THE MATRIX OF A SLIM SEMIMODULAR LATTICE

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Abstract. A finite lattice L is called *slim* if no three join-irreducible elements of L form an antichain. *Slim semimodular lattices* play the main role in G. Czédli and E. T. Schmidt [5], where lattice theory is applied to a purely group theoretical problem. Here we develop a unique matrix representation for these lattices.

1. Introduction

By a *slim lattice* we mean a finite lattice M such that $J_0(M)$, the poset (partially ordered set) of its join-irreducible elements, contains no three-element antichain. In virtue of R. P. Dilworth [7], a finite lattice M is slim iff $J_0(M)$ is the union of two chains. By [5, Lemma 6], slim lattices are *planar*. A lattice L is called (upper) *semimodular*, if $b \vee c$ covers or equals $a \vee c$ for all $a, b, c \in L$ such that b covers a (in notation, $a \prec b$). For the rudiments of lattice theory the reader is referred to G. Grätzer [8].

Because of their links to combinatorics and geometry, these lattices constitute an important branch of Lattice Theory; see M. Stern [14] for an overview. Semimodular and slim semimodular lattices have recently proved to be useful in strengthening a classical group theoretical result, the Jordan-Hölder theorem; see G. Grätzer and J. B. Nation [11] for the start and [5] for the final result. Motivated by [11] and [5], two visual recursive methods of constructing slim semimodular lattices have recently been given in G. Czédli and E. T. Schmidt [6]. See also Remark 35 later.

Our goal is to present a somewhat less pictorial but equally or even more useful approach to these lattices. Namely, a unique "matrix representation" and a "vector representation" of slim semimodular lattices will be given, see Theorem 24 together with Remarks 30-32. A possible benefit is that a slim semimodular lattice can be determined by a very little amount of data, see Remark 32.

All lattices occurring in the present paper are assumed to be *finite*. Since chains would usually cause unpleasant problems (like a matrix without rows) and they do not need any representation, our lattices are usually assumed not to be chains; in short, we deal mostly with "*non-chain*" lattices.

2. Some preliminary facts

Let K and L be lattices. We say that $\varphi\colon K \to L$ is a *cover-preserving join-homomorphism*, if $\varphi(a \vee b) = \varphi(a) \vee \varphi(b)$ and, in addition, $x \preceq y$ implies $\varphi(x) \preceq \varphi(y)$ for all $a, b, x, y \in K$. Notice that this is the "right" morphism concept for finite

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semimodular lattices, because cover-preserving join-homomorphic images of these lattices are semimodular by G. Grätzer and E. Knapp [9, Lemma 16]; see also [14, Section 6.3]. On the other hand, cover-preserving join-homomorphic images of slim lattices are trivially slim.

A join-congruence α of L, that is, a congruence of the join-semilattice $(L;\vee)$, is called a *cover-preserving join-congruence*, if the canonical $L \to L/\alpha$, $x \mapsto [x]\alpha$ mapping is a cover-preserving join-homomorphism. Due to finiteness, L/α is not only a join-semilattice but it is a lattice. Hence it will be called a quotient *lattice* instead of a quotient join-semilattice. As a reformulation of the previously mentioned result for a cover-preserving join-congruence α, we note that L/α is slim and semimodular, provided L is slim and semimodular, respectively.

Let L be a slim *semimodular* lattice. By a *covering square* of L we mean a quadruple $\varpi = (a\wedge b, a, b, a\vee b)$ such that $a\wedge b \prec a$ and $a\wedge b \prec b$, and, consequently, $a \prec a\vee b$ and $b \prec a \vee b$. With this notation, a and b are called the *corners* of ϖ, while $a\wedge b$ and $a\vee b$ are called the *top (element)* and the *bottom* of ϖ, respectively.

Consider a join-congruence α of L. By an *α-forbidden covering square* we mean a covering square $(a \wedge b, a, b, a \vee b)$ such that the α-classes $[a]\alpha$, $[b]\alpha$, $[a \wedge b]\alpha$ are pairwise distinct but $[a]\alpha = [a \vee b]\alpha$. The importance of this notion is revealed by

Lemma 1 (G. Czédli and E. T. Schmidt [3])**.** *Let α be a join-congruence of a finite semimodular lattice L. Then α is cover-preserving iff L has no α-forbidden covering square.*

As usual, the principal ideal $\{x \in L : x \le a\}$ and the principal filter $\{x \in L : x \ge a\}$ are denoted by $\downarrow a$ and $\uparrow a$, respectively. A lattice L is called *glued sum indecomposable*, or briefly, *indecomposable*, if $L \ne \downarrow a \cup \uparrow a$ for all $a \in L \setminus \{0,1\}$.

3. Grid systems and minimal grid systems

By a *grid lattice* we mean a non-chain lattice that is the direct product of two finite chains. For a grid lattice G, we fix the notation as follows: $J_0(G)$ is the union of two chains, $C = \{0 = c_0 \prec c_1 \prec \cdots \prec c_m\}$ and $D = \{0 = d_0 \prec d_1 \prec \cdots \prec d_n\}$, whose intersection is $\{0\}$. In figures, C and D will usually be on the southwest and on the southeast boundary, respectively. Here $m, n \in \mathbb{N} = \{1, 2, \ldots\}$, and *we always assume* that $m \le n$. Each element of G can uniquely be written in the form

$$c_i \vee d_j, \text{ where } 0 \le i \le m \text{ and } 0 \le j \le n.$$

The pair (m, n) is called the *type* of the grid lattice. Grid lattices of type $(1, n)$ are also called *strips*.

Definition 2. By a *grid system* we mean a pair $K = (G; \alpha)$ such that G is a grid lattice, α is a cover-preserving join-congruence of G, and G/α is not a chain. Some notation and terminology:

- $\mathrm{typ}(K) := \mathrm{typ}(G) = (m, n)$;
- $\mathrm{typ}_\ell(K) = \mathrm{typ}_\ell(G) := m$ and $\mathrm{typ}_r(K) = \mathrm{typ}_r(G) := n$;
- $\mathrm{typ}(K') \le \mathrm{typ}(K'')$ iff $\mathrm{typ}_\ell(K') \le \mathrm{typ}_\ell(K'')$ and $\mathrm{typ}_r(K') \le \mathrm{typ}_r(K'')$;

Remark 3. Although $m = \mathrm{typ}_\ell(K) \le \mathrm{typ}_r(K) = n$ is always assumed, many of the forthcoming arguments do not use this assumption. Then we are allowed to refer to *left-right symmetry*, explicitly or implicitly. Notice also that we often assume that G/α is indecomposable; then it is not a chain iff it consists of at least three elements.

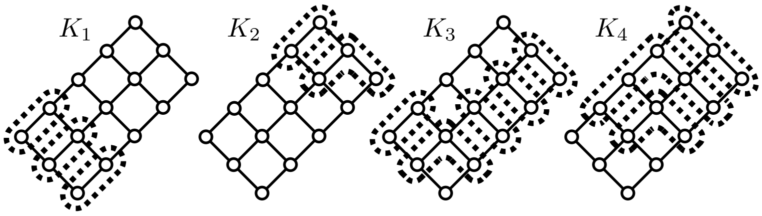


FIGURE 1. K_i satisfies each of (gr1), ..., (gr4) but (gr$i$)

Let $\hat{K} = (\hat{G}; \hat{\alpha})$ be another grid system. Then K and $\hat{K}$ are said to be *isomorphic*, if there is a lattice isomorphism $\varphi\colon G \to \hat{G}$ such that $\varphi(\alpha) = \hat{\alpha}$. We are interested in grid systems only up to isomorphism.

We are motivated by [3, Corollary 2], see also G. Grätzer and E. Knapp [10], which asserts that each slim semimodular non-chain lattice L is (isomorphic to) G/α for an appropriate grid system $K = (G; \alpha)$. This K is far from being unique. In order to provide a much deeper insight into the theory of slim semimodular lattices, we will soon designate a unique grid system K to L. Notice that we will not use [3, Corollary 2]; in fact, it will become a corollary.

Definition 4. By a *minimal grid system* we mean a grid system $K = (G; \alpha)$ such that

- G/α is indecomposable;
- whenever $K' = (G'; \alpha')$ is a grid system with $G'/\alpha' \cong G/\alpha$, then $\mathrm{typ}(K) \leq \mathrm{typ}(K')$.

The above definition does not make it easy to recognize minimal grid systems or to work with them. In particular, it is not clear whether minimal grid systems are determined by their quotient lattices; see Remark 29 later. See also Remark 28 for an alternative definition of minimality.

