This paper is ***different*** from its precursor http://arxiv.org/abs/2112.12498 ; even the titles are different!

LATTICES OF RETRACTS OF DIRECT PRODUCTS OF TWO FINITE CHAINS AND NOTES ON RETRACTS OF LATTICES

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Dedicated to the memory of my scientific advisor, András P. Huhn (1947–1985)

Abstract. Ordered by set inclusion, the *retracts* of a lattice L together with the empty set form a bounded poset $\mathrm{Ret}(L)$. By a *grid* we mean the direct product of two non-singleton finite chains. We prove that if G is a grid, then $\mathrm{Ret}(G)$ is a lattice. We determine the number of elements of $\mathrm{Ret}(G)$. Some easy properties of retracts, *retractions*, and their kernels called *retraction congruences* of (mainly distributive) lattices are found. Also, we present several examples, including a 12-element modular lattice M such that $\mathrm{Ret}(M)$ is not a lattice.

1. Introduction

Idempotent endomorphisms are called *retractions*. So a retraction of a lattice L is a *lattice* homomorphism $f\colon L \to L$ such that $f(f(x)) = f(x)$ for all $x \in L$. A sublattice of L is a *retract* of L if it is of the form $f(L) := \{f(x) : x \in L\}$ for some retraction f of L. Congruence kernels of retractions are called *retraction congruences*. These concepts are meaningful for other algebras, not only for lattices. For an algebra A, $\mathrm{Ret}(A)$ will stand for the set consisting of of all retracts of A and the empty set; $\mathrm{Ret}(A) = \big(\mathrm{Ret}(A), \subseteq\big)$ is a bounded poset (partially ordered set). By a *grid* we mean the direct product of two finite non-singleton chains.

1.1. **Outline.** The rest of this section surveys some earlier results on retracts of lattices and explains our motivation to study these retracts. Section 2 proves that, in presence of a majority term, retraction kernels and quasiorders have the Fraser–Horn property, that is, they cannot be "skew" on direct products. In Section 3, we state and prove the main result, Theorem 3.1. This theorem states that, for any grid G, $\mathrm{Ret}(G)$ is a lattice. Also, the theorem describes the retracts of G and gives their number, $|\mathrm{Ret}(G)|$. In Section 4, we prove some properties that retracts or retraction congruences of some lattices, mainly distributive lattices, have. Section 5 presents several examples. These examples indicate that neither the properties presented in

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Section 4, nor Theorem 3.1 can be extended to arbitrary lattices. In particular, we give a 12-element modular lattice M such that $\mathrm{Ret}(M)$ is not a lattice.

1.2. **A mini-survey and our motivation.** We recall a concept: assume that L belongs to a given category $\mathcal{V}$ of lattices and for every embedding $g\colon L \to K \in \mathcal{V}$ such that g is a morphism in $\mathcal{V}$, $g(L)$ is a retract of K witnessed by a retraction that is a morphism in $\mathcal{V}$; in this case we say that L is an *absolute retract* in $\mathcal{V}$.

From time to time, retractions, retracts, and absolute retracts appeared in the literature of lattice theory. In 1970, Fofanova [11] described the lattices L all sublattices or ideals of which are retracts. In a recent paper, Czédli [4] presents two properties of retracts of some (rather special) lattices. The lattices investigated by Freese and Nation [13] and Ploščica [21, Theorem 1.4(i)] are the same as the retracts of the free lattices in the appropriate varieties of lattices.

Absolute retracts in some categories of lattices were described by Czédli [5], Czédli and Molkhasi [8], and Schmid [22]. Retracts of lattices play an important role in Bezhanishvili, Harding, and Jibladze [2]. Some results on absolute retracts of general algebras have corollaries for lattices; see Ouwehand [20] and Jenner, Jipsen, Ouwehand, and Rose [17].

In spite of all these sources, we hardly know anything about the retracts of a lattice L. With few exceptions occurring in Fofanova [11] and Czédli [4] (both are mentioned above), we cannot describe or enumerate the retracts of L and we do not know what properties these retracts have. Even if we are far from answering these questions, the purpose of the present paper is to widen our knowledge related to these questions.

In addition to the lattice theoretic preliminaries listed so far, our research is also motivated by two papers; one of them is outside lattice theory while the other is not about retracts. First, we have learned from Jakubíková–Studenovská and Pócs [16] that the retracts of a monounary algebra A (together with $\emptyset$) form a lattice, and this lattice is semimodular if A is connected; in fact, this result is where the lion's share of our motivation comes. The second one is Fraser and Horn [12], from which we know that lattices have the Fraser–Horn property, that is, the direct product of two lattices cannot have a skew congruence. Even if examples show that none of the results just recalled from [16] and [12] extend to retracts of lattices, we are going to present some sort of extensions in particular cases.

There are papers in which lattices are considered posets and their retracts are understood in a different, order-theoretic sense; see, for example, Li [19]. Retracts occur in algebras that are not far from lattices; see, for example, Wehrung [23]. Also, there are many papers devoted to the retracts of structures that have not much to do with lattices. In fact, the concept of retracts seems to originate from topology; see Borsuk [3].

Finally, there are motivations at elementary level. Subalgebras and homomorphic images have been of particular importance for long, at least since Birkhoff's HSP-theorem. Retracts are both, so it is natural to study them. When Θ is a congruence of an algebra A and we make computations in the quotient algebra A/Θ, then we can pick an element a_X from each Θ-block X and use it as the representative of X. In view of Observation 3.2, the representatives can be chosen optimally if Θ is a *retraction congruence*.

2. Retraction congruences of algebras with a majority term

For lattices L_1 and L_2, every retraction congruence Θ of $L_1 \times L_2$ is of the form $\Theta_1 \times \Theta_2$ with Θ_i being a congruence of L_i for $i = 1, 2$ since lattices satisfy the Fraser–Horn property; see Lemma 2.1 later. This section addresses the question whether Θ_i is a *retraction* congruence of L_i, $i \in \{1, 2\}$. For convenience and to state slightly more than what can be found in the literature, we present a short proof of the following lemma. Its part (B) is due to Fraser and Horn [12, Corollary 1]. But first we need some definitions. If ρ_1 and ρ_2 are relations of algebras A_1 and A_2, respectively, then $\rho_1 \times \rho_2$ denotes $\{((x_1, x_2), (y_1, y_2)) : (x_1, y_1) \in \rho_1 \text{ and } (x_2, y_2) \in \rho_2\}$. A *majority term* for a variety $\mathcal{V}$ of algebras is a ternary term $m(x, y, z)$ such that $\mathcal{V}$ satisfies the identities $m(x, x, y) = m(x, y, x) = m(y, x, x) = x$. The variety of all lattices has majority terms since, say, $m(x, y, z) := (x \vee y) \wedge (x \vee z) \wedge (y \vee z)$ is such a term. A *quasiorder* (in other words, a *preorder*) is a reflexive and transitive relation. The set (in fact, the lattice) of congruences and that of compatible quasiorders of an algebra A are denoted by $\mathrm{Con}(A)$ and $\mathrm{Quo}(A)$, respectively.

Lemma 2.1. *If A_1 and A_2 are algebras in a variety with a majority term, then the following two assertions hold.*

(A) $\mathrm{Quo}(A_1 \times A_2) = \{\rho_1 \times \rho_2 : \rho_1 \in \mathrm{Quo}(A_1)$ *and* $\rho_2 \in \mathrm{Quo}(A_2)\}$.

(B)*(Fraser and Horn* [12, Corollary 1] *combined with Jónsson* [18, Example 1]*)* $\mathrm{Con}(A_1 \times A_2) = \{\Theta_1 \times \Theta_2 : \Theta_1 \in \mathrm{Con}(A_1)$ *and* $\Theta_2 \in \mathrm{Con}(A_2)\}$.

