STRONGER ASSOCIATION RULES FOR POSITIVE ATTRIBUTES

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Abstract. By a context we mean a binary table with crosses at some entries, i.e. a relation between two sets. The elements of these sets are called objects (= row labels) and attributes (= column labels). Each context determines a pair of Galois closure operators. This gives rise to formal concept analysis, cf. Ganter and Wille [6], and also to studying strong association rules in data mining, cf. Agrawal, Imielinski and Swami [1]; the term "association rule" being kept for the fuzzy version. There are cases where the Galois closure is too large or, in other words, even the strong association rules challenge decision making with too many choices.

In [3], some stronger association rules (i.e., a smaller pair of closure operators) have been introduced. Their mathematical features and possible further applicability have been studied in [4] and [5]. While [3] makes it clear that the new operator is useful in (pure) algebra, [4] and [5] point out that we expect its use in applied fields only when all the attributes are advantageous or good or useful, shortly, if the attributes are *positive*.

The goal of this paper is to introduce a more general pair of closure operators, smaller than the Galois one, such that the corresponding stronger association rules take into account that not all the attributes are positive.

The main result confirms that our definition gives indeed closure operators. A lot of emphasis is put on detailing how and why the new stronger association rules promise future applications although no concrete database has been analyzed from this aspect yet.

The history of science has several examples showing that a proper treatment, arrangement or visualization of information can be the source of new information. Many of these examples witness that the mathematical tool was developed much before any application of this kind. For the classical periodic system of chemical elements Mendeleyev resorted to the ancient "mathematical" notion of binary tables. Formal concept analysis, cf. Wille [9] and Ganter and Wille [6], uses an old concept that goes back to Évariste Galois.

The mathematical tool we intend to generalize in order to make it more applicable is quite recent. It was introduced and successfully used in [3]. However, the argument pro its applicability in [4] and [5] relies on an assumption which does not always holds. Our goal is to drop this restricting assumption. We will explain in details what sort of applications in information processing and decision making is kept in mind, and we strongly hope that this dream will come true. However,

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developing real applications in information theory or in other sciences will remain a task for specialists of these fields.

Although the terminology of formal concept analysis is frequently used throughout, as long as no real applications are available at hand, we cannot say that this work has a citizenship in the realm of formal concept analysis. In what follows, the mathematics and our motivations will be developed simultaneously.

Following Wille's terminology, cf. [9] or [6], a triplet

$$(A^{(0)}, A^{(1)}, \rho)$$

is called a *context* if $A^{(0)}$ and $A^{(1)}$ are nonempty sets and $\rho \subseteq A^{(0)} \times A^{(1)}$ is a binary relation. It is often, especially in the finite case, convenient to depict our context in the usual form: a binary table with row labels from $A^{(0)}$, column labels from $A^{(1)}$, and a cross in the intersection of the x-th row and the y-th column iff $(x, y) \in \rho$. We will refer to this table as the *context table.* For example, a context is given by Table 1. (We should disregard from the + signs at this stage.) We think of the elements of $A^{(0)}$, i.e. the row labels, as *objects* while the elements of $A^{(1)}$, i.e. the column labels, are called attributes. Then $(x, y) \in \rho$ means that the object x has the attribute y.

	b_1	$+b_2$	$+b_3$	$+b_4$
$+a_1$	×	×		
$+a_2$	×		×	
$+a_3$	×	×		×
$+a_4$	×			
$+a_5$		×	×	×

Table 1

For example, $A^{(0)}$ may consist of courses offered by a university, chemical compounds in pharmacy, patients of a psychologist, types of cars, etc. Then the respective $A^{(1)}$ may consist of certain skills or prerequisites, certain physiological effects, certain symptoms, some technical attributes (like having an automatic gearshift), etc. Here we think of *finite* $A^{(0)}$ and $A^{(1)}$ but our forthcoming theorem will be valid for the infinite case as well.

From what follows, we fix a context $(A^{(0)}, A^{(1)}, \rho)$ and let

$$\rho_0 = \rho \text{ and } \rho_1 = \rho^{-1}.$$

Unless otherwise stated, i will be an arbitrary element of $\{0, 1\}$. So whatever we say including i without specification, it will be understood as prefixed by $\forall i$. The set of all subsets of $A^{(i)}$ will be denoted by $P(A^{(i)})$.

