SOME NEW CLOSURES ON ORDERS

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Abstract. For each of the relations "less than or equal to", "less than", "covered by", and "covered by or equal to", we characterize finite orders (also called posets) with the property that the pair of Galois closure operators induced by the relation in question coincides with the pair of closure operators introduced and applied in our previous paper in 2007. We also consider the "less than or equal to" relation between the set of join-irreducible elements and the set of meet-irreducible elements, and we show that the above-mentioned pairs of closure operators coincide for finite modular lattices.

1. Introduction and the results

It goes back to Galois that each (binary) relation $\rho \subseteq A^{(0)} \times A^{(1)}$ determines a pair $\vec{\mathcal{G}} = \vec{\mathcal{G}}(A^{(0)}, A^{(1)}, \rho)$ of closure operators. Another pair, $\vec{\mathcal{C}} = \vec{\mathcal{C}}(A^{(0)}, A^{(1)}, \rho)$, of closure operators has been introduced in [1]; its definition is postponed to the next section. For the relation considered in [1], $\vec{\mathcal{C}} \neq \vec{\mathcal{G}}$, and this is the main reason that $\vec{\mathcal{C}}$ was so useful there. This leads to

Problem 1. Characterize relations ρ with $\vec{\mathcal{C}}(A^{(0)}, A^{(1)}, \rho) = \vec{\mathcal{G}}(A^{(0)}, A^{(1)}, \rho)$.

A reasonable answer can be expected only for particular classes of relations. The present paper deals with some familiar relations for orders (also called posets). Only the rudiments of lattice theory is assumed to be known by the reader.

For a finite lattice L, let $J(L) = \big\{x \in L \setminus \{0\} : x \text{ is join-irreducible}\big\}$ and $M(L) = \big\{x \in L \setminus \{1\} : x \text{ is meet-irreducible}\big\}$. Notice that $\big(J(L), M(L), \leq\big)$ is known to be the least amount of data to describe an arbitrary finite lattice L, see Wille [4], and it is called the *standard context of* L in the literature of Formal Concept Analysis.

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Proposition 2. *Let L be a finite lattice. If L is modular, then*

$$\vec{\mathcal{C}}\big(J(L), M(L), \leq\big) = \vec{\mathcal{G}}\big(J(L), M(L), \leq\big).$$

We will point out in Remark 7 that the converse is not true.

For a finite order $Q = (Q, \leq)$, let $\max(Q)$ resp. $\min(Q)$ denote the set of maximal resp. minimal elements of Q. The length of Q, denoted by $\mathrm{length}(Q)$, is defined to be $\max\{\mathrm{length}(C) : C \subseteq Q$ and C is a chain$\}$. For $X \subseteq Q$, let $L(X)$ denote the set $\{y \in Q : y \leq x$ for all $x \in X\}$ of lower bounds of X. Dually, $U(X)$ denotes the set of upper bounds of X. Note that $U(\emptyset) = L(\emptyset) = Q$. We will write $U(a, b)$ rather than $U(\{a, b\})$, and the same convention applies for L.

The following statements deal with the strict ordering relation, the covering relation, the ordering relation, and the "covers or equal" relation of finite orders, respectively.

Proposition 3. *Let $Q = (Q, \leq)$ be a finite order. Then $\vec{\mathcal{C}}(Q, Q, <)$ is equal to $\vec{\mathcal{G}}(Q, Q, <)$ if and only if $U\big(Q \setminus \max(Q)\big) \neq \emptyset$ and $L\big(Q \setminus \min(Q)\big) \neq \emptyset$.*

Corollary 4. *Let $Q = (Q, \leq)$ be a finite order. Then $\vec{\mathcal{C}}\big(Q, Q, \prec\big) = \vec{\mathcal{G}}\big(Q, Q, \prec\big)$ if and only if $\mathrm{length}(Q) \leq 1$, $U\big(Q \setminus \max(Q)\big) \neq \emptyset$ and $L\big(Q \setminus \min(Q)\big) \neq \emptyset$.*

The disjoint union (or cardinal sum) of the orders $(Q_1, \leq_1)$ and $(Q_2, \leq_2)$ is $(Q_1 \cup Q_2, \leq_1 \cup \leq_2)$ where Q_1 is assumed to be distinct from Q_2. For example, an n-element antichain is the disjoint union of n chains of length 0.

Theorem 5. *Let $Q = (Q, \leq)$ be a finite order. Then $\vec{\mathcal{C}}(Q, Q, \leq) = \vec{\mathcal{G}}(Q, Q, \leq)$ if and only if either $|\max(Q)| = |\min(Q)| = 1$, or $|\max(Q)| \geq 2$, $|\min(Q)| \geq 2$ and*

$$\big(\forall x, y, z, t \in \max(Q)\big)\ \big(x \neq y \quad \textit{and} \quad z \neq t \quad \textit{imply} \quad L(x, y) = L(z, t)\big),$$
$$\big(\forall x, y, z, t \in \min(Q)\big)\ \big(x \neq y \quad \textit{and} \quad z \neq t \quad \textit{imply} \quad U(x, y) = U(z, t)\big).$$

As one may expect, Theorem 5 will be needed in the proof of the following theorem; the orders mentioned in this theorem are defined by Figure 1.