Note that if an algebra A belongs to a variety with a majority term, in particular, if A is a lattice, then $\mathrm{Quo}(A)$ is a distributive lattice by Czédli and Lenkehegyi [7, Corollary 5.2]. There is also another way to see that $\mathrm{Quo}(L)$ for a lattice L is distributive since $\mathrm{Quo}(L)$ was described by Czédli, Huhn, and Szabó [6]. Later, a shorter proof of this description was given in Czédli and Szabó [9], and the history of the topic was thoroughly surveyed by Davey [10].

Proof of Lemma 2.1. While reading the proof, observe that a *symmetric* ρ will trivially yield symmetric ρ_1 and ρ_2. The "$\supseteq$" inclusion in place of the equality "=" in part (A) is trivial. To prove the converse inclusion, let $A := A_1 \times A_2$ and $\rho \in \mathrm{Quo}(A)$. Define $\rho_1 := \{(x, y) \in A_1^2 : (\exists z \in A_2)\, ((x, z), (y, z)) \in \rho\}$. We claim that

$$\text{if } (x, y) \in \rho_1, \text{ then for all } t \in A_2,\ ((x, t), (y, t)) \in \rho. \tag{2.1}$$

To see this, assume that $(x, y) \in \rho_1$ is witnessed by $((x, z), (y, z)) \in \rho$. Let $t \in A_2$, and let m be a majority term in the variety containing A_1 and A_2. Since ρ is reflexive, $((x, t), (x, t)) \in \rho$ and $((y, t), (y, t)) \in \rho$. Since ρ is closed with respect to m, we obtain that $((x, t), (y, t)) = ((m(x, x, y), m(z, t, t)), (m(y, x, y), m(z, t, t))) = m(((x, z), (y, z)), ((x, t), (x, t)), ((y, t), (y, t))) \in \rho$, proving (2.1).

Clearly, ρ_1 is reflexive. Its compatibility and transitivity follows trivially by (2.1), which allows us to use the *same* element $z \in A_2$ witnessing that several pairs belong to ρ_1. Hence, $\rho_1 \in \mathrm{Quo}(A_1)$. By symmetry, the analogously defined ρ_2 belongs to $\mathrm{Quo}(A_2)$. Next, we show that, for any $x_1, x_2 \in A_1$ and $y_1, y_2 \in A_2$,

$$((x_1, x_2), (y_1, y_2)) \in \rho \iff \big((x_1, y_1) \in \rho_1 \text{ and } (x_2, y_2) \in \rho_2\big). \tag{2.2}$$

Assume that $((x_1, x_2), (y_1, y_2)) \in \rho$. By reflexivity, $((x_1, y_2), (x_1, y_2)) \in \rho$ and $((y_1, y_2), (y_1, y_2)) \in \rho$. Hence,

$$\big((x_1, y_2), (y_1, y_2)\big)$$

$$= \big((m(x_1,x_1,y_1), m(x_2,y_2,y_2)), (m(y_1,x_1,y_1), m(y_2,y_2,y_2))\big)$$
$$= m\big(((x_1,x_2),(y_1,y_2)), ((x_1,y_2),(x_1,y_2)), ((y_1,y_2),(y_1,y_2))\big) \in \rho,$$

implying that $(x_1,y_1) \in \rho_1$. We obtain similarly that $(x_2,y_2) \in \rho_2$. Thus, the "$\Rightarrow$" part of (2.2) holds. Conversely, assume that $(x_1,y_1) \in \rho_1$ and $(x_2,y_2) \in \rho_2$. Using (2.1) and its counterpart for the other component, we obtain that $((x_1,x_2),(y_1,x_2)) \in \rho$ and $((y_1,x_2),(y_1,y_2)) \in \rho$. Thus, the transitivity of ρ yields that $((x_1,x_2),(y_1,y_2))$ belongs to ρ, completing the argument for (2.2).

Since $\rho = \rho_1 \times \rho_2$ by (2.2), we have proved part (A) of the lemma. Part (B) follows from the argument above and the first sentence of the proof. ☐

For an algebra A, let $\mathrm{RCon}(A)$ denote the set of *retraction congruences* of A. That is, $\mathrm{RCon}(A)$ consists of the kernels of retractions of A.

Proposition 2.2. *If A_1 and A_2 are algebras in a variety $\mathcal{V}$ with a majority term and each of A_1 and A_2 has a singleton subalgebra, then*

$$\mathrm{RCon}(A_1 \times A_2) = \{\Psi_1 \times \Psi_2 : \Psi_1 \in \mathrm{RCon}(A_1) \text{ and } \Psi_2 \in \mathrm{RCon}(A_2)\}. \tag{2.3}$$

The nontrivial part of Proposition 2.2 is "$\subseteq$" in place of "$=$" in (2.3); this "$\subseteq$" means that algebras with a majority term have no skew retraction congruences.

Proof of Proposition 2.2. Let m be a majority term in $\mathcal{V}$, and denote $A_1 \times A_2$ by A. For $i \in \{1,2\}$, let $\{c_i\}$ be a one-element subalgebra of A_i. We need the following maps

$$\pi_i \colon A \to A_i \text{ defined by } (x_1,x_2) \mapsto x_i \text{ for } i \in \{1,2\},$$
$$\iota_1 \colon A_1 \to A \text{ defined by } x_1 \mapsto (x_1,c_2) \text{ and}$$
$$\iota_2 \colon A_2 \to A \text{ defined by } x_2 \mapsto (c_1,x_2).$$

We claim that

$$\left.\begin{array}{l}\text{if } f\colon A \to A \text{ is a retraction, then so are } f_1 := \pi_1 \circ f \circ \iota_1 \colon A_1 \to A_1\\ \text{and } f_2 := \pi_2 \circ f \circ \iota_2 \colon A_2 \to A_2, \text{ and } \ker f = \ker f_1 \times \ker f_2.\end{array}\right\} \tag{2.4}$$

As a composite of homomorphisms, f_1 is a homomorphism, in fact, an endomorphism of A_1. For $x \in A_1$, let $(u,v) := f(x,c_2)$ and $(u',v') := f(u,c_2)$. Then $u := f_1(x)$ and $u' = f_1(u)$. Let $\Theta = \ker f \in \mathrm{Con}(A)$. Since A has the Fraser–Horn property by Lemma 2.1, $\Theta = \Theta_1 \times \Theta_2$ with $\Theta_1 \in \mathrm{Con}(A_1)$ and $\Theta_2 \in \mathrm{Con}(A_2)$. Using that f is idempotent, we have that $f(x,c_2) = (u,v) = f(u,v)$. This gives that $((x,c_2),(u,v)) \in \Theta$, whereby $(c_2,v) \in \Theta_2$. Since $(u,u) \in \Theta_1$, we have that $((u,c_2),(u,v)) \in \Theta_1 \times \Theta_2 = \Theta$. Hence, $(u',v') = f(u,c_2) = f(u,v) = (u,v)$, whereby $u' = u$. Hence, $f_1(f_1(x)) = u' = u = f_1(x)$, implying that f_1 is a retraction of A_1. By symmetry, f_2 is a retraction of A_2.

To complete the argument for (2.4), we need to show that

$$\text{for } i \in \{1,2\}, \quad \ker f_i = \Theta_i. \tag{2.5}$$

By symmetry, it suffices to deal with $i = 1$. Assume that $(x,x') \in \ker f_1$. Then $u := f_1(x) = f_1(x')$, $f(x,c_2) = (u,v)$, and $f(x',c_2) = (u,v')$ for some $v,v' \in A_2$. Since f is idempotent, $f(u,v) = (u,v)$. This equality and $f(x,c_2) = (u,v)$ give that $((x,c_2),(u,v)) \in \Theta$, whereby $(x,u) \in \Theta_1$. Similarly, $(x',u) \in \Theta_1$. By transitivity and symmetry, we obtain that $(x,x') \in \Theta_1$. Thus, $\ker f_1 \subseteq \Theta_1$.