As usual, a mapping $\mathcal{D}^{(i)} : P(A^{(i)}) \to P(A^{(i)})$ is called a *closure operator* if it is *extensive* (i.e., $X \subseteq \mathcal{D}^{(i)}(X)$ for all $X \in P(A^{(i)})$), *monotone* (i.e., $X \subseteq Y$ implies $\mathcal{D}^{(i)}(X) \subseteq \mathcal{D}^{(i)}(Y)$), and *idempotent* (i.e., $\mathcal{D}^{(i)}(\mathcal{D}^{(i)}(X)) = \mathcal{D}^{(i)}(X)$ for all $X \in P(A^{(i)})$). If $\mathcal{D}^{(i)}$ is a closure operator for $i = 0, 1$ then $\mathcal{D} = (\mathcal{D}^{(0)}, \mathcal{D}^{(1)})$ is called a *pair of closure operators.* If $\mathcal{E} = (\mathcal{E}^{(0)}, \mathcal{E}^{(1)})$ is another such pair then let $\mathcal{D} \leq \mathcal{E}$ mean that $\mathcal{D}^{(i)}(X) \subseteq \mathcal{E}^{(i)}(X)$ for all $i \in \{0, 1\}$ and all $X \in P(A^{(i)})$.

From the perspective of applied mathematics it is worth noting that closure operators have been playing an important role in the theory of relational databases and knowledge systems for a long time, cf. e.g., Caspard and Monjardet [2] for a survey. Nowadays most investigations of this kind belong to formal concept analysis, cf. Ganter and Wille [6] for an extensive survey. Closure operators are also important in the theory of mining association rules, which goes back to Agrawal, Imielinski and Swami [1]; Lakhal and Stumme [7] gives a good account on the present status of this field.

Now, associated with $(A^{(0)}, A^{(1)}, \rho)$, we define a pair of closure operators. For $X \in P(A^{(i)})$ let

$$X\rho_i = \{y \in A^{(1-i)} : \text{ for all } x \in X,\ (x, y) \in \rho_i\},$$

and, again for $X \in P(A^{(i)})$, define

$$\mathcal{G}^{(i)}(X) := (X\rho_i)\rho_{1-i} = \bigcap_{y \in X\rho_i} (\{y\}\rho_{1-i})\ .$$

Then $\mathcal{G} = (\mathcal{G}^{(0)}, \mathcal{G}^{(1)})$ is the well-known *pair of Galois closure operators*, which plays the main role in formal concept analysis, cf. Wille [9] and Ganter and Wille [6]. The visual meaning of

$$\mathcal{G} = \mathcal{G}(A^{(0)}, A^{(1)}, \rho)$$

is the following. The maximal subsets of ρ of the form $U^{(0)} \times U^{(1)}$ with $U^{(i)} \subseteq A^{(i)}$ are called the (formal) *concepts*, cf. [9] or [6]. Pictorially, they are the maximal full rectangles $U^{(0)} \times U^{(1)}$ of the context table. (Full means that each entry is a cross.) For $X_i \in P(A^{(i)})$ take all maximal full rectangles $U^{(0)} \times U^{(1)}$ such that $X \subseteq U^{(i)}$, then $\mathcal{G}^{(i)}(X)$ is the intersection of all the $U^{(i)}$'s.

Now, to develop our motivations further, we think of a (huge) context which is typical in warehouse basket analysis. Let $A^{(0)}$ be the set of costumers' baskets (i.e, the set of costumers) and let $A^{(1)}$ be the set of items sold in the warehouse. Data miners want to compute which items are frequently bought together. This information, expressed by so-called "association rules", can help the warehouse in developing appropriate marketing strategies. For example,

$$\{\text{cereal, coffee}\} \to \{\text{milk}\}$$

is an association rule (in many real warehouses), and this association rule says that, with a given probability p, costumers buying cereal and coffee also buy milk. When the probability is 1 then we speak about *strong association rules* but this is only a technical reformulation of the Galois closure. Indeed, for $Y \subseteq A^{(1)}$ and $y \in A^{(1)}$, the strong association rule $Y \to y$ is defined by $y \in \mathcal{G}^{(1)}(Y)$.

This example shows how we associate from a set of attributes to another attribute, but it is equally frequent to associate from a set $X \subseteq A^{(0)}$ of objects to another object $x \in A^{(0)}$; then the strong association rule $X \to x$ means $x \in \mathcal{G}^{(0)}(X)$, i.e. that x has all the common attributes of the members in X. It is needless to say that this kind of associations is typical for human thinking and it is crucial in decision making. However, modeling human thinking in the above way is not perfect, for there are *positive* objects and attributes, which we like for some reason. (The real scale could be even larger, including more or less positive, neutral, or even negative etc. attributes but now we restrict our considerations to "positive" and the "not necessarily positive".)