Therefore we define an equivalent notion. The goal of the current section is to prove that the two notions are equivalent. The least congruence, that is, the equality relation, of G will be denoted by ω_G. By a *principal congruence* we mean a congruence generated by a single pair of elements. Clearly, the adjective "principal" can be omitted from (gr1) below.

Definition 5. Let $K = (G; \alpha)$ be a grid system. Then K is called a *regular grid system* iff it satisfies the following four conditions:

(gr1): ω_G is the only (principal) *lattice* congruence included in α;
(gr2): c_m is the least element of $[c_m]\alpha$, and d_n is the least element of $[d_n]\alpha$;
(gr3): for $k = 1, \ldots, m-1$, $(c_k, d_k) \notin \alpha$;
(gr4): $(c_m, 1) \notin \alpha$ and $(1, d_n) \notin \alpha$.

As Figure 1 indicates, the conditions listed in Definition 5 are independent. Concerning (gr3), notice that (gr2) evidently implies that $(c_m, d_m) \notin \alpha$.

In order to prove that minimal grid systems and regular grid systems are the same, we need some auxiliary statements.

Lemma 6. *Let γ be a principal lattice congruence generated by a pair of covering elements in the grid lattice $G = C \times D$. Then the set of two-element γ-blocks is either $\big\{\{c_{i-1} \vee d_s, c_i \vee d_s\} : 0 \leq s \leq n\big\}$, for some $i \in \{1, \ldots, m\}$, or $\big\{\{c_t \vee d_{j-1}, c_t \vee$*

$d_j\}: 0 \le t \le m\}$, for some $j \in \{1, \dots, n\}$. The rest of the γ-blocks are one-element. Finally, γ is a cover-preserving join-congruence.

Proof. The last statement follows easily from Lemma 1. The rest is evident. $\square$

Lemma 7. *If α and β are cover-preserving join-congruences of a finite semimodular lattice L and $\alpha \subseteq \beta$, then β/α is a cover-preserving join-congruence of the quotient lattice L/α.*

Proof. By way of contradiction, assume that $(C = A \wedge B, A, B, D = A \vee B)$ is a β/α-forbidden covering square in L/α. In particular, $(A, D) \in \beta/\alpha$. Since the α-blocks are convex join-subsemilattices (see, e.g., G. Czédli and E. T. Schmidt [4, Lemma 10]), $c := \bigvee C$ belongs to C. Since $x \vee c \in A$ for any $x \in A$, there is a $y \in A \cap \uparrow c$. Choose an element a such that $c \prec a \le y$. Then $C = [c]\alpha < [a]\alpha \le [y]\alpha = A$, because c is the largest element of C. But $C \prec A$ in L/α, so $[a]\alpha = A$. Similarly, c has a cover $b \in B$. It follows from semimodularity that $(c = a \wedge b, a, b, d := a \vee b)$ is a covering square of L. Clearly, $d \in D$, and (c, a, b, d) is a β-forbidden covering square. This contradiction completes the proof by Lemma 1. $\square$

Corollary 8. *Let $K = (G; \alpha)$ be a minimal grid system or a regular grid system. Then, for $0 \le i < j \le m$ and $0 \le k < t \le n$, $(c_i, c_j) \notin \alpha$ and $(d_k, d_t) \notin \alpha$.*

Proof. Suppose the contrary, say, $(c_i, c_j) \in \alpha$. Then $(c_i, c_{i+1}) \in \alpha$, because the α-blocks are convex. Let γ be the join-congruence generated by (c_i, c_{i+1}). Clearly, it is a lattice congruence, and $\gamma \subseteq \alpha$. This settles the case when K is regular, so we assume that K is a minimal grid system. By Lemmas 6 and 7, $(G/\gamma, \alpha/\gamma)$ is a grid system. The Second Isomorphism Theorem, see S. Burris and H. P. Sankappanavar [2, Thm. 6.15], gives that $G/\alpha \cong (G/\gamma)/(\alpha/\gamma)$. This together with $\operatorname{typ}(G/\gamma) < \operatorname{typ}(G)$ contradict the minimality of K. $\square$

Lemma 9. *Minimal grid systems are regular.*

Proof. Let $K = (G; \alpha)$ be a minimal grid system. Then (gr1) follows from Lemma 6 and Corollary 8.

By way of contradiction, we next suppose that, say, c_m is not the least element of $[c_m]\alpha$. Since the α-block $[c_m]\alpha$ is a convex set and closed with respect to joins, there is a $j \in \{0, \dots, n\}$ such that $c_{m-1} \vee d_j \in [c_m]\alpha$, and so $(c_{m-1} \vee d_k, c_m \vee d_k) \in \alpha$ for all $k \in \{j, \dots, n\}$. Let $G' = \downarrow(c_{m-1} \vee d_n)$. Clearly, $\operatorname{typ}(G') < \operatorname{typ}(G)$. Let α' be the restriction of α to G'. By Lemma 1, α' is cover-preserving. The Third Isomorphism Theorem, see [2, Thm. 6.18], yields that $G/\alpha \cong G'/\alpha'$. This together with $\operatorname{typ}(K') = \operatorname{typ}(G') < \operatorname{typ}(G) = \operatorname{typ}(K)$ contradict the minimality of K. Hence we get (gr2) by left-right symmetry.

For $1 \le k \le m-1$, let $e_k = c_k \vee d_k$. By way of contradiction, suppose that $(c_k, d_k) \in \alpha$. Then $[c_k]\alpha = [d_k]\alpha = [e_k]\alpha$, and we claim that

(1) $[e_k]\alpha$ is comparable with all elements of G/α.

To show (1), let $[c_i \vee d_j]\alpha$ be an arbitrary element of G/α. If $k \le i$ and $k \le j$, or $k \ge i$ and $k \ge j$, then $[c_i \vee d_j]\alpha$ is clearly comparable with $[e_k]\alpha$. Assume that $j \le k \le i$. Then $c_k \vee d_j \in [e_k]\alpha$ by convexity, whence $c_i \vee d_j = c_i \vee d_j \vee c_k \vee d_j \equiv c_i \vee d_j \vee e_k \pmod{\alpha}$ yields that $[c_i \vee d_j]\alpha = [c_i \vee d_j]\alpha \vee [e_k]\alpha \ge [e_k]\alpha$. Since the case $j \ge k \ge i$ is analogous, (1) follows.

We get from Corollary 8 that $[e_k]\alpha = [c_k]\alpha$ is distinct from $[0]\alpha$. So (1) and the indecomposability of K/α yield that $[c_k]\alpha = [1]\alpha$. Since $[c_k]\alpha$ is convex, $(c_k, c_m) \in \alpha$. This contradicts Corollary 8. Hence left-right symmetry yields (gr3).

Assume that (gr4) fails. Then, say, $(d_n, e_m \vee d_n) \in \alpha$. Let j be the smallest index in $\{0, \dots, n\}$ such that $(d_j, c_m \vee d_j) \in \alpha$. Observe that $j \neq 0$ by Corollary 8. Let $i \in \{0, \dots, m\}$ be the smallest index with $(c_i \vee d_{j-1}, c_m \vee d_{j-1}) \in \alpha$. The minimality of j gives that $0 < i$. Since the covering square with top $c_i \vee d_j$ is not an α-forbidden square by Lemma 1, we obtain that $(c_i \vee d_{j-1}, c_i \vee d_j) \in \alpha$. Hence $c_i \vee d_{j-1} \in [d_j]\alpha$. Since $c_i \vee d_{j-1} \not\geq d_n$, (gr2) implies $[d_j]\alpha \neq [d_n]\alpha = [1]\alpha$. Clearly, $[d_j]\alpha = [c_m \vee d_j]\alpha$ is comparable with all elements of G/α. Therefore, the indecomposability of G/α yields that $[d_j]\alpha = [0]\alpha$. This contradicts Corollary 8. □

Given a (fixed) planar diagram of a slim lattice L, the *left boundary chain*, the *right boundary chain*, and the *boundary* of L are denoted by $\mathcal{B}_{\mathrm{left}}(L)$ and $\mathcal{B}_{\mathrm{right}}(L)$, and $\mathcal{B}(L) = \mathcal{B}_{\mathrm{left}}(L) \cup \mathcal{B}_{\mathrm{right}}(L)$, respectively. For example, for L in Figure 2 we have $\mathcal{B}_{\mathrm{left}}(L) = \{c_0^*, \dots, c_4^*, e_1, e_2, 1\}$, $\mathcal{B}_{\mathrm{right}}(L) = \{d_0^*, \dots, d_6^*, 1\}$, and the elements of $J_0(L)$ are the black-filled ones. Note that $\mathcal{B}_{\mathrm{left}}(L)$ and $\mathcal{B}_{\mathrm{right}}(L)$ are maximal chains in L.

