SUMS OF LATTICES AND A RELATIONAL CATEGORY

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Abstract. We introduce a new relational category of lattices, and an analogous category of complete lattices. These categories allow us to construct sums of (complete) lattices. While previous constructions used two functors (or, for complete lattices, a single functor that had an adjoint), we need only a single functor (and no additional property when complete lattices are considered). In the finite case, the present construction is easy to visualize.

1. Introduction

Roughly speaking, *sum* refers to a construction that, for any lattice L and a congruence Θ of L, produces L from $K = L/\Theta$ and the system of Θ-classes. As surveyed in Section 5, this construction and its particular cases and analogous constructions were considered by several papers, including [1], [2], [5], [6], [7], [15], [16] and [20], Graczyńska [5] being the first. (Details and further authors will be mentioned in Section 5.) Except for some particular cases ([1], [2] and [16], to be discussed later), the system of lattices to be summed is described by *two* functors from K into the category of lattices with certain *mappings* as morphisms.

The chief goal of this paper is to give an equivalent, easy-to-visualize construction based on a *single* functor into an appropriate category of (complete) lattices with certain *relations* as morphisms. "Easy-to-visualize" means that, in the finite case, sums can be perfectly described by diagrams like Figure 1. (Figure 1 will be explained later.)

In order to introduce the key notion of the present paper, let $L_1 = (L_1, \leq_1)$ and $L_2 = (L_2, \leq_2)$ be lattices. Roughly speaking, a relation $\rho \subseteq L_1 \times L_2$ will be called an *atop relation*, if taking disjoint copies of L_1 and L_2 and putting L_2 atop L_1 modulo ρ (that is, adding ρ to the union of $\leq_1$ and $\leq_2$) we obtain a lattice. (For example, $\rho = L_1 \times L_2$ is an atop relation, since putting L_2 atop L_1 modulo ρ gives a lattice, the ordinal sum of L_1 and L_2. Another example: if $L_1 = L_2 = L$, then ι_L, the usual ordering $\leq_L$ of L, is an atop relation, since putting L atop L modulo ι_L yields $L \times \mathbf{2}$, where $\mathbf{2}$ is the two-element lattice.)

More precisely, for $i = 1, 2$, let ι_i stand for $\leq_i$, and let $\iota_i' = \big\{\big((x, i), (y, i)\big) \colon (x, y) \in \iota_i\big\}$. Then a relation $\rho \subseteq L_1 \times L_2$ will be called an *atop relation*, if

$$\begin{aligned} &L = (L, \iota), \text{ where } L = L_1' \cup L_2', \ L_1' = L_1 \times \{1\} \ L_2' = L_2 \times \{2\}, \\ &\rho' = \big\{\big((x, 1), (y, 2)\big) \colon (x, y) \in \rho\big\}, \text{ and } \iota = \iota_1' \cup \iota_2' \cup \rho', \end{aligned} \tag{1}$$

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is a lattice. If, in addition, L_1, L_2 and L are complete lattices, then ρ is called a *complete atop relation.*

Atop resp. complete atop relations will be the morphisms of a new category $\mathcal{L}_r^\bullet$ of lattices resp. $\mathcal{C}_r^\bullet$ of complete lattices. The class of finite lattices determines exactly the same subcategory of $\mathcal{L}_r^\bullet$ as that of $\mathcal{C}_r^\bullet$. Lemma 2 will give a reasonable description for the morphisms of $\mathcal{C}_r^\bullet$; this allows reasonable pictorial representations. Proposition 5 will show that $\mathcal{C}_r^\bullet$ is embeddable into a standard category of complete lattices.

Suppose that K is a complete lattice, L is a lattice, and $S\colon K \to \mathcal{C}_r^\bullet$ and $T\colon L \to \mathcal{L}_r^\bullet$ are functors. We will define the sum ΣS of S and the sum ΣT of T. While ΣS is always a complete lattice, ΣT is an ordered set (also called "partially ordered set" or, recently, "order") in general, not necessarily a lattice. However, if all the $T(x)$, $x \in L$, are *bounded* lattices, then ΣT is a lattice. If ΣT is a lattice, then it is isomorphic to the sum constructed by Graczyńska and Grätzer [6].