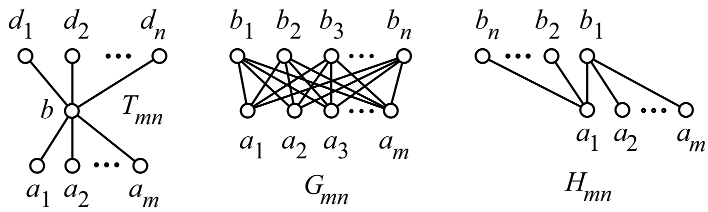

FIGURE 1. T_{mn}, G_{mn} and H_{mn}

Theorem 6. *Let $Q = (Q, \leq)$ be a finite order. Then $\vec{\mathcal{C}}(Q, Q, \preceq) = \vec{\mathcal{G}}(Q, Q, \preceq)$ if and only if one of the following possibilities holds:*

- *Q is (isomorphic to) T_{mn} for some $m, n \geq 1$;*
- *Q is G_{mn} for some $m, n \geq 2$;*
- *Q is H_{mn} for some $m, n \geq 2$;*
- $\mathrm{length}(Q) \leq 1$ *and Q is a disjoint union of chains.*

2. More about $\vec{\mathcal{C}}(A^{(0)}, A^{(1)}, \rho)$

While $\vec{\mathcal{G}}(A^{(0)}, A^{(1)}, \rho)$ is very important in mathematics and it has applications even outside mathematics, see Wille [4], the definitions below will look neither friendly nor natural at the first sight. However, the proof of the main result of [1] is based on $\vec{\mathcal{C}}$, although the result itself has nothing to do with closure operators. Some hopes of further applications of $\vec{\mathcal{C}}$ are mentioned in [2] and [3].

The notation $(A^{(0)}, A^{(1)}, \rho)$ will express that $A^{(0)}$ and $A^{(1)}$ are nonempty sets and $\rho \subseteq A^{(0)} \times A^{(1)}$ is a binary relation. Let us fix $(A^{(0)}, A^{(1)}, \rho)$, and let

$$\rho_0 = \rho \quad \text{and} \quad \rho_1 = \rho^{-1} \tag{1}$$

throughout the paper. The set of all subsets of $A^{(i)}$ will be denoted by $P(A^{(i)})$.

It is often convenient to depict $(A^{(0)}, A^{(1)}, \rho)$ in the usual form: a binary table with row labels from $A^{(0)}$, column labels from $A^{(1)}$, and a cross in the intersection of the x-th row and the y-th column iff $(x, y) \in \rho$. For example,

$$A^{(0)} = \{a_1, \ldots, a_5\}, \quad A^{(1)} = \{b_1, \ldots, b_4\}, \quad \rho : \begin{array}{|c||c|c|c|c|} \hline & b_1 & b_2 & b_3 & b_4 \\ \hline\hline a_1 & \times & \times & & \\ \hline a_2 & \times & & \times & \\ \hline a_3 & \times & \times & & \times \\ \hline a_4 & \times & & & \\ \hline a_5 & & \times & \times & \times \\ \hline \end{array} . \tag{2}$$

As usual, a mapping $\mathcal{D}^{(i)} : P(A^{(i)}) \to P(A^{(i)})$ is called a *closure operator* on $A^{(i)}$, if $X \subseteq \mathcal{D}^{(i)}(X) \subseteq \mathcal{D}^{(i)}(Y) = \mathcal{D}^{(i)}\big(\mathcal{D}^{(i)}(Y)\big)$ holds for all $X \subseteq Y \subseteq A^{(i)}$. If the $\mathcal{D}^{(i)}$, $i = 0, 1$, are closure operators, then $\vec{\mathcal{D}} = (\mathcal{D}^{(0)}, \mathcal{D}^{(1)})$ is called a *pair of closure operators*. If $\vec{\mathcal{E}}$ is another such pair, then $\vec{\mathcal{D}} \leq \vec{\mathcal{E}}$ means that $\mathcal{D}^{(i)}(X) \subseteq \mathcal{E}^{(i)}(X)$ for all $i \in \{0, 1\}$ and all $X \in P(A^{(i)})$.

For $X \in P(A^{(i)})$, let

$$X\rho_i = \{y \in A^{(1-i)} : \quad \text{for all} \quad x \in X,\ (x, y) \in \rho_i\},$$

and, again for $X \in P(A^{(i)})$, define

$$\mathcal{G}^{(i)}(X) := (X\rho_i)\rho_{1-i} = \bigcap_{y \in X\rho_i} (\{y\}\rho_{1-i}) \ .$$

Then $\vec{\mathcal{G}} = (\mathcal{G}^{(0)}, \mathcal{G}^{(1)})$ is the well-known *pair of Galois closure operators.*

Next, we define a sequence $\vec{\mathcal{C}}_i$, $i = 0, 1, 2, \dots$, of pairs of of closure operators. For $X \in P(A^{(i)})$, let

$$X\psi_i := \{Y \in P(A^{(1-i)}) : \quad \text{there is a surjection} \quad \varphi : X \to Y \quad \text{with} \quad \varphi \subseteq \rho_i\}.$$

Clearly, $\emptyset\psi_i = \{\emptyset\}$, so $\emptyset\psi_i$ is never empty. Using the table of ρ, the elements of $X\psi_i$ are easy to imagine pictorially. For example, let $i = 0$, that is, let $X \subseteq A^{(0)}$ be a set of rows. Select a cross in each row of X, then the collection of the columns of the selected crosses is an element of $X\psi_0$, and each element of $X\psi_0$ is obtained this way. For example, if $X = \{a_1, a_2\}$ in the table given in (2), then $X\psi_0$ consists of $\{b_1\}$, $\{b_1, b_2\}$, $\{b_1, b_3\}$ and $\{b_2, b_3\}$.

