A SELFDUAL EMBEDDING OF THE FREE LATTICE OVER COUNTABLY MANY GENERATORS INTO THE THREE-GENERATED ONE

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Dedicated to E. Tamás Schmidt on the occasion of his eightieth birthday

ABSTRACT. By an 1941 result of Whitman, the free lattice $\mathrm{FL}(3) = \mathrm{FL}(x, y, z)$ includes a sublattice $\mathrm{FL}(\omega)$ freely generated by infinitely many elements. Let δ denote the unique dual automorphism of $\mathrm{FL}(x, y, z)$ that acts identically on the set $\{x, y, z\}$ of generators. We prove that $\mathrm{FL}(x, y, z)$ has a sublattice S isomorphic to $\mathrm{FL}(\omega)$ such that $\delta(S) = S$.

1. Introduction and our result

For a nonempty set X, the free lattice over X will be denoted by $\mathrm{FL}(X)$ or $\mathrm{FL}(|X|)$. We write $\mathrm{FL}(\omega)$ rather than $\mathrm{FL}(\aleph_0)$. Finally, $\mathrm{FL}(x, y, z)$ or $\mathrm{FL}(3)$ stands for $\mathrm{FL}(\{x, y, z\})$. By a classical result of Whitman [7], $\mathrm{FL}(\omega)$ is isomorphic to a sublattice of $\mathrm{FL}(3)$. Actually, there are many copies of $\mathrm{FL}(\omega)$ in $\mathrm{FL}(3)$, because every infinite interval of $\mathrm{FL}(3)$ includes a sublattice isomorphic to $\mathrm{FL}(\omega)$ by Tschantz [6].

Since both $\mathrm{FL}(3)$ and $\mathrm{FL}(\omega)$ are selfdual lattices, it is natural to ask if $\mathrm{FL}(\omega)$ can be embedded in $\mathrm{FL}(3)$ in a *selfdual way*. Let $\delta\colon \mathrm{FL}(x, y, z) \to \mathrm{FL}(x, y, z)$ be the unique dual automorphism such that $\delta(x) = x$, $\delta(y) = y$, and $\delta(z) = z$. Our aim is to prove the following generalization of Whitman's result.

Theorem 1.1. *The free lattice* $\mathrm{FL}(x, y, z)$ *includes a sublattice* S *such that* S *is isomorphic to* $\mathrm{FL}(\omega)$ *and* $\delta(S) = S$.

1.1. **Method.** Our proof is based on the following result of Whitman; it can also be found in Crawley and Dilworth [1, 16.5], in Freese, Ježek and Nation [3, Corollary 1.13], and in Grätzer [4, Theorem 546]. To reduce the number of parentheses in the paper, the join and meet operations will be written as addition and multiplication, respectively.

Lemma 1.2 (Whitman [8]). *A nonempty subset* Y *of* $\mathrm{FL}(X)$ *generates a sublattice isomorphic to a free lattice if and only if for all* $h \in Y$ *and all finite subsets* $Z \subseteq Y$*, the following condition and its dual hold.*

$$h \notin Z \quad \textit{implies} \quad h \nleq \sum_{z \in Z} z. \tag{1.1}$$

Apart from Lemma 1.2, our approach is self-contained and elementary.

2000 *Mathematics Subject Classification.* 06B25. Version: September 22, 2015.
Key words and phrases. free lattice, sublattice, dual automorphism.

2. Ternary terms

We borrow the following ternary lattice terms from Grätzer [4, Subsection VII.2.4].

$$
\begin{aligned}
a(x,y,z) &= x(y+z) + y(x+z)\\
b(x,y,z) &= x(y+z) + z(x+y),\\
c(x,y,z) &= (x+yz)\cdot(y+xz) = \delta(a),\\
d(x,y,z) &= (x+yz)\cdot(z+xy) = \delta(b).
\end{aligned} \tag{2.1}
$$

Unless otherwise stated, $a := a(x,y,z)$, ..., $d := d(x,y,z)$ are the corresponding elements of $\mathrm{FL}(x,y,z)$. As usual, $[a,b,c,d]$ denotes the sublattice of $\mathrm{FL}(x,y,z)$ generated by these elements. Part (A) of the following lemma is taken from [4, Subsection VII.2.4], where the easy proof is left to the reader.

Lemma 2.1 (Grätzer [4])**.**

(A) $[a,b,c,d]$ *is isomorphic to* $\mathrm{FL}(4)$ *and it is freely generated by* $\{a,b,c,d\}$.

