Representing homomorphisms of distributive lattices as restrictions of congruences of rectangular lattices

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Abstract. Let φ be a $\{0,1\}$-homomorphism of a finite distributive lattice D into the congruence lattice $\operatorname{Con} L$ of a rectangular (whence finite, planar, and semimodular) lattice L. We prove that L is a filter of an appropriate rectangular lattice K such that $\operatorname{Con} K$ is isomorphic with D and φ is represented by the restriction map from $\operatorname{Con} K$ to $\operatorname{Con} L$. The particular case where φ is an embedding was proved by E. T. Schmidt. Our result implies that each $\{0,1\}$-lattice homomorphism between two finite distributive lattices can be represented by the restriction of congruences of an appropriate rectangular lattice to a rectangular filter.

1. Introduction and the main result

Congruence lattices of lattices are distributive (and algebraic), see N. Funayama and T. Nakayama [8]. While the natural converse fails by a deep result of F. Wehrung [30], a classical result of R. P. Dilworth (see [7] and [10]) states that each *finite* distributive lattice D is isomorphic to the congruence lattice $\operatorname{Con} L$ of an appropriate finite lattice L. As surveyed in G. Grätzer [10, Chapter III], many improvements of this theorem yield an L with some nice additional properties. For example,

Theorem 1.1 (G. Grätzer, H. Lakser and E. T. Schmidt [20] and G. Grätzer and E. Knapp [14] and [15]). *Each finite distributive lattice D is* (*isomorphic to*) *the congruence lattice of a planar semimodular lattice L. If, in addition, D is non-trivial* (*that is,* $|D| \geq 2$)*, then it is the congruence lattice an appropriate rectangular lattice.*

We adopt the convention that a *planar* lattice is *finite* by definition; see G. Grätzer and E. Knapp [13]. Hence, *all* lattices occurring in the paper are assumed to be *finite*, unless otherwise stated. A finite lattice M is called *semimodular* if $x \prec y$ implies that $x \vee z \preceq y \vee z$ for all $x, y, z \in M$.

By a *left weak corner* (resp., *right weak corner*) of a planar lattice M we mean a doubly-irreducible element of $M - \{0,1\}$ on the left (resp., right) boundary of M. This concept was introduced in G. Grätzer and E. Knapp [14]. Since we need three different "corner" concepts, we usually add an adjective

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such as “weak”. As in G. Grätzer and E. Knapp [14], a *rectangular lattice* means a planar semimodular lattice M such that $|M| \geq 3$ and, in addition, M has exactly one left weak corner, w^L_M, and exactly one right weak corner, w^R_M, and they are complementary, that is, $w^L_M \vee w^R_M = 1$ and $w^L_M \wedge w^R_M = 0$. For example, S_7 and S_8 in Figure 2 are rectangular lattices. Another example is M_3, the five-element modular, nondistributive lattice.

A lattice L is called *isoform* if any two blocks of each congruence of L are isomorphic sublattices. If L is a convex sublattice (in particular, a filter or an ideal) of another lattice K, then the *restriction map* $\varrho^K_L \colon \operatorname{Con} K \to \operatorname{Con} L$ is a lattice homomorphism preserving 0 and 1, a $\{0,1\}$-*lattice homomorphism*, for short. Every lattice homomorphism between two finite distributive lattices can be represented this way. Moreover, even some nice properties of K and L can be stipulated:

Theorem 1.2 (G. Grätzer and H. Lakser [17], [18], and [19])**.** *Let $\mathfrak{K}$ be one of the following three classes:*

(i) *the class of planar lattices with no nontrivial automorphisms;*
(ii) *the class of finite isoform lattices;*
(iii) *the class of finite sectionally complemented lattices.*

Let D and E be finite distributive lattices, and let $\varphi \colon D \to E$ be a $\{0,1\}$-lattice homomorphism. Then there are lattices $K, L \in \mathfrak{K}$ and isomorphisms $\alpha \colon D \to \operatorname{Con} K$ and $\beta \colon E \to \operatorname{Con} L$ such that L is an ideal of K and $\varphi = \beta^{-1} \circ \varrho^K_L \circ \alpha$. By the duality principle, if $\mathfrak{K}$ is one of the first two classes, then L can be chosen to be a filter of K instead of an ideal.

