Acta Sci. Math., 56 (1992), 15—18

Some nontrivial implications in congruence varieties

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Dedicated to Professor Béla Csákány on his 60th birtday

A congruence variety is a lattice variety generated by the class of congruence lattices of all members of some variety of algebras. The most known examples are $\mathcal{V}(R)$, the lattice varieties generated by congruence (or submodule) lattices of R-modules for rings R with 1. Given a lattice identity α and a set Γ of lattice identities, we write $\Gamma \models_c \alpha$ if every congruence variety satisfying Γ also satisfies α (cf. JÓNSSON [8]). The implication $\Gamma \models_c \alpha$ is called nontrivial if $\Gamma \not\models \alpha$ (in the class of all lattices). For $\Gamma = \{\gamma\}$ we will write γ rather than $\{\gamma\}$.

There are many results stating that $\gamma \models_c \alpha$ without $\gamma \models \alpha$ for certain pairs (γ, α) of lattice identities. These results are surveyed in JÓNSSON [8]; for a further development cf. FREESE, HERRMANN and HUHN [3]. However, all the known results are located at distributivity or modularity in the sense that either $\gamma \models_c \alpha \models_c$ distributivity $\models_c \gamma$ or $\gamma \models_c \alpha \models_c$ modularity $\models_c \gamma$. Now [1] offers an easy way to achieve $\gamma \models_c \alpha$ results of a different nature.

For an integer $n > 2$ and a modular lattice L, a system

$$\bar{f} = (a_i, c_{ij}\colon\ 1 \leqq i \leqq n,\ 1 \leqq j \leqq n,\ i \neq j)$$

of elements of L is called a (von Neumann) n-frame in L if $a_j \sum_{i \neq j} a_i = 0_{\bar{f}}$, $c_{jk} = c_{kj}$, $a_j c_{jk} = 0_{\bar{f}}$, $a_j + c_{jk} = a_j + a_k$ and $c_{jk} = (a_j + a_k)(c_{jl} + c_{lk})$ for all distinct $j, k, l \in \{1, 2, \ldots, n\}$ where $0_{\bar{f}}$ resp. $1_{\bar{f}}$ are the meet resp. join of all elements of $\bar{f}$ (cf. VON NEUMANN [9]). We write $x + y$ and xy for the join and meet of x and y.

Given $m \geqq 0$ and $n \geqq 1$, a lattice identity $\Delta(m, n)$ is defined in [7, page 289] such that, for any ring R with 1, $\Delta(m, n)$ holds in $\mathcal{V}(R)$ iff the divisibility condition $(\exists r)(m \cdot r = n \cdot 1)$, abbreviated by $D(m, n)$, holds in R (cf. [7, Prop. 6]). What else

Research partially supported by Hungarian National Foundation for Scientific Research grant no. 1813.

Received January 9, 1990.

we need to know about $\Delta(m, n)$ is that $\Delta(m, n)$ is of the form

$$(x_1+x_2)(x_3+x_4) \leqq q_{m,n}(x_1, x_2, x_3, x_4).$$

Frames are projective in the variety of modular lattices. This was proved in two steps; first for (Huhn) diamonds in HUHN [6] (for a more explicit statement cf. FREESE [2]) and then frames and diamonds turned out to be equivalent in HERRMANN and HUHN [5, page 104]. Therefore there are lattice terms $b_i(\vec{x})$ and $d_{ij}(\vec{x})$ in variables $\vec{x}=(x_i, x_{ij}\colon 1\leqq i, j\leqq k,\ i\neq j)$ such that these terms produce a k-frame $(b_i(\vec{y}), d_{ij}(\vec{y})\colon 1\leqq i, j\leqq k,\ i\neq j)$ from any system $\vec{y}$ of elements of a modular lattice L and, in addition, if $\vec{f}=(a_i, c_{ij}\colon 1\leqq i, j\leqq k,\ i\neq j)$ is a k-frame in L then $b_i(\vec{f})=a_i$ and $d_{ij}(\vec{f})=c_{ij}$ for every $i\neq j$.