Section 5 presents a brief survey of earlier sum constructions. The paper is concluded with a problem and a result on products of lattice varieties or, equivalently, on lattice properties preserved by forming sums.

2. Atop and complete atop relations

We need some notation. Let $L_1 = (L_1, \le_1) = (L_1, \iota_1)$ and $L_2 = (L_2, \le_2) = (L_2, \iota_2)$ be lattices. For $a \in L_i$, let $\downarrow a = \{x \in L_i \colon x \le_i a\}$ and, dually, $\uparrow a = \{x \in L_i \colon a \le_i x\}$. For $\rho \subseteq L_1 \times L_2$, $X \subseteq L_1$ and $Y \subseteq L_2$, let

$$X^\rho = \{y \in L_2 \colon (x, y) \in \rho \text{ for all } x \in X\} \text{ and}$$
$${}^\rho Y = \{x \in L_1 \colon (x, y) \in \rho \text{ for all } y \in Y\}.$$

Recall that ideals and filters are non-empty by definition. Our notation and terminology are standard, see Grätzer [9].

Lemma 1. *Let $L_1 = (L_1, \le_1)$ and $L_2 = (L_2, \le_2)$ be lattices, and let $\rho \subseteq L_1 \times L_2$. Then ρ is an atop relation iff for every $a_1 \in L_1$ and $a_2 \in L_2$, $\{a_1\}^\rho$ is a filter of L_2, ${}^\rho\{a_2\}$ is an ideal of L_1, $\{a_1\}^\rho \cap \uparrow a_2$ has a least element (that is, $\{a_1\}^\rho \cap \uparrow a_2$ is a principal filter of L_2), and $\downarrow a_1 \cap {}^\rho\{a_2\}$ has a greatest element.*

Proof. To show the "if" part, let L denote the ordered set defined in (1), and assume that $c = a \vee_1 b$ in L_1, $d \in L_2$, and $a, b \in {}^\rho\{d\}$. Then $\downarrow c \cap {}^\rho\{d\}$ is a principal ideal, so $\downarrow c \cap {}^\rho\{d\} = \downarrow e$ for some $e \in L_1$. Observe that $e \in \downarrow c$ and $c = a \vee_1 b \in \downarrow c \cap {}^\rho\{d\} = \downarrow e$ imply $e = c$, whence $(a, 1) \vee (b, 1) = (c, 1)$ in L. The rest is even easier. ☐

Lemma 2. *Let $L_1 = (L_1, \le_1) = (L_1, \iota_1)$ and $L_2 = (L_2, \le_2) = (L_2, \iota_2)$ be complete lattices, and let $\rho \subseteq L_1 \times L_2$. Then the following four conditions are equivalent, and (d) implies* (e).

- (a) *ρ is a complete atop relation.*
- (b) *For all subsets $X \subseteq L_1$ and $Y \subseteq L_2$, X^ρ is a principal filter of L_2 and ${}^\rho Y$ is a principal ideal of L_1.*
- (c) *There exist a complete meet-subsemilattice E_1 of L_1, a complete join-subsemilattice E_2 of L_2 and an order isomorphism $\varphi\colon E_1 \to E_2$ such that $\rho = \{(x_1, x_2) \colon x_1 \le_1 y \text{ and } y\varphi \le_2 x_2 \text{ for some } y \in E_1\}$.*

(d) *There are* E_1, E_2 *and* φ *as in Condition* (c) *such that, with the notation*

$$x^{*\rho} = \bigwedge\{y \in E_1 \colon x \leq_1 y\} \quad \text{and} \quad y_{*\rho} = \bigvee\{x \in E_2 \colon x \leq_2 y\},$$

we have $\rho = \{(x,y) \in L_1 \times L_2 \colon (x^{*\rho})\varphi \leq_2 y\}$
$= \{(x,y) \in L_1 \times L_2 \colon x\varphi \leq_2 y_{*\rho}\}$.