Next, $E = \operatorname{Con} L$ is given but we are not allowed to choose L freely:

Main Theorem 1.3. *Let L be a rectangular lattice, and let φ be an arbitrary $\{0,1\}$-lattice homomorphism of a finite distributive lattice D to $\operatorname{Con} L$. Then φ can be represented by a restriction map in the following sense: there is a rectangular lattice K and there is a lattice isomorphism $\alpha \colon D \to \operatorname{Con} K$ such that L is a filter of K and $\varphi = \varrho^K_L \circ \alpha$.*

Note that E. T. Schmidt [29] proved this result for the special case when φ is injective. Theorem 1.3 extends the “filter variant” of Theorem 1.2 to the class of rectangular lattices (see Corollary 1.5), and offers an interesting new proof of Theorem 1.1 (see Corollary 1.4; see also Section 8).

Corollary 1.4 (G. Grätzer and E. Knapp [14])**.** *Every finite non-trivial distributive lattice D is* (*isomorphic to*) *the congruence lattice of a rectangular lattice.*

First proof. The Prime Ideal Theorem yields a $\{0,1\}$-homomorphism $D \to \operatorname{Con} M_3$. Hence, Theorem 1.3 applies. $\square$

The next statement follows obviously from Theorem 1.3 and Corollary 1.4.

Corollary 1.5. *Assume that D and E are finite distributive lattices. Assume also that $\varphi\colon D \to E$ is a $\{0,1\}$-lattice homomorphism. Then there are rectangular lattices K and L and isomorphisms $\alpha\colon D \to \operatorname{Con} K$ and $\beta\colon E \to \operatorname{Con} L$ such that L is a* filter *of K and $\varphi = \beta^{-1} \circ \varrho^K_L \circ \alpha$.*

Method. This work was motivated and influenced by G. Grätzer [10] and E. T. Schmidt [29]. The new features are as follows. We introduce the concept of *quasi-colorings*, whose ranges are quasiordered sets rather than orders. (An *order* $(A; \varrho)$ is a nonempty set A with an *ordering* $\varrho \subseteq A^2$. Orders are also called partially ordered sets or posets.) The advantage is that, as opposed to orderings, the quasiorderings of a set form a lattice. This allows us to construct the desired lattice by a sequence of elementary steps. Each step is accompanied by a quasiordering. If several steps are carried out, then the join of the corresponding quasiorderings gives some insight into the construction. It is the *left adjoint* of the homomorphism $\varphi\colon D \to \operatorname{Con} L$ that extends this insight to a proper understanding of where to navigate with the elementary steps.

Outline. In Section 2, we introduce the notion of quasi-colorings. We describe congruence-preserving extensions of lattices by means of certain extensions of quasi-colorings in Section 3. Based on G. Grätzer and E. Knapp [13] and G. Czédli and E. T. Schmidt [5], Section 4 provides the structure theory of planar semimodular lattices that we shall need later. The longest part of the present paper is Section 5, which describes some important elementary lattice extensions by means of quasi-colorings. After exploring the structure of rectangular lattices in Section 6, we prove two auxiliary results on their congruence-preserving rectangular filters in Section 7. Utilizing the previous sections, the key construction is given in Section 8, where we prove that each extension of a quasi-coloring of a rectangular lattice L is realized by a "filter extension" of L to a rectangular lattice K. The strength of this construction is illustrated by a new proof of Lemma 1.4. In Section 9, we deal with the left adjoint of φ, which leads to the proof of Theorem 1.3.

Notation and terminology. For the basic concepts and notation, the reader is referred to G. Grätzer [10]. The Glossary of Notation of [10] is available as a pdf file at

`http://mirror.ctan.org/info/examples/Math_into_LaTeX-4/notation.pdf`

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FIGURE 1. Congruence-perspectivities

2. Quasi-colored lattices

For a finite lattice M, let $\mathrm{Ji}(M)$ denote the set of (non-zero) join-irreducible elements of M. The set of non-unit meet-irreducible elements is denoted by $\mathrm{Mi}(M)$. The following well-known property of a finite *distributive* lattice D_0, see G. Grätzer [9, proof of Thm. II.1.9], will frequently be used:

$$\text{if } a \in \mathrm{Ji}(D_0),\ X \subseteq D_0 \text{ and } a \le \bigvee X, \text{ then } a \le x \text{ for some } x \in X. \tag{2.1}$$