Conversely, assume that $(x,x') \in \Theta_1$. Denote $f(x,c_2)$ and $f(x',c_2)$ by (u,v) and (u',v'), respectively. Since $((x,c_2),(x',c_2)) \in \Theta_1 \times \Theta_2 = \Theta = \ker f$, we have that

$(u, v) = (u', v')$. Hence, $f_1(x) = u = u' = f_1(x')$, whence $(x, x') \in \ker f_1$. Thus, $\Theta_1 \subseteq \ker f_1$, and we have obtained the validity of (2.5) and that of (2.4).

Next, armed with (2.4), denote $\{\Psi_1 \times \Psi_2 : \Psi_1 \in \mathrm{RCon}(A_1) \text{ and } \Psi_2 \in \mathrm{RCon}(A_2)\}$ occurring in (2.3) by H. If $\Psi \in \mathrm{RCon}(A)$, then we can pick a retraction $f\colon A \to A$ with $\ker f = \Psi$, and it follows from (2.4) that $\Psi \in H$. Therefore, $\mathrm{RCon}(A) \subseteq H$.

Conversely, assume that $\Psi = \Psi_1 \times \Psi_2 \in H$. For $i \in \{1, 2\}$, pick a retraction $g_i\colon A_i \to A_i$ with $\ker g_i = \Psi_i$. It is obvious that $g_1 \times g_2\colon A \to A$, defined by $(x_1, x_2) \mapsto (g_1(x_1), g_2(x_2))$ is a retraction of A. Since

$$\begin{aligned}
&((x_1, x_2), (y_1, y_2)) \in \ker (g_1 \times g_2)\\
&\qquad\iff \big((x_1, y_1) \in \ker g_1 \text{ and } (x_2, y_2) \in \ker g_2\big)\\
&\qquad\iff \big((x_1, y_1) \in \Psi_1 \text{ and } (x_2, y_2) \in \Psi_2\big)\\
&\qquad\iff ((x_1, x_2), (y_1, y_2)) \in \Psi_1 \times \Psi_2 = \Psi,
\end{aligned}$$

we have that $\Psi = \ker (g_1 \times g_2) \in \mathrm{RCon}(A)$. Thus, $H \subseteq \mathrm{RCon}(A)$. Consequently, $\mathrm{RCon}(A) = H$, and the proof of Proposition 2.2 is complete. □

Remark 2.3. As opposed to retraction congruences, retracts and retractions of direct products of two lattices are not factorizable in general. This is exemplified by the direct square L of the two-element chain $\mathsf{C}_2 = \{0, 1\}$, its retraction map $f\colon L \to L$ defined by $(x, y) \mapsto (x, x)$, and the retract $f(L) = \{(0, 0), (1, 1)\}$.

By this remark, the converse of the following observation does not hold.

Observation 2.4. *Let A_1 and A_2 be algebras. For $i \in \{1, 2\}$, let S_i be a retract of A_i and let $f_i\colon A_i \to A_i$ be a retraction. Then $S_1 \times S_2$ is a retract of $A := A_1 \times A_2$, and $f_1 \times f_2\colon A \to A$ defined by $(x_1, x_2) \mapsto (f_1(x_1), f_2(x_2))$ is a retraction.*

Proof. We can assume that $S_i = f_i(A_i)$. Denote $f_1 \times f_2$ by f; it is clearly a retraction and $f(A) = S_1 \times S_2$. □

The following remark is trivial and does not assume the existence of a majority term, but it will be useful later.

Remark 2.5. If A is an algebra and $f\colon A \to A$ is a retraction of A, then $f(A) = \{x \in A : f(x) = x\}$.

Proof. If $f(x) = x$, then $x = f(x) \in f(A)$ is clear. Conversely, if $x \in f(A)$, then x is of the form $x = f(y)$, whereby $f(x) = f(f(y)) = (f \circ f)(y) = f(y) = x$. □

3. The main result

In addition to the notations and concepts given in the first paragraph of the Introduction, we need some additional ones. For $n \in \mathbb{N}^+ := \{1, 2, 3, \dots\}$, the n-element chain will be denoted by C_n. If $G = \mathsf{C}_m \times \mathsf{C}_n$ is a grid, then its subsets of the form $A_1 \times A_2$ with $A_1 \subseteq \mathsf{C}_m$ and $A_2 \subseteq \mathsf{C}_n$ are said to be *straight* while the rest of the subsets are *skew*. The restriction of a map (= function) g to a set Y is denoted by $g\rceil_Y$. The *first projection* $G \to \mathsf{C}_m$ defined by $(x_1, x_2) \mapsto x_1$ is denoted by π_1 while $\pi_2\colon G \to \mathsf{C}_n$ stands for the *second projection*. A subset X of G is *left injective* if $\pi_1\rceil_X$ is injective. Similarly, if $\pi_2\rceil_X$ is injective then X is *right injective*.

We say that $X \subseteq G$ is an *injective subset* if it is left injective or right injective. Subsets that are both left and right injective are *doubly injective*. We let

$$\mathrm{Sts}(G) := \{X : X \text{ is a } \textbf{st}\text{raight } \textbf{s}\text{ubset of } G\}$$
$$\mathrm{Isc}(G) := \{X : X \text{ is an } \textbf{i}\text{njective } \textbf{s}\text{kew } \textbf{c}\text{hain in } G\}.$$

For integers $m, n \ge 2$, we define the following two numbers:

$$t(m,n) = 1 + (2^m - 1)(2^n - 1) \qquad \text{and} \tag{3.1}$$

$$w(m,n) = \sum_{s=2}^{\max\{m,n\}} \Bigg(\binom{m}{s} \cdot \binom{n+s-1}{s} + \binom{n}{s} \cdot \binom{m+s-1}{s} - \binom{m}{s} \cdot \binom{n}{s} - n \cdot \binom{m}{s} - m \cdot \binom{n}{s} \Bigg). \tag{3.2}$$

Now we are in the position to state the main result of the paper.

Theorem 3.1. *For integers $m, n \ge 2$ and $G = \mathsf{C}_m \times \mathsf{C}_n$, the following assertions hold.*

(A) $\mathrm{Ret}(G) = \big(\mathrm{Ret}(G), \subseteq\big)$ *is a lattice in which the meet operation is the same as forming intersection.*

(B) $\mathrm{Ret}(G) = \mathrm{Sts}(G) \cup \mathrm{Isc}(G)$ *and* $\mathrm{Sts}(G) \cap \mathrm{Isc}(G) = \emptyset$.

(C) $|\mathrm{Sts}(G)| = t(m,n)$, $|\mathrm{Isc}(G)| = w(m,n)$, *and so* $|\mathrm{Ret}(G)| = t(m,n) + w(m,n)$.

(D) $\mathrm{Ret}(G)$ *has maximal chains E_1 and E_2 such that* $|E_1| = \max\{m,n\} + 2$ *and* $|E_2| = m + n$.