Lemma 10 ([6, Lemmas 2, 4, 6 and 7])**.** *Let L be a slim lattice.*

(1) *$\mathcal{B}(L)$ is uniquely determined, and $J_0(L) \subseteq \mathcal{B}(L)$. Furthermore, $\mathcal{B}_{\mathrm{left}}(L)$ and $\mathcal{B}_{\mathrm{right}}(L)$ are maximal chains in L.*
(2) *If $x, y \in \mathcal{B}_{\mathrm{left}}(L)$ and $x \prec y$, then x is meet-irreducible or y is join-irreducible.*
(3) *If, in addition, L is indecomposable, then the* set *$\{\mathcal{B}_{\mathrm{left}}(L), \mathcal{B}_{\mathrm{right}}(L)\}$ is uniquely determined, that is, $\mathcal{B}_{\mathrm{left}}(L)$ and $\mathcal{B}_{\mathrm{right}}(L)$ are unique up to left-right symmetry.*
(4) *Each element of L has at most two covers.*

In virtue of part (3), we consider our diagrams up to left-right symmetry. That is, as if they were drawn on a transparent sheet, and no matter which side of the sheet is up.

Lemma 11. *Let $K = (G; \alpha)$ be a grid system such that G/α is indecomposable. Then, up to left-right symmetry, $[c_i]\alpha \in \mathcal{B}_{\mathrm{left}}(G/\alpha)$ for $i = 0, \dots, m$, and $[d_j]\alpha \in \mathcal{B}_{\mathrm{right}}(G/\alpha)$ for $j = 0, \dots, n$.*

Proof. We know that $[c_m]\alpha \neq [0]\alpha$, because G/α is not a chain. Let i_0 be the smallest index such that $[c_{i_0}]\alpha$ is distinct from $[0]\alpha$. Since α is cover-preserving, $[c_{i_0}]\alpha$ is an atom of G/α. By Lemma 10 and indecomposability, G/α has exactly two atoms. They are necessarily on the boundary of G/α. So, up to left-right symmetry, $[c_{i_0}]\alpha$ belongs to $\mathcal{B}_{\mathrm{left}}(G/\alpha)$. Clearly, so does $[0]\alpha$. We prove $[c_i]\alpha \in \mathcal{B}_{\mathrm{left}}(G/\alpha)$ by induction on i.

We next assume that $i_0 < i \leq m$ and $[c_{i-1}]\alpha \in \mathcal{B}_{\mathrm{left}}(G/\alpha)$. By way of contradiction, we also assume that

$$[c_i]\alpha \notin \mathcal{B}_{\mathrm{left}}(G/\alpha). \tag{2}$$

Then, since α is cover-preserving, $[0]\alpha \neq [c_{i-1}]\alpha \prec [c_i]\alpha$. Observe that

$$[c_i]\alpha \text{ is join-reducible in } G/\alpha. \tag{3}$$

Indeed, otherwise (2) together with Lemma 10 would imply that $[c_i]\alpha$ belongs to $\mathcal{B}_{\mathrm{right}}(G/\alpha)$, whence its unique lower cover, $[c_{i-1}]\alpha$, would also belong to $\mathcal{B}_{\mathrm{right}}(G/\alpha)$.

Then, belonging to both boundary chains, $[c_{i-1}]\alpha$ would be comparable with all elements of G/α, contradicting the indecomposability of G/α.

Based on (3), there are indices s and t such that $[c_{i-1}]\alpha \neq [c_s \vee d_t]\alpha \prec [c_i]\alpha$. Clearly, $s < i$ and $0 < t$. Since $[c_{i-1} \vee d_t]\alpha = [c_{i-1} \vee c_s \vee d_t]\alpha = [c_{i-1}]\alpha \vee [c_s \vee d_t]\alpha = [c_i]\alpha$, there is a smallest j such that $[c_{i-1} \vee d_j]\alpha = [c_i]\alpha$. We know that $0 < j$, because $[c_{i-1}]\alpha \neq [c_i]\alpha$. Since $[c_{i-1}]\alpha \leq [c_{i-1} \vee d_{j-1}]\alpha \leq [c_{i-1} \vee d_j]\alpha = [c_i]\alpha$ and the second inequality is strict by the minimality of j, we obtain that $[c_{i-1}]\alpha = [c_{i-1} \vee d_{j-1}]\alpha$. Let us consider an arbitrary $z \in G$ with $[c_{i-1}]\alpha < [z]\alpha$. Then, for $y := z \vee c_{i-1} \vee d_{j-1}$, we have $[z]\alpha = [z]\alpha \vee [c_{i-1}]\alpha = [z]\alpha \vee [c_{i-1} \vee d_{j-1}]\alpha = [y]\alpha$. We have $y \neq c_{i-1} \vee d_{j-1}$, because otherwise $[z]\alpha = [y]\alpha = [c_{i-1} \vee d_{j-1}]\alpha = [c_{i-1}]\alpha$. Hence $c_{i-1} \vee d_{j-1} < y$, that is, $c_{i-1} \vee d_j \leq y$ or $c_i \vee d_{j-1} \leq y$. In the first case, $[c_i]\alpha = [c_{i-1} \vee d_j]\alpha \leq [y]\alpha = [z]\alpha$, while $[c_i]\alpha \leq [z]\alpha$ is even more evident in the second case. This shows that $[c_i]\alpha$ is the only cover of $[c_{i-1}]\alpha$. Therefore the unique element covering $[c_{i-1}]\alpha$ in $\mathcal{B}_{\text{left}}(G/\alpha)$ is $[c_i]\alpha$. This contradicts (2). ☐

Let L be a slim, indecomposable non-chain lattice. Then $J_0(L)$ has exactly two maximal elements, which we call the *top corners* of L. Denoting them by $\hat{c}$ and $\hat{d}$ such that $h(\hat{c}) \leq h(\hat{d})$ holds for their heights (understood in L, not in $J_0(L)$), we define the *type* of L as

$$\operatorname{typ}(L) = (\operatorname{typ}_\ell(L), \operatorname{typ}_r(L)) := (h(\hat{c}), h(\hat{d})).$$

Notice that $\hat{c}$ and $\hat{d}$ are on the boundary of L by Lemma 10. Since they are incomparable, they belong to distinct boundary chains. This explains the terminology and allows us to speak of the *top left corner* and the *top right corner* of L. (Usually, $\hat{c}$ is on the left.)

Lemma 12. *Let $K = (G;\alpha)$ be a grid system such that G/α is indecomposable. Then* $\operatorname{typ}(G/\alpha) \leq \operatorname{typ}(K)$. *Further, if there is a principal* lattice *congruence γ of G such that $\omega_G \neq \gamma \subseteq \alpha$, then* $\operatorname{typ}(G/\alpha) < \operatorname{typ}(K)$.

Proof. As always, let $(m, n) = \operatorname{typ}(K)$. Since each element of G/α is of the form $[c_i \vee d_j]\alpha = [c_i]\alpha \vee [d_j]\alpha$, we infer that

$$J_0(G/\alpha) \subseteq \{[c_i]\alpha : 0 \leq i \leq n\} \cup \{[d_j]\alpha : 0 \leq j \leq m\}. \tag{4}$$

Since α is cover-preserving, $h([c_i]\alpha) \leq i$ and $h([d_j]\alpha) \leq j$. This implies $\operatorname{typ}(G/\alpha) \leq \operatorname{typ}(K)$.

To prove the second part, observe that $K' := (G/\gamma, \alpha/\gamma)$ is a grid system with $\operatorname{typ}(K') < \operatorname{typ}(K)$ by Lemmas 6 and 7. By the Second Isomorphism Theorem, $(G/\gamma)/(\alpha/\gamma) \cong G/\alpha$, whence $\operatorname{typ}(G/\alpha) = \operatorname{typ}((G/\gamma)/(\alpha/\gamma)) \leq \operatorname{typ}(K') < \operatorname{typ}(K)$ by the first part of the lemma. ☐

Lemma 13. *If $K = (G;\alpha)$ is a regular grid system, then G/α is indecomposable.*

Proof. Let $K = (G;\alpha)$ be a regular grid system. Assume, by way of contradiction, that $v = c_i \vee d_j \in G$ such that $[0]\alpha \neq [v]\alpha \neq [1]\alpha$ and $[v]\alpha$ is comparable with all elements of G/α. We can also assume that v is the largest element of $[v]\alpha$.