Let $\vec{\mathcal{C}}_0 = \vec{\mathcal{G}}$. If $\vec{\mathcal{C}}_n$ is already defined then let

$$\mathcal{C}^{(i)}_{n+1}(X) := \mathcal{C}^{(i)}_n(X) \cap \bigcap_{Y \in X\psi_i} \; \bigcup_{y \in \mathcal{C}^{(1-i)}_n(Y)} \{y\}\rho_{1-i} \; . \tag{3}$$

This defines the pair $\vec{\mathcal{C}}_{n+1} = (\mathcal{C}^{(0)}_{n+1}, \mathcal{C}^{(1)}_{n+1})$.

The easiest way to digest formula (3) is to think of it pictorially. For example, let $i = 0$ and $X \subseteq A^{(0)}$, and suppose that $\vec{\mathcal{C}}_n = (\mathcal{C}^{(0)}_n, \mathcal{C}^{(1)}_n)$ is already well-understood. Then a row z belongs to $\mathcal{C}^{(0)}_{n+1}(X)$ if and only if $z \in \mathcal{C}^{(0)}_n(X)$ and, in addition, for each set $Y \in X\psi_0$ of columns there is a column y in $\mathcal{C}^{(1)}_n(Y)$ such that y intersects the row z at a cross. (Notice that y need not be unique and it depends on Y.)

Finally, let

$$\vec{\mathcal{C}} = (\mathcal{C}^{(0)}, \mathcal{C}^{(1)}) := \Big(\bigwedge_{n=0}^{\infty} \mathcal{C}^{(0)}_n, \bigwedge_{n=0}^{\infty} \mathcal{C}^{(1)}_n \Big),$$

which means that, for all $X \in P(A^{(i)})$ and $i \in \{0, 1\}$, $\mathcal{C}^{(i)}(X) = \bigcap_{n=0}^{\infty} \mathcal{C}^{(i)}_n(X)$.

It was routine to prove in [1] that we have indeed defined pairs of closure operators. Clearly, $\vec{\mathcal{G}} = \vec{\mathcal{C}}_0 \geq \vec{\mathcal{C}}_1 \geq \vec{\mathcal{C}}_2 \geq \cdots \geq \vec{\mathcal{C}}_n \geq \cdots \geq \vec{\mathcal{C}}$ holds. It follows from [1] in a straightforward way that, for each $n \in \mathbb{N}$, there is a finite $(A^{(0)}, A^{(1)}, \rho)$ such that $\vec{\mathcal{C}}_0 > \vec{\mathcal{C}}_1 > \vec{\mathcal{C}}_2 > \cdots > \vec{\mathcal{C}}_n$. (For $n = 6$, this is witnessed by $(A^{(0)}, A^{(1)}, \rho)$ of Figure 2 in [1].)

Notice that while $\vec{\mathcal{G}}(A^{(0)}, A^{(1)}, \rho)$ induces two dually isomorphic lattices, this fails for $\vec{\mathcal{C}}(A^{(0)}, A^{(1)}, \rho)$ in general. Indeed, in case of the table given in (2), a straightforward but tedious calculation[1] shows $|\{X \in P(A^{(0)}) : \mathcal{C}^{(0)}(X) = X\}| = 12$ while $|\{X \in P(A^{(1)}) : \mathcal{C}^{(1)}(X) = X\}| = 10$.

[1]Alternatively, the 2006 computer program in the author's web site can do this calculation.

3. Proofs

Notice that $\vec{\mathcal{C}} = \vec{\mathcal{G}}$ iff $\vec{\mathcal{C}}_1 = \vec{\mathcal{G}}$, and this fact will be used implicitly in our proofs. Given a context $(A^{(0)}, A^{(1)}, \rho)$, by the *dual context* we mean

$$(A^{(1)}, A^{(0)}, \rho^{-1}).$$

Clearly, if $L_d = (L_d, \leq_d)$ denotes the dual of L, then $\big(J(L_d), M(L_d), \leq_d\big)$ is the dual of the context $\big(J(L), M(L), \leq\big)$. Since the conditions in all of our statements are self-dual, *we will show only* that $\mathcal{C}_1^{(0)} = \mathcal{G}^{(0)}$, since this will imply $\vec{\mathcal{C}}_1 = \vec{\mathcal{G}}$ by duality. Remember that $\rho = \rho_0$ always denotes the relation in question, and ρ_1 stands for ρ^{-1}. Formula (1) will be used often without referring to it.

Proof of Proposition 2. Let L be a finite modular lattice. Denote $J(L)$ and $M(L)$ by J and L. The restriction of the lattice ordering to $J \times M$ will also be denoted by $\rho = \rho_0$. Since modularity is a self-dual lattice property, by the duality principle it suffices to show that $\mathcal{C}_1^{(0)} = \mathcal{G}^{(0)}$, that is, $\mathcal{C}_1^{(0)}(X) = \mathcal{G}^{(0)}(X)$ for all $X \subseteq J$.

If $X = \emptyset$, then $\emptyset\rho_0 = M$ implies $\mathcal{G}^{(0)}(\emptyset) = M\rho_1 = \emptyset$, whence $\mathcal{C}_1^{(0)}(\emptyset) = \mathcal{G}^{(0)}(\emptyset)$. Next, let $X = \{a_1, \ldots, a_n\} \subseteq J$ with $|X| = n \geq 1$, and let

$$x \in \mathcal{G}^{(0)}(X) = (X\rho_0)\rho_1 = ([a_1 \vee \cdots \vee a_n) \cap M)\rho_1 = (a_1 \vee \cdots \vee a_n] \cap J$$

be an arbitrary element. Let $Y = \{b_1, \ldots, b_n\} \in X\psi_0$. This means that $a_j \leq b_j \in M$ for $j = 1, \ldots, n$ (but the b_j are not necessarily distinct). Then, dually to the displayed formula above, $\mathcal{G}^{(1)}(Y) = [b_1 \wedge \cdots \wedge b_n) \cap M$. Let $b = b_1 \wedge \cdots \wedge b_n$. According to formula (3), we have to show that

$$\text{there exists a} \quad y \in [b) \cap M \quad \text{such that} \quad x \leq y.$$

