The congruence variety of metaabelian groups is not self-dual

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For a ring R with unit element let $\mathcal{L}(R)$ denote the class of lattices embeddable in submodule lattices of R-modules. Then $\mathbf{H}\mathcal{L}(R)$, the variety generated by $\mathcal{L}(R)$, is a self-dual congruence variety by Hutchinson [6, Thm. 7, and 5]. On the other hand, non-modular congruence varieties need not be self-dual by Day and Freese [2]. The $\mathbf{H}\mathcal{L}(R)$ have been the only known congruence varieties for a long time, leading to the impression that the congruence variety of Abelian groups, alias $\mathbf{H}\mathcal{L}(\mathbf{Z})$, could be the largest modular congruence variety. This picture was refuted in two steps. First, an unpublished work of Kiss and Pálfy [7] showed that the congruence lattice of a certain metaabelian group cannot be embedded in the congruence lattice of any Abelian group. Developing these ideas further, Pálfy and Szabó [8, 9] have recently shown that the congruence variety of certain group varieties are not subvarieties of $\mathbf{H}\mathcal{L}(\mathbf{Z})$. This leads to the problem whether every modular congruence variety is self-dual, cf. Pálfy and Szabó [9, Problem 4.2] for a slightly different formulation. The aim of the present paper is to give a negative solution.

For a variety V let $\mathbf{Con}(V)$ denote the congruence variety of V, i.e., the lattice variety generated by the congruence lattices of all algebras in V. Let $\mathbf{M}$ be the variety of metaabelian groups. $\mathbf{M}$ is defined by the identity $[x,y]z = z[x,y]$ where $[x,y] = x^{-1}y^{-1}xy$. By the elementary properties of the commutator (cf., e.g., Gorenstein [3, Ch. 2.2]) it is easy to see that $\mathbf{M}$ satisfies the identities

$$\begin{aligned} [a,b]^{-1} &= [b,a] = [a^{-1},b] = [a,b^{-1}] \\ ba &= ab[a,b]^{-1} \\ [ab,c] &= [a,c][b,c], \quad [a,bc] = [a,b][a,c] \\ b^l a^k &= a^k b^l [a,b]^{-kl} \quad (k,l \in \mathbf{Z}). \end{aligned} \tag{1}$$

Let $\mathbf{A}$ be the variety of Abelian groups and let $\mathbf{M}_4$ be the variety generated by the quaternion group. Then $\mathbf{M}_4$ is a subvariety of $\mathbf{M}$, and it is defined by the identities $[x,y]z = z[x,y]$, $x^4 = 1$ and $[x,y]^2 = 1$. Pálfy and Szabó [8, 9] gave identities satisfied in $\mathbf{Con}(\mathbf{A})$ but not in $\mathbf{Con}(\mathbf{M}_4)$. However, the duals of their identities do the same, so we have to consider another identity. In the variables $\alpha_1, \alpha_2, \ldots, \alpha_{13}$ let us consider the following lattice terms:

$$p = (\alpha_{12} + \alpha_{13})\big(\alpha_4 + \alpha_5 + (\alpha_1 + \alpha_6 + \alpha_7)(\alpha_2 + \alpha_8 + \alpha_9)(\alpha_3 + \alpha_{10} + \alpha_{11})\big)$$

$$q_1 = \alpha_1 + \alpha_2 + \alpha_3, \quad q_2 = \alpha_6 + \alpha_7 + \alpha_{12} + \alpha_{13}, \quad q_3 = \alpha_1 + \alpha_4 + \alpha_5 + \alpha_{10}$$

$$q_4 = \alpha_3 + \alpha_8 + \alpha_9, \quad q_5 = \alpha_2 + \alpha_4 + \alpha_{10} + \alpha_{11} + \alpha_{12}, \quad q_6 = \alpha_2 + \alpha_{11} + \alpha_{12} + \alpha_{13}$$