By a *context with positivity domains*, or *p-context* for short, we mean a 5-tuple $(A^{(0)}, A^{(1)}, B^{(0)}, B^{(1)}, \rho)$ where $(A^{(0)}, A^{(1)}, \rho)$ is a context, $B^{(0)} \subseteq A^{(0)}$ and $B^{(1)} \subseteq A^{(1)}$. The elements of $B^{(0)}$ resp. $B^{(1)}$ are called *positive objects* resp. *positive* *attributes*; however, we do not call the rest of objects and attributes as negative ones. For example, a p-context is given by Table 1, where all the attributes but b_1 and all the objects are positive. (Notice that, in a sense detailed in [5] but not relevant here, this is the smallest context.) Associated with $(A^{(0)}, A^{(1)}, B^{(0)}, B^{(1)}, \rho)$ we intend to define a new pair $\mathcal{C}$ of closure operators such that $\mathcal{C} \le \mathcal{G}$, i.e. $\mathcal{C}$ should determine stronger association rules than $\mathcal{G}$, and $\mathcal{C}$ should take the positivity domains into account somehow. We could obtain $\mathcal{C}^{(0)}(X)$ via omitting certain objects from $\mathcal{G}^{(0)}(X)$ that have too few positive attributes but this hint is, of course, far from being sufficient, for there is a criterion: we want that $\mathcal{C}$ should be uniquely defined for each p-context and should properly depend on every component of $(A^{(0)}, A^{(1)}, B^{(0)}, B^{(1)}, \rho)$. Further motivations will be supplied at the end of the paper.

To accomplish our goal first we define a sequence $\mathcal{C}_i$, $i = 0, 1, 2, \dots$, of pairs of closure operators associated with $(A^{(0)}, A^{(1)}, B^{(0)}, B^{(1)}, \rho)$ such that $\mathcal{G} = \mathcal{C}_0 \ge \mathcal{C}_1 \ge \mathcal{C}_2 \ge \mathcal{C}_3 \ge \cdots$, and $\mathcal{C}$ will be the meet of this sequence. (Notice that for any $i \in \mathbf{N}$, $\mathcal{C}_i$ would also be appropriate for our purposes; however, we feel that $\mathcal{C}$ is better, for it gives stronger association rules.) For $X \in P(A^{(i)})$ let

$$X\psi_i := \{Y \in P(B^{(1-i)}) : \text{ there is a surjection } \varphi : X \to Y \text{ with } \varphi \subseteq \rho_i\}.$$

Pictorially, the elements of $X\psi_i$ are easy to imagine. Let us call a cross in the table *column-positive* if its column is positive (i.e., belongs to $B^{(1)}$). Row-positive crosses are defined dually. Let $i = 0$ for example, i.e., let $X \subseteq A^{(0)}$ be a set of rows. Select a column-positive cross in each row of X, then the collection of the columns of the selected crosses is an element of $X\psi_0$, and each element of $X\psi_0$ is obtained this way. Notice that $X\psi_0$ is empty iff there is a row in X that does not contain any column-positive cross. For example, if $X = \{a_1, a_2\}$ in Table 1 then $X\psi_0 = \{\{b_2, b_3\}\}$ while $\{a_4, a_5\}\psi_0 = \emptyset$.

Let $\mathcal{C}_0 = \mathcal{G}$. If $\mathcal{C}_n$ is already defined then let

$$\mathcal{C}^{(i)}_{n+1}(X) := \mathcal{C}^{(i)}_n(X) \cap \bigcap_{Y \in X\psi_i} \ \bigcup_{y \in B^{(1-i)} \cap \mathcal{C}^{(1-i)}_n(Y)} \{y\}\rho_{1-i} \ . \tag{1}$$

Although the $\cap$ operation in the above formula clearly gives $\mathcal{C}^{(i)}_{n+1} \le \mathcal{C}^{(i)}_n$, as requested, it is reasonable to digest formula (1) by thinking of it pictorially. For example, let $i = 0$ and $X \subseteq A^{(0)}$, and suppose that $\mathcal{C}_n = (\mathcal{C}^{(0)}_n, \mathcal{C}^{(1)}_n)$ is already well-understood. Then a row z belongs to $\mathcal{C}^{(0)}_{n+1}(X)$ if and only if $z \in \mathcal{C}^{(0)}_n(X)$ and, in addition, for each set $Y \in X\psi_0$ of columns there is a positive column y in $\mathcal{C}^{(1)}_n(Y)$ such that y intersects the row z at a cross. (Notice that $X\psi_0$ has already been explained pictorially, $\mathcal{C}^{(0)}_n(X)$ and $\mathcal{C}^{(1)}_n(Y)$ are already well-known by assumption, and y need not be unique and it depends on Y.)