For $n \in \{1, \dots, 10\}$, Table 1 gives $|\mathrm{Ret}(\mathsf{C}_n \times \mathsf{C}_n)|$, $|\mathrm{Sts}(\mathsf{C}_n \times \mathsf{C}_n)| = t(n,n)$, and $|\mathrm{Isc}(\mathsf{C}_n \times \mathsf{C}_n)| = w(n,n)$; $\mathsf{C}_n \times \mathsf{C}_n$ is abbreviated to C_n^2. It took less than a millisecond to compute these numbers with computer algebra, namely, Maple V Release 5 (of Nov. 27, 1997) on a desktop computer with Intel(R) Core(TM) i5-4440 CPU, 3.10 GHz was used. The computation for, say, $n = 2021$ took 73 seconds; to save space, we only give the first 36 digits (the 37-th digit is less than 5 in each cases):

$$|\mathrm{Sts}(\mathsf{C}_{2021}^2)| \approx 5.797\,522\,914\,036\,970\,546\,568\,254\,329\,481\,553\,09 \cdot 10^{1216}$$
$$|\mathrm{Isc}(\mathsf{C}_{2021}^2)| \approx 2.255\,329\,749\,845\,851\,410\,792\,165\,541\,679\,387\,92 \cdot 10^{1545}$$
$$|\mathrm{Ret}(\mathsf{C}_{2021}^2)| \approx 2.255\,329\,749\,845\,851\,410\,792\,165\,541\,679\,387\,92 \cdot 10^{1545}$$

As opposed to $\mathrm{Ret}(A)$ for a connected monounary algebra, see Jakubíková-Studenovská and Pócs [16], part (D) indicates that the lattice $\mathrm{Ret}(G)$ in the theorem is not semimodular in general. The rest of this section is devoted to the proof of Theorem 3.1; however, some of the observations needed in the proof can be of separate interest. In particular, the following observation is trivial, but it is important. If Θ is a congruence of an algebra A and $a \in A$, then a/Θ stands for the Θ-*block* $\{x \in A : (a, x) \in \Theta\}$ of a.

Observation 3.2. *If A is an algebra, then the following two assertions hold.*

(A) *A subalgebra S of A is a retract of A if and only if there exists a congruence* $\Theta \in \mathrm{Con}(A)$ *such that*

$$\textit{for each block } X \textit{ of } \Theta, \textit{ we have that } |X \cap S| = 1. \tag{3.3}$$

n	1	2	3	4	5
$\lvert\mathrm{Sts}(\mathsf{C}_n^2)\rvert$	2	10	50	226	962
$\lvert\mathrm{Isc}(\mathsf{C}_n^2)\rvert$	0	1	22	209	1 466
$\lvert\mathrm{Ret}(\mathsf{C}_n^2)\rvert$	2	11	72	435	2 428
n	6	7	8	9	10
$\lvert\mathrm{Sts}(\mathsf{C}_n^2)\rvert$	3 970	16 130	65 026	261 122	1 046 530
$\lvert\mathrm{Isc}(\mathsf{C}_n^2)\rvert$	9 027	52 466	297 481	1 670 554	9 354 899
$\lvert\mathrm{Ret}(\mathsf{C}_n^2)\rvert$	12 997	68 596	362 507	1 931 676	10 401 429

TABLE 1. $|\mathrm{Sts}(\mathsf{C}_n^2)|$, $|\mathrm{Isc}(\mathsf{C}_n^2)|$, and $|\mathrm{Ret}(\mathsf{C}_n^2)|$ for $n \in \{1, \dots, 10\}$

(B) *A congruence* $\Theta \in \mathrm{Con}(A)$ *is a retraction congruence of* A *if and only if there exists a subalgebra* S *of* A *such that* (3.3) *holds.*

Proof. To prove (A), assume that S is a retract. Take a retraction $f\colon A \to A$ with $f(A) = S$, and let $\Theta := \ker f$. For a Θ-block X, let $u_X := f(x_0)$ for some (equivalently, for any) $x_0 \in X$. Since $f(u_X) = f(f(x_0)) = f(x_0) = u_X$ gives that $(u_X, x_0) \in \Theta$, we have that $u_X \in X$ and $X = \{y \in A : f(y) = u_X\}$. By Remark 2.5, $X \cap S = \{y \in A : f(y) = u_X \text{ and } f(y) = y\} = \{u_X\}$. Hence, (3.3) holds. Conversely, if (3.3) holds for a subalgebra S, then $f\colon A \to S$, defined by the rule $\{f(x)\} = S \cap (x/\Theta)$ is a retraction and $S = f(A)$ is a retract.

To prove (B), let $\Theta \in \mathrm{Con}(A)$. Assuming that $\Theta \in \mathrm{RCon}(A)$, pick a retraction $f\colon A \to A$ with $\ker f = \Theta$, and let $S := f(A)$. Then S is a retract of A and we are in the same situation as after the second sentence of the proof of part (A), whereby (3.3) holds. Conversely, assume that there is a subalgebra S of A such that (3.3) holds. Then Θ is the kernel of $f\colon A \to S$, defined by the rule $\{f(x)\} = S \cap (x/\Theta)$. Since f is a retraction, $\Theta \in \mathrm{RCon}(A)$, as required. □

Observation 3.3. *If* C *is a finite chain, then each of its nonempty subsets is a retract of* C*. (The empty set is not a retract but it belongs to* $\mathrm{Ret}(C)$ *by definition.) Furthermore, every congruence of* C *is a retraction congruence.*

Proof. It is well known that the blocks of a congruence of a lattice are convex sublattices. This implies easily that

$$\left.\begin{array}{l}\text{an equivalence } \Theta \text{ of } C \text{ is a congruence of the finite chain}\\ C \text{ if and only if every } \Theta\text{-block is an interval of } C.\end{array}\right\} \tag{3.4}$$

Now if S is a nonempty subset of C, then (3.4) makes it is easy to find a congruence Θ of C such that (3.3) holds. If Θ is a congruence of C, then there is a sublattice S satisfying (3.3) since every nonempty subset is a sublattice. In both cases, Observation 3.2 applies, and we conclude Observation 3.3. □

Corollary 3.4. *If a lattice* L *is the direct product of finitely many finite chains, then* $\mathrm{RCon}(L) = \mathrm{Con}(L)$*, whence* $\mathbf{RCon}(L) = (\mathrm{RCon}(L), \subseteq)$ *is a boolean lattice.*

Proof. Combine Proposition 2.2, Observation 3.3, and the fact that the congruence lattice of a chain (and that of any finite modular lattice) is boolean; see, for example, Grätzer [14, Theorem 357]. □

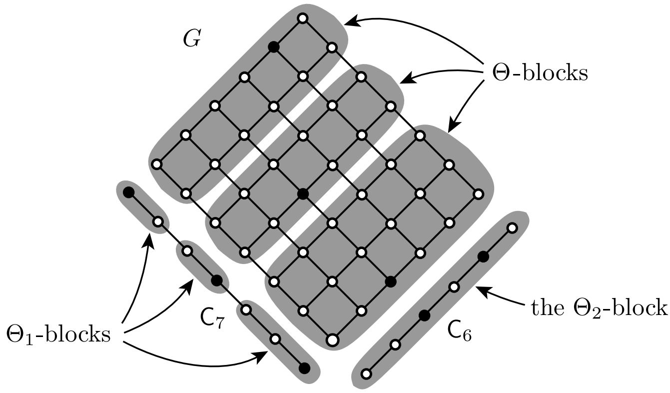


FIGURE 1. Illustration of a left injective chain and also for Case 1

Proof of Theorem 3.1. Even if $m = n$, C_m denotes the first direct factor while C_n stands for the second direct factor of the direct product $G = \mathsf{C}_m \times \mathsf{C}_n$.