Let $\mathrm{Int}(M)$ denote the set of all intervals of a lattice M. For $\mathfrak{p} = [a, b] \in \mathrm{Int}(M)$, let $\mathrm{con}_M(\mathfrak{p}) = \mathrm{con}_M(a, b)$ stand for the smallest congruence collapsing a and b. When there is no danger of confusion, we drop the subscript M. Sometimes we write $0_\mathfrak{p}$ and $1_\mathfrak{p}$ instead of a and b, respectively. The set of prime (that is, two-element) intervals of M will be denoted by $\mathrm{Pri}(M)$. Prime intervals are also called *edges*. By the folklore, see e.g. G. Grätzer [10, Sect. I.3.2],

$$\mathrm{Ji}(\mathrm{Con}\, M) = \{\mathrm{con}(\mathfrak{p}) : \mathfrak{p} \in \mathrm{Pri}(M)\}. \tag{2.2}$$

Let $\mathfrak{p}_1 = [x_1, y_1]$ and $\mathfrak{p}_2 = [x_2, y_2]$ be intervals of M. Following the terminology of G. Grätzer [11], we say that $\mathfrak{p}_1$ is *up congruence-perspective* to $\mathfrak{p}_2$, in notation $\mathfrak{p}_1 \overset{\text{up}}{\twoheadrightarrow} \mathfrak{p}_2$, if $y_1 \vee x_2 = y_2$ and $x_1 \le x_2$; see Figure 1 for illustration. If $\mathfrak{p}_1 \overset{\text{up}}{\twoheadrightarrow} \mathfrak{p}_2$ and $y_1 \wedge x_2 = x_1$, then we say that $\mathfrak{p}_1$ *transposes up to* $\mathfrak{p}_2$, or $\mathfrak{p}_2$ *transposes down to* $\mathfrak{p}_1$, and we also say that $\mathfrak{p}_1$ and $\mathfrak{p}_2$ are *transposed intervals*. *Down congruence-perspectivity* is defined dually: $\mathfrak{p}_1 \overset{\text{dn}}{\twoheadrightarrow} \mathfrak{p}_2$, if $x_1 \wedge y_2 = x_2$ and $y_1 \ge y_2$. We say that $\mathfrak{p}_1$ is *congruence-perspective* to $\mathfrak{p}_2$, in notation $\mathfrak{p}_1 \twoheadrightarrow \mathfrak{p}_2$, if $\mathfrak{p}_1 \overset{\text{up}}{\twoheadrightarrow} \mathfrak{p}_2$ or $\mathfrak{p}_1 \overset{\text{up}}{\twoheadrightarrow} \mathfrak{p}_2$. If $\mathfrak{p}_1 \twoheadrightarrow \mathfrak{p}_2$, then there are two possibilities: either $\mathfrak{p}_2$ is a subinterval of $\mathfrak{p}_1$ and we speak of a *comparable congruence-perspectivity*, or $\mathfrak{p}_2$ is not a subinterval of $\mathfrak{p}_1$ and we speak of a *parallel congruence-perspectivity*. The transitive closure of congruence-perspectivity is called *congruence-projectivity*. In this paper, it will be denoted by $\mathfrak{p} \Rrightarrow \mathfrak{q}$. Let $\mathfrak{p} \Lleftarrow\!\!\Rrightarrow \mathfrak{q}$ (to be read as *congruence-equivalent*) stand for the conjuction of $\mathfrak{p} \Rrightarrow \mathfrak{q}$ and $\mathfrak{q} \Rrightarrow \mathfrak{p}$. Sometimes we will use subscripts like $\mathfrak{p}_1 \twoheadrightarrow_M \mathfrak{p}_2$ and $\mathfrak{p} \Rrightarrow_M \mathfrak{q}$ to avoid ambiguity.

We will often rely, usually implicitly, on the fact that

$$\text{for } \mathfrak{p}, \mathfrak{q} \in \mathrm{Pri}(M),\ \ \mathfrak{p