Continuing the example $X = \{a_1, a_2\}$ at Table 1, $\mathcal{G}^{(0)}(X) = \{a_1, \dots, a_4\}$. Since $Y = \{b_2, b_3\} \in X\psi_0$ but there is no $y \in B^{(1)} \cap \mathcal{G}^{(1)}(Y) = \{b_2, b_3, b_4\}$ with $a_4 \in \{y\}\rho_1$ (i.e., with $(a_4, y) \in \rho$), formula (1) gives $a_4 \notin \mathcal{C}^{(0)}_1(X)$. After the trivial and therefore omitted details we can easily see that $\mathcal{C} = \mathcal{C}_1$ and $\mathcal{C}^{(0)}_1(X) = \mathcal{C}^{(0)}_2(X) = \cdots = \mathcal{C}^{(0)}(X) = \{a_1, a_2, a_3\}$.

Now (1) defines the pair $\mathcal{C}_{n+1} = (\mathcal{C}^{(0)}_{n+1}, \mathcal{C}^{(1)}_{n+1})$ and, finally, let

$$\mathcal{C} = (\mathcal{C}^{(0)}, \mathcal{C}^{(1)}) := (\bigwedge_{n=0}^{\infty} \mathcal{C}^{(0)}_n, \bigwedge_{n=0}^{\infty} \mathcal{C}^{(1)}_n),$$

which means that, for all $X \in P(A^{(i)})$,

$$\mathcal{C}^{(i)}(X) = \bigcap_{n=0}^{\infty} \mathcal{C}^{(i)}_n(X).$$

The main result, in fact the only purely mathematical result, of the present paper is the following.

Theorem 1. *$\mathcal{C}$ and $\mathcal{C}_n$, $n = 0, 1, \ldots$, are pairs of closure operators. Further,*

$$\mathcal{C}_0 \geq \mathcal{C}_1 \geq \mathcal{C}_2 \geq \cdots \geq \mathcal{C}\,.$$

Proof. We prove the theorem via induction on n. It is well-known that $\mathcal{C}_0 = \mathcal{G}$ is a pair of closure operators. Suppose that $\mathcal{C}_n$ is a pair of closure operators.

Let $X \subseteq U \in P(A^{(i)})$ and let u belong to $\mathcal{C}^{(i)}_{n+1}(X)$, i.e. to the righthand side of (1). Since $\mathcal{C}^{(i)}_n(X) \subseteq \mathcal{C}^{(i)}_n(U)$ by the induction hypothesis, it suffices to show that u belongs to the "big" intersection in

$$\mathcal{C}^{(i)}_{n+1}(U) = \mathcal{C}^{(i)}_n(U) \cap \bigcap_{V \in U\psi_i} \bigcup_{y \in B^{(1-i)} \cap \mathcal{C}^{(1-i)}_n(V)} \{y\}\rho_{1-i}\;. \tag{2}$$

Let $V \subseteq B^{(1-i)}$ be an arbitrary member of $U\psi_i$ by means of a surjection $\varphi : U \to V$ with $(x, x\varphi) \in \rho_i$ for all $x \in U$. Then $Y := X\varphi|_X$ is clearly in $X\psi_i$, and $Y = X\varphi \subseteq U\varphi = V$. Since $\mathcal{C}^{(1-i)}_n(Y) \subseteq \mathcal{C}^{(1-i)}_n(V)$ by the induction hypothesis,

$$u \in \bigcup_{y \in B^{(1-i)} \cap \mathcal{C}^{(1-i)}_n(Y)} \{y\}\rho_{1-i} \subseteq \bigcup_{y \in B^{(1-i)} \cap \mathcal{C}^{(1-i)}_n(V)} \{y\}\rho_{1-i}.$$

This shows that $u \in \mathcal{C}^{(i)}_{n+1}(U)$, whence $\mathcal{C}^{(i)}_{n+1}$ is monotone.