(B) *Furthermore,* $\delta(a) = c$, $\delta(c) = a$, $\delta(b) = d$, $\delta(d) = b$ *and, consequently,* $\delta([a,b,c,d]) = [a,b,c,d]$.

Proof. The equations $\delta(a) = c$, ..., $\delta(d) = b$ are obvious. Let $S = [a,b,c,d]$. For a lattice term t, let t^* denote the dual term. If $u \in S$, then $u = t(a,b,c,d)$ for an appropriate quaternary lattice term t. Since

$$\delta(u) = \delta\big(t(a,b,c,d)\big) = t^*(\delta(a),\delta(b),\delta(c),\delta(d)) = t^*(c,d,a,b) \in S,$$

we obtain that $\delta(S) \subseteq S$ and, consequently, $S = \delta\big(\delta(S)\big) \subseteq \delta(S)$. This proves part (B).

To prove part (A), we use Lemma 1.2. It suffices to show that (1.1) holds for $\{a,b,c,d\}$; then its dual also holds, because, say, $a \geq bcd$ would imply $c = \delta(a) \leq \delta(b)+\delta(c)+\delta(d) = d+a+b$. The elements x,y,z in L_1 given in Figure 1 witness that the lattice identity $a(x,y,z) \leq b(x,y,z)+c(x,y,z)+d(x,y,z)$ fails in L_1. This proves that $a \not\leq b+c+d$ in $\mathrm{FL}(x,y,z)$. Similarly, L_2 in Figure 1 witnesses that $c \not\leq a+b+d$ in $\mathrm{FL}(x,y,z)$. The rest follows by interchanging y and z. $\square$

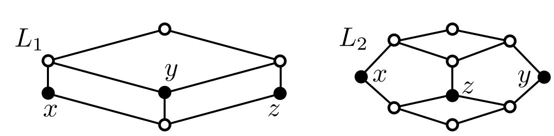

Figure 1. L_1 and L_2

3. Quaternary terms

We define the following terms, which are also elements of $\mathrm{FL}(w,x,y,z)$.

$$
\begin{aligned}
g_1 &= g_1(w,x,y,z) = \big((w+x)y+z\big)w + x(y+z),\\
g_2 &= g_2(w,x,y,z) = \big((xw+z)y+x\big)\cdot(w+zy),\\
g_3 &= g_3(w,x,y,z) = \big((x+y)z+w\big)x + y(z+w),\\
g_4 &= g_4(w,x,y,z) = \big((wz+y)x+w\big)\cdot(z+yx),\\
g_5 &= g_5(w,x,y,z) = \big((y+z)w+x\big)y + z(w+x),
\end{aligned}
$$

$$g_6 = g_6(w,x,y,z) = \big((zy+x)w+z\big)\cdot(y+xw).$$

Besides the dual automorphism δ_4 of $\mathrm{FL}(w,x,y,z)$ with $w\mapsto w, \dots, z\mapsto z$, we also consider the lattice automorphism α of $\mathrm{FL}(w,x,y,z)$ with $w\mapsto x$, $x\mapsto w$, $y\mapsto z$, $z\mapsto y$. Note that α and δ_4 commute and $\alpha\circ\delta_4$ is a dual automorphism. The aim of this section is to prove the following lemma.

Lemma 3.1. *In $\mathrm{FL}(w,x,y,z)$, let S denote the sublattice generated by $\{g_1,\dots,g_6\}$. Then $\{g_1,\dots,g_6\}$ freely generates S, that is, $S\cong \mathrm{FL}(6)$. Furthermore, the dual automorphism $\alpha\circ\delta_4$ acts on the generators of S as follows*

$$\begin{pmatrix} g_1 & g_2 & g_3 & g_4 & g_5 & g_6\\ g_2 & g_1 & g_4 & g_3 & g_6 & g_5\end{pmatrix}. \tag{3.1}$$

Consequently, $(\alpha\circ\delta_4)(S)=S$.