For $k\geqq 4$ the conjugation of the modular law and the identity

$$\big(d_{13}(\vec{x})+d_{23}(\vec{x})\big)\big(d_{14}(\vec{x})+d_{24}(\vec{x})\big) \leqq q_{m,n}\big(d_{13}(\vec{x}), d_{23}(\vec{x}), d_{14}(\vec{x}), d_{24}(\vec{x})\big),$$

where $\vec{x}=(x_i, x_{ij}\colon 1\leqq i, j\leqq k,\ i\neq j)$, will be denoted by $\Delta(m, n, k)$. Clearly, $\Delta(m, n, k)$ is equivalent to a single lattice identity modulo lattice theory.

Theorem. *Consider arbitrary integers* $m', m_i\geqq 0, n', n_i\geqq 1$, *and* $k', k_i\geqq 4$ $(i\in I)$ *where I is an index set. Then* $\{\Delta(m_i, n_i, k_i)\colon i\in I\}\models_c \Delta(m', n', k')$ *if and only if* $\{D(m_i, n_i)\colon i\in I\}$ *implies* $D(m', n')$ *in the class of rings with* 1.

In particular, if $m\nmid n$ and $k\geqq 5$ then $\Delta(m, n, k)\models_c \Delta(m, n, k-1)$. This is a nontrivial implication, for we have the following

Proposition. *If* $m\nmid n$, $m\geqq 0$, $n\geqq 1$ *and* $k\geqq 5$ *then* $\Delta(m, n, k)\not\models \Delta(m, n, k-1)$.

To point out that the $\Delta(m, n, k)$ in the proposition are essentially distinct we present the following.

Remark. The set $\{\Delta(p, 1, \mathrm{k})\colon p \text{ prime}\}$, where $k\geqq 4$, is independent in congruence varieties in the sense that for every prime q

$$\{\Delta(p, 1, k)\colon p \text{ prime},\ p\neq q\}\not\models_c \Delta(q, 1, k).$$

Proof of the theorem. Since frames and diamonds are equivalent (cf. HERRMANN and HUHN [5, page 104]), the identities $\Delta(m, n, k)$ are diamond identities in the sense of [1]. What we need from [1] is only its Theorem 2, which we reformulate less technically as follows: For any diamond identity α, $\Gamma\models_c\alpha$ iff for any ring R with 1 Γ implies α in $\mathcal{V}(R)$. Therefore it suffices to show that $\Delta(m, n, k)$ and $\Delta(m, n)$ are equivalent in any $\mathcal{V}(R)$. Clearly, $\Delta(m, n)$ implies $\Delta(m, n, k)$ and $\Delta(m, n)$ are equivalent in any $\mathcal{V}(R)$. Clearly, $\Delta(m, n)$ implies $\Delta(m, n, k)$ in $\mathcal{V}(R)$. Conversely, assume that $\Delta(m, n, k)$ holds in $\mathcal{V}(R)$. Let $M=M(u_1, u_2, \ldots, u_k)$

denote the R-module freely generated by $\{u_1, u_2, \ldots, u_k\}$. Then $\Delta(m, n, k)$ holds Sub (M), the submodule lattice of M. It is easy to see (or cf. NEUMANN [9]) that the cyclic submodules $(Ru_i, R(u_i - u_j)\colon 1 \leqq i, j \leqq k, i \neq j)$ constitute a k-frame in Sub (M). (In fact, this is the most typical example of a k-frame.) Therefore

$$(1) \qquad \big(R(u_1-u_3)+R(u_2-u_3)\big)\big(R(u_1-u_4)+R(u_2-u_4)\big) \leqq$$
$$\leqq q_{m,n}\big(R(u_1-u_3), R(u_2-u_3), R(u_1-u_4), R(u_2-u_4)\big)$$

holds in Sub (M) and even in Sub $\big(M(u_1, u_2, u_3, u_4)\big)$. Now the theory of Mal'tsev conditions (cf. WILLE [11] or PIXLEY [10]) together with the canonical isomorphism between Sub $\big(M(u_1, u_2, u_3, u_4)\big)$ and the congruence lattice of $M(u_1, u_2, u_3, u_4)$ yield easily that $\Delta(m, n)$ holds in $\mathscr{V}(R)$. (Note that the first nine rows in the proof of [7, Prop. 6] supply a detailed proof of the fact that (1) implies the satisfaction of $\Delta(m, n)$ in $\mathscr{V}(R)$.)

