SOME LATTICE HORN SENTENCES FOR SUBMODULES OF PRIME POWER CHARACTERISTIC

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April, 15, 1992

Abstract. For a unital ring R of prime power characteristic p^k let the class, in fact the quasivariety, of lattices embeddable in the submodule lattices of R modules be denoted by $\mathcal{L}(R)$. Let $\mathbf{W}(p^k) = \{\, \mathcal{L}(R) : \operatorname{char} R = p^k \,\}$. Hutchinson in [2] gave a necessary condition, leading to interesting consequences, for the inclusion $\mathcal{L}(R_1) \subseteq \mathcal{L}(R_2)$ when $\mathcal{L}(R_1), \mathcal{L}(R_2) \in \mathbf{W}(p^k)$. However, it is not known if this condition is sufficient. Another open problem from [2] is whether $\mathbf{W}(p^k)$ is closed with respect to arbitrary joins. Using certain appropriate lattice Horn sentences, the present paper shows that at least one of the above-mentioned two problems has a negative solution.

For a ring R with unit the class of lattices embeddable in the submodule lattices of R modules is known to be a quasivariety (cf. Makkai and McNulty [6]). This quasivariety will be denoted by

$$\mathcal{L}(R) = \{\, \mathrm{Su}({}_R M) : {}_R M \text{ is an } R\text{-module} \,\}.$$

We will consider rings with prime power characteristic p^k where $k > 1$. All the rings in the sequel, unless otherwise stated, will be assumed to be of characteristic p^k. Let $\mathbf{W}(p^k)$ denote the class $\{\, \mathcal{L}(R) : \operatorname{char} R = p^k \,\}$. While the *variety* $\mathbf{H}\,\mathcal{L}(R)$ depends only on p^k, the characteristic of R (cf. [5]), and $\mathbf{W}(p)$ is a singleton (cf. [3, p. 88]), $\mathbf{W}(p^k)$ consists of continuously many quasivarieties $\mathcal{L}(R)$, cf. [2] . The proof of this result was proved by the following powerful tool. Let τ denote the similarity type consisting of operation symbols $\vee, \wedge, \cdot, \uparrow, \downarrow, \mathbf{0}, \mathbf{1}$ with respective arities 2,2,2,1,1,0,0. The set $\mathcal{I}(R)$ of two-sided ideals of R becomes a τ-algebra in a natural way: $\vee, \wedge$ are the lattice operations, $\mathbf{0} = \{0\}$, $\mathbf{1} = R$, $\cdot$ is the usual product of ideals, $\downarrow X = \{\, p\,x : x \in X \,\}$, and $\uparrow X = \{\, x : p\,x \in X \,\}$. Let $K(R)$ denote the set of all nullary τ-terms σ such that $\sigma = \mathbf{1}$ $(= R)$ holds in $\mathcal{I}(R)$, and let $\Sigma(R)$ denote the set of (universal) lattice Horn sentences satisfied in $\mathcal{L}(R)$.

Theorem A. *(Hutchinson [2]) If $\mathcal{L}(R_1) \subseteq \mathcal{L}(R_2)$ then $K(R_1) \supseteq K(R_2)$.*

The proof of this theorem is based on the following

This work was partially supported by Hungarian National Foundation for Scientific Research Grant No. 1903.

Theorem B. *(Hutchinson [3] and [4]) $\mathcal{L}(R_1) \subseteq \mathcal{L}(R_2)$ is equivalent to the existence of an exact embedding functor R_1-**Mod** $\to R_2$-**Mod**.*

Note that $\Sigma(R_1) \supseteq \Sigma(R_2)$ is also equivalent to $\mathcal{L}(R_1) \subseteq \mathcal{L}(R_2)$. Therefore our present investigation based on Horn sentences might be interesting from abelian cathegory theoretical point of view, too.

Our goal is to deal with the following two open problems, the first of them is related to the converse of Theorem A.