(e) $\{x\}^\rho = \uparrow((x^{*\rho})\varphi)$ *and* ${}^\rho\{y\} = \downarrow((y_{*\rho})\varphi^{-1})$ *for every* $x \in L_1$ *and* $y \in L_2$.

In connection with Conditions (c) and (d), note that $1 = \bigwedge \emptyset$ belongs to E_1 and $0 = \bigvee \emptyset$ belongs to E_2.

Proof of Lemma 2. Assume (b). Let X be an arbitrary subset of L_1, and let a be the join of X in L_1. Then there is a (unique) $b \in L_2$ such that $X^\rho = \uparrow b$. Since $x \in {}^\rho\{b\}$ for all $x \in X$ and ${}^\rho\{b\}$ is a principal ideal of L_1, we obtain that $a \in {}^\rho\{b\}$. This implies that a is the join of X in L. Now let $\emptyset \neq X_i \subseteq L_i$ with joins a_i in L_i, $i = 1, 2$. Since (L, ι) is a lattice by Lemma 1, $a_1 \vee a_2$ exists in L, and it is clearly the join of $X_1 \cup X_2$ in L. Hence (b) implies (a).

Still assuming (b), let us keep the rudiments of Galois connections (for example, pages 68-69 of Grätzer [8]) in mind. Then we see that $E_1' := \{{}^\rho Y \colon Y \subseteq L_2\}$ and $E_2' := \{X^\rho \colon X \subseteq L_1\}$ are closure systems and $\varphi' \colon E_1' \to E_2'$, $X \mapsto X^\rho$ is a dual isomorphism. Since E_1' consists of principal ideals and E_2' consists of principal filters, we conclude easily that (b) implies (c).

Clearly, (c) and (d) are equivalent.

Assume (d). Let $X \subseteq L_1$, and let a denote the join of X in L_1. To see that a is the join of X in L, assume that $b \in X^\rho$. Then $(x^{*\rho})\varphi \leq_2 b$ for all $x \in X$. The join of $\{(x^{*\rho})\varphi \colon x \in X\}$, taken in L_2, belongs to E_2, so it is of the form $y\varphi$ for some $y \in E_1$. Clearly, $y\varphi \leq_2 b$. Since φ is an isomorphism, $x \leq_1 x^{*\rho} \leq_1 y$ for all $x \in X$. Hence $a \leq_1 y$, and $a^{*\rho} \leq_1 y$. Therefore, $(a^{*\rho})\varphi \leq_2 y\varphi \leq_2 b$ gives that $(a, b) \in \rho$, so $a \leq b$ in L, as requested. Now, it is easy to see that (d) implies (a).

The rest of the proof is trivial, whence it is omitted. □

3. Two relational categories

Our morphisms will be relations. The product of two morphisms, α and β, is the usual $\alpha\beta = \big\{(x, y) \colon \text{ there is an } z \text{ with } (x, z) \in \alpha \text{ and } (z, y) \in \beta\big\}$.

Theorem 3. *The class* $\mathcal{L}_r^\bullet$ *of all lattices, as objects, together with all atop relations, as morphisms and the lattice orderings* ι_L *(also denoted by* $\leq_L$*), as identities, constitute a category.*

Proof. Clearly, the ι_L act as identity morphisms. Assume that $\rho \subseteq L_1 \times L_2$ and $\tau \subseteq L_2 \times L_3$ are atop relations. To ease the notations, we can assume that L_1, L_2 and L_3 are pairwise disjoint; then instead of L_i' and ι_i' of (1), we can work with L_i and ι_i in the natural way. Let $L_{ij} = L_i \cup L_j$ for $i \neq j$. Let $\leq_{12}$ be the union of $\leq_1$, $\leq_2$ and ρ. Similarly, let $\leq_{23}$ be the union of $\leq_2$, $\leq_3$ and τ. We know for $(i, j) \in \{(1, 2), (2, 3)\}$ that $L_{ij} = (L_{ij}, \leq_{ij})$ is a lattice. Let $\leq_{13}$ denote the union of $\leq_1$, $\leq_3$ and $\rho\tau$. Notice that L_i is clearly a convex sublattice of L_{ij} and $\leq_i$ is the restriction of $\leq_{ij}$ to L_i for all $i \neq j \in \{1, 2, 3\}$. We have to show that $(L_{13}, \leq_{13})$ is a lattice; then the statement will follow.