$$q_7 = \alpha_4 + \alpha_5 + \alpha_7 + \alpha_8 + \alpha_9, \quad q_8 = \alpha_1 + \alpha_3 + \alpha_5, \quad q_9 = \alpha_6 + \alpha_7 + \alpha_8 + \alpha_{10} + \alpha_{11}$$

$$q_{10} = \alpha_3 + \alpha_6 + \alpha_9 + \alpha_{12} + \alpha_{13}, \quad q_{11} = \alpha_4 + \alpha_5 + \alpha_{10} + \alpha_{11}$$

$$q_{12} = \alpha_1 + \alpha_2 + \alpha_{13}, \quad q_{13} = \alpha_6 + \alpha_7 + \alpha_8 + \alpha_9, \quad \text{and}$$

$$q = q_1 + (q_2q_3 + q_4q_5)(q_6q_7 + q_8q_9)(q_{10}q_{11} + q_{12}q_{13}).$$

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Let μ_{13} denote the identity

$$p \leq q,$$

and let μ_{13}^d denote the dual of μ_{13} . Note that μ_{13} was found by modifying, in fact weakening, the dual of the identity in Pálfy and Szabó [8].

Theorem.
(A) μ_{13} holds in $\mathbf{Con}(\mathbf{M})$.
(B) μ_{13}^d fails in $\mathbf{Con}(\mathbf{M})$.

We will actually show that μ_{13}^d fails even in $\mathbf{Con}(\mathbf{M}_4)$. Therefore the modular congruence varieties $\mathbf{Con}(\mathbf{M})$ and $\mathbf{Con}(\mathbf{M}_4)$ are not self-dual.

PROOF. (B) The rather long calculations required by this part of the proof were done by a personal computer; here we outline the algorithm only. (A Pascal program, Borland's Turbo Pascal 6.0, is available from the author upon request.) The Wille — Pixley algorithm [10, 11] offers a standard way to check if a lattice identity holds in the congruence variety of a variety with permuting equivalences. Like in [6], we can construct a strong Mal'cev condition (MC) such that (MC) holds in $\mathbf{M}_4$ iff μ_{13}^d holds in $\mathbf{Con}(\mathbf{M}_4)$. This Mal'cev condition is a finite collection of n-ary term symbols f_k and equations of the form

$$f_l(x_{1C}, x_{2C}, \ldots, x_{nC}) = f_r(x_{1C}, x_{2C}, \ldots, x_{nC}) \qquad \text{or} \tag{2}$$

$$f_l(x_{1C}, x_{2C}, \ldots, x_{nC}) = x_j \tag{2'}$$

where C is a partition on the set $\{1, 2, \ldots, n\}$ and iC denotes the smallest element of the C-block containing i. Suppose μ_{13}^d holds in $\mathbf{Con}(\mathbf{M}_4)$, then there exist group terms f_k such that all the equations (2) and (2′) of (MC) are valid identities in $\mathbf{M}_4$. By Pálfy and Szabó [9] or the identities (1) each n-ary group term $g(x_1, \ldots, x_n)$ in $\mathbf{M}_4$ can be uniquely represented in the form

$$\prod_{i=1}^{n} x_i^{a_i} \prod_{i<j} [x_i, x_j]^{b_{ij}} \tag{3}$$

where $a_i \in \mathbf{Z}_4 = \{0, 1, 2, 3\}$ and $b_{ij} \in \mathbf{Z}_2 = \{0, 1\}$. Here $\prod_{i=1}^{n} x_i^{a_i}$ and $\prod_{i<j} [x_i, x_j]^{b_{ij}}$ are called the Abelian part and the commutator part of g, respectively.

The variety of Abelian groups of exponent four is a subvariety of $\mathbf{M}_4$, whence (MC) holds in it. Since $\mathbf{M}_4$ is term equivalent to the variety of modules over $\mathbf{Z}_4$, we can use the algorithm described in [6] to determine the $a_i^{(k)}$, the exponents occurring in the Abelian part of f_k according to (3). Luckily enough, these $a_i^{(k)}$ are uniquely determined by (MC).