To prove the "$\supseteq$" inclusion for part (B), assume that $S \in \mathrm{Sts}(G) \cup \mathrm{Isc}(G)$; we need to show that S is a retract. We can assume that $|S| \geq 2$ since otherwise S is trivially a retract. If $S \in \mathrm{Sts}(G)$, then S is a retract by Observations 2.4 and 3.3. Thus, we can assume that $S \in \mathrm{Isc}(G)$. Let, say, S be a left injective skew chain; see Figure 1 where $m = 7$, $n = 6$, and S consists of the black-filled elements of G. Then $\pi_1(S)$ is a retract of C_m by Observation 3.3, whereby Observation 3.2 allows us to pick a retraction congruence $\Theta_1 \in \mathrm{Con}(\mathsf{C}_m)$ such that for each block X of Θ_1, we have that $|X \cap \pi_1(S)| = 1$. Let $\Theta_2 = \nabla_{\mathsf{C}_n}$, the largest congruence of C_n, and define $\Theta = \Theta_1 \times \Theta_2 \in \mathrm{Con}(G)$. Since π_1 is injective, each block of Θ has exactly one element of S. Hence S is a retract of G by Observation 3.2, and we have verified the '$\supseteq$" inclusion for part (B).

To prove the converse inclusion, let $S \in \mathrm{Ret}(G) \setminus \mathrm{Sts}(G)$; we have to show that $S \in \mathrm{Isc}(G)$. Since $S \notin \mathrm{Sts}(G)$, we know that $|S| \geq 2$. Observation 3.2(A) allows us to pick a congruence $\Theta \in \mathrm{Con}(G)$ such that for each Θ-block X, we have that $|S \cap X| = 1$. By the Fraser–Horn property, see Lemma 2.1(B), there are $\Theta_1 \in \mathrm{Con}(\mathsf{C}_m)$ and $\Theta_2 \in \mathrm{Con}(\mathsf{C}_n)$ such that $\Theta = \Theta_1 \times \Theta_2$. Clearly, Θ_1 and Θ_2 are uniquely determined by Θ. There are two cases.

Case 1. We assume that $\Theta_1 = \nabla_{\mathsf{C}_m}$ or $\Theta_2 = \nabla_{\mathsf{C}_n}$. Both equalities cannot simultaneously hold since otherwise $\Theta = \nabla_G$ would contradict that $|S| > 1$. Hence, we can assume that $\Theta_1 \neq \nabla_{\mathsf{C}_m}$ but $\Theta_2 = \nabla_{\mathsf{C}_n}$; see Figure 1 where S consists of the black-filled elements of G. If $x, y \in S$ such that $\pi_1(x) = \pi_1(y)$, then $(\pi_1(x), \pi_1(y)) \in \Theta_1$ and $(\pi_2(x), \pi_2(y)) \in \nabla_{\mathsf{C}_n} = \Theta_2$ gives that $(x, y) \in \Theta_1 \times \Theta_2 = \Theta$, that is, $y \in x/\Theta$, whence $x, y \in S \cap x/\Theta$ yields that $x = y$. Therefore, $\pi_1\rceil_S$ is injective, that is, S is a left injective subset of G. If we had that $|\pi_2(S)| = 1$, then $S = \pi_1(S) \times \pi_2(S) \in \mathrm{Sts}(G)$ would contradict our assumption that $S \in \mathrm{Ret}(G) \setminus \mathrm{Sts}(G)$. Hence, $|\pi_2(S)| > 1$. (In the figure, $\pi_2(S)$ consists of the two black-filled elements of $\pi_2(G) = \mathsf{C}_n = \mathsf{C}_6$.) By way of contradiction, we are going to prove that S is a chain. Suppose to the contrary that this is not so, and pick two incomparable elements $x = (x_1, x_2)$ and $y = (y_1, y_2)$ from S. The components of x and y belong to chains, whereby $x \parallel y$ is only possible if either $x_1 > y_1$ and $x_2 < y_2$, or $x_1 < y_1$ and $x_2 > y_2$.

By symmetry, we can assume the first alternative, that is, $x_1 > y_1$ and $x_2 < y_2$. Let $z := x \vee y = (x_1, y_2)$. Since S is a sublattice, $z \in S$. Since $\pi_1(x) = x_1 = \pi_1(z)$, we have that $(\pi_1(x), \pi_1(z)) \in \Theta_1$. We also have that $(\pi_2(x), \pi_2(z)) \in \nabla_{\mathsf{C}_m} = \Theta_2$. Thus, $(x, z) \in \Theta_1 \times \Theta_2 = \Theta$, which gives that $f(x) = f(z)$. Hence, using that f is order-preserving and $y \le z$, we have that $y = f(y) \le f(z) = f(x) = x$, contradicting that $x \parallel y$. Therefore, S is a chain, so it is an injective chain belonging to $\mathrm{Isc}(G)$, as required. This completes Case 1.

Case 2. We assume that $\Theta_1 \neq \nabla_{\mathsf{C}_m}$ and $\Theta_2 \neq \nabla_{\mathsf{C}_n}$; see Figure 2, where $m = 7$, $n = 6$, and S consists of the black-filled elements. Then

$$C_1/\Theta_1 \text{ is a non-singleton chain } \{U_0 \prec U_1 \prec \dots \prec U_{s-1}\}, \tag{3.5}$$

where $U_0, \dots, U_{s-1}$ are the Θ_1-blocks. Similarly, $C_2/\Theta_2 = \{V_0 \prec V_1 \prec \dots \prec V_{t-1}\}$ where the V_j's are the Θ_2-blocks. Since $\Theta = \Theta_1 \times \Theta_2$, the Θ-blocks are the $U_i \times V_j$'s, $i \in \{0, 1, \dots, s-1\}$ and $j \in \{0, 1, \dots, t-1\}$. In the figure, the Θ-blocks are grey-filled. Let $w_{i,j}$ denote the unique element of $S \cap (U_i \times V_j)$. We claim that, for $i, i' \in \{0, 1, \dots, s-1\}$ and $j, j' \in \{0, 1, \dots, t-1\}$,

$$w_{i,j} \wedge w_{i',j'} = w_{\min\{i,i'\},\min\{j,j'\}} \text{ and } w_{i,j} \vee w_{i',j'} = w_{\max\{i,i'\},\max\{j,j'\}}. \tag{3.6}$$

To verify (3.6), observe that $(U_i \times V_j) \wedge (U_{i'} \times V_{j'})$, computed in L/Θ, contains $w_{i,j} \wedge w_{i',j'} \in S$ and equals $U_{\min\{i,i'\}} \times V_{\min\{j,j'\}}$. Since this Θ-block only contains one element from S, we obtain the first half of (3.6). Hence, (3.6) follows by duality.

Since C_m and C_n are chains, it follows from (3.5), its counterpart for the V_j's, $w_{i,j} \in U_i \times V_j$, $w_{i',j'} \in U_{i'} \times V_{j'}$, and (3.6) that

$$\text{if } i \le i' \text{ and } j \le j', \text{ then } \pi_1(w_{i,j}) \le \pi_1(w_{i',j'}) \text{ and } \pi_2(w_{i,j}) \le \pi_2(w_{i',j'}) \tag{3.7}$$

for $i, i' \in \{0, 1, \dots, s-1\}$ and $j, j' \in \{0, 1, \dots, t-1\}$.