Let $a_1, b_1 \in L_1$. To show that they have a join in L_{13}, let $c_3 \in L_3$ such that $a_1, b_1 \leq_{13} c_3$. Then there are $a_2, b_2 \in L_2$ such that $a_1 \leq_{12} a_2 \leq_{23} c_3$ and $b_1 \leq_{12}$

$b_2 \le_{23} c_3$. Then $a_1 \vee_1 b_1 \le_{12} a_2 \vee_2 b_2 \le_{23} c_3$ since L_{12} and L_{23} are lattices. Hence $a_1 \vee_1 b_1 = a_1 \vee_{13} b_1$.

Next, let $a_1 \in L_1$ and $b_3 \in L_3$; we want to show that they have a join. Choose a $c_2 \in L_2$ such that $c_2 \le_{23} b_3$. (For example, $c_2 := u_2 \wedge_{23} b_3$ where $u_2 \in L_2$.) Define $d_2 := a_1 \vee_{12} c_2$ and $e_3 := d_2 \vee_{23} b_3$, and notice that e_3 is an upper bound of $\{a_1, b_3\}$ in L_{13}. Let $x_3 \in L_{13}$ be an arbitrary upper bound. Then $b_3 \le_{13} x_3$ gives $x_3 \in L_3$. Since $a_1 \le_{13} x_3$, there is a $y_2 \in L_2$ such that $a_1 \le_{12} y_2 \le_{23} x_3$. Let $y_2' = y_2 \wedge_2 d_2$, and notice that $a \le_{12} y_2' \le_2 d_2$. Hence $d_2 = a_1 \vee_{12} c_2 \le_{12} y_2' \vee_2 c_2 \le_2 d_2$ implies $d_2 = y_2' \vee_2 c$. Consequently, $e_3 = d_2 \vee_{23} b_3 = y_2' \vee_{23} c_2 \vee_{23} b_3 = y_2' \vee_{23} b_3 \le_{23} y_2 \vee_{23} b_3 \le_{23} x_3$. Therefore, $e_3 \le_3 x_3$ and $e_3 \le_{13} x_3$, proving that $e_3 = a_1 \vee_{13} b_3$. □

Proposition 4. *The class $\mathcal{C}_r^\bullet$ of complete lattices, as objects, with all the complete atop relations, as morphisms, and the lattice orderings ι_L, as identities, constitute a category.*

Proof. We keep several notations of the previous proof, but we use a different technique for the sake of a later reference to (2).

Let $X \subseteq L_1$. By (b) of Lemma 2, we want to show that

$$X^{\rho\tau} = {\uparrow} b_3, \tag{2}$$

$$\text{where } a_2 = \bigwedge X^\rho, \quad X^\rho = {\uparrow} a_2,$$

$$b_3 = \bigwedge \{a_2\}^\tau, \text{ and } \{a_2\}^\tau = {\uparrow} b_3.$$

Clearly, ${\uparrow} b_3 \subseteq X^{\rho\tau}$. To show the reverse inclusion, assume that $y_3 \in X^{\rho\tau}$. Then, for each $x \in X$, there is a $z_x \in \{x\}^\rho \cap {}^\tau\{y_3\}$. Set $z = \bigvee\{z_x \colon x \in X\}$ in L_2. Then $z \in {}^\tau\{y_3\}$, which gives $y_3 \in \{z\}^\tau$, and $z \in X^\rho = {\uparrow} a_2$, which gives $a_2 \le_2 z$. Hence $y_3 \in \{z\}^\tau \subseteq \{a_2\}^\tau = {\uparrow} b_3$, indeed. Thus, (2) is shown, and $\rho\tau$ is a complete atop relation by (b) of Lemma 2 and duality. □