Suppose that $i < m$. If we had $[c_{i+1}]\alpha \leq [v]\alpha$, then $(v, c_{i+1} \vee v) \in \alpha$ would contradict the maximality of v in its α-block. Hence $[c_{i+1}]\alpha > [v]\alpha$. This and the maximality of v yield

$$(c_{i+1}, c_{i+1} \vee v) \in \alpha \text{ and } (v, c_{i+1} \vee v) \notin \alpha, \tag{5}$$

Applying Lemma 1 (to the whole G) we infer that the strip $[c_i, c_{i+1} \vee v] = [c_i, c_{i+1} \vee d_j]$ does not contain any α-forbidden covering square. This fact together with (5) imply that each edge of the chain $[c_i, v]$ is collapsed by α. Hence, by transitivity, $(c_i, v) \in \alpha$. Let $G' = \downarrow v$, and let α' be the restriction of α to G'. It is clear by Lemma 1 that α' is a cover-preserving join-congruence. Therefore the already mentioned [9, Lemma 16] yields that G'/α' is a semimodular lattice. In this lattice, the height of $[v]\alpha' = [c_i]\alpha'$ is i by Corollary 8. Similarly, the height of $[d_j]\alpha'$ is j. We conclude from the Jordan-Hölder theorem applied to G'/α' that

$$\text{if } i < m, \text{ then the length of the interval } \big[[d_j]\alpha', [v]\alpha'\big] \text{ is } i - j. \tag{6}$$

In particular, if $i < m$, then $j \le i$. By left-right symmetry,

$$\text{if } j < n, \text{ then } (d_{j+1}, d_{j+1} \vee v) \in \alpha \text{ and } (v, d_{j+1} \vee v) \notin \alpha, \tag{7}$$

and

$$\text{if } j < n, \text{ then the length of the interval } \big[[c_i]\alpha', [v]\alpha'\big] \text{ is } j - i. \tag{8}$$

In particular, if $j < n$, then $i \le j$.

Suppose that $i < m$. Then $j \le i$, and $j < m \le n$ implies $i \le j$. Hence $i = j < m$, whence (6) and (8) imply $[c_i]\alpha = [v]\alpha = [d_i]\alpha$, which is excluded by (gr3). Therefore $i = m$. Since $j < m = i \le n$ would imply its opposite, $i \le j$, we know that $m \le j$. From $v \ne 1$ we get that $j < n$. Then (7) yields that $(d_n, 1) = (d_n \vee d_{j+1}, d_n \vee d_{j+1} \vee v) \in \alpha$, which contradicts (gr4). □

Lemma 14. *Assume that $K = (G; \alpha)$ is a regular grid system. Then $[c_m]\alpha$ and $[d_n]\alpha$ are the top corners of G/α.*

Proof. Let $L = G/\alpha$; remember, it is not a chain. Assume that $[c_m]\alpha$ is join-reducible. Then there are $x, y \in G$ such that $[x]\alpha < [c_m]\alpha$, $[y]\alpha < [c_m]\alpha$, and $[x \vee y]\alpha = [x]\alpha \vee [y]\alpha = [c_m]\alpha$. Clearly, $x, y \notin \uparrow c_m$. Hence $x \vee y \notin \uparrow c_m$ as well, and $x \vee y \in [c_m]\alpha$ contradicts (gr2). This proves that $[c_m]\alpha \in J_0(L)$.

If $[c_m]\alpha$ not maximal in $J_0(L)$, then $[c_m]\alpha < [d_n]\alpha$ by Lemma 13 and (4). Hence $J_0(L)$ has a largest element, $[d_n]\alpha$. It is the largest element of L, which contradicts the indecomposability of L, ensured by Lemma 13. □

Proposition 15. *A grid system is minimal if and only if it is regular.*

Proof. We know from Lemma 9 that minimal grid systems are regular.

Conversely, let $K = (G; \alpha)$ be a regular grid system. Then G/α is an indecomposable lattice by Lemma 13. Corollary 8 together with Lemma 14 yield that for $\operatorname{typ}(K) = \operatorname{typ}(G)$ we have

$$\operatorname{typ}(K) = \operatorname{typ}(G/\alpha). \tag{9}$$

Therefore, if $K' = (G'; \alpha')$ is another grid system with $G'/\alpha' \cong G/\alpha$, then Lemma 12 implies $\operatorname{typ}(K') \ge \operatorname{typ}(G'/\alpha') = \operatorname{typ}(G/\alpha) = \operatorname{typ}(K)$. Thus, K is a minimal grid system. For a later reference we record an evident consequence of $\operatorname{typ}(K') \ge \operatorname{typ}(K)$:

$$|G'| \ge |G|. \tag{10}$$ □

In what follows, the terms *minimal* grid system and *regular* grid system will be equivalent.

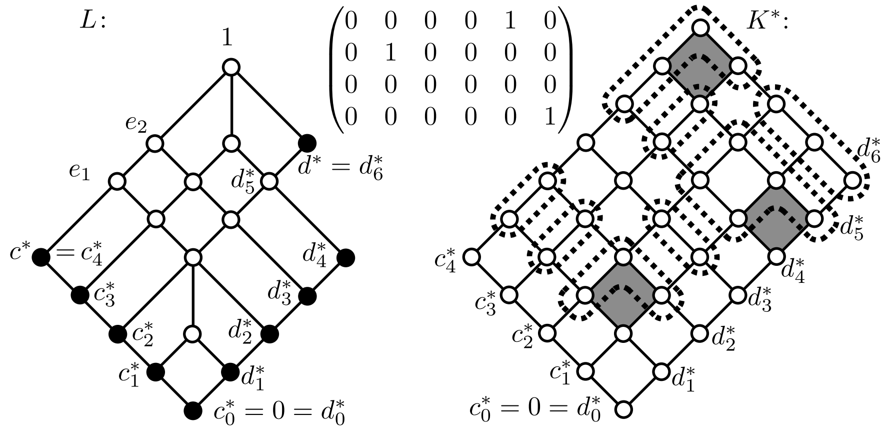


FIGURE 2. L, the matrix of L, and $K^* = \psi(L)$

Definition 16. Let $K = (G; \alpha)$ be a regular grid system. Let L be an indecomposable slim semimodular non-chain lattice, and let $\mathcal{B}_{\text{left}}(L)$ and $\mathcal{B}_{\text{right}}(L)$ denote its boundary chains according to Lemma 10.

- Let $\varphi(K)$ be the lattice G/α.
- Let c^* and d^* be the top left corner and the top right corner of L. Let $C^* = \{0 = c_0^* \prec \cdots \prec c_{m^*}^* = c^*\}$ be the chain $\mathcal{B}_{\text{left}}(L) \cap {\downarrow} c^*$, and let $D^* = \{0 = d_0^* \prec \cdots \prec d_{n^*}^* = d^*\}$ be the chain $\mathcal{B}_{\text{right}}(L) \cap {\downarrow} d^*$. Define a grid $G^* := C^* \times D^*$. Let $\alpha^* \subseteq G^* \times G^*$, a relation on G^*, be defined by $(c_i^*, d_j^*) \equiv (c_h^*, d_k^*) \pmod{\alpha^*}$ iff $c_i^* \vee d_j^* = c_h^* \vee d_k^*$. Finally, let[1]
$$\psi(L) = K^* = (G^*; \alpha^*).$$

A part of the above definition is visualized in Figure 2, where the α-blocks are indicated by dotted thick curves. (The matrix and the grey cells will be relevant only later). It is not hard to see (and it follows trivially from [6]) that L in the figure is an indecomposable slim semimodular lattice. Here, d_5^* does not belong to $J_0(L)$, the set of the black-filled elements; this indicates that C^* and D^* need not be subsets of $J_0(L)$ in general.

Proposition 17. *Let $K = (G; \alpha)$ be a regular grid system, and let L be an indecomposable slim semimodular non-chain lattice. Then*

- *$\varphi(K)$ is an indecomposable slim semimodular non-chain lattice;*
- *$\psi(L)$ is a regular grid system;*
- *$\psi(\varphi(K)) \cong K$;*
- *$\varphi(\psi(L)) \cong L$.*

Proof. As already mentioned in Section 2, $\varphi(K)$ is a slim semimodular lattice by [9, Lemma 16]. By Lemma 13, it is indecomposable. It is not a chain by definitions. This proves the first part of the proposition.

Let $\eta \colon G^* \to L$, $(c_i^*, d_j^*) \mapsto c_i^* \vee d_j^*$. Using part (1) of Lemma 10, we obtain that η is a *surjective* join-homomorphism. Hence its kernel, α^*, is a join-congruence. It

[1] If $m^* = n^*$, then $\psi(L)$ is defined only up to isomorphism.

follows that $G^*/\alpha^* = \varphi(\psi(L))$ is isomorphic to L. This proves the fourth part of the proposition.