Now let $C_1, \ldots, C_w$ be the blocks of a partition C such that the minimal representatives $c_i \in C_i$ satisfy $c_1 < c_2 < \ldots < c_w$. For a term g of the form (3) the term $g(x_{1C}, \ldots, x_{nC})$ can be written in the (unique) form

$$\prod_{i=1}^{w} x_{c_i}^{d_i} \prod_{i<j} [x_{c_i}, x_{c_j}]^{t_{ij}}$$

Here $d_i = \sum_{j\in C_i} a_j$. To determine the t_{ij} for $i < j$ let us consider an $u \in C_i$ and a $v \in C_j$. If $u < v$ then $[x_u, x_v]^{b_{uv}}$ turns into $[x_{c_i}, x_{c_j}]^{b_{uv}}$. If $u > v$ then $[x_v, x_u]^{b_{vu}}$ turns into $[x_{c_j}, x_{c_i}]^{b_{vu}} = [x_{c_i}, x_{c_j}]^{-b_{vu}}$ and exchanging the places of $x_{c_j}^{a_v}$ and $x_{c_i}^{a_u}$ in the Abelian part enters $[x_{c_i}, x_{c_j}]^{-a_u a_v}$ as well. Combining all these effects we obtain that

$$t_{ij} = \sum_{\substack{u<v \\ u\in C_i,\ v\in C_j}} b_{uv} - \sum_{\substack{u>v \\ u\in C_i,\ v\in C_j}} (b_{vu} + a_u a_v). \tag{5}$$

Therefore, if the a_i and b_{ij} for f_k are denoted by $a_i^{(k)}$ and $b_{ij}^{(k)}$, (2) implies

$$\begin{aligned} &\sum_{\substack{u<v \\ u\in C_i,\ v\in C_j}} b_{uv}^{(l)} - \sum_{\substack{u>v \\ u\in C_i,\ v\in C_j}} (b_{vu}^{(l)} + a_u^{(l)} a_v^{(l)}) = \\ &\qquad \sum_{\substack{u<v \\ u\in C_i,\ v\in C_j}} b_{uv}^{(r)} - \sum_{\substack{u>v \\ u\in C_i,\ v\in C_j}} (b_{vu}^{(r)} + a_u^{(r)} a_v^{(r)}) \end{aligned} \tag{6}$$

for all meaningful $i < j$. The equations (6) and the analogous equations derived from (2′) constitute a system of linear equations over the two-element field with the $b_{uv}^{(k)}$ being the unknowns. Using some reductions, including the one offered by [1, Prop. 2] or its special case for groups [9, Lemma 1.1], the system eventually considered consists of 130 equations for 108 unknowns. Since this system proved to be unsolvable, μ_{13}^d fails in $\mathbf{Con}(\mathbf{M})$.

(A) Assume that $\alpha_1, \alpha_2, \ldots, \alpha_{13}$ are congruences of a metaabelian group $G \in \mathbf{M}$ and y_1 is an element of $[1]p$, the $p(\alpha_1, \alpha_2, \ldots, \alpha_{13})$-block of the group unit 1. From the permutability of group congruences and $(1, y_1) \in p$ we infer that there exists an element $y_2 \in G$ such that $(1, y_2) \in \alpha_{12}$ and $(y_2, y_1) \in \alpha_{13}$. Parsing the lattice term p further we obtain elements $y_3, y_4, \ldots, y_{13} \in G$ such that

$$\begin{gathered} (1, y_4) \in \alpha_4, \quad (y_4, y_3) \in \alpha_5, \quad (y_3, y_5) \in \alpha_1, \quad (y_5, y_7) \in \alpha_6, \\ (y_7, y_1) \in \alpha_7, \quad (y_3, y_6) \in \alpha_2, \quad (y_6, y_9) \in \alpha_8, \quad (y_9, y_1) \in \alpha_9, \\ (y_3, y_8) \in \alpha_3, \quad (y_8, y_{10}) \in \alpha_{10}, \quad (y_{10}, y_1) \in \alpha_{11}. \end{gathered}$$