Next, let $x_{s-1} := \pi_1(w_{s-1,0})$, $y_0 := \pi_2(w_{s-1,0})$, $x_0 := \pi_1(w_{0,t-1})$, and $y_{t-1} := \pi_2(w_{0,t-1})$. Then $w_{s-1,0} = (x_{s-1}, y_0)$ and $w_{0,t-1} = (x_0, y_{t-1})$. We know from (3.7) that $x_0 \le x_{s-1}$ and $y_0 \le y_{t-1}$. These inequalities and (3.6) give that $w_{0,0} = w_{s-1,0} \wedge w_{0,t-1} = (x_{s-1}, y_0) \wedge (x_0, y_{t-1}) = (x_0, y_0)$. Hence, $\pi_1(w_{0,0}) = x_0 = \pi_1(w_{0,t-1})$ and $\pi_2(w_{0,0}) = y_0 = \pi_2(w_{s-1,0})$. Thus, (3.7) gives that $\pi_2(w_{i,0}) = y_0$ and $\pi_1(w_{0,j}) = x_0$ for all meaningful i and j. Therefore, letting $x_i = \pi_1(w_{i,0})$ and $y_j := \pi_2(w_{0,j})$,

$$w_{i,0} = (x_i, y_0) \quad \text{and} \quad w_{0,j} = (x_0, y_j) \tag{3.8}$$

for $i \in \{0, \dots, s-1\}$ and $j \in \{0, \dots, t-1\}$. We obtain from (3.7) that $x_0 \le x_1 \le \dots \le x_{s-1}$ and $y_0 \le y_1 \le \dots \le y_{t-1}$. Since $w_{0,0,}$, $w_{1,0}$, $\dots$, $w_{s-1,0}$ belong to different Θ-blocks, we have that

$$x_0 < x_1 < \dots < x_{s-1} \text{ and, similarly, } y_0 < \dots < y_{t-1}. \tag{3.9}$$

Let $X := \{x_0, \dots, x_{s-1}\}$ and $Y := \{y_0, \dots, y_{s-1}\}$. Combining (3.6), (3.8), and (3.9), we obtain that, for all $i \in \{0, \dots, s-1\}$ and $j \in \{0, \dots, t-1\}$,

$$w_{i,j} = w_{i,0} \vee w_{0,j} = (x_i, y_0) \vee (x_0, y_j) = (x_i, y_j).$$

Hence, $S = \{w_{i,j} : 0 \le i < s \text{ and } 0 \le j < t\} = X \times Y$. This contradicts the assumption that $S \notin \mathrm{Sts}(G)$, whereby Case 2 cannot occur.

Now that we have excluded Case 2, Case 1 can only hold. Therefore, we conclude the converse inclusion for part (B). Thus, $\mathrm{Ret}(G) = \mathrm{Sts}(G) \cup \mathrm{Isc}(G)$, as required. Since $\mathrm{Sts}(G) \cap \mathrm{Isc}(G) = \emptyset$ is trivial by definition, part (B) of the theorem has been proved.

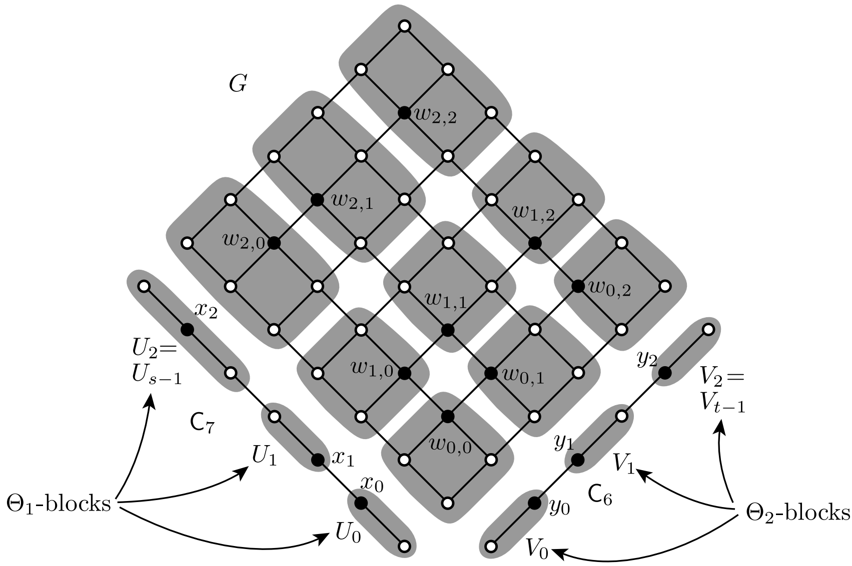


FIGURE 2. Illustration for Case 2

Next, we deal with part (A). The rule $(X_1\times X_2)\cap(Y_1\times Y_2)=(X_1\cap Y_1)\times(X_2\cap Y_2)$ shows that $\mathrm{Sts}(G)$ is closed with respect to intersection. So if $X,Y\in\mathrm{Sts}(G)$, then $X\cap Y\in\mathrm{Sts}(G)$, whence part (B) gives that $X\cap Y\in\mathrm{Ret}(G)$. Now let $X,Y\in\mathrm{Ret}(G)$ but, say, $X\notin\mathrm{Sts}(G)$. Then X is an injective skew chain; say, it is left injective. Since $X\cap Y$ is a subset of X, we obtain that $X\cap Y$ is a left injective chain. If it is not a straight subset, then $X\cap Y\in\mathrm{Isc}(G)\subseteq\mathrm{Ret}(G)$ by part (B). If $X\cap Y$ is a straight subset, then $X\cap Y\in\mathrm{Sts}(G)\subseteq\mathrm{Ret}(G)$ again. Therefore, $\mathrm{Ret}(G)$ is closed with respect to the binary intersection. By finiteness and since $\mathrm{Ret}(G)$ has a largest member, G, we conclude part (A).

The argument for part (D) relies on part (B) again. We use the notation $\mathsf{C}_m=\{0=c_0\prec c_1\prec\cdots\prec c_{m-1}=1\}$ and $\mathsf{C}_n=\{0=d_0\prec d_1\prec\cdots\prec d_{n-1}=1\}$. The principal ideals $\downarrow c_i$ and $\downarrow d_j$ are understood in C_m and C_n, respectively. Without loss of generality, we can assume that $m\leq n$. Take the following two chains in $\mathrm{Ret}(G)$:

$$H_1:=\Big\{\emptyset,\{(c_0,d_0)\},\,\{(c_0,d_0),(c_1,d_1)\},\,\dots,\,\{(c_0,d_0),\dots,(c_{m-1},d_{m-1})\},\\ \mathsf{C}_m\times{\downarrow}d_{m-1},\,\mathsf{C}_m\times{\downarrow}d_m,\,\dots,\mathsf{C}_m\times{\downarrow}d_{n-1}\Big\}\quad\text{and}$$

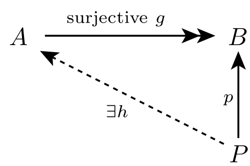
$$H_2:=\Big\{\emptyset,\,{\downarrow}c_0\times{\downarrow}d_0,\,c_1\times{\downarrow}d_0,\,\dots,\,{\downarrow}c_{m-1}\times{\downarrow}d_0=\mathsf{C}_m\times\{d_0\},\\ \mathsf{C}_m\times{\downarrow}d_1,\dots,\mathsf{C}_m\times{\downarrow}d_{n-1}=\mathsf{C}_m\times\mathsf{C}_n\Big\}.$$

Based on part (B), it is straightforward to see that both H_1 and H_2 are maximal chains in $\mathrm{Ret}(G)$. Since $|H_1|=n+2=\max\{m,n\}+2$ and $|H_2|=m+n$, we obtain the validity of part (D).

To prove part (C), it suffices to show that $|\mathrm{Isc}(G)|=w(m,n)|$ since it is trivial that $|\mathrm{Sts}(G)|=t(m,n)$. To obtain an s-element left injective chain $X=\{(x_1,y_1),\dots,(x_s,y_s)\}$, we need to select $(x_1,\dots,x_s)$ and $(y_1,\dots,y_s)$ independently so that $x_1 < x_2 < \cdots < x_s$ and $y_1 \leq y_2 \leq \cdots \leq y_s$. We can do this in $\binom{m}{s} \cdot \binom{n+s-1}{s}$ ways. This explains the first summand after the big $\sum$ sign in (3.1). Note that $\binom{m}{s}$ is 0 if $s > m$. Similarly, the next summand is the number of right injective chains. The sum of the first two summands has to be corrected; first with the number of doubly (that is, both left and right) injective skew chains, then with the number of left injective straight chains, and finally with the number of right injective straight chains; this is where the three subtrahends in (3.1) come from. (For chains, since $s \geq 2$, the properties "doubly injective", "left injective and straight", and "right injective and straight" mutually exclude each other.) Therefore, $|\mathrm{Isc}(G)| = w(m, n)$, completing the proof of the theorem. □

4. Some easily provable facts

This section collects some easily provable facts about retracts and related concepts. Some other facts are presented in Czédli [4] and in other sections of the present paper. Recall that an algebra P in a variety $\mathcal{V}$ is *projective* in $\mathcal{V}$ if for any algebras $A, B \in \mathcal{V}$, any homomorphism $p\colon P \to B$ and any surjective homomorphism $g\colon A \to B$, there is a homomorphism $h\colon P \to A$ such that $p = g \circ h$. This is visualized by the commutativity of the triangle below.