Suppose that $(c_i^*, d_j^*) \prec (c_h^*, d_k^*)$ in G^*, and keep the above definition of η. Then $(h,k) = (i+1, j)$ or $(h,k) = (i, j+1)$, and the semimodularity of L yields that $\eta(c_i^*, d_j^*) \preceq \eta(c_h^*, d_k^*)$. Hence η is cover-preserving, and so is its kernel, α^*. Since $G^*/\alpha^* \cong L$ is not a chain, $\psi(L) = K^*$ is a grid system.

Next, we assume that $\gamma \neq \omega_{G^*}$ is a principal lattice congruence of G^* such that $\gamma \subseteq \alpha^*$. We already know that $G^*/\alpha^* = \varphi(\psi(K)) \cong L$ is indecomposable. Hence, using Lemma 12, we conclude that $\operatorname{typ}(L) = \operatorname{typ}(G^*/\alpha^*) < \operatorname{typ}(K^*) = \operatorname{typ}(G^*) = \operatorname{typ}(L)$. This contradiction proves that (gr1) holds for K^*.

To prove (gr2) for K^*, assume that $(c^*, 0) \equiv (c_i^*, d_j^*) \pmod{\alpha^*}$. Then, by the definition of α^*, $c^* = c_i^* \vee d_j^*$ holds in L. But c^* is join-irreducible, whence $c^* = c_i^*$ or $c^* = d_j^*$. If $c^* = d_j^*$, then $c^* \leq d^*$ yields that $1 = c^* \vee d^* = d^* \in J_0(L)$, which contradicts the indecomposability of L. Hence $c^* = c_i^*$, and $(c^*, 0) \leq (c_i^*, d_j^*)$. This proves (gr2).

Similarly, if $(0, d^*) \equiv 1_{G^*} = (c^*, d^*) \pmod{\alpha^*}$, then $d^* = c^* \vee d^*$ gives $c^* \leq d^*$, and we get the same contradiction as above. Hence K^* satisfies (gr4).

If $1 \leq k \leq m-1$ and $(c_k^*, 0) \equiv (0, d_k^*) \pmod{\alpha^*}$, then $c_k^* = c_k^* \vee 0 = 0 \vee d_k^* = d_k^*$ belongs to both border chains of L, which contradicts its indecomposability. Hence $\psi(L) = K^*$ satisfies (gr3), and it is a *regular* grid system.

To prove the third part of the proposition, let $L' := \varphi(K) = G/\alpha$. Its top corners are $[c_m]\alpha$ and $[d_n]\alpha$ by Lemma 14. Since α is cover-preserving, Corollary 8 yields that $[c_{i-1}]\alpha \prec [c_i]\alpha$ for $1 \leq i \leq m$. All the $[c_i]\alpha$ are on the left boundary chain by Lemma 11. Hence $\mathcal{B}_{\text{left}}(L') \cap {\downarrow}[c_m]\alpha = \{[c_0]\alpha \prec \cdots \prec [c_m]\alpha\}$ and, similarly, $\mathcal{B}_{\text{right}}(L') \cap {\downarrow}[d_n]\alpha = \{[d_0]\alpha \prec \cdots \prec [d_n]\alpha\}$. Hence, with the notation $K' = (G'; \alpha') := \psi(L') = \psi(\varphi(K))$, we have $G' = \bigl\{([c_i]\alpha, [d_j]\alpha) : 0 \leq i \leq m,\ 0 \leq j \leq n\bigr\}$ and $([c_i]\alpha, [d_j]\alpha) \equiv ([c_h]\alpha, [d_k]\alpha) \pmod{\alpha'}$ iff $[c_i]\alpha \vee [d_j]\alpha = [c_h]\alpha \vee [d_k]\alpha$ in L', that is, iff $c_i \vee d_j \equiv c_h \vee d_k \pmod{\alpha}$. Hence

$$\tau\colon \psi(\varphi(K)) \to K,\ ([c_i]\alpha, [d_j]\alpha) \mapsto c_i \vee d_j$$

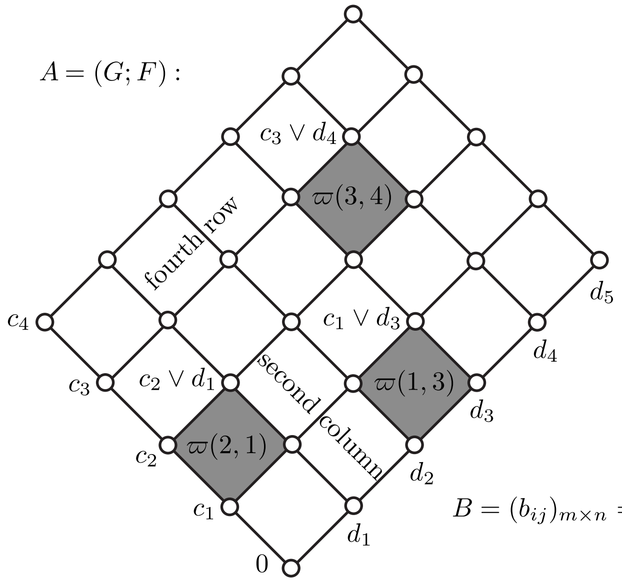
is clearly an isomorphism. $\square$

4. Grid matrices and source cells

Let L be a slim semimodular non-chain lattice. Since it is planar, the edges of the (fixed) planar diagram divide the plane into *regions*. The minimal regions are called *cells*. Let $\operatorname{Cells}(L)$ denote the set of all cells of L. We know from [9] and [6, Proposition 1] that the cells (called 4-cells in [6]) and the covering squares of L are the same. By an *α-forbidden cell* we mean an *α-forbidden covering square*, of course.

Next, let G be a grid lattice. It will be convenient to refer to the cells of G by their top elements. For $1 \leq i \leq m := \operatorname{typ}_\ell(G)$ and $1 \leq j \leq n := \operatorname{typ}_r(G)$, let $\varpi(i,j)$ denote the cell with top $c_i \vee d_j$. That is, $\varpi(i,j)$ is the covering square $(c_{i-1} \vee d_{j-1}, c_i \vee d_{j-1}, c_{i-1} \vee d_j, c_i \vee d_j)$. With $u = c_i \vee d_j$, the cell $\varpi(i,j)$ is also denoted by $\varpi(u)$.

By a *grid matrix* we mean a pair $(G; F)$, where G is a grid and F is unary relation on (that is, a subset of) the set $\operatorname{Cells}(G)$. The elements of F will be called *F-cells*. In figures, the F-cells will be the grey-filled cells. The *type* of a grid matrix $A = (G; F)$ is, of course, defined to be $\operatorname{typ}(G)$. Similarly, $\operatorname{typ}_\ell(A) := \operatorname{typ}_\ell(G)$ and



$$B = (b_{ij})_{m\times n} = \begin{pmatrix} 0 & 0 & 1 & 0 & 0\\ 1 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

FIGURE 3. A grid matrix and the corresponding 0-1 matrix

$\mathrm{typ}_r(A) := \mathrm{typ}_r(G)$. Another grid matrix, $(G'; F')$, is said to be isomorphic with $(G; F)$, if there is a lattice isomorphism $G \to G'$ that maps F onto F'. We are interested in grid matrices only up to isomorphism. There is an obvious connection between grid matrices and (ordinary) 0-1 matrices, illustrated by Figure 3 and to be discussed later. This connection allows us to speak of *rows* and *columns* of grid matrices; they are southwest-northeast and southeast-northwest strips, respectively, see Figure 3. For example, the i-th row is the set $\{\varpi(i,j) : 1 \le j \le n\}$. For another example of a grid matrix and the corresponding 0-1 matrix see Figure 2.

Given a cover-preserving join-congruence α of a slim semimodular lattice L, by a *source cell* of α we mean a covering square $\varpi = (a \wedge b, a, b, a \vee b)$ such that $\{a, b\} \subseteq [a \vee b]\alpha$ but $a \wedge b \notin [a \vee b]\alpha$. By a *source element* of α we mean the top of a source cell of α. If there is no danger of ambiguity, we will simply say "source cell" and "source element" without mentioning α. The set of source cells and that of source elements of α will be denoted by

$$\mathrm{Sc}(\alpha) \text{ and } \mathrm{Se}(\alpha), \text{ respectively.} \tag{11}$$

The purpose of the following definition is to describe regular (that is, minimal) grid systems by grid matrices.