Now let $z \in X \in P(A^{(i)})$ and let $Y \in X\psi_i$ by means of a surjection $\varphi : X \to Y$ with $(x, x\varphi) \in \rho_i$ for all $x \in X$. In particular for $y := z\varphi \in B^{(1-i)}$ we have $(y, z) \in \rho_{1-i}$, i.e., $z \in \{y\}\rho_{1-i}$. Since $\mathcal{C}^{(1-i)}_n$ is extensive by the induction hypothesis, $y \in Y \subseteq \mathcal{C}^{(1-i)}_n(Y)$ shows that this y actually occurs in the righthand-side of (1). Therefore, from $z \in \{y\}\rho_{1-i}$ and $z \in X \subseteq \mathcal{C}^{(i)}_n(X)$ we obtain $x \in \mathcal{C}^{(i)}_{n+1}(X)$, showing that $\mathcal{C}^{(i)}_{n+1}$ is extensive.

Now, to show that $\mathcal{C}^{(i)}_{n+1}$ is idempotent, let $X \in P(A^{(i)})$, $U = \mathcal{C}^{(i)}_{n+1}(X)$ and $v \in \mathcal{C}^{(i)}_{n+1}(U)$. We need to show that $v \in U$. Beside the induction hypothesis we will use without further notice that $\mathcal{C}_{n+1} \leq \mathcal{C}_n$, which is evident, and $\mathcal{C}^{(i)}_{n+1}$ is monotone and extensive (shown so far). The easy part is as follows:

$$v \in \mathcal{C}^{(i)}_{n+1}(U) = \mathcal{C}^{(i)}_{n+1}(\mathcal{C}^{(i)}_{n+1}(X)) \subseteq \mathcal{C}^{(i)}_{n+1}(\mathcal{C}^{(i)}_n(X)) \subseteq \mathcal{C}^{(i)}_n(\mathcal{C}^{(i)}_n(X)) = \mathcal{C}^{(i)}_n(X).$$

To deal with the other part of the righthand-side of (1), let $Y \subseteq B^{(1-i)}$ be an arbitrary member of $X\psi_i$ by means of a surjection $\varphi : X \to Y$ with $(x, x\varphi) \in \rho_i$ for all $x \in X$. We know from (1), which determines U, that for each $z \in U \setminus X$ we can choose an element $y_z \in B^{(1-i)} \cap \mathcal{C}_n^{(1-i)}(Y)$ with $z \in \{y_z\}\rho_{1-i}$, i.e. $(z, y_z) \in \rho_i$. We define a map

$$\mu : U \to B^{(1-i)} \cap \mathcal{C}_n^{(1-i)}(Y), \quad z \mapsto \begin{cases} z\varphi \text{ if } z \in X \\ y_z \text{ if } z \in U \setminus X \ . \end{cases}$$

Let $V := U\mu \subseteq B^{(1-i)} \cap \mathcal{C}_n^{(1-i)}(Y)$. Clearly, $V \in U\psi_i$, so V takes part in (2). Hence

$$v \in \bigcup_{y \in B^{(1-i)} \cap \mathcal{C}_n^{(1-i)}(V)} \{y\}\rho_{1-i} \, .$$

So $v \in \{y\}\rho_{1-i}$ for a suitable $y \in B^{(1-i)} \cap \mathcal{C}_n^{(1-i)}(V)$. Using the induction hypothesis we obtain $y \in \mathcal{C}_n^{(1-i)}(V) \subseteq \mathcal{C}_n^{(1-i)}(\mathcal{C}_n^{(1-i)}(Y)) = \mathcal{C}_n^{(1-i)}(Y)$, and of course $y \in B^{(1-i)}$. Since $Y \in X\psi_i$ was arbitrary, this entails that $v \in U = \mathcal{C}_{n+1}^{(i)}(X)$. Hence $\mathcal{C}_{n+1}^{(i)}$ is idempotent, and so it is a closure operator.

Finally, $\mathcal{C}^{(i)}$ is clearly extensive and monotone. For any $n \in \mathbf{N}_0$ and $X \in P(A^{(i)})$,

$$\mathcal{C}^{(i)}(\mathcal{C}^{(i)}(X)) \subseteq \mathcal{C}^{(i)}(\mathcal{C}_n^{(i)}(X)) \subseteq \mathcal{C}_n^{(i)}(\mathcal{C}_n^{(i)}(X)) = \mathcal{C}_n^{(i)}(X),$$

which gives $\mathcal{C}^{(i)}(\mathcal{C}^{(i)}(X)) \subseteq \mathcal{C}^{(i)}(X)$. Therefore $\mathcal{C}$ is a pair of closure operators. □

When $(B^{(0)}, B^{(1)}) = (A^{(0)}, A^{(1)})$ then the p-context reduces to the context $(A^{(0)}, A^{(1)}, \rho)$ and Theorem 1 implies Lemma 1 in [3]. Hence we can say that (even this particular case of) our new notion has a proper application in pure mathematics, cf. [3]. "Proper" means that $\mathcal{C}$ was heavily used when proving a theorem which has nothing to do with the notion of $\mathcal{C}$.