This is evident when $x \vee b \neq 1$, since $[x \vee b) \cap M$ is not empty in this case. So, by way of contradiction assume that $x \vee b = 1$. Then

$$\begin{aligned}
1 = x \vee b &= (b_1 \wedge \cdots \wedge b_n) \vee x \leq b_1 \vee \cdots \vee b_n \vee x \\
&= (a_1 \vee b_1) \vee \cdots (a_n \vee b_n) \vee x = (b_1 \vee \cdots \vee b_n) \vee (a_1 \vee \cdots \vee a_n \vee x) \\
&= (b_1 \vee \cdots \vee b_n) \vee (a_1 \vee \cdots \vee a_n) \\
&= (a_1 \vee b_1) \vee \cdots (a_n \vee b_n) = b_1 \vee \cdots \vee b_n.
\end{aligned}$$

Hence $b_1 \vee \cdots \vee b_n = 1$ and this happens in the interval $[b, 1] = [b, b \vee x]$. Since L is modular, this interval is isomorphic to the interval $[b \wedge x, x]$. But $x \in J$, so x is join irreducible also in the interval $[b \wedge x, x]$, whence 1 is join irreducible in $[b, 1]$, and we conclude that $b_j = 1$ for some j. But this is a contradiction, since $b_j \in M$ and $1 \notin M$. Thus we have shown that $\mathcal{C}_1^{(0)}(X) = \mathcal{G}^{(0)}(X)$. $\square$

Remark 7. There are finite <u>non</u>-modular lattices L and K such that

$$\vec{\mathcal{C}}\big(J(L), M(L), \leq\big) = \vec{\mathcal{G}}\big(J(L), M(L), \leq\big) \quad \text{and}$$
$$\vec{\mathcal{C}}\big(J(K), M(K), \leq\big) \neq \vec{\mathcal{G}}\big(J(K), M(K), \leq\big).$$

Indeed, it is easy to check that both five-element non-modular lattices, N_5 and M_3, can serve as L. The simplest appropriate K is probably the n-crown, for $n \geq 4$, with additional 0 and 1. That is, we can choose K as the $(2n+2)$-element lattice $(\{0, a_0, \dots, a_{n-1}, b_0, \dots, b_{n-1}, 1\}, \leq)$ where the $\{a_0, \dots, a_{n-1}\}$ is the set of atoms, $\{b_0, \dots, b_{n-1}\}$ is the set of coatoms, and $a_j < b_k$ iff $k \in \{j, j+1\}$ (here $j+1$ is understood modulo n). For $n = 4$, a trivial computation shows that K does the job. Since, for any $n \geq 4$, K is just the straightforward lattice theoretic reformulation of $(A^{(0)}, A^{(1)}, \rho)$ of Figure 2 in [1]; the details about K are omitted.

Proof of Proposition 3. Let us suppose that $\vec{\mathcal{C}}_1 = \vec{\mathcal{C}}_1(Q, Q, <)$ coincides with $\vec{\mathcal{G}} = \vec{\mathcal{G}}(Q, Q, <)$. Let $A = Q \setminus \max(Q)$ and $B = Q \setminus \min(Q)$. By way of contradiction, suppose that $U(A)$ or $L(B)$ is empty. By the duality principle, it suffices to consider the case when $U(A)$ is empty. Then $A \neq \emptyset$, $A\rho_0 \subseteq U(A) = \emptyset$, whence $\mathcal{G}^{(0)}(A) = (A\rho_0)\rho_1 = \emptyset\rho_1 = Q$. Let $x \in \max(Q) \subseteq Q = \mathcal{G}^{(0)}(A) = \mathcal{C}_1^{(0)}(A)$. Clearly, $A\psi_0$ is not empty, so we can choose a $Y \in A\psi_0$. However, since x is a maximal element, $x \in \{y\}\rho_1$, i.e. $x < y$, holds for no $y \in \mathcal{G}^{(1)}(Y)$. Hence $x \notin \mathcal{C}_1^{(0)}(A)$, a contradiction.

To prove the converse, suppose that A has an upper bound a and B has a lower bound b. We can assume that $a \in \max(Q)$ and $b \in \min(Q)$. If A or B is empty, then Q is an antichain, $\rho = \emptyset$, and $\vec{\mathcal{C}}_1 = \vec{\mathcal{G}}$ follows easily from the fact that $X\psi_i$ is empty when X is nonempty. Hence we assume that neither A nor B is empty.

Clearly, $x < a$ for all $x \in A$, whence $a \notin \min Q$, that is, $a \in B$. Similarly, $b < y$ for all $y \in B$ and $b \in A$. In particular, $b < a$. Notice that, for any $\emptyset \neq U \subseteq Q$, $U\rho_0 \subseteq B$ and $U\rho_1 \subseteq A$.