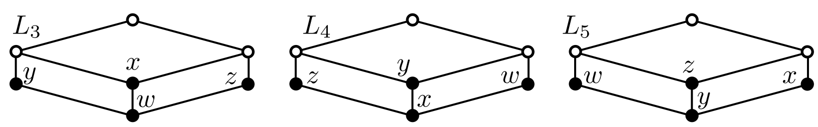

FIGURE 2. L_3, L_4, and L_5

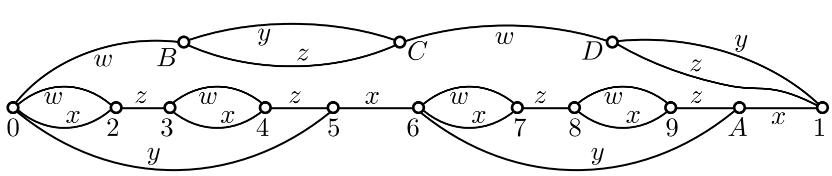

FIGURE 3. The equivalences w,x,y,z that generate L_6

Proof. The proof is similar to that of Lemma 2.1 in the sense that we use concrete finite lattices to establish "$\not\leq$" as (1.1) requires. The statement on $\alpha\circ\delta_4$ is evident. If, say, $g_1\geq g_2g_3g_4g_5g_6$, then $\alpha\circ\delta$ turns this inequality into $g_2\leq g_1+g_4+g_3+g_6+g_5$. This shows that it suffices to verify the validity of (1.1) from Lemma 1.2 for $\{g_1,\dots,g_6\}$, and we do not have to deal with the dual condition. The black-filled elements of L_3, L_4, and L_5 in Figure 2 show that none of g_1, g_3, and g_5, respectively, is below the join of the remaining elements.

Next, to show the same for g_2, we define a larger lattice, L_6. The set of the first fourteen hexadecimals will be denoted by $U=\{0,1,2,\dots,9,A,B,C,D\}$. Let $\mathrm{Equ}(U)$ stand for the lattice of equivalences on U. To describe and denote an equivalence Θ on U, we adopt the following convention. To give a non-singleton block of Θ, we simply list the elements of this block in increasing order, without commas. Then, to describe Θ, we list its non-singleton blocks separated by vertical dashes, $|$, in the increasing order of their first elements. The least and greatest element of $\mathrm{Equ}(U)$ will be denoted by Δ and ∇, respectively. We consider the following equivalences on U.

$$\begin{aligned} w = w_6 &= 02B|34|67|89|CD, \\ x = x_6 &= 02|1A|34|567|89, \\ y = y_6 &= 05|1D|6A|BC, \\ z = z_6 &= 1D|23|45|78|9A|BC. \end{aligned}$$

These equivalences are visualized by the graph given in Figure 3 so that, for $t \in \{w_6, x_6, y_6, z_6\}$ and $u, v \in \{0, 1, \dots, D\}$, $\langle u, v\rangle \in t$ iff there is a t-colored path connecting u and v in the graph. However, the calculations below need not rely on this visualization. Let L_6 be the sublattice of $\mathrm{Equ}(U)$ generated by $\{w_6, x_6, y_6, z_6\}$. To ease our formulas and the figure, we usually write w, x, y, z, g_i rather than $w_6, x_6, y_6, z_6, g_i(w_6, x_6, y_6, z_6)$; this will cause no confusion, because we work in L_6, not in $\mathrm{FL}(w, x, y, z)$. To help the reader in following the computations, we often reference earlier equalities or inequalities over the relation signs.

To show that $g_2 \le g_1 + g_3 + g_4 + g_5 + g_6$ fails in $\mathrm{FL}(w, x, y, z)$, it suffices to show that, for our equivalences, $\langle 0, 1\rangle \in g_2$ but

$$\langle 0, 1\rangle \notin g_1 + g_3 + g_4 + g_5 + g_6. \tag{3.2}$$

The containment $\langle 0, 1\rangle \in g_2$ is clear. To make our argument easier to read, we often only write "$\le$" where "$=$" also holds, because inequalities are easier to verify and the equalities in question are not needed. Compute in L_6 as follows.

$$w + x \le 02B|1A|34|567|89|CD, \tag{3.3}$$

$$(w + x)y \overset{3.3}{=} \Delta, \quad \big((w + x)y + z\big)w = zw = \Delta, \tag{3.4}$$

$$y + z \le 045|1D|23|69A|78|BC, \tag{3.5}$$

$$g_1 \overset{3.4}{=} x(y + z) \overset{3.5}{=} \Delta = yx, \tag{3.6}$$

$$x + y \le 012567AD|34|89|BC, \tag{3.7}$$

$$(x + y)z \overset{3.7}{\le} 1D|BC, \quad (x + y)z + w \le 012BCD|34|67|89 \tag{3.8}$$