Definition 18. Let $A = (G; F)$ be a grid matrix of type (m, n). (Remember that $m \le n$.) Then A is called a *regular grid matrix*, if the following five conditions hold:

- **(mr1):** every row of A contains at most one F-cell, and the same holds for every column;
- **(mr2):** $|F| < m$;
- **(mr3):** for $k = 1, \dots, m-1$, $|F \cap \mathrm{Cells}(\downarrow(c_k \vee d_k))| < k$;
- **(mr4):** if $\varpi(i, n) \in F$, then there is an i' such that $1 \le i' < i$ and there is no F-cell in the i'-th row;

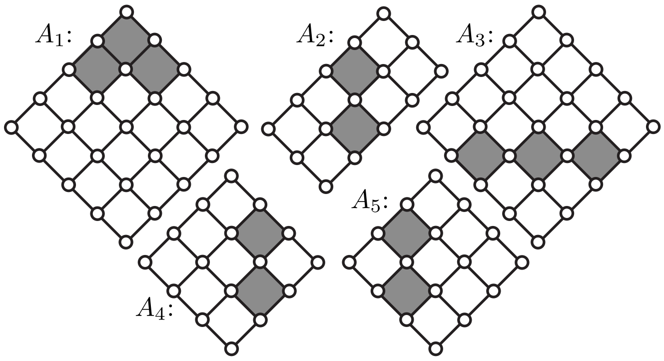


FIGURE 4. A_i satisfies each of (mr1),...,(mr5) but (mr$i$)

(mr5): if $\varpi(m, j) \in F$, then there is a j' such that $1 \le j' < j$ and there is no F-cell in the j'-th column.

Figure 4 witnesses that the conditions in Definition 18 are independent. Given a grid G, let $u = c_i \vee d_j$ be the top of a cell of G, and let

$$\vartheta(u) \tag{12}$$

denote the smallest join-congruence that collapses u with its two lower covers. The following definition presupposes (11) and (12).

Definition 19.
- Let $K = (G; \alpha)$ be a regular grid system. Then $\mu(K) = \mu(G; F)$, the *grid matrix associated with* K, is defined to be $(G; \mathrm{Sc}(\alpha))$.
- Let $A = (G; F)$ be a regular grid matrix. Then the *grid system associated with* A is defined to be $\sigma(A) = \sigma(G; F) := (G; \beta)$ where

$$\beta := \bigvee\{\vartheta(u) : \varpi(u) \in F\}. \tag{13}$$

(The join is taken in the congruence lattice of $(G; \vee)$.)

Proposition 20. *Let $K = (G; \alpha)$ be a regular grid system, and let $A = (G; F)$ be a regular grid matrix. Then*
- *$\mu(K)$ is a regular grid matrix;*
- *$\sigma(A)$ is a regular grid system;*
- *$\sigma(\mu(K)) = K$;*
- *$\mu(\sigma(A)) = A$.*

The proof of Proposition 20 will need the following auxiliary statements.

Lemma 21 ([4, Lemma 11])**.** *Let α_i, $i \in I$, be congruences of a join-semilattice $(L; \vee)$, and let β denote their join in the congruence lattice of $(L; \vee)$. Then, for each x, y in L, $(x, y) \in \beta$ iff there is a $k \in \mathbb{N}_0$ and there are elements $x = z_0 \le z_1 \le \cdots \le z_k = v_k \ge v_{k-1} \ge \cdots \ge v_0 = y$ in L such that $\{(z_{j-1}, z_j), (v_{j-1}, v_j)\} \subseteq \bigcup_{i \in I} \alpha_i$ for $j = 1, \ldots, k$.*

The following statement is an evident consequence of Lemma 21, and it will be illustrated in Figure 5.

Corollary 22. *Let $\vartheta(u)$ be the join-congruence defined at* (12)*. Then*
- $[u]\vartheta(u) = \{c_{i-1} \vee d_j, u, c_i \vee d_{j-1}\}$,

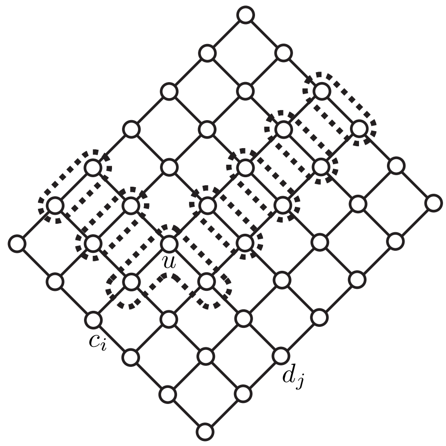


FIGURE 5. The join-congruence $\vartheta(u)$

- $[c_k \vee d_j]\vartheta(u) = \{c_k \vee d_j, c_k \vee d_{j-1}\}$ *for* $i < k \leq m$,
- $[c_i \vee d_t]\vartheta(u) = \{c_i \vee d_t, c_{i-1} \vee d_t\}$ *for* $j < t \leq n$,
- *and all other* $\vartheta(u)$*-blocks are singletons.*

Lemma 23. *Let* $K = (G; \alpha)$ *be a regular grid system. Then, in the congruence lattice of* $(G; \vee)$, $\alpha = \bigvee\{\vartheta(u) : u \in \mathrm{Se}(\alpha)\}$.

Proof. Let β denote the join in question. Clearly, $\beta \subseteq \alpha$. To show the reverse inclusion, it suffices to show, by induction on the height $h(x)$ of $x \in G$, that for any $x \prec y$, either $(x, y) \notin \alpha$, or $(x, y) \in \vartheta(u)$ for some $u \in \mathrm{Se}(\alpha)$.

If $h(x) = 0$, that is, $x = 0$, then $(x, y) \notin \alpha$ by Corollary 8. So, assume that $h(x) > 0$ and $(x, y) \in \alpha$. We conclude from Corollary 8 that $x \prec y$ is one of the upper edges of a cell ϖ. If $\varpi \in \mathrm{Sc}(\alpha)$ and u denotes the top of ϖ, then $(x, y) \in \vartheta(u)$, as desired. Otherwise Lemma 1 implies that the opposite edge $x' \prec y'$ of ϖ belongs to α. Since $h(x') < h(x)$, the induction hypothesis implies $(x', y') \in \vartheta(v)$ for some $v \in \mathrm{Se}(\alpha)$, which clearly yields that $(x, y) \in \vartheta(v)$. This completes the induction. □

Proof of Proposition 20. First we prove that the grid matrix $\mu(K)$ is regular. Suppose that $u \in \mathrm{Se}(\alpha)$, $u < v \in G$, and $\varpi(u)$ and $\varpi(v)$ are in the same row or column. Then Corollary 22 yields that $v \notin \mathrm{Se}(\alpha)$. This shows that $\mu(K)$ satisfies (mr1). If we had $|F| \geq m$, then (mr1), Corollary 22, and Lemma 23 would yield that $(d_n, 1) \in \alpha$, contradicting (gr4). Hence $\mu(K)$ satisfies (mr2). Similarly, (mr1), Corollary 22, Lemma 23, and (gr3) imply that $\mu(K)$ satisfies (mr3). Finally, (mr4) and (mr5) follow from (gr2). This proves the first part of Proposition 20.

Next, with β defined in (13), we want to verify that $\sigma(A) = (G; \beta)$ is a regular grid system. We first show that

$$\text{if } (x, y) \in \beta \text{ and } x \prec y, \text{ then } (x, y) \in \bigcup\{\vartheta(u) : \varpi(u) \in F\}. \tag{14}$$

Let $x \prec y$, $(x, y) \in \beta$,

$$\gamma := \bigcup\{\vartheta(u) : \varpi(u) \in F\}, \tag{15}$$

and let us use the notation of Lemma 21, with $\vartheta(u)$ instead of α_i. Then there is a smallest $i \in \mathbb{N}$ such that $z_{i-1} \notin \uparrow y$ but $z_i \in \uparrow y$; see Figure 6 for an illustration.

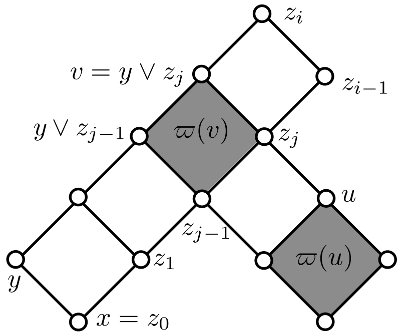


FIGURE 6. The contradiction derived from $F \cap \mathrm{Cells}(I) \neq \emptyset$

Clearly, $[x, z_{i-1}]$ is a chain (a singleton, if $i = 1$), and all edges of this chain belong to γ. Hence all edges of the chain $[y, z_i]$ belong to γ as well. Consider the interval $I = [x, z_i]$. Clearly, it is a strip. So, without loss of generality, we can think of $\mathrm{Cells}(I)$ as a subset of a row; then the edges $x \prec y$ and $z_{i-1} \prec z_i$ are southeast-northwest oriented.