From now on *we always assume* that $(A^{(0)}, A^{(1)}, B^{(0)}, B^{(1)}, \rho)$ is *finite*. Then there are only finitely many pairs of operators, whence there is a smallest n with $\mathcal{C} = \mathcal{C}_n = \mathcal{C}_{n+1} = \mathcal{C}_{n+2} = \cdots$. This raises the natural question how large this n can be. It is pointed out in [4] that n can be arbitrarily large even in the particular case $(B^{(0)}, B^{(1)}) = (A^{(0)}, A^{(1)})$. Another question is that how often $\mathcal{C}$ is different from $\mathcal{G}$; [4] and [5] make it clear that $\mathcal{C} \neq \mathcal{G}$ is not a rare phenomenon even in the particular case $(B^{(0)}, B^{(1)}) = (A^{(0)}, A^{(1)})$.

We close the paper by outlining a possible application of $\mathcal{C}$. First of all let us mention that the importance of looking for the hidden regularities and rules is not restricted only to *huge* databases. Indeed, the previously mentioned Mendeleyev's example or many concrete small contexts in Ganter and Wille [6] show that exploring some rules in *small* databases may also lead to important results. This is good, for it is not clear in the moment how one could compute $\mathcal{C}$ for large databases; the fixed point method of [5] for *large* contexts is not appropriate even in the particular case $(B^{(0)}, B^{(1)}) = (A^{(0)}, A^{(1)})$.

Hence the idea of applications will be explained via the small Table 1, the smallest possible table for this purpose, but this idea is clearly valid to many larger tables as well. Suppose the objects are something to learn, investigate or accomplish and the attributes are appropriately chosen. In our concrete example the objects are juggling tricks [1]. However, the reader need not know anything about juggling and

[1] The concrete meaning of objects and that of attributes in Table 1 are available, partially via video clips, at http://www.math.u-szeged.hu/∼czedli/jtable.html, but our argument will be clear even without this web site. The interested reader can also resort to Polster [8] for information on juggling.

one can imagine many other examples where the objects mean, say, courses offered by a university, musical compositions to learn, mountain peaks to reach, dishes to cook, dances or languages to learn, dangers to avoid, places to visit, books to read, etc. Suppose a person P has already learnt (or accomplished, etc.) a_1 and a_2 but not the rest of the objects, and she/he has to decide which single one of the rest she/he wants to learn (or accomplish, etc.) next. Denoting $\{a_1, a_2\}$ by X we can say that P has to associate an object with X. Suppose that $B^{(0)}$ resp. $B^{(1)}$ denotes the set of objects resp. attributes which P considers positive from his own aspect. For example, in case of the attributes, "positive" can somehow mean that each of these attributes are easy to learn, difficult to accomplish, cheap, near, useful, etc., depending on P's attitude. In our concrete example about juggling all the positive attributes mean that the trick is difficult and therefore, in other words, each attribute (if holds) makes the trick more attractive.

The first natural idea is to use $\mathcal{G}$ and associate an element $\mathcal{G}^{(0)}(X) \setminus X$ with X. However, this does not solve the problem, for $\mathcal{G}^{(0)}(X) = \{a_1, \ldots, a_4\}$ whence $\mathcal{G}^{(0)}(X) \setminus X$ has more than one element. Hence it is quite natural to consider stronger association rules, i.e. the smaller $\mathcal{C}$, and indeed, $\mathcal{C}^{(0)}(X) \setminus X = \{a_3\}$ has only one element, and P can choose a_3. And this is a good choice for P, for a_3 enjoys more positive attributes than a_4. Of course the juggling student P may have the opposite taste and may want to learn something easy, then either he/she can follow the opposite strategy of choosing from $\mathcal{G}^{(0)}(X) \setminus \mathcal{C}^{(0)}(X)$ or he/she can build a new p-context where "negated attributes" occur.

Of course, "more positive attributes" does not necessarily mean "greater number of positive attributes", which we cannot expect at this level of generality. However, we offer a tool of decision making which, except for stochastic algorithms, is more promising than relying on coin tossing or horoscopes.

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