$$\big((x + y)z + w\big)x \overset{3.8}{\le} 02|34|67|89, \tag{3.9}$$

$$z + w \le 012345BCD|6789A, \quad y(z + w) \le 05|1D|6A|BC, \tag{3.10}$$

$$g_3 \overset{3.9,3.10}{\le} 025|1D|34|67A|89|BC, \tag{3.11}$$

$$(y + z)w \overset{3.5}{=} \Delta, \quad \big((y + z)w + x\big)y = xy \overset{3.6}{=} \Delta, \tag{3.12}$$

$$z(w + x) \overset{3.3}{=} \Delta = wz, \tag{3.13}$$

$$g_5 \overset{3.12,3.13}{=} \Delta, \tag{3.14}$$

$$g_4 \overset{3.13}{=} (yx + w) \cdot (z + yx) \overset{3.6}{=} wz \overset{3.13}{=} \Delta, \tag{3.15}$$

$$xw = 02|34|67|89, \quad y + xw \le 025|1D|34|67A|89|BC, \tag{3.16}$$

$$g_6 \overset{3.16}{\le} 025|1D|34|67A|89|BC. \tag{3.17}$$

By (3.6), (3.11), (3.14), (3.15), and (3.17), we obtain

$$g_2 + \cdots + g_6 \le 025|1D|34|67A|89|BC,$$

which implies (3.2). This settles (1.1) for g_2.

Next, extend the permutation

$$\begin{pmatrix} w & x & y & z \\ x & y & z & w \end{pmatrix}$$

to an automorphism γ_4 of $\mathrm{FL}(w,x,y,z)$, and let $g'_i = \gamma_4(g_i)$ for $i \in \{1,\dots,6\}$. Observe that

$$\begin{aligned} &g'_1 = g_3, \quad g'_3 = g_5, \quad g'_4 = g_2, \quad g'_6 = g_4, \\ &g'_2 = \big((yx+w)z+y\big)(x+wz), \;\; g'_5 = \big((z+w)x+y\big)z + w(x+y). \end{aligned} \tag{3.18}$$

Hence, instead of verifying

$$g_4 \nleq g_1 + g_2 + g_3 + g_5 + g_6, \tag{3.19}$$

it suffices to verify $g'_4 \nleq g'_1 + g'_2 + g'_3 + g'_5 + g'_6$. Thus, taking $\langle 0,1\rangle \in g_2 = g'_4$ and (3.18) into account, it suffices to show that

$$\langle 0,1\rangle \notin g_3 + g'_2 + g_5 + g'_5 + g_4. \tag{3.20}$$

We have to compute again.

$$g'_2 \overset{3.6}{=} (wz+y)(x+wz) \overset{3.13}{=} yx \overset{3.6}{=} \Delta, \tag{3.21}$$

$$(z+w)x \overset{3.10}{\leq} 02|34|67|89, \quad (z+w)x+y \leq 025|1D|34|67A|89|BC, \tag{3.22}$$

$$\big((z+w)x+y\big)z \overset{3.22}{\leq} 1D|BC, \quad w(x+y) \overset{3.7}{\leq} 02|34|67|89, \tag{3.23}$$

$$g'_5 \overset{3.23}{\leq} 02|1D|34|67|89|BC. \tag{3.24}$$

We obtain from (3.11), (3.14), (3.15), (3.21), and (3.24) that

$$g_3 + g'_2 + g_5 + g'_5 + g_4 \leq 025|1D|34|67A|89|BC.$$

This implies (3.20) and settles (1.1) for g_4.

Finally, extend the permutation

$$\begin{pmatrix} w & x & y & z \\ y & z & w & x \end{pmatrix}$$

to an automorphism γ_6 of $\mathrm{FL}(w,x,y,z)$, and let $g''_i = \gamma_6(g_i)$ for $i \in \{1,\dots,6\}$. Observe that

$$g''_1 = g_5, \quad g''_2 = g_6, \quad g''_3 = g'_5, \quad g''_4 = g'_2, \quad g''_5 = g_1, \quad g''_6 = g_2 \ni \langle 0,1\rangle.$$