Assume, by way of contradiction, that $F \cap \mathrm{Cells}(I) \neq \emptyset$. Hence there is a $j \in \{1, \dots, i-1\}$ such that, with the notation $v := y \vee z_j$, $\varpi(v) = (z_{j-1}, y \vee z_{j-1}, z_j, v) \in F$. But $(z_{j-1}, z_j) \in \gamma$, whence Corollary 22 yields an $u \in G$ such that $\varpi(u) \in F$ and $(z_{j-1}, z_j) \in \vartheta(u)$. We get from Corollary 22 that $u < v$ and, moreover, $\varpi(u)$ and $\varpi(v)$ are distinct F-cells in the same column, contradicting (mr1). Hence we conclude that $F \cap \mathrm{Cells}(I) = \emptyset$.

Since the southeast-northwest oriented edge (z_{i-1}, z_i) belongs to γ, Corollary 22 yields an F-cell $\varpi(w)$ to the southwest of this edge (in the row of $\varpi(z_i)$). But I contains no F-cell, so $\varpi(w)$ is to the southwest of the edge (x, y). Hence $(x, y) \in \vartheta(w) \subseteq \gamma$, proving (14).

Next, assume that $(d_n, 1) \in \beta$. By the convexity of $[1]\beta$, all the edges $c_{i-1} \vee d_n \prec c_i \vee d_n$, $i = 1, \dots, m$, are collapsed by β. It follows from Corollary 22 and (14) that $|F| \geq m$, contradicting (mr2). Since $(c_m, 1) \in \beta$ would similarly lead to $|F| \geq n \geq m$, we have shown that $(G; \beta)$ satisfies (gr4).

If we had $[c_m]\beta \leq [d_n]\beta$, then $[c_m \vee d_n]\beta$ would equal $[d_n]\beta$, whence $(d_n, 1) \in \beta$ would contradict (gr4). Since $[d_n]\beta \leq [c_m]\beta$ would lead to $(c_m, 1) \in \beta$, a contradiction again, we infer that $[c_m]\beta \parallel [d_n]\beta$. Hence G/β is not a chain, and $(G; \beta)$ is a grid system.

It follows from Corollary 22 and (14) that no two elements of C are collapsed by β, and the same holds for D. Hence $(G; \beta)$ satisfies (gr1) by Lemma 6.

Assume that c_m is not the least element of $[c_m]\beta$. The convexity of $[c_m]\beta$ yields a j such that $c_{m-1} \vee d_j \in [c_m]\beta$; we can assume that j is minimal with this property. Since $j = 0$ would clearly contradict the already established (gr1), $1 \leq j \leq n$. Then Corollary 22, (14) and the minimality of j implies that $\varpi(c_m \vee d_j) \in F$. Consider j' guaranteed by (mr5). Since there is no F-cell in the j'-th column, Corollary 22 and (14) imply that $(c_m \vee d_{j'-1}, c_m \vee d_{j'}) \notin \beta$. On the other hand, since $[c_m]\beta$ contains $c_m \vee d_j$ and $c_m \leq c_m \vee d_{j'-1} < c_m \vee d_{j'} < c_m \vee d_j$ and $[c_m]\beta$ is convex, we also get the opposite relation, $(c_m \vee d_{j'-1}, c_m \vee d_{j'}) \in \beta$. This contradiction shows that c_m is the least element of $[c_m]\beta$, and (gr2) for $(G; \beta)$ follows by left-right symmetry.

Finally, assume that $1 \le k \le m-1$ and $(c_k, d_k) \in \beta$. Then $(c_k, c_k \vee d_k) \in \beta$, whence $(c_k \vee d_{j-1}, c_k \vee d_j) \in \beta$ for $j = 1, \dots, k$. Using Corollary 22 and (14) as before, we derive that $\mathrm{Cells}(\downarrow(c_k \vee d_k))$ contains at least k distinct F-cells. This contradicts (mr3), and we conclude that $(G; \beta)$ satisfies (gr3). We have proved the second part of Proposition 20: $\sigma(A) = (G; \beta)$ is indeed a regular grid system.

The third part, $\sigma(\mu(K)) = K$ is just the statement of Lemma 23.

To prove the fourth part, we have to show that, for β defined in (13), $F = \mathrm{Sc}(\beta)$. Let $\varpi(u)$ be a cell of G with left corner a and right corner b, that is, $\varpi(u) = (a \wedge b, a, b, u = a \vee b)$.

Assume first that $\varpi(u) \in F$. Then $(a, u), (b, u) \in \vartheta(u) \subseteq \beta$. If $(a \wedge b, a) \notin \beta$ and $(a \wedge b, b) \notin \beta$, then $\varpi(u) \in \mathrm{Sc}(\beta)$, as intended. Otherwise we have, say, $(a \wedge b, a) \in \beta$. It follows from Corollary 22 and (14) that the row of $\varpi(u)$ contains an F-cell distinct from $\varpi(u)$, contradicting (mr1). This proves that $F \subseteq \mathrm{Sc}(\beta)$.

Conversely, assume that $\varpi(u) \in \mathrm{Sc}(\beta)$. Then $(a, u) \in \beta$, whence it is in γ by (14). Hence there is an F-cell $\varpi(v)$ in the column of $\varpi(u)$ to the southeast of $\varpi(u)$ by Corollary 22. However, $\varpi(v)$ is not to the southeast of the edge $a \wedge b \prec b$, because $(a \wedge b, b) \notin \beta$. Hence $v = u$, implying $\varpi(u) = \varpi(v) \in F$. Thus, $\mathrm{Sc}(\beta) \subseteq F$. □

Keeping Definitions 16 and 19 in mind, the composite of Propositions 17 and 20 reads as follows.

Theorem 24 (Main Theorem)**.** *Let L be a (glued sum) indecomposable slim semimodular non-chain lattice, and let $A = (G; F)$ be a regular grid matrix. Then*

- $\mathrm{Mtx}(L) := \mu(\psi(L))$ *is a regular grid matrix;*
- $\mathrm{Lat}(A) := \varphi(\sigma(A))$ *is an indecomposable slim semimodular non-chain lattice;*
- $\mathrm{Lat}(\mathrm{Mtx}(L)) \cong L$*;*
- $\mathrm{Mtx}(\mathrm{Lat}(A)) \cong A$*.*

As an illustration, Figure 2 depicts an L in the scope of this theorem together with the associated grid matrix $\mathrm{Mtx}(L) = (G^*; F)$, where F is the collection of the grey cells, and the corresponding 0-1 matrix.

5. A category theoretical version

Next, we briefly give a slightly strengthened formulation of the Main Theorem. Readers not familiar with the rudiments of category theory may want to skip over this section.

Corollary 25. *Consider the category $\mathcal{S}$ of indecomposable slim semimodular non-chain lattices with isomorphisms, the category $\mathcal{R}$ of regular grid systems with isomorphism, and the category $\mathcal{M}$ of regular grid matrices with isomorphisms. These three categories are equivalent.*

Proof. For $K = (G; \alpha)$ and $K' = (G'; \alpha')$ in $\mathcal{R}$ and an isomorphism $f\colon K \to K'$, we define $\varphi(f)$ to be the mapping $G/\alpha \to G'/\alpha'$, $[x]\alpha \mapsto [f(x)]\alpha'$. It is easy to check that this way φ becomes a functor $\mathcal{R} \to \mathcal{S}$. In order to prove that this functor is an equivalence, S. Mac Lane [12, Theorem IV.4.1] makes it sufficient to show that φ is full, it is faithful, and each object L of $\mathcal{S}$ is isomorphic to the φ-image of an appropriate object K of $\mathcal{R}$.

The last condition is evident, because L is isomorphic to $\varphi(\psi(L))$ by Proposition 17.

Since top corners are necessarily mapped to top corners, it is easy to see that each object of $\mathcal{R}$ has at most two automorphisms, and the same holds for each object of $\mathcal{S}$. (Note that if an object has two automorphisms then $m = n$.) Therefore the hom sets, both in $\mathcal{R}$ and $\mathcal{S}$, consist of at most two morphisms. Hence it is straightforward to check that, for $K, K' \in \mathcal{R}$, $\varphi\colon \hom(K, K') \to \hom(\varphi(K), \varphi(K'))$ is surjective and injective. That is, φ is full and faithful. This proves that $\varphi\colon \mathcal{R} \to \mathcal{S}$ is an equivalence functor.