Let $\mathcal{C}_\vee$ resp. $\mathcal{C}_\wedge$ denote the category of complete lattices with complete join-homomorphisms resp. complete meet-homomorphisms, respectively. A functor is called an *embedding functor* if it sends distinct morphisms to distinct morphisms. Define

$$F_\vee(L) = F_\wedge(L) = L, \quad F_\vee(\rho) \colon x \mapsto \bigwedge \{x\}^\rho, \quad F_\wedge(\rho) \colon y \mapsto \bigvee {}^\rho\{y\}.$$

Proposition 5. *$F_\vee \colon \mathcal{C}_r^\bullet \to \mathcal{C}_\vee$ is a covariant embedding functor, and $F_\wedge \colon \mathcal{C}_r^\bullet \to \mathcal{C}_\wedge$ is a contravariant embedding functor.*

Proof. (2) for singleton X shows that $F_\vee$ is a functor. The rest is trivial. □

Remark 6. Let $\mathbf{1}$ denote the one-element lattice. Then there is only one morphism $\rho\colon \mathbf{1} \to (\mathbf{Z}, \le)$ in $\mathcal{L}_r^\bullet$. However, there are infinitely many lattice homomorphisms from $\mathbf{1}$ to $(\mathbf{Z}, \le)$, but none of them has to do anything with ρ. This explains why we do not try to present a similar statement for $\mathcal{L}_r^\bullet$.

Remark 7. It is evident by Proposition 5 that two objects of $\mathcal{C}_r^\bullet$ are isomorphic (in the sense of category theory) iff they are isomorphic complete lattices.

Remark 8. Let ${\mathcal{C}_\wedge}^{\mathrm{op}}$ denote the category opposite to $\mathcal{C}_\wedge$, and let $I_{\mathrm{op}} \colon \mathcal{C}_\wedge \to {\mathcal{C}_\wedge}^{\mathrm{op}}$ be the contravariant functor which acts identically on objects and reverses all morphisms. Notice that we compose functors from left to right, that is, $(F_\wedge \circ I_{\mathrm{op}})(\rho)$

means $I_{\mathrm{op}}(F_\wedge(\rho))$. Then, by Proposition 5, ${F_\wedge}^{\mathrm{op}} = F_\wedge \circ I_{\mathrm{op}}$ is a covariant $\mathcal{C}_r^\bullet \to {\mathcal{C}_\wedge}^{\mathrm{op}}$ embedding functor, and so is $F_\vee \times {F_\wedge}^{\mathrm{op}} : \mathcal{C}_r^\bullet \to \mathcal{C}_\vee \times {\mathcal{C}_\wedge}^{\mathrm{op}}$. However, $F_\vee \times {F_\wedge}^{\mathrm{op}}$ is not an isomorphism. Indeed, if $\mathbf{2}$ denotes the two-element lattice, then there are nine $\mathbf{2} \to \mathbf{2}$ morphisms in $\mathcal{C}_\vee \times {\mathcal{C}_\wedge}^{\mathrm{op}}$ but there are only two in $\mathcal{C}_r^\bullet$.

4. The sum of atop and complete atop systems

Let K be a lattice, and consider it a small category with the usual sense: the objects are the elements of L, for $x \le y$ there is exactly one morphism $x \to y$, and there is no $x \to y$ morphism if $x \not\le y$. By a *K-indexed atop system* resp. *complete atop system* we mean a functor $S : K \to \mathcal{L}_r^\bullet$ resp. $S : K \to \mathcal{C}_r^\bullet$. Notice that S captures the following information: the set $\{S(x) : x \in K\}$ of (complete) lattices and the set $\{\rho_{xy} : x \le y \in K\}$ of (complete) atop relations, and we know that $\rho_{xy}\rho_{yz} = \rho_{xz}$ whenever $x \le y \le z \in K$. We call S a *complete atop system* resp. an *atop system*, if we do not want to specify K.

The *sum* ΣS of S is defined to be the ordered set $\Sigma S = (\Sigma S, \le)$, where $\Sigma S = \bigcup_{x\in K} \big(S(x) \times \{x\}\big)$ and $(u, x) \le (v, y)$ iff $x \le y$ and $(u, v) \in \rho_{xy}$.