Let X be a subset of Q. If $X = \emptyset$, then $\mathcal{G}^{(0)}(\emptyset) = (\emptyset\rho_0)\rho_1 = Q\rho_1 = \emptyset$ yields $\mathcal{C}_1^{(0)}(\emptyset) = \mathcal{G}^{(0)}(\emptyset)$. If $X \not\subseteq A$ then $X\psi_0 = \emptyset$ yields $\mathcal{C}_1^{(0)}(X) = \mathcal{G}^{(0)}(X)$ again. Hence we can assume that $\emptyset \neq X \subseteq A$. Then $X\rho_0 \supseteq \{a\}$ yields

$$\mathcal{G}^{(0)}(X) = (X\rho_0)\rho_1 \subseteq \{a\}\rho_1 = L(a) \setminus \{a\} \subseteq A.$$

Suppose that $x \in \mathcal{G}^{(0)}(X)$ and let $Y \in X\psi_0$ be arbitrary. Then $Y\rho_1 \subseteq A$ gives $\mathcal{G}^{(1)}(Y) = (Y\rho_1)\rho_0 \supseteq A\rho_0 \ni a$. Since $x \in A$, $x \in \{a\}\rho_1$. Hence a can play the role of y in formula (3), and we obtain that $x \in \mathcal{C}_1^{(0)}(X)$. This shows that $\mathcal{C}_1^{(0)} = \mathcal{G}^{(0)}$. $\square$

Proof of Corollary 4. Suppose length$(Q) \geq 2$. Then we can choose $a, b, c \in Q$ such that $a \prec b$, $b \prec c$ and $c \in \max(Q)$. Let $X = \{a, b\}$. Then $\mathcal{G}^{(0)}(X) = (\{a, b\})\rho_0)\rho_1 = \emptyset\rho_1 = Q$. If $\mathcal{C}_1^{(0)} = \mathcal{G}^{(0)}$, then $c \in Q = \mathcal{G}^{(0)}(X) = \mathcal{C}_1^{(0)}(X)$ and $Y = \{b, c\} \in X\psi_0$ imply that $c \in \{y\}\rho_1$, i.e. $c \prec y$, for some $y \in \mathcal{G}^{(1)}(Y)$, which contradicts $c \in \max(Q)$. Hence $\mathcal{C}_1^{(0)} = \mathcal{G}^{(0)}$ implies length$(Q) \leq 1$. Then $(Q, Q, \prec)$ is exactly the same context as $(Q, Q, <)$, and the rest of the statement follows from Proposition 3. □

Proof of Theorem 5. Assume that $\vec{\mathcal{C}}_1 = \vec{\mathcal{G}}$. Suppose first that $|\min(Q)| = 1$, i.e., Q has a unique least element 0. Let $X = \emptyset$. Then $\mathcal{C}_1^{(0)}(X) = \mathcal{G}^{(0)}(X) = Q\rho_1 = \{0\}$ and $Y = \emptyset \in X\psi_0$ yields that there is a $y \in \mathcal{G}^{(1)}(Y)$ with $0 \in \{y\}\rho_1$. Thus $\mathcal{G}^{(1)}(Y) = \mathcal{G}^{(1)}(\emptyset) = (\emptyset\rho_1)\rho_0 = Q\rho_0 = \{z \in Q : t \leq z \text{ for all } t \in Q\}$ is nonempty. Therefore Q has a greatest element and $|\max(Q)| = 1$. The duality principle gives that $|\min(Q)| = 1$ iff $|\max(Q)| = 1$, and the condition of the theorem holds.

Next, suppose that $|\min(Q)| > 1$. Then $|\max(Q)| > 1$ either. By way of contradiction, let us assume that $L(u, v)$ (where $u \neq v$) is not constant on $\max(Q)$. Then we can choose a three-element subset $\{a, b, c\}$ of $\max(Q)$ such that $L(a, b) \not\subseteq L(a, c)$. Then there is an element $x \in L(a, b) \setminus L(a, c)$. Let $X = \{a, b\}$. We obtain $\mathcal{G}^{(0)}(X) = (X\rho_0)\rho_1 = L\big(U(a, b)\big) = L(\emptyset) = Q$, so $c \in \mathcal{G}^{(0)}(X)$. Let $Y = X$. Then $Y \in X\psi_0$ and $\vec{\mathcal{C}}_1 = \vec{\mathcal{G}}$ imply that there is an element $y \in \mathcal{G}^{(1)}(Y)$ with $c \in \{y\}\rho_1$, i.e, $c \leq y$. Since $c \in \max(Q)$, $c = y \in \mathcal{G}^{(1)}(Y) = (Y\rho_1)\rho_0 = U\big(L(a, b)\big)$. This and $x \in L(a, b)$ yield $x \leq c$, contradicting $x \in L(a, b) \setminus L(a, c)$. We have shown that L is constant on $\{(u, v) : u, v \in \max(Q) \text{ and } u \neq v\}$. It follows from the duality principle that U is constant on $\{(u, v) : u, v \in \min(Q) \text{ and } u \neq v\}$.

In order to prove the converse, suppose first that $0, 1 \in Q$, i.e., $|\max(Q)| = |\min(Q)| = 1$. Then $1 \in U(Q) = U\big(L(\emptyset)\big) = \mathcal{G}^{(1)}(\emptyset)$. Since $\mathcal{G}^{(1)}$ is monotone, $1 \in \mathcal{G}^{(1)}(Y)$ for any $Y \subseteq Q$. Moreover, $\{1\}\rho_1 = Q$. Hence 1 can always serve as y in formula (3), and we conclude that $\mathcal{C}_1^{(0)} = \mathcal{G}^{(0)}$.

From now on we suppose that $|\max(Q)| = |\min(Q)| \geq 2$, L is constant on $\{(u, v) : u, v \in \max(Q) \text{ and } u \neq v\}$, and U is constant on $\{(u, v) : u, v \in \min(Q) \text{ and } u \neq v\}$. Then $\mathcal{G}^{(0)}(\emptyset) = L\big(U(\emptyset)\big) = L(Q) = \emptyset$ gives $\mathcal{C}_1^{(0)}(\emptyset) = \mathcal{G}^{(0)}(\emptyset)$. So, it suffices to consider a nonempty subset X of Q.