Problem C. *Does $K(R_1) \supseteq K(R_2)$ imply $\mathcal{L}(R_1) \subseteq \mathcal{L}(R_2)$?*

Problem D. *Is $\mathbf{W}(p^k)$ closed with respect to arbitrary joins (taken in the lattice of all latice quasivarieties)?*

Note that $\mathbf{W}(p^k)$ is closed with respect to finite joins. It is shown in [2] that $(\mathbf{W}(p^k); \subseteq)$ contains large chains and antichains and it has a nontrivial automorphism, namely $\mathcal{L}(R) \mapsto \mathcal{L}(R^{\mathrm{op}})$, but we do not know if it is a lattice. An affirmative answer to Problem D or (much less trivially!) to Problem C would imply that $\mathbf{W}(p^k)$ is a lattice. The analogous problems for the *set* of lattice varieties $\mathbf{H}\mathcal{L}(S)$, where the S are rings of any characteristic, have positive solutions (cf.[5]).

Main Theorem. *At least one of Problems C and D has a negative answer.*

The *proof* of the Main Theorem is based on certain lattice Horn sentences $\chi(m,p)$, which might be of separate interest. Note that $\chi(2,2)$ appeared in [1] but without any application that time. Our proof is divided into several lemmas.

First we define appropriate rings. The ring of integers modulo p^k will be denoted by $\mathbf{Z}_{p^k}$. For a given n let F_n denote the polynomial ring

$$\mathbf{Z}_{p^k}[\xi_1, \dots, \xi_n, \eta_1, \dots, \eta_n].$$

Let I_n be the ideal generated by

$$\begin{aligned}&\{\, \xi_i\eta_i - p^{k-1}\xi_{i-1} : 1 \le i \le n \,\} \cup \{\, p\eta_i : 1 \le i \le n \,\} \cup \\ &\quad \{\, p^{k-1}\xi_n \,\} \cup \{\, \xi_i\xi_j : 1 \le i \le n,\ 1 \le j \le n \,\} \cup \{\, \eta_i\eta_j : 1 \le i \le n,\ 1 \le j \le n \,\} \\ &\qquad \cup \{\, \xi_i\eta_j : 1 \le i \le n,\ 1 \le j \le n,\ i \ne j \,\},\end{aligned}$$

where $\xi_0 = 1$. Put $R_n = F_n/I_n$, $\quad x_i = \xi_i + I_n$, $\quad y_i = \eta_i + I_n$. Note that $x_0 = 1$. By the definition of R_n we have

$$\begin{aligned}(1)\quad & x_iy_i = p^{k-1}x_{i-1}, \quad y_iy_j = 0, \quad x_ix_j = 0, \quad x_iy_l = 0, \quad p^kx_i = 0, \\ & p^{k-1}x_n = 0, \quad p\,y_i = 0 \qquad \text{for } i,j,l \in \{\,1,2,\dots n\,\},\ i \ne l.\end{aligned}$$

Lemma 1. *The elements x_i $(i = 0,1,\dots,n-1)$, x_n and y_i $(i = 1,2,\dots,n)$ are of respective additive order p^k, p^{k-1} and p. Further, the additive group of R_n is the direct sum of the additive cyclic subgroups generated by these elements. In other words, each element of R_n has a unique canonical form*

$$\sum_{i=0}^{n-1} \alpha_i x_i + \beta x_n + \sum_{i=1}^{n} \gamma_i y_i \tag{2}$$

where $\alpha_i \in \{0,1,\ldots,p^k-1\}$, $\beta \in \{0,1,\ldots,p^{k-1}-1\}$ *and* $\gamma_i \in \{0,1,\ldots,p-1\}$. *The rules of computation in* R_n *are* (1) *together with the axioms of unital commutative rings of characteristic* p^k*.*

Proof. It suffices to show the uniqueness of (2); the rest is clear. Assume that $0 \in R_n$ is of the form (2). Then, by the definition of I_n, we have

$$\sum_{i=0}^{n-1}\alpha_i\xi_i + \beta\xi_n + \sum_{i=1}^{n}\gamma_i\eta_i = \sum_{i=1}^{n} f_i\cdot(\xi_i\eta_i - p^{k-1}\xi_{i-1}) + \sum_{i=1}^{n} g_i\cdot p\eta_i + g_0p^{k-1}\xi_n + \sum_{i=1}^{n}\sum_{j=1}^{n} h_{ij}\cdot\xi_i\xi_j + \sum_{i=1}^{n}\sum_{j=1}^{n} r_{ij}\cdot\eta_i\eta_j + \sum_{i=1}^{n}\sum_{\substack{l=1\\ l\neq i}}^{n} s_{ij}\cdot\xi_i\eta_l \tag{3}$$

where f_i, g_i, h_{ij}, r_{ij}, $s_{ij} \in F_n$. We treat the elements of F_n as polynomials in the usual canonical form. Hence these polynomials are sums of uniquely determined summands and each summand consists of uniquely determined factors (i.e, powers of indeterminants) and a unique coefficient (from $\mathbf{Z}_{p^k}$). Suppose we have performed the operations on the right-hand-side of (3). Then each summand on the right-hand-side in which η_i is the only indeterminant has a coefficient divisible by p. Therefore $\gamma_i = 0$ for all i. We obtain $\beta = 0$ similarly.