Theorem 9. *Let K be a lattice, and let S be a K-indexed complete atop system. Then ΣS is a lattice. If K is a complete lattice, then so is ΣS.*

Proof. Let $H = \{(u_i, x_i) : i \in I\}$ be a subset of ΣS. Let $x = \bigvee_{i\in I} x_i$. For each i, S defines a unique complete atop relation $\rho = \rho_{x_i x} \subseteq S(x_i) \times S(x)$. Applying the notations of Lemma 2 to this ρ, let $v_i = (u_i^{*\rho})\varphi$. Armed with our previous statements, it is straightforward to check that $(\bigvee_{i\in I} v_i, x)$ is the join of H in K. $\square$

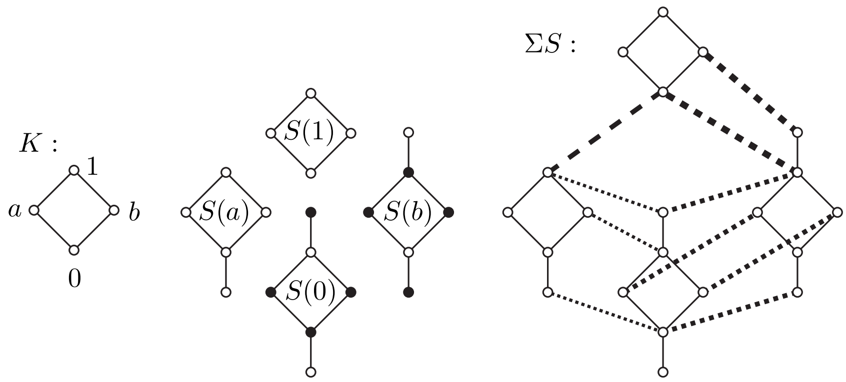

Figure 1. An example for a complete atop sum

Remark 10. It is an advantage of our approach that, in the finite case, we can visualize S very easily: for each covering pair $x \prec y$ in K, we add the graph of φ from Lemma 2 (that is, the $(e, e\varphi)$ edges) to the diagram between $S(x)$ and $S(y)$. This way we also get the diagram of ΣS. For example, see Figure 1. For ρ_{0b}, E_1 and E_2 from Condition (c) of Lemma 2 are denoted by black-filled elements. S is fully given in the right-hand side such that the graph of φ are drawn by thick dotted lines. (Different line styles are used in case of distinct complete atop relations.) If we change the dotted lines into solid lines (of usual thickness), then we obtain ΣS.

We define *atop systems* as $K \to \mathcal{L}_r^\bullet$ functors where K is a lattice. The sum of an atop system S is defined the same way as it was done for a complete atop system.

Remark 11. The sum of an atop system is an ordered set but not necessarily a lattice. For example, let $K = \{0, a, b, 1\}$ be the four-element boolean lattice, let $S(0) = S(a) = S(b)$ be the one-element lattice, and let $S(1) = (\mathbf{Z}, \leq)$. Then there is only one way to give the atop relations ρ_{xy}, $x \leq y \in K$, and ΣS is not a lattice.

Remark 12. If $S\colon K \to \mathcal{L}_r^\bullet$ is an atop system such that ΣS happens to be a lattice, then it is trivial to see that

$$\Theta = \big\{\big((u, x), (v, y)\big) \colon x = y\big\}$$

is a congruence of ΣS with blocks isomorphic to the $S(x)$, $x \in K$, and $(\Sigma S)/\Theta \cong K$. Conversely, if L is a lattice and Θ is a congruence of L, then L is (isomorphic to) the sum of the naturally defined L/Θ-indexed atop system of the Θ-blocks. Hence our construction gives the same lattice as earlier sums.