Proof of the proposition. Let $\underset{\sim}{Z}$ denote the ring of integers. Since $m \nmid n$ and $\Delta(m, n, k-1)$ implies $\Delta(m, n)$ in $\mathscr{V}(\underset{\sim}{Z})$ by the proof above, $\Delta(m, n, k-1)$ fails in $\mathscr{V}(\underset{\sim}{Z})$. It is shown in HERRMANN and HUHN [4, Satz 7] that $\mathscr{V}(\underset{\sim}{Z})$ is generated by its finite members. Therefore there is a finite modular lattice L with minimal number of elements such that $\Delta(m, n, k-1)$ fails in L. We intend to show that $\Delta(m, n, k)$ holds in L. Assume the contrary. Then there is a k-frame $\vec{f}=(a_i, c_{ij}\colon 1 \leqq i, j \leqq k, i \neq j)$ such that $\Delta(m, n)$ fails when $c_{13}, c_{23}, c_{14}, c_{24}$ are substituted for its variables. It is known that either all elements of a frame are equal or $a_1, a_2, \ldots, a_k$ are distinct atoms of a Boolean sublattice of length k (cf., e.g., HERRMANN and HUHN [5, (iii) on page 101 and page 104]). Now only the latter is possible since the one element lattice satisfies any identity. Hence the subframe $\vec{g}=(a_i, c_{ij}\colon 1 \leqq i, j \leqq k-1, i \neq j)$ lies in the interval $L'=[0_{\vec{g}}, 1_{\vec{g}}]$. From $1_{\vec{g}}=a_1+\ldots+a_{k-1}<a_1+\ldots+a_k=1_{\vec{f}}$ we obtain $|L'|<|L|$. The frame $\vec{g}$ witnesses that $\Delta(m, n, k-1)$ fails in L', which contradicts the choice of L.

The remark is concluded from the theorem quite easily; we need only to consider the ring of those rational numbers whose denominator is not divisible by q.

References

[1] G. CZÉDLI, How are diamond identities implied in congruence varieties, *Algebra Universalis*, to appear.

[2] R. FREESE, Planar sublattices of $FM(4)$, *Algebra Universalis*, **6** (1976), 69—72.

[3] R. FREESE, C. HERRMANN and A. P. HUHN, On some identities valid in modular congruence varieties, *Algebra Universalis*, **12** (1981), 322—334.

[4] C. HERRMANN and A. P. HUHN, Zum Wortproblem für freie Untermodulverbände, *Archiv der Mathematik*, **26** (1975), 449—453.

[5] C. Herrmann and A. P. Huhn, Lattices of normal subgroups generated by frames, in *Coll. Math. Soc. J. Bolyai 14* (1974), pp. 97—136.
[6] A. P. Huhn, Schwach distributive Verbände. I, *Acta Sci. Math.*, **23** (1972), 297—305.
[7] G. Hutchinson and G. Czédli, A test for identities satisfied in lattices of submodules, *Algebra Universalis*, **8** (1978), 296—309.
[8] B. Jónsson, Congruence varieties, *Algebra Universalis*, **10** (1980), 355—394.
[9] J. von Neumann, *Continuous Geometry*, Princeton University Press (Princeton, N.J., 1960).
[10] A. F. Pixley, Local Mal'cev conditions, *Canadian Math. Bull.*, **15** (1972), 559—568.
[11] R. Wille, *Kongruenzklassengeometrien*, Lecture Notes in Math. 113, Springer-Verlag (Berlin—Heidelberg—New York, 1970).

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