Let $Y \in X\psi_0$. Then Y is nonempty either. We distinguish two cases according to $U(Y)$.

First, suppose that $U(Y)$ is nonempty, and let us fix an element $z \in U(Y)$. Since $Y \in X\psi_0$, $U(X) \supseteq U(Y)$, so $U(X) \supseteq \{z\}$, whence $\mathcal{G}^{(0)}(X) = L\big(U(X)\big) \subseteq L(\{z\}) = \{z\}\rho_1$. On the other hand, the transitivity of the ordering gives

$U(Y) \subseteq U\big(L(Y)\big) = \mathcal{G}^{(1)}(Y)$, whence $z \in \mathcal{G}^{(1)}(Y)$. Now it is clear from formula (3) that $\mathcal{C}_1^{(0)}(X) = \mathcal{G}^{(0)}(X)$.

Secondly, we suppose that $U(Y)$ is empty. Then there are $y_1, y_2 \in Y$ and $z_1, z_2 \in \max(Q)$ such that $y_1 \le z_1$, $y_2 \le z_2$ and $z_1 \ne z_2$. Since $\mathcal{G}^{(1)}(Y) = U\big(L(Y)\big)$ is an order filter including Y, $\{z_1, z_2\} \subseteq \mathcal{G}^{(1)}(Y)$. Now let x be an arbitrary element of $\mathcal{G}^{(0)}(X)$, and choose an element $\tilde{x} \in \max(Q)$ such that $x \le \tilde{x}$. If $\tilde{x} = z_j$ for some $j \in \{1,2\}$, then we can chose $y = \tilde{x} = z_j$ in formula (3). Hence we can assume that $|\{\tilde{x}, z_1, z_2\}| = 3$. Using the assumption that L is constant for distinct maximal elements we obtain

$$\begin{aligned}\tilde{x} \in \mathcal{G}^{(1)}(\{\tilde{x}, z_1\}) &= U\big(L(\tilde{x}, z_1)\big) = U\big(L(z_1, z_2)\big)\\ &= \mathcal{G}^{(1)}(\{z_1, z_2\}) \subseteq \mathcal{G}^{(1)}\big(\mathcal{G}^{(1)}(Y)\big) = \mathcal{G}^{(1)}(Y),\end{aligned}$$

and therefore the choice $y = \tilde{x}$ for formula (3) works again. This shows that $\mathcal{C}_1^{(0)}(X) = \mathcal{G}^{(0)}(X)$ for any $X \in P(A^{(0)})$. So, $\mathcal{C}_1^{(0)} = \mathcal{G}^{(0)}$. □

Proof of Theorem 6. Consider $(Q, Q, \preceq)$, and suppose Q is one of the orders listed in the theorem. We have to show that $\mathcal{C}_1^{(0)} = \mathcal{G}^{(0)}$. If $\mathrm{length}(Q) \le 1$, then $(Q, Q, \preceq)$ coincides with $(Q, Q, \le)$, whence Theorem 5 easily implies that $\mathcal{C}_1^{(0)} = \mathcal{G}^{(0)}$. So we can assume that $\mathrm{length}(Q) \ge 2$. Then Q is T_{mn} for some $m, n \ge 1$. Let

$$K = \big\{X \in P(Q) : \big(\forall Z \in P(Q)\big)\ \big(Z \subset X \Rightarrow \mathcal{G}^{(0)}(Z) \subset \mathcal{G}^{(0)}(X)\,\big)\big\}.$$

If $\mathcal{C}_1^{(0)}$ and $\mathcal{G}^{(0)}$ agreed on K, then, for any $X \in P(Q)$, we could take a minimal element Z of $\{X' \in P(Q) : X' \subseteq X \text{ and } \mathcal{G}^{(0)}(X') = \mathcal{G}^{(0)}(X)\}$, and from $Z \in K$ we could deduce

$$\mathcal{C}_1^{(0)}(X) \supseteq \mathcal{C}_1^{(0)}(Z) = \mathcal{G}^{(0)}(Z) = \mathcal{G}^{(0)}(X),$$

implying $\mathcal{C}_1^{(0)} = \mathcal{G}^{(0)}$.

Hence it suffices to show that $\mathcal{C}_1^{(0)}(X) = \mathcal{G}^{(0)}(X)$ holds for all $X \in K$. Moreover, it suffices to consider a small subset K' of K with the following property: for each X in K, there is an automorphism of Q that maps X to an element of K'. Let $A = \{a_1, \dots, a_m\}$, $A^+ = A \cup \{b\}$, $D = \{d_1, \dots, d_n\}$, $D^+ = D \cup \{b\}$, and assume that $m \ge 2$ and $n \ge 2$. (The case $m = 1$ or $n = 1$ is simpler and will not be detailed.)

Let us compute $\mathcal{G}^{(0)}(X)$ for "all" $X \subseteq T_{mn}$ with $|X| \le 2$; "all" means that "all apart from automorphisms of T_{mn}". The possible subsets X are listed in the first row of Table (4) below with the abbreviation x and xy for $\{x\}$ and $\{x, y\}$, respectively. The corresponding values $\mathcal{G}^{(0)}(X)$ in the second row imply easily that each member of K consists of at most two elements. Hence the third row of the table defines an appropriate K'. (The fourth row, which is useful for later computations, comes easily from the second row by duality.)