The standard category theoretic approach would be to only require that g is an epimorphism. Although there are varieties in which epimorphism need not be surjective, we go after, say, Freese an Nation [13] and require g to be surjective rather than just stipulating that g is an epimorphism. There is a well-known connection between retracts and projective algebras; see Freese and Nation [13] or Ploščica [21]. Below, we present another connection.

Observation 4.1. *If Θ is a congruence of an algebra A such that A/Θ is projective in the variety generated by A, then Θ is a retraction congruence.*

Proof. Let $g\colon A \to A/\Theta$ be the natural homomorphism defined by $u \mapsto u/\Theta$; it is surjective. Let p be the identity map $\mathrm{id}_{A/\Theta}\colon A/\Theta \to A/\Theta$. Since A/Θ is projective, there is a homomorphism $h\colon A/\Theta \to A$ such that $\mathrm{id}_{A/\Theta} = g \circ h$. Now if X is a Θ-block, that is, $X \in A/\Theta$, then $h(X) \in X$ since $X = \mathrm{id}_{A/\Theta}(X) = g(h(X)) = h(X)/\Theta$. Furthermore, the homomorphic image $\{h(X) : X \in A/\Theta\}$ is a subalgebra of A. Hence, $\Theta \in \mathrm{RCon}(A)$ by Observation 3.2(B). □

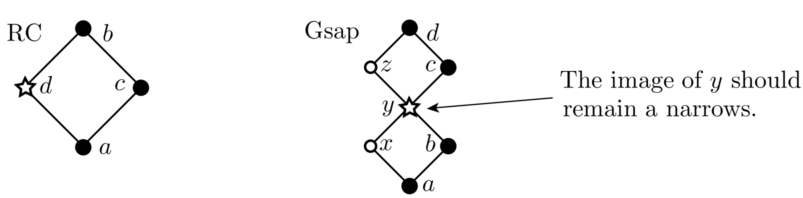


Figure 3. RC and Gsap

Definition 4.2. Assume that $A^\bullet$ and $X^\star$ are subsets of a lattice K and Γ is a property of possible lattice embeddings with domain K. We say that a retract S of a lattice L satisfies the *absorption property* $\mathrm{AP}(K, A^\bullet, X^\star, \Gamma)$ if for every embedding $g\colon K \to L$ such that g satisfies Γ and $g(A^\bullet) \subseteq S$, we have that $g(X^\star) \subseteq S$. If Γ automatically holds for any embedding, then we omit it from the notation. If all retracts of L satisfy an absorption property, then we say that *the retracts of* L *satisfy* the absorption property in question. In Figures 3 and 4, each of the Hasse-diagrams defines an absorption property so that the diagram determines K while $A^\bullet$ and $X^\star$ are the sets of the black-filled elements and that of the star-shaped elements, respectively. The property Γ, if relevant, is written in the figure. If an absorption property $\mathrm{AP}(K, A^\bullet, X^\star)$ is denoted by a string $\vec{\boldsymbol{\sigma}}$, then $K(\vec{\boldsymbol{\sigma}})$, $A^\bullet(\vec{\boldsymbol{\sigma}})$, and $X^\star(\vec{\boldsymbol{\sigma}})$ stand for its ingredients, K, $A^\bullet$, and $X^\star$, respectively.

The concept of absorption properties (without Γ) was introduced in Czédli [4]. Among the reasonable absorption properties, the simplest one is probably RC, which is given on the left of Figure 3. Another way of saying that a sublattice S of a lattice L satisfies RC is to say that S is *closed with respect to taking relative complements.*

Observation 4.3. *The retracts of a distributive lattice are closed with respect to taking relative complements, that is, they satisfy RC.*

Proof. Let S be a retract of a distributive lattice L and let $f\colon L \to L$ be a retraction with $f(L) = S$. Assume that $a, b, c, d \in L$ form a sublattice isomorphic to the four-element boolean lattice with bottom a and top b, and $a, b, c \in S$. Then $f(d) \wedge c = f(d) \wedge f(c) = f(d \wedge c) = f(a) = a$, and we similarly obtain that $f(d) \vee c = b$. Hence, both d and $f(d)$ are complements of c in the interval $[a, b]_L$, which is a distributive lattice. The uniqueness of complements in a distributive lattice yields that $f(d) = d$, implying that $d \in S$, as required. ☐

If x is an element of a lattice L, $x \neq 0_L$, $x \neq 1_L$, and x is comparable with every element of L, then x is called a *narrows* (of L). If we form the glued sum of two squares (i.e., four-element boolean lattices) to obtain a seven-element lattice K, then the middle element y of K is a narrows of K. However, y need not remain a narrows if we embed K into another lattice. The condition Γ on the embedding g we consider in $\mathrm{Gsap} := \mathrm{AP}^+(K, A^\bullet, X^\star, \Gamma)$ given by Figure 3 is that $g(y)$ should be a narrows. (The acronym comes from ***G***lued ***s***quares ***a***bsorption ***p***roperty.)

Observation 4.4. *The retracts of every lattice satisfy the absorption property* Gsap.

Proof. Let $K := K(\mathrm{Gsap})$, $A^\bullet := A^\bullet(\mathrm{Gsap})$, and $X^\star := X^\star(\mathrm{Gsap})$. Assume that K is a sublattice and S is a retract of a lattice L, y is a narrows of L, and $\{a, b, c, d\} = A^\bullet \subseteq S$. Pick a retraction $f\colon L \to L$ such that $S = f(L)$. In fact, by Remark 2.5, $S = \{u \in L : f(u) = u\}$. If we had that $f(x) \geq b$, then $a = f(a) = f(x \wedge b) = f(x) \wedge f(b) = f(x) \wedge b = b$ would be a contradiction. Hence, $f(x) \not\geq b$, implying that $f(x) \not\geq y$. But $f(x)$ and y are comparable since y is a narrows, whence we obtain that $f(x) \leq y$. Thus, $f(y) = f(x \vee b) = f(x) \vee f(b) = f(x) \vee b \leq y$. A dual argument that uses z instead of x yields that $f(y) \leq y$. Therefore, $f(y) = y$ implies that $X^\star = \{y\} \subseteq S$, as required. ☐

The absorption properties $P(8, 3)$ and $P(9, 4)$ are given by Figure 4. While $P(8, 3)$ is a selfdual property, the dual of $P(9, 4)$ will be denoted by $P(9, 4)^{\mathrm{dual}}$.

FIGURE 4. The absorption properties occurring in Observation 4.5

Observation 4.5. *The retracts of* planar *distributive lattices satisfy each of the absorption properties* $P(8,3)$, $P(9,4)$, *and* $P(9,4)^{\text{dual}}$.