Suppose $\alpha_i \neq 0$ for some i. The only source of ξ_i on the right is $f_{i+1}\cdot(\xi_{i+1}\eta_{i+1} - p^{k-1}\xi_i)$. Since p^k does not divide α_i, the constant δ in f_{i+1} is not divisible by p. But then $\delta\xi_{i+1}\eta_{i+1}$ cannot be cancelled by other summands. This contradiction completes the proof.

Before describing $K(\mathbf{Z}_{p^k})$ we make the set $\{0,1,2,\ldots k\}$ into an algebra of type τ via putting $x\vee y = \max\{x,y\}$, $x\wedge y = \min\{x,y\}$, $\uparrow x = \min\{x+1,k\}$, $\downarrow x = \max\{x-1,0\}$, $\mathbf{0} = 0$, $\mathbf{1} = k$ and $x\cdot y = \max\{x+y-k,0\}$. (To avoid confusion, the ordinary product of x and y will be denoted by the concatenation xy.) Denoting the set of nullary τ-terms by $\mathcal{P}_0$, let h be the map associating with any element of $\mathcal{P}_0$ its value in the above-defined algebra $\{0,1,2,\ldots k\}$.

Lemma 2. $K(\mathbf{Z}_{p^k}) = \{\sigma \in \mathcal{P}_0 : h(\sigma) = k\}$.

Proof. An easy induction on the length of σ yields that the value of σ in $\mathcal{I}(\mathbf{Z}_{p^k})$ is $p^{k-h(\sigma)}\mathbf{Z}_{p^k} = \downarrow^{k-h(\sigma)} \mathbf{Z}_{p^k}$, whence the lemma follows.

Lemma 3. $\bigcap_{n=1}^{\infty} K(R_n) = K(\mathbf{Z}_{p^k})$

Proof. For $0 \le t \le n-1$ and $0 \le j \le k$ we consider the following subsets of R_n:

$$\begin{aligned}
A_{j,t}^{(n)} &= \{p^ix_l : 1 \le l < n-t,\ i \ge k-j,\ i \ge 0\},\\
B_{j,t}^{(n)} &= \{p^ix_l : n-t \le l \le n-1,\ i+l \ge n-t+k-j-1,\ i \ge 0\},\\
C_{j,t}^{(n)} &= \{p^ix_n : i \ge k-j-1,\ i \ge 0\},\\
D_{j,t}^{(n)} &= \{y_l : 1 \le l \le n,\ j > 0\} \quad \text{and}\\
E_{j,t}^{(n)} &= \{p^i : i \ge k-j\} \cup A_{j,t}^{(n)} \cup B_{j,t}^{(n)} \cup C_{j,t}^{(n)} \cup D_{j,t}^{(n)}.
\end{aligned}$$