Consider the group elements

$$\begin{aligned} f_1 &= y_1 y_5^{-1} y_6 [y_1, y_2][y_1, y_6]^{-1}[y_2, y_5][y_3, y_6]^{-1}[y_3, y_9][y_6, y_9]^{-1}, \\ f_2 &= y_3^{-1} y_5^{-1} y_6 y_8 [y_1, y_3][y_1, y_5]^{-1}[y_2, y_5][y_2, y_8]^{-1}[y_3, y_5]^{-1}[y_3, y_9][y_6, y_9]^{-1}, \\ f_3 &= y_3^{-1} y_6 y_8 [y_2, y_3][y_2, y_8]^{-1}[y_3, y_9][y_6, y_9]^{-1}, \\ f_4 &= y_1 y_5^{-1} y_8 [y_2, y_5][y_2, y_8]^{-1}. \end{aligned}$$

We claim that

$$\begin{aligned} &(1, f_2) \in q_1, \quad (f_2, f_1) \in q_{11}, \quad (f_2, f_1) \in q_{10}, \\ &(f_1, y_1) \in q_{12}, \quad (f_1, y_1) \in q_{13}, \quad (f_2, f_3) \in q_3, \\ &(f_2, f_3) \in q_2, \quad (f_3, y_1) \in q_4, \quad (f_3, y_1) \in q_5, \quad (f_2, f_4) \in q_6, \\ &(f_2, f_4) \in q_7, \quad (f_4, y_1) \in q_8, \quad (f_4, y_1) \in q_9. \end{aligned} \tag{7}$$

Each of the relations of (7) follows easily from (1) and the definitions. E.g., to verify $(f_2, f_1) \in q_{11}$ we can compute as follows. Since 1, y_4 and y_3 are pairwise congruent modulo q_{11} and so are y_1 and y_8 we obtain

$$
\begin{aligned}
f_2 \; q_{11} \; & 1^{-1} y_5^{-1} y_6 y_1 [y_1, 1][y_1, y_5]^{-1}[y_2, y_5][y_2, y_1]^{-1}[1, y_5]^{-1}[1, y_9][y_6, y_9]^{-1} = \\
& y_5^{-1} y_6 y_1 [y_1, y_5^{-1}][y_2, y_5][y_1, y_2][y_6, y_9]^{-1} = \\
& y_5^{-1} y_1 y_6 [y_1, y_6]^{-1}[y_1, y_5^{-1}][y_2, y_5][y_1, y_2][y_6, y_9]^{-1} = \\
& y_1 y_5^{-1}[y_1, y_5^{-1}]^{-1} y_6 [y_1, y_6]^{-1}[y_1, y_5^{-1}][y_2, y_5][y_1, y_2][y_6, y_9]^{-1} = \\
& y_1 y_5^{-1} y_6 [y_1, y_6]^{-1}[y_2, y_5][y_1, y_2][y_6, y_9]^{-1} \qquad \text{and} \\
f_1 \; q_{11} \; & y_1 y_5^{-1} y_6 [y_1, y_2][y_1, y_6]^{-1}[y_2, y_5][1, y_6]^{-1}[1, y_9][y_6, y_9]^{-1} = \\
& y_1 y_5^{-1} y_6 [y_1, y_6]^{-1}[y_2, y_5][y_1, y_2][y_6, y_9]^{-1},
\end{aligned}
$$

showing $(f_2, f_1) \in q_{11}$. From (7) it follows that $(1, y_1) \in q$. Therefore the p-class of 1 is included in the q-class of 1. By the canonical bijection between group congruences and normal subgroups we conclude that μ_{13} holds in **Con**(**M**).

Problem. Note that, in spite of some particular positive results of Haiman [4], it is still an open question if the variety generated by all linear lattices is self-dual. Thus it would be interesting to know if μ_{13} holds in every linear lattice, but we do not know even if it holds in the normal subgroup lattice of any group.

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