Proof. Since the class of planar distributive lattices is selfdual, we need not deal with $P(9,4)^{\text{dual}}$. Let $P'(8,3)$ and $P'(9,4)$ denote the absorption properties that we obtain from $P(8,3)$ and $P(9,4)$ by changing $X^\star(P(8,3))$ and $X^\star(P(9,4))$ to $\{y\}$ and $\{x,y\}$, respectively. It is proved in Czédli [4] that the retracts of lattices belonging to a particular class satisfy $P'(8,3)$ and $P'(9,4)$. The class considered there contains all planar distributive lattices, whereby the retracts of planar distributive lattices satisfy $P'(8,3)$ and $P'(9,4)$. Applying Observation 4.3, we obtain that they satisfy $P(8,3)$ and $P(9,4)$. □

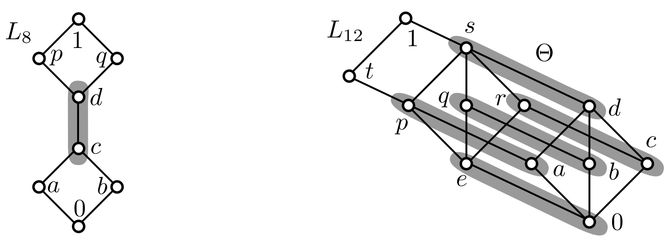


FIGURE 5. L_8 and L_{12}

5. EXAMPLES

Example 5.1. For the modular lattice L_{12} given in Figure 3, the poset $\text{Ret}(L_{12}) = (\text{Ret}(L_{12}), \subseteq)$ is not a lattice.

Proof. With $M_3 = [e,s]$, L_{12} is a Hall–Dilworth gluing of $M_3 \times \mathsf{C}_2$ and $\mathsf{C}_2 \times \mathsf{C}_2$. This implies the modularity of L_{12} since we know from Hall and Dilworth [15, Lemma 4.1] that gluing preserves modularity.

Observe that $[t,1] = \{t,1\} \notin \text{Ret}(L_{12})$. Suppose the contrary and take a retraction $f\colon L_{12} \to L_{12}$ such that $f(L_{12}) = \{t,1\}$. Then $\ker f$ collapses two distinct elements of the "diamond" $[e,s]$. Since the diamond is a simple lattice, $(e,s) \in \ker f$. Hence, $t = f(t) = f(t \vee e) = f(t) \vee f(e) = f(t) \vee f(s) = f(t \vee s) = f(1) = 1$, which is a contradiction. Thus, $\{t,1\} \notin \text{Ret}(L_{12})$.

Let $S_1 := [e,1]$ and $S_2 := [t,1] \cup [0,d]$. Both are retracts with the same retraction congruence, the non-singleton blocks of which are given by grey ovals in the diagram of L_{12}. We claim that $\{S_1, S_2\}$ has no greatest lower bound in $\text{Ret}(L_{12})$. Since any of their lower bounds is a subset of $S_1 \cap S_2 = \{t,1\}$ but $\{t,1\} \notin \text{Ret}(L_{12})$, there are at most three lower bounds, $\emptyset$, $\{t\}$, and $\{1\}$. They are retracts, whence there are exactly three lower bounds, $\emptyset$, $\{t\}$ and $\{1\}$. Since none of these three sets is larger than the other two, $S_1 \wedge S_2$ does not exist in $\mathrm{Ret}(L_{12})$. Therefore, $\mathrm{Ret}(L_{12})$ is not a lattice. □

Remark 5.2. $\mathrm{RCon}(L_{12}) = \mathrm{Con}(L_{12})$, and it is the eight-element boolean lattice.

Proof. Since the congruence lattice of a finite modular lattice is boolean by Grätzer [14, Theorem 357], $\mathrm{Con}(L_{12})$ is a boolean lattice. The atoms in $\mathrm{Con}(L_{12})$ are the principal congruences $\mathrm{con}(p,t)$, $\mathrm{con}(t,1)$, and $\mathrm{con}(0,e)$, whereby $|\mathrm{Con}(L_{12})| = 8$ and it is easy to list the congruences of L_{12}. For each congruence $\Psi \neq \nabla_{L_{12}}$, there are two easy ways to conclude that $\Psi \in \mathrm{RCon}(L_{12})$. First, we can easily give a retraction with kernel Ψ. Second, we can use the criterion given by Balbes [1] to see that L_{12}/Ψ is projective in the variety of distributive lattices, and then $\Psi \in \mathrm{RCon}(L_{12})$ follows from Observation 4.1. □

Although we do not know whether, for a lattice L, $\mathrm{RCon}(L)$ is always a lattice or when it is a lattice, the following example points out that the situation is usually different from what Corollary 3.4 and Remark 5.2 may suggest.

Example 5.3. For L_8 given in Figure 5, $\mathrm{RCon}(L_8) \neq \mathrm{Con}(L_8)$, and $\mathrm{RCon}(L_8)$ is a non-distributive lattice.

Proof. Let $\Theta = \mathrm{con}(c,d)$ be the principal congruence indicated in the figure. Except for $\{c,d\}$, its blocks are singletons. Hence, $L_8 \setminus \{c\}$ and $L_8 \setminus \{d\}$ are the only candidates for S in Observation 3.2(B) but none of them is a sublattice. Thus, $\Theta \notin \mathrm{RCon}(L_8)$, witnessing that $\mathrm{RCon}(L_8) \neq \mathrm{Con}(L_8)$.

Applying Grätzer [14, Theorem 357], it is easy to see that $\mathrm{Con}(L_8)$ is a boolean lattice consisting of 32 elements. Using Observation 4.1 and the criterion of Balbes [1], it is not hard to see that all other congruences are retraction congruences. That is, $\mathrm{RCon}(L_8) = \mathrm{Con}(L_8) \setminus \{\mathrm{con}(c,d)\}$. Hence, $\mathrm{RCon}(L_8)$ is obtained from a finite boolean lattice by omitting an atom. By the Duality Principle, it suffices to show that

$$\text{if } d \text{ is a coatom of a boolean lattice } K \text{ with } |K| \geq 8, \text{ then the subposet } (K \setminus \{d\}, \leq) \text{ is a non-distributive lattice.} \tag{5.1}$$

Indeed, the join-irreducible elements (that is, the elements with exactly one lower cover) are the same in K and $K \setminus \{d\}$, and these elements are antichains in both cases. If $K \setminus \{d\}$ was a distributive lattice, then the structure theorem of finite distributive lattices, see Grätzer [14, Theorem 107], would give that K and $K \setminus \{d\}$ are isomorphic, contradicting that $|K \setminus \{d\}| < |K|$. Since d is meet-irreducible, $K \setminus \{d\}$ is meet-closed. Also, $K \setminus \{d\}$ contains $1 = 1_K$, whereby it is a lattice. □

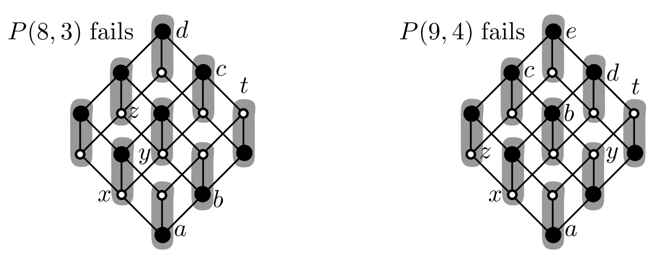


FIGURE 6. Where $P(8,3)$ and $P(9,4)$ fail

Example 5.4. The distributive lattice $L = \mathsf{C}_3\times\mathsf{C}_3\times\mathsf{C}_2$ has a retract S that satisfies none of $P(8,3)$, nor $P(9,4)$. Moreover, no matter how we reduce $X^\star$ to some of its nonempty subsets, the weaker absorption property we obtain from $P(8,3)$ or $P(9,4)$ in this way is not satisfied by S. In Figure 6, L is diagrammed twice; S consists of the black-filled elements.

Proof. Using Θ given by the grey-filled ovals in Figure 6, Observation 3.2(A) shows that S is indeed a retract of L. The embedding is defined by the labeling. □

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