Hence, to complete the proof by verifying $g_6 \nleq g_1 + g_2 + g_3 + g_4 + g_5$, it suffices to show that $\langle 0,1\rangle \notin g_5 + g_6 + g'_5 + g'_2 + g_1$. But this is clear, because (3.6), (3.14), (3.17), (3.21), and (3.24) yield that

$$g_5 + g_6 + g'_5 + g'_2 + g_1 \leq 025|1D|34|67A|89|BC. \qquad \square$$

4. A diagonal construct and the end of the proof

First, we describe our construction. In $\mathrm{FL}(x,y,z)$, we let

$$y_1 = a, \quad y_2 = c, \quad y_3 = b, \quad y_4 = d, \quad \text{and} \quad S_0 = [y_1, y_2, y_3, y_4];$$

see (2.1). We are going to work in S_0. As usual, $\mathbb{N}$ stands for the set $\{1, 2, \dots\}$ of positive integers. For $n \in \mathbb{N}$ and $i \in \{1, 2, \dots, 6\}$, we define an element $x_i^n \in S_0$ by induction as follows. First, with the quaternary terms from Section 3, we let

$$x_i^1 = g_i(y_1, y_2, y_3, y_4), \text{ for } i \in \{1, \dots, 6\}.$$

Then, for $n \in \mathbb{N}$, we let

$$x_i^{n+1} = g_i(x_3^n, x_4^n, x_5^n, x_6^n), \text{ for } i \in \{1, \dots, 6\}.$$

Finally, consider the following "diagonal" sublattice of S_0:

$$S_\infty = [x_1^1, x_2^1,\ x_1^2, x_2^2, \dots, x_1^n, x_2^n,\ x_1^{n+1}, x_2^{n+1}, \dots].$$

"Diagonal" refers to the following natural arrangement; since no quaternary term is applied for x_1^n and x_2^n, they have no "descendants" below themselves. However, the proof does not rely on this arrangement.

$$\begin{array}{cccccccccc}
x_1^1 & x_2^1 & x_3^1 & x_4^1 & x_5^1 & x_6^1 & & & & \\
 & & x_1^2 & x_2^2 & x_3^2 & x_4^2 & x_5^2 & x_6^2 & & \\
 & & & & x_1^3 & x_2^3 & x_3^3 & x_4^3 & x_5^3 & x_6^3 \\
 & & & & & \ddots & & & &
\end{array}$$

Clearly, the following lemma completes the proof of Theorem 1.1.

Lemma 4.1. *S_∞ is freely generated by the set $\{x_1^n : n \in \mathbb{N}\} \cup \{x_2^n : n \in \mathbb{N}\}$. Furthermore, $\delta(x_1^n) = x_2^n$ and $\delta(x_2^n) = x_1^n$ holds for all $n \in \mathbb{N}$ and, therefore, $\delta(S_\infty) = S_\infty$.*

Proof. By Lemma 2.1, S_0 is freely generated by $\{y_1, \dots, y_4\}$ and

$$\delta(y_1) = y_2, \quad \delta(y_2) = y_1, \quad \delta(y_3) = y_4, \quad \delta(y_4) = y_3. \tag{4.1}$$

Combining (3.1) with (4.1) and the definition of the x_i^1, we obtain that δ acts on the set $\{x_1^1, \dots, x_6^1\}$ as indicated with $n = 1$ by the matrix

$$\begin{pmatrix} x_1^n & x_2^n & x_3^n & x_4^n & x_5^n & x_6^n \\ x_2^n & x_1^n & x_4^n & x_3^n & x_6^n & x_5^n \end{pmatrix}. \tag{4.2}$$

Clearly, $\{x_3^1, \dots, x_6^1\}$ freely generates a sublattice, because, by Lemma 3.1, so does a larger set, $\{x_1^1, \dots, x_6^1\}$ and the validity of (1.1) is inherited by subsets. We claim that for all $n \geq 1$,

$$\begin{array}{l}\delta \text{ acts on the set } \{x_1^n, \dots, x_6^n\} \text{ as given by (4.2)} \\ \text{and } \{x_3^n, \dots, x_6^n\} \text{ freely generates a sublattice.}\end{array} \tag{4.3}$$

For $n = 1$, we have already seen this. For an induction step, assume that (4.3) holds for an $n \in \mathbb{N}$. Since the validity of (1.1) is inherited by subsets, $\{x_3^n, \dots, x_6^n\}$ freely generates a sublattice T. By the induction hypothesis, δ acts on the free generators of T in the same way as α does on the generators of $\mathrm{FL}(w, x, y, z)$ in Lemma 3.1. Hence, Lemma 3.1 yields the validity of (4.2) for $n + 1$ and so (4.3) holds for $n + 1$. This proves (4.3) for all $n \in \mathbb{N}$.

As a consequence of (4.3), we obtain $\delta(x_1^n) = x_2^n$ and $\delta(x_2^n) = x_1^n$ for all $n \in \mathbb{N}$, and so $\delta(S_\infty) = S_\infty$.