Proposition 13. *Let K be a lattice, and let $S\colon K \to \mathcal{L}_r^\bullet$ be an atop system such that $S(x)$ is a bounded lattice for every $x \in K$. Then ΣS is a lattice.*

Proof. Let $(a, x), (b, y) \in \Sigma S$. Take $z = x \vee y$, and let 0_z denote the least element of $S(z)$. By Lemma 1, $\{a\}^\rho \cap {\uparrow} 0_z$ has a unique least element a'. Define b' similarly. It is easy to see that $(a' \vee b', z)$ is the join of (a, x) and (b, y) in ΣS. □

5. Historical overview

Graczyńska [5] introduced her concept of sums of lattices closely related to Płonka sums [18]. Speaking in the present terminology, she defines systems of lattices by means of *two* functors; this is why her systems are called *double* systems. Since there is no stipulation how the two functors are related, the sum in [5] is a only a bisemilattice in general, see also Romanowska [20].

Following [5], Graczyńska and Grätzer [6] imposed some additional conditions, including some compatibility conditions on the two functors, to guarantee that the sum be a lattice. One can also see from [6] how complex the situation becomes when arbitrary lattices are treated. (Our construction is simpler, but sometimes it gives only an ordered set, not a lattice.) As a generalization of [6], double systems of ordered sets were considered by Höft [15].

Bandelt [1] points out that the situation is much better if one considers complete lattices only. Then the mappings in his system are residuated mappings. That is, he uses only one functor, but he assumes that this functor has a right adjoint. (We can also say that he uses two functors, but each of them is determined by the other one, so "half of the double system" can be disregarded.)

Notice that, for complete lattices, instead of our functor $S\colon K \to \mathcal{C}_r^\bullet$, [6] and [1] use the two components of the product functor $S \circ (F_\vee \times {F_\wedge}^{\mathrm{op}})$, and [1] points out that one of the components is sufficient.

There are approaches motivated by and used in structural descriptions of some concrete lattices. (Notice that they, [2] and [16], do not recognize that they are rediscovering particular cases of a previously known sum construction.)

The construction given in Jedlička [16] is used to describe some lattices arising from the Coxeter group. In effect, [16] uses only one system of mappings (which is not a functor but does something similar), but his system consists of isomorphic copies of a fixed lattice, indexed by another lattice.

The basic motivation of all the above-mentioned lattice constructions is that one starts from a congruence Θ (a complete congruence in [1]) of a lattice L, and wants to reconstruct L from $K = L/\Theta$ and the system of the Θ-blocks, indexed by K. (Hence these constructions automatically give a *decomposition* result for L). Opposed to these constructions, congruences are not even mentioned in [2], where sums of **2**-indexed systems are defined, and these sums are used to give a structure theorem for coalition lattices. Similarly, the purpose of an atop or a complete atop system is rather to *build* than to decompose a lattice. We believe that, in the practice, building needs simpler and easier-to-visualize tools than decomposing. This is why we do not add complicated further conditions to the definition of an atop system, and we do not go further than Proposition 13.

Although the present paper is based on relations rather than mappings, several earlier ideas from Bandelt [1], [2], Graczyńska [5], and Graczyńska and Grätzer [6] are encoded into Lemmas 1 and 2.

Sums are particularly useful when gluings are considered, see Herrmann [14] and Day and Herrmann [4]. Particular gluings were studied in Grygiel [11], [12], and in her further papers listed in [13]. Even the simplest gluing, the Dilworth gluing, is better understood if we consider it a quotient lattice of the sum of a special **2**-indexed system.

6. On properties preserved by sums

Let $\mathcal{U}$ and $\mathcal{V}$ be two classes of lattices, and keep Remark 12 in mind. Then the *product* $\mathcal{U} \circ \mathcal{V}$, which is due to Mal'cev [17] (see also Day [3] and Grätzer and Kelly [10]) is defined as follows:

$$\begin{aligned}\mathcal{U} \circ \mathcal{V} = \{\Sigma S \colon \Sigma S \text{ is a lattice, } S \colon K \to \mathcal{L}_r^\bullet,\\ K \in \mathcal{V}, \text{ and } S(x) \in \mathcal{U} \text{ for all } x \in K\}.\end{aligned}$$

It is natural to ask which lattice properties are preserved by forming sums. According to the above formula, it is convenient to ask this question in terms of products of classes. By a proper class we mean a class that has a lattice with more than one element.

