(4)} |\mathcal{D}| - 1 =^{(5)} \\ |\mathcal{C}_1| + \cdots + |\mathcal{C}_k| + (|\operatorname{out}(\mathcal{H})| + m + n + 1) - (m + n + 2) =^{(6)} \\ |\mathcal{C}_1| + \cdots + |\mathcal{C}_k| + \mu(\mathcal{G}) - (m + n + 2) =^{(7)} \\ -(m + n + 2) + \sum_{i=1}^{k} |\mathcal{C}_i| + \mu(\mathcal{E}) + \sum_{i=1}^{k} \mu(R_i^* \setminus R_i) = \\ -(m + n + 2) + \mu(\mathcal{E}) + \sum_{i=1}^{k} \big(|\mathcal{C}_i| + \mu(R_i^* \setminus R_i)\big) \geq^{(9)} \\ -(m + n + 2) + \mu(\mathcal{E}) + \sum_{i=1}^{k} \lceil \mu(R_i^*)/2 + 1 \rceil = \\ -(m + n + 2) + \mu(\mathcal{E}) - \lceil \mu(\mathcal{E})/2 \rceil + \lceil \mu(\mathcal{E})/2 \rceil + \sum_{i=1}^{k} \lceil \mu(R_i^*)/2 + 1 \rceil \geq \end{gathered}$$

$$-(m+n+2)+\mu(\mathcal{E})-\lceil\mu(\mathcal{E})/2\rceil+\Big\lceil\mu(\mathcal{E})/2+\sum_{i=1}^{k}(\mu(R_i^*)/2+1)\Big\rceil =^{(8)}$$

$$-(m+n+2)+\mu(\mathcal{E})-\lceil\mu(\mathcal{E})/2\rceil+\lceil\mu(T^*)/2\rceil+\sum_{i=1}^{k}1=$$

$$-(m+n+2)+\mu(\mathcal{E})-\lceil\mu(\mathcal{E})/2\rceil+\lceil(m+1)(n+1)/2\rceil+k=$$

$$\big(\mu(\mathcal{E})-\lceil\mu(\mathcal{E})/2\rceil+k-2\big)+\lceil(m+1)(n+1)/2-(m+n)\rceil=$$

$$\big(\mu(\mathcal{E})-\lceil\mu(\mathcal{E})/2\rceil+k-2\big)+\lceil(mn+1-m-n)/2\rceil.$$

Hence the only thing we have to show now is that

$$\mu(\mathcal{E})-\lceil\mu(\mathcal{E})/2\rceil+k-2\geq 0. \tag{10}$$

Since $\mu(\mathcal{E})-\lceil\mu(\mathcal{E})/2\rceil\geq 0$, it suffices to deal with the case of $k\leq 1$. If $k=0$ then $\mathcal{E}=T^*$ by (8), so $\mu(\mathcal{E})=(m+1)(n+1)=mn+m+n+1\geq 5+2+2+1=10$ yields (10). If $k=1$ then $\mu(\mathcal{E})\geq\min\{m+1,n+1\}\geq 3$, whence $\mu(\mathcal{E})-\lceil\mu(\mathcal{E})/2\rceil\geq 1$ implies (10). □

Notice that most of our auxiliary statements are valid for the analogous three dimensional problem. However, it seems to be only a lucky peculiarity of the planar configuration that the right hand side of the key formula (9) depends only on $\mu(R_i^*)$ but not on R_i^*. Finally, at the time of final revision, we mention Horváth, Németh and Pluhár [6] as a related research motivated by the present paper.

Acknowledgment. *The author expresses his thanks to János Barát for a general discussion on combinatorics.*

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University of Szeged, Bolyai Institute, Szeged, Aradi vértanúk tere 1, HUNGARY 6720

E-mail address: `czedli@math.u-szeged.hu`

URL: `http://www.math.u-szeged.hu/∼czedli/`