X	$\emptyset$	a_1	a_1, a_2	b	a_1, b	a_1, d_1	b, d_1	d_1	d_1, d_2
$\mathcal{G}^{(0)}(X)$	$\emptyset$	a_1	A^+	b	A^+	Q	b, d_1	b, d_1	Q
$X \in K'$?	yes	yes	yes	yes	yes	yes	no	yes	yes
$\mathcal{G}^{(1)}(X)$	$\emptyset$	a_1, b	Q	b	a_1, b	Q	D^+	d_1	D^+

(4)

Now, we can easily list all possible Y's from formula (3) (up to isomorphism, again), and then we can check that $\mathcal{C}_1^{(0)}(X) = \mathcal{G}^{(0)}(X)$ for $X \in K'$; the tedious details will be omitted.

In order to prove the converse direction, assume that $\vec{\mathcal{C}}_1 = \vec{\mathcal{G}}$. If $\text{length}(Q) = 0$ then Q is an antichain, which is a disjoint union of chains, and there is nothing to prove.

Next, assume that $\text{length}(Q) = 1$ and Q is not a disjoint union of chains. Then Theorem 5 applies, so $2 \le |\max(Q)|$, $2 \le |\min(Q)|$, L is constant on $\{(u, v) : u, v \in \max(Q),\ u \ne v\}$ and U is constant on $\{(u, v) : u, v \in \min(Q),\ u \ne v\}$. Since Q is not a disjoint union of chains, there are $a_1, b_1, b_2 \in Q$ such that $a_1 < b_1$ and $a_1 < b_2$, or dually. So we can assume that $a_1 < b_1$ and $a_1 < b_2$. If we had an element $c \in \max(Q) \cap \min(Q)$, then $\emptyset = L(b_1, c) \ne L(b_1, b_2) \supseteq \{a_1\}$ would lead to a contradiction. Therefore, taking $\text{length}(Q) = 1$ into account, we obtain that

$$Q \quad \text{is the disjoint union of} \quad \max(Q) \quad \text{and} \quad \min(Q). \tag{5}$$

Notice also that the diagram of Q is connected as a graph, since otherwise we could find an $x \in \max(Q)$ with $L(b_1, x) = \emptyset$. Let

$$B = \{x \in \max(Q) : a_1 \le x\}, \quad \text{and remember that} \quad b_1, b_2 \in B.$$

Since a_1 is connected with all elements of $\min(Q)$ in the graph and $|\min(Q)| \ge 2$, there is an $a_2 \in \min(Q) \setminus \{a_1\}$ which is less than some element of B. So we can assume that $a_2 < b_1$. Let

$$A := \{x \in \min(Q) : x < b_1\}, \quad \text{and notice that} \quad a_1, a_2 \in A.$$

If we had an element $c \in \max(Q) \setminus B$, then $a_1 \notin L(b_1, c) = L(b_1, b_2) \supseteq \{a_1\}$ would be a contradiction. Hence $B = \max(Q)$, and we obtain $A = \min(Q)$ similarly. Hence, by (5), Q is the disjoint union of A and B.

Let $m = |A|$ and $n = |B|$. If, for $a \in A$ and $b \in B$, $a < b$ holds only when $\{a_1, b_1\} \cap \{a, b\} \ne \emptyset$, then Q is H_{mn}. Otherwise we may suppose that $a_2 < b_2$. Then, for any $b \in B \setminus \{b_1\}$, $a_2 \in L(b_1, b_2) = L(b_1, b)$ yields $a_2 < b$. Hence $U(a_1, a_2) = B$, and for any $a \in A \setminus \{a_1\}$ we have $U(a_1, a) = U(a_1, a_2) = B$. This means that $Q = G_{mn}$, and the case $\text{length}(Q) = 1$ is settled.

Next, suppose that $\text{length}(Q) \ge 2$, and introduce the notation

$$\text{mid}(Q) = Q \setminus \big(\max(Q) \cup \min(Q)\big).$$

Let us observe that for any $u, v \in Q$,

$$\text{if}\quad u \prec v \quad\text{then}\quad \mathcal{G}^{(0)}(\{u,v\}) = \{x : x \preceq v\}$$
$$\text{and}\quad \mathcal{G}^{(1)}(\{u,v\}) = \{x : u \preceq x\}. \tag{6}$$

Indeed, $\mathcal{G}^{(0)}(\{u,v\}) = (\{u,v\}\rho_0)\rho_1 = \{v\}\rho_1 = \{x : x \preceq v\}$, and the other equation follows by duality.

First of all, we consider the case when $\text{length}(Q) \geq 3$. Then there are elements $a, b, c \in Q$ and $d \in \max(Q)$ such that $a \prec b \prec c \prec d$. Let $X = \{b, d\}$. Then $\mathcal{C}_1^{(0)}(X) = \mathcal{G}^{(0)}(X) = (X\rho_0)\rho_1 = \emptyset\rho_1 = Q$. Let $Y = \{c, d\} \in X\psi_0$. Then, for all $y \in \mathcal{G}^{(1)}(Y)$, we have $c \leq y$ by (6), so $a \not\succeq y$, whence $a \notin \{y\}\rho_1$, and $a \notin \mathcal{C}_1^{(0)}(X) = Q$ by formula (3), a contradiction. Hence $\text{length}(Q) \geq 3$ is excluded, and from now on we assume that $\text{length}(Q) = 2$. Clearly, then $\text{mid}(Q)$ is an antichain.