For an $\mathcal{M}$-morphism f, let $\sigma(f) = f$ (the same mapping). This way σ becomes a functor $\mathcal{M} \to \mathcal{R}$. Essentially the same argument as above proves that σ is an equivalence.

Finally, the composite functor $\mathrm{Lat} = \sigma \circ \varphi$ is an $\mathcal{M} \to \mathcal{S}$ equivalence. $\square$

Remark 26. In contrast with φ, ψ is *not* a functor. The reason is that $\psi(L)$ is defined only up to isomorphism in general. Indeed, if $m^* = n^*$, then left and right, that is, c^* and d^*, can be interchanged. Similarly, μ is not a functor either.

6. Concluding remarks

Remark 27. It is known that each slim semimodular lattice L is a join-homomorphic image of a grid; see [10], and see also [3] and M. Stern [14, Theorem 6.3.14]. Since we have not used this result in our proofs, now it becomes an easy corollary of Proposition 17. Indeed, L is clearly a join-homomorphic image of the grid $\psi(L)$, provided L is indecomposable, while the non-indecomposable case follows easily from the indecomposable one.

Remark 28. Replacing "$\mathrm{typ}(K) \le \mathrm{typ}(K')$" by "$|G| \le |G'|$" in Definition 4, we get an alternative notion of a minimal grid system. Let us call the new variant a *size minimal grid system*, while the original notion is referred to as a *type minimal grid system*. It is not hard to see that size minimal grid systems and type minimal grid systems are the same. Indeed, "size minimal" evidently implies "type minimal", which implies "regular" in virtue of Lemma 9. Finally, (10) in the proof of Proposition 15 takes care of the implication "regular" $\Rightarrow$ "size minimal".

Remark 29. Combining Propositions 15 and 17 we obtain that, for each indecomposable slim semimodular non-chain lattice L, there exists a *minimal* grid system $K = (G; \alpha)$ with $L \cong G/\alpha$, and this K is *unique* up to isomorphism. While the existence part seems to be a consequence of [10], the uniqueness part does not.

Remark 30. Grid matrices can easily be described by *0-1 matrices*. Namely, the grid matrix $(G; F)$ of type (m, n) is described by the 0-1 matrix $B = (b_{ij})_{m\times n}$, where $b_{ij} = 1$ for $\varpi(i, j) \in F$ and $b_{ij} = 0$ for $\varpi(i, j) \notin F$. Pictorially, B is derived in the following way: first we write 1 into every grey cell and we write 0 elsewhere, then we turn the diagram by 45 degrees clockwise, and finally reflect it with respect to a horizontal axis. However, we should be careful, because the 0-1 matrix is not unique when $m = n$. (To make it unique we should make no distinction between a matrix and its transposed matrix.)

Remark 31. Another way to describe matrices is to use vectors. Given a grid matrix $A = (G; F)$, its *grid vector* $\vec{v}(A) = \vec{v}(G; F)$ is defined to be

$$\vec{v}(A) =:= (\mathrm{typ}_\ell(A), \mathrm{typ}_r(A) : (i_1, j_1), \ldots, (i_t, j_t)),$$

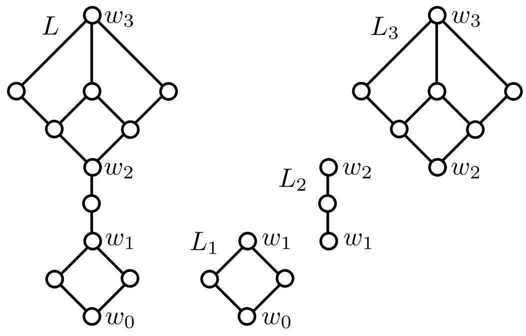


FIGURE 7. The glued sum decomposition of L

where $F = \{\varpi(i_1, j_1), \ldots, \varpi(i_t, j_t)\}$. The ": $(i_1, j_1), \ldots, (i_t, j_t)$" part is missing if $F = \emptyset$. For example, the grid vector of the matrix in Figure 3 is

$$(4, 5 : (1, 3), (2, 1), (3, 4)),$$

while that of the matrix in Figure 2 is

$$(4, 6 : (1, 5), (2, 2), (4, 6)).$$

As the first attempt to make the grid vector unique, we can try to stipulate that (i_1, j_1), …, (i_t, j_t) should be listed in lexicographic order; however, it helps only when $m < n$. Indeed, the grid vectors $(4, 4 : (2, 4), (3, 1))$ and $(4, 4 : (1, 3), (4, 2))$ determine the same grid matrix. (To make the grid vector of a grid matrix unique, we should make no distinction between two grid vectors if one of them can be "flipped" and then rearranged to the other one.)

In virtue of Theorem 24, an indecomposable slim semimodular non-chain lattice L is perfectly described by its grid vector $\vec{v}(\mathrm{Mtx}(L))$. Observe that $\vec{v}(\mathrm{Mtx}(L))$ provides a very *concise* description of L up to isomorphism.

Remark 32. Each slim semimodular lattice L can be decomposed as a glued sum

$$L_1 + \cdots + L_t \quad (t \in \mathbb{N}) \tag{16}$$

such that each L_i is either a chain or an indecomposable slim semimodular non-chain lattice and each chain summand in (16) is as long as possible. See Figure 7 for an illustration. This decomposition is unique. Although $\mathrm{Mtx}(L_i)$ is undefined if L_i happens to be a chain, let us agree that then $\vec{v}(\mathrm{Mtx}(L_i))$ simply means the length of the chain L_i, which is a nonnegative integer. With this convention, L can satisfactorily be described by the *grid hypervector*

$$\big(\vec{v}(\mathrm{Mtx}(L_1)); \ldots; \vec{v}(\mathrm{Mtx}(L_t))\big) \tag{17}$$

up to isomorphism. This is the *most concise* known description of a slim semimodular lattice L. For example, L from Figure 7 is described by the grid hypervector

$$\big((1, 1); 2; (2, 2 : (2, 2))\big).$$

Note that for *arbitrary* finite lattices the most concise known description is given by R. Wille [15].

Remark 33. According to [9] (cited in [6, Proposition 9]), all planar semimodular lattices can easily be obtained from slim ones. This together with (17) can be used to describe a planar semimodular lattice with a very little amount of data; the straightforward technical details will be omitted.

Remark 34. It is natural to ask what happens if $J_0(L)$ is the union of three but not fewer chains; however, there is not much hope for a satisfactory answer. While some of our auxiliary statements could be saved for this situation, the difficulty is well exposed by the following consideration. Let G_3 be the direct product of three non-singleton chains, and let α be a cover-preserving join-congruence of G_3. We can assume that $J_0(G_3/\alpha)$ is not the union of two chains. Then $K_3 = (G_3;\alpha)$ is a "three-dimensional" grid system, and we know from [3] that L is a cover-preserving join-homomorphic image of an appropriate K_3. Define $\mathrm{Se}(\alpha)$ as the set of $v \in G_3 \setminus J_0(G_3)$ satisfying

- v is not the least element of $[v]\alpha$; and
- for any covering square ϖ with top v and for any corner x of ϖ, α does not collapse x with the bottom of ϖ.

It follows from Lemma 1 that α collapses each $v \in \mathrm{Se}(\alpha)$ with all its lower covers. Then, as it is easy to see, (gr1) still implies that $\mathrm{Se}(\alpha)$ determines α. The real problem is how to characterize those subsets X of $G_3 \setminus J_0(G_3)$ that are of the form $\mathrm{Se}(\alpha)$, $(G_3;\alpha)$ being a three-dimensional grid. The key to the "two-dimensional case" is the easy observation that X is of this form iff it is an antichain. However, this condition is not sufficient even when G_3 is the eight-element Boolean lattice. Indeed, then two coatoms form a subset that cannot be $\mathrm{Se}(\alpha)$.

Remark 35. Let $\mathcal{S}_2$ and $\mathcal{M}_2$ denote the class of slim semimodular lattices and that of modular lattices generated by two chains, respectively. An anonymous referee has pointed out that there is an interesting analogy between these two classes. Indeed, semimodularity is a generalization of modularity, and slimness means being join-generated by two chains. A classical result of G. Birkhoff [1, Section III.7] states that all members of $\mathcal{M}_2$ are distributive. Note that after presenting his proof, Birkhoff acknowledges that similar arguments were used by O. Schreier [13] and and H. Zassenhaus [16]. Now that $\mathcal{M}_2$ has this very strong property, it is not so surprising that the analogous class, $\mathcal{S}_2$, has some nice properties as well. Like [9], [10] and [6], the present work is a part of the search for these properties.

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