Next, for $n \in \mathbb{N}$, consider the following "semi-diagonal" set

$$X_n = \{x_1^1, x_2^1,\ x_1^2, x_2^2, \dots, x_1^{n-1}, x_2^{n-1},\ x_1^n, x_2^n, x_3^n, x_4^n, x_5^n, x_6^n\}.$$

We claim that, for $n \in \mathbb{N}$,

$$X_n \text{ freely generates the sublattice } [X_n] \text{ of } S_0. \tag{4.4}$$

For $n = 1$, $X_1 = \{x_1^1, \dots, x_6^1\}$, and since S_0 is freely generated by $\{y_1, \dots, y_4\}$, (4.4) follows by Lemma 3.1. For the induction step, assume (4.4) for some $n \in \mathbb{N}$. Let $z_i = g_i(x_3^n, x_4^n, x_5^n, x_6^n)$, for $i \in \{1, \dots, 6\}$. Since the problem is selfdual, it suffices to show the validity of (1.1) for

$$X_{n+1} = \{x_1^1, x_2^1,\ x_1^2, x_2^2, \dots, x_1^{n-1}, x_2^{n-1},\ x_1^n, x_2^n,\ z_1, \dots, z_6\}.$$

That is, we have to show that no element u of X_{n+1} is below the join of the rest. Suppose, for a contradiction, that $u \in X_{n+1}$ and

$$u \le \sum_{s \in X_{n+1} \setminus \{u\}} s. \tag{4.5}$$

There are two cases; either $u \notin \{z_1, \dots, z_6\}$, or $u \in \{z_1, \dots, z_6\}$.

First, assume $u \notin \{z_1, \dots, z_6\}$. Observe that $u \in X_n$. Let $\mathbf{2} = \{0, 1\}$ denote the 2-element lattice. Since X_n freely generates $[X_n]$ by the induction hypothesis, there is a unique homomorphism $\psi\colon \mathrm{FL}(X_n) \to \mathbf{2}$ such that $\psi(u) = 1$ and, for all $v \in X_n \setminus \{v\}$, $\psi(v) = 0$. In particular, since $u \in X_{n+1}$ but $x_3^n, \dots, x_6^n \notin X_{n+1}$, these four elements are distinct from u and we have $\psi(x_3^n) = \dots = \psi(x_6^n) = 0$. Hence, for $i \in \{1, \dots, 6\}$, $\psi(z_i) = g_i(\psi(x_3^n), \dots, \psi(x_6^n)) = g_i(0, \dots, 0) = 0 \in \mathbf{2}$. Thus, ψ maps all the joinands s in (4.5) onto 0 while $\psi(u) = 1$. This is impossible since ψ is order-preserving.

Hence, we can assume that $u = z_j$ for some $j \in \{1, \dots, 6\}$. Since X_n is a free generating set by the induction hypothesis and $\{x_3^n, \dots, x_6^n\} \subseteq X_n$, we obtain (trivially or by Lemma 1.2) that $\{x_3^n, \dots, x_6^n\}$ freely generates a sublattice H in S_0. Let $\tau\colon [X_n] \to H$ be the unique homomorphism that extends the map $X_n \to \{x_3^n, \dots, x_6^n\}$ defined by

$$r \mapsto \begin{cases} r, & \text{if } r \in \{x_3^n, \dots, x_6^n\}, \\ x_3^n x_4^n x_5^n x_6^n = 0_H, & \text{if } r \notin \{x_3^n, \dots, x_6^n\}. \end{cases}$$

For $i \in \{1, \dots, 6\}$, $\tau(z_i) = g_i\big(\tau(x_3^n), \dots, \tau(x_6^n)\big) = g_i(x_3^n, \dots, x_6^n) = z_i$. If $r \in X_{n+1} \setminus \{z_1, \dots, z_6\}$, then $\tau(r) = 0_H$. Hence, applying τ for (4.5), we conclude that $z_j = u \le \sum_{i \ne j} z_i$. This is a contradiction, because $\{z_1, \dots, z_6\}$ freely generates a sublattice in the free lattice $H \cong \mathrm{FL}(4)$ by Lemma 3.1. This completes the induction step, and we have shown (4.4) for all $n \in \mathbb{N}$.

Finally, let $X_\infty = \{x_1^n : n \in \mathbb{N}\} \cup \{x_2^n : n \in \mathbb{N}\}$. Observe that every finite subset of X_∞ is a subset of some X_n. Hence, trivially or by Lemma 1.2, X_∞ freely generates S_∞. □

As a final remark, we note that the elementary proof of Lemma 3.1 was motivated by [2]. For another application of [2], see Skublics [5].

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