Note that $D_{j,t}^{(n)} = \{y_1,\ldots,y_n\}$ for $j > 0$ and $D_{0,t}^{(n)} = \emptyset$. Let $I_{j,t}^{(n)}$ be the additive subgroup of R_n generated by $E_{j,t}^{(n)}$. With the help of Lemma 1 it is not hard to see that the $I^{(n)}_{j,t}$ are ideals of R_n, $I^{(n)}_{k,t} = R_n$, $0 \le t_1 \le t_2 \le n-1$ implies $I^{(n)}_{j,t_1} \subseteq I^{(n)}_{j,t_2}$, and $0 \le j_1 \le j_2 \le k$ implies $I^{(n)}_{j_1,t} \subseteq I^{(n)}_{j_2,t}$. Further, $\downarrow I^{(n)}_{j,t} \subseteq I^{(n)}_{\downarrow j,t}$, and $\uparrow I^{(n)}_{j,t} \subseteq I^{(n)}_{\uparrow j,t}$. Now we claim that $I^{(n)}_{j,t} \cdot I^{(n)}_{s,t} \subseteq I^{(n)}_{j\cdot s,t+1}$. Suppose $a \in E^{(n)}_{j,t}$ and $b \in E^{(n)}_{s,t}$. It suffices to check $ab \in E^{(n)}_{j\cdot s,t+1}$. We omit the straightforward but long details and consider only the case $a \in B^{(n)}_{j,t}$ and $b \in D^{(n)}_{s,t}$. Then $a = p^i x_l$, $n-t \le l \le n-1$, $i+l \ge n-t+k-j-1$ and $s > 0$. We may assume that $b = y_l$ as otherwise $ab = 0$. We conclude $ab = p^{i+k-1}x_{l-1}$, $n-(t+1) \le l-1 \le n-1$ and $(i+k-1)+(l-1) = i+l+k-2 \ge n-t+k-j-1+k-2 = n-(t+1)+k-(j+1-k)-1 \ge n-(t+1)+k-(j+s-k)-1 \ge n-(t+1)+k-j\cdot s-1$, yielding $ab \in B^{(n)}_{j\cdot s,t+1} \subseteq E^{(n)}_{j\cdot s,t+1}$.

For a τ-term $\sigma \in \mathcal{P}_0$ let σ_{R_n} denote the value of σ in $\mathcal{I}(R_n)$. The *length* $|\sigma|$ of σ is defined via induction: $|\mathbf{0}| = |\mathbf{1}| = 1$, $|\uparrow\sigma| = |\downarrow\sigma| = |\sigma|+1$, $|\sigma_1 \vee \sigma_2| = |\sigma_1 \wedge \sigma_2| = |\sigma_1 \cdot \sigma_2| = |\sigma_1| + |\sigma_2| + 1$. The inclusions among the $I^{(n)}_{j,t}$ we have already established yield

$$\sigma_{R_n} \subseteq I^{(n)}_{h(\sigma),|\sigma|}\,, \qquad \text{provided } |\sigma| < n\,, \tag{4}$$

via an easy induction on $|\sigma|$.

Now the proof of Lemma 3 will be completed easily. Suppose that $\sigma \notin K(\mathbf{Z}_{p^k})$. Then $h(\sigma) \le k-1$ by Lemma 2. Choose an n with $n > |\sigma|+2$. Then, by (4) and Lemma 1,

$$\sigma_{R_n} \subseteq I^{(n)}_{h(\sigma),|\sigma|} \subseteq I^{(n)}_{k-1,|\sigma|} \subseteq I^{(n)}_{k-1,n-2} \not\ni 1,$$

whence $\sigma \notin K(R_n)$. Therefore $\bigcap_{l=1}^{\infty} K(R_l) \not\supset K(\mathbf{Z}_{p^k})$.

Conversely, an easy induction on $|\sigma|$ yields $\sigma_{R_n} \supseteq \downarrow^{k-h(\sigma)} R_n$. In particular, if $h(\sigma) = k$ then $\sigma_{R_n} = R_n$. Hence Lemma 2 yields $\bigcap_{l=1}^{\infty} K(R_l) \supseteq K(\mathbf{Z}_{p^k})$, proving Lemma 3.

Now let $m = p^{k-1}$. On the set of variables $\{x, y, z, t\}$ we define the following lattice terms:

$$\begin{aligned} r &= (x \vee y) \wedge (z \vee t), & h_0 &= g_0 = t, & h'_i &= (h_i \vee y) \wedge (x \vee z)\\ h_{i+1} &= (h'_i \vee r) \wedge (x \vee t), & g'_i &= (g_i \vee x) \wedge (y \vee z), & g_{i+1} &= (g'_i \vee r) \wedge (y \vee t),\\ r_0 &= (h_{m-1} \vee z) \wedge y, & q_0 &= x \vee z \vee g_{p-1}, & q &= r_0 \vee x. \end{aligned}$$