The first step in the case $\text{length}(Q) = 2$ is to show that, for any $b \neq c$,

$$\text{if}\quad b, c \in \text{mid}(Q), \quad\text{then}\quad |L(b,c)| \leq 1 \quad\text{and}\quad |U(b,c)| \leq 1. \tag{7}$$

Suppose, by way of contradiction, that $d_1, d_2 \in U(b,c)$ and $d_1 \neq d_2$. Let $X = \{d_1, d_2\}$, and choose an element a such that $a \prec b$. Since $X \subseteq \max(Q)$, we obtain that $a \in Q = \emptyset\rho_1 = \mathcal{G}^{(0)}(X) = \mathcal{C}_1^{(0)}(X)$. Let $Y = X \in X\psi_0$. By formula (3) there is a $y \in \mathcal{G}^{(1)}(Y)$ with $a \preceq y$. But $Y\rho_1 \supseteq \{b, c\}$ implies $y \in \mathcal{G}^{(1)}(Y) = (Y\rho_1)\rho_0 \subseteq \{b,c\}\rho_0$, that is, $b \preceq y$ and $c \preceq y$. Since $b \parallel c$, we obtain $b \prec y$. As $a \prec b$, the covering $b \prec y$ contradicts $a \preceq y$. This and the duality principle prove (7).

Now, to sharpen the previous assertion, we prove that, for any $b \neq c$,

$$\text{if}\quad b, c \in \text{mid}(Q), \quad\text{then}\quad L(b,c) = U(b,c) = \emptyset. \tag{8}$$

Suppose the contrary. By the duality principle, we can assume that $L(b,c)$ is nonempty. Let $L(b,c) = \{a\}$. We can choose $d_1, d_2 \in \max(Q)$ such that $b \prec d_1$ and $c \prec d_2$. If possible, then we choose them equal: $d_1 = d_2$. Let $X = \{b, c\}$. If $U(b,c)$ is nonempty, then $d_1 = d_2$, $X\rho_0 = \{d_1\}$, and we have $d_1 \in \mathcal{G}^{(0)}(X) = \mathcal{C}_1^{(0)}(X)$. If $U(b,c)$ is empty, then so is $X\rho_0$, and we have $d_1 \in Q = \mathcal{G}^{(0)}(X) = \mathcal{C}_1^{(0)}(X)$ again. Let $Y = X = X\psi_0$, and notice that $Y\rho_1 = \{a\}$ by (7). Then, by formula (3), $d_1 \preceq y$ for some $y \in \mathcal{G}^{(1)}(Y)$. Since $d_1 \in \max(Q)$, $d_1 = y \in \mathcal{G}^{(1)}(Y) = (Y\rho_1)\rho_0 = \{a\}\rho_0$. This gives $a \preceq d_1$, which contradicts $a \prec b \prec d_1$. This shows (8).

Based on (8) we can prove even more: for arbitrary elements of Q, we have

$$\text{if}\quad c \in Q, \quad b \in \text{mid}(Q) \quad\text{and}\quad b \parallel c, \quad\text{then}\quad L(b,c) = U(b,c) = \emptyset. \tag{9}$$

Suppose the contrary. By (8), $c \notin \text{mid}(Q)$. By the duality principle we can assume that $c \in \max(Q)$. Then $U(b,c) = \emptyset$. Let $a \in L(b,c)$ and choose an element $d \in \max(Q)$ with $b \prec d$. Let $X = \{a, d\}$. We obtain from $X\rho_0 = \emptyset$

that $c \in Q = \mathcal{G}^{(0)}(X) = \mathcal{C}_1^{(0)}(X)$. Let $Y = \{b, d\} \in X\psi_0$. Then $c \preceq y$ for some $y = \mathcal{G}^{(1)}(Y)$ by (3), and $c \in \max(Q)$ yields that $y = c$. This contradicts $b \parallel c$, since $b \preceq y = c$ by (6). This proves (9).

Now, we are in the position to show that

$$\text{if} \quad b \in \mathrm{mid}(Q), \quad \text{then there is no} \quad c \in Q \quad \text{with} \quad b \parallel c. \tag{10}$$

Suppose the contrary, and choose $a, d \in Q$ with $a \prec b \prec d$. Let $X = \{a, d\}$ and $Y = \{b, d\} \in X\psi_0$. Like in the previous step, $c \in Q = \mathcal{G}^{(0)}(X)$ implies the existence of an element $y \in \mathcal{G}^{(1)}(Y)$ with $c \preceq y$. Then $y \in U(b, c)$ by (6), contradicting (9). This proves (10).

Since $\mathrm{length}(Q) = 2$, we can choose elements $a_1 \prec b \prec d_1$ in Q. It follows from (10) that, for any further element x, we have either $x < b$ or $b < x$. Let $m = |\{x \in Q : x < b\}|$ and $n = |\{x \in Q : b < x\}|$. Clearly, Q is T_{mn}. $\square$

We conclude this paper with a problem, which is more concrete than Problem 1. For motivation (and a possible application if the answer is affirmative) see [1].

Problem (on non-degenerate triangles) 8. Let $\vec{\mathcal{C}} = \vec{\mathcal{C}}(A^{(0)}, A^{(1)}, \rho)$ such that

- ρ is indecomposable, that is, for every nonempty sets $B^{(i)}$ and $C^{(i)}$ with $B^{(i)} \cup C^{(i)} = A^{(i)}$ and $B^{(i)} \cap C^{(i)} = \emptyset$, ρ is distinct from the relation $\big(\rho \cap (B^{(0)} \times B^{(1)})\big) \cup \big(\rho \cap (C^{(0)} \times C^{(1)})\big)$;
- ρ is uniform, that is, $|\{x\}\rho_i| = |\{y\}\rho_i|$ for all $x, y \in A^{(i)}$; and
- $A^{(0)}$ and $A^{(1)}$ are finite, and both have at least three elements.

Do these assumptions imply that there exists an $i \in \{0, 1\}$ and there are $x, y, z \in A^{(i)}$ such that $\mathcal{C}^{(i)}(\{x, y\}) \cap \mathcal{C}^{(i)}(\{y, z\}) \cap \mathcal{C}^{(i)}(\{z, x\}) = \emptyset$?

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