Remark 14. There are continuumly many proper quasivarieties $\mathcal{W}$ of lattices such that $\mathcal{W} \circ \mathcal{W} \subseteq \mathcal{W}$.

Indeed, take a finite simple lattice M, and let $\mathcal{W}_M$ be the class of lattices that have no sublattice isomorphic to M. A Horn sentence defining $\mathcal{W}_M$ is described as follows: consider the elements of M as variables, let $a \neq b \in M$, and let T be the operation table of M, then $T \Rightarrow a = b$ is the desired Horn sentence. Hence $\mathcal{W}_M$ is a quasivariety. We can clearly see, or read in the introduction of Grätzer and Kelly [10], that $\mathcal{W}_M \circ \mathcal{W}_M \subseteq \mathcal{W}_M$. For a prime number p, let $M(p)$ be the subspace lattice of the projective plane over the p-element field; it is a finite simple lattice. If H is a subset of the set of all primes, then $\mathcal{W}_H = \bigcap\{\mathcal{W}_{M(p)} \colon p \in H\}$ is a quasivariety, and $\mathcal{W}_H \circ \mathcal{W}_H \subseteq \mathcal{W}_H$. Finally, $H_1 \neq H_2$ implies $\mathcal{W}_{H_1} \neq \mathcal{W}_{H_2}$.

Opposed to Remark 14, the situation for varieties is entirely different.

Remark 15. If $\mathcal{V}$ is a nontrivial variety of lattices such that $\{\mathbf{1}, \mathbf{2}\} \circ \mathcal{V} \subseteq \mathcal{V}$, then $\mathcal{V}$ is the variety of all lattices. (Although the main result of Day [3] says that $\mathcal{V} \circ \mathcal{V} \subseteq \mathcal{V}$ implies $\mathcal{V} = \{\text{all lattices}\}$, the proof uses only that $\{\mathbf{1}, \mathbf{2}\} \circ \mathcal{V} \subseteq \mathcal{V}$.)

Motivated by the above remark, we formulate

Problem 16. Let $\mathcal{W}$ be a non-trivial lattice variety such that $\mathcal{W} \circ \{\mathbf{1}, \mathbf{2}\} \subseteq \mathcal{W}$. Equivalently, such that $\mathcal{W}$ is closed with respect to **2**-indexed atop sums. Does it follow that $\mathcal{W}$ is the largest variety of lattices?

Even if we do not know if $\mathcal{W}$ is the largest, next we show that it is large.

Proposition 17. *Let $\mathcal{W}$ be a non-trivial lattice variety closed with respect to* **2**-*indexed atop sums. Then the free lattice of $\mathcal{W}$ on three generators is infinite.*

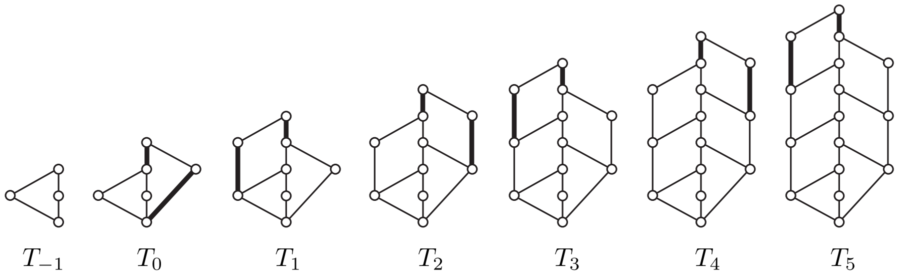

FIGURE 2. Poguntke lattices

Proof. Consider the herringbone like lattices T_n of Poguntke [19]. Figure 2 depicts the first few of these lattices. By [19], T_n is three-generated for all $0 \leq n < \infty$. It suffices to show that $T_n \in \mathcal{W}$ for all $-1 \leq n$. Since T_{-1} is distributive, it is in $\mathcal{W}$. For $0 \leq n$, T_n is a **2**-indexed sum of T_{n-1} and **2**; the graph of φ from Part (c) of Lemma 2 is indicated by thick lines. Hence all the T_n belongs to $\mathcal{W}$. □

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