Let $\chi(m,p)$ denote the lattice Horn sentence

$$r_0 \le q_0 \implies r \le q\,.$$

Lemma 4. *$\chi(m,p)$ does not hold in $\mathcal{L}(\mathbf{Z}_{p^k})$.*

Proof. Let M be the $\mathbf{Z}_{p^k}$-module freely generated by $\{f_1, f_2, f_3\}$. Consider the submodules $x = [f_2]$, $y = [f_1 - f_2]$, $z = [f_3]$, $t = [f_1 - f_3]$. An easy calculation gives $r = [f_1]$. (We do not make a notational distinction between lattice terms and the submodules obtained from them by substituting the *submodules* x, y, z, t for their variables.) It is not hard to check, via induction on i, that $h'_i = [(i+1)f_2 - f_3]$, $h_i = [f_1 + if_2 - f_3]$, $g'_i = [(i+1)f_1 - (i+1)f_2 - f_3]$, $g_i = [(i+1)f_1 - if_2 - f_3]$. These equations yield $r_0 = \{\alpha(f_1 - f_2) : m\alpha = 0\} = [p(f_1 - f_2)]$, $q_0 = [pf_1,\ f_2,\ f_3]$, $q = [pf_1,\ f_2]$. Therefore $\chi(m,p)$ does not hold in $\mathrm{Su}(M)$.

Lemma 5. *$\chi(m,p)$ holds in $\mathcal{L}(R_n)$ for every $n \geq 1$.*

Proof. Assume that x, y, z, t are submodules of an R_n-module M such that $r_0 \subseteq q_0$, and let $f_1 \in M$ be an arbitrary element of r. Our aim is to show $f_1 \in q$. Since $f_1 \in r = (x+y) \cap (z+t)$, we can choose $f_2, f_3 \in M$ such that $f_2 \in x$, $f_1 - f_2 \in y$, $f_3 \in z$, $f_1 - f_3 \in t$. An easy calculation, essentially the same as in the previous lemma, gives $(i+1)f_2 - f_3 \in h'_i$, $f_1 + if_2 - f_3 \in h_i$, and $\{\alpha(f_1 - f_2) : m\alpha = 0\} \subseteq r_0$. In particular, $x_n(f_1 - f_2) \in r_0$.

Now let us suppose that $x_j(f_1 - f_2) \in r_0$ for some $j > 0$. We intend to show $x_{j-1}(f_1 - f_2) \in r_0$; then $f_1 - f_2 = x_0(f_1 - f_2) \in r_0$ follows by (downward) induction on j. From $r_0 \subseteq q_0$ we infer $x_j(f_1 - f_2) \in q_0 = x + z + g_{p-1}$. Hence there exist elements e_0 and e_1 in M such that $e_0 \in x$, $e_1 - e_0 \in z$ and $x_j(f_1 - f_2) - e_1 \in g_{p-1} = (g'_{p-2} + r) \cap (y+t)$. This implies the existence of two elements, say e_2^{p-1} and $e_4^{p-1} \in M$ such that $e_1 - e_4^{p-1} \in y$, $x_j(f_1 - f_2) - e_4^{p-1} \in t$, $e_1 - e_2^{p-1} \in g'_{p-2}$, and $x_j(f_1 - f_2) - e_2^{p-1} \in r$. Continuing this parsing and denoting $x_j(f_1 - f_2)$ by e_1^p we obtain that there exist elements $e_l^i \in M$ for $i = 1, 2, \ldots, p-1$ and $l = 1, 2, \ldots, 6$ such that for $i \in \{1, 2, \ldots, p-1\}$

$$e_1 - e_3^i \in y, \quad e_1 - e_4^i \in y, \quad e_2^i - e_3^i \in z, \quad e_4^i - e_1^{i+1} \in t, \quad e_1^i - e_2^i \in x,$$
$$e_2^i - e_5^i \in x, \quad e_1^{i+1} - e_5^i \in y, \quad e_2^i - e_6^i \in z, \quad e_1^{i+1} - e_6^i \in t, \quad e_1 - e_1^1 \in t.$$

Clearly, $e_1^p = x_j(f_1 - f_2) \in y$. Let us observe that x contains $u_0 = x_j f_2 + e_0 + \sum_{i=1}^{p-1}(e_2^i - e_1^i)$. But $u_0 = \sum_{i=1}^{p-2}(e_2^i - e_6^i) + \sum_{i=1}^{p-2}(e_6^i - e_1^{i+1}) - (x_j(f_1 - f_2) - e_4^{p-1}) + x_j(f_1 - f_3) + x_j f_3 + (e_0 - e_1) + (e_1 - e_1^1) + (e_2^{p-1} - e_6^{p-1}) + (e_6^{p-1} - e_1^p) + (e_1^p - e_4^{p-1})$, whence $u_0 \in r$. Now $u_0 \in x$ and $u_0 \in r$ imply $u_0 \in h_i$ for all $i > 0$. In particular, $u_0 \in h_{m-1}$. Let $u_i = e_0 - e_1 - e_2^i + e_3^i$ for $1 \leq i \leq p-1$. We have, for $i > 0$, $u_i = e_0 - (e_1 - e_3^i) - e_1^p + \sum_{l=i}^{p-1}(e_1^{l+1} - e_5^l) + \sum_{l=i}^{p-1}(e_5^l - e_2^l) + \sum_{l=i+1}^{p-1}(e_2^l - e_1^l) \in x + y$ and $u_i = (e_0 - e_1) - (e_2^i - e_3^i) \in z$, whence $u_i \in r$. Let $v_i = e_1 + e_1^i - e_3^i$. Since $e_1^1 - e_3^1 = (e_1^1 - e_1) + (e_1 - e_3^1) \in y + t$ and, for $i > 1$, $e_1^i - e_3^i = (e_1^i - e_4^{i-1}) - (e_1 - e_4^{i-1}) + (e_1 - e_3^i) \in y + t$, we have $v_i = (e_1 - e_4^{p-1}) + (e_4^{p-1} - e_1^p) + e_1^p + (e_1^i - e_3^i) \in y + t$. But $v_i = e_0 - (e_0 - e_1) + (e_1^i - e_2^i) + (e_2^i - e_3^i) \in x + z$, whence $v_i \in h'_0 \quad (i = 1, 2, \ldots, p-1)$. For $1 \leq i \leq p-1$ let $w_i = e_0 + e_1^i - e_2^i$. From $w_i = v_i + u_i \in h'_0 + r$ and $w_i = e_0 + (e_1^i - e_2^i) \in x$ we infer that $w_i \in h_1$. This together with $w_i \in x$ yield $w_i \in h_{m-1}$.

Now $x_{j-1}(f_1 - f_2) \in y$ and, by $y_j x_j = m x_{j-1}$ and $p y_j = 0$, $x_{j-1}(f_1 - f_2) = x_{j-1}(f_1 + (m-1)f_2 - f_3) - y_j u_0 - \sum_{i=1}^{p-1} y_j w_i + x_{j-1} f_3 \in h_{m-1} + z$. Thus $x_{j-1}(f_1 - f_2) \in r_0$, as intended.

Finally, $f_1 = (f_1 - f_2) + f_2 \in r_0 + x = q$ completes the proof of Lemma 5.

Proof of the Main Theorem. Let us assume that Problem C has an affirmative answer. We claim that

$$\bigvee_{n=1}^{\infty} \mathcal{L}(R_n) = \mathcal{L}(\mathbf{Z}_{p^k}) \tag{5}$$

where the join is formed in $(\mathbf{W}(p^k); \subseteq)$. Since $K(R_n) \supseteq K(\mathbf{Z}_{p^k})$ by Lemma 3, we obtain $\mathcal{L}(R_n) \subseteq \mathcal{L}(\mathbf{Z}_{p^k})$, for every n, by the assumption. (Note that $\mathcal{L}(R_n) \subseteq \mathcal{L}(\mathbf{Z}_{p^k})$ also follows from Theorem B.) On the other hand, suppose $\mathcal{L}(S) \in \mathbf{W}(p^k)$

and, for all n, $\mathcal{L}(R_n) \subseteq \mathcal{L}(S)$. Theorem A yields $K(R_n) \supseteq K(S)$. From Lemma 3 we conclude $K(\mathbf{Z}_{p^k}) = \bigcap_{n=1}^{\infty} K(R_n) \supseteq K(S)$, and the assumption on Problem C gives $\mathcal{L}(\mathbf{Z}_{p^k}) \subseteq \mathcal{L}(S)$. This proves (5).

Now if Problem D had an affirmative answer then (5) would be true even in the lattice of all quasivarieties of lattices. But this would contradict to Lemmas 4 and 5.

Acknowledgement: The author is grateful to George Hutchinson for his helpful comments. In particular, the present result was initiated by his idea which, although not used in this paper, is necessary to prove that an affirmative answer to Problem C yields that $\mathbf{W}(p^k)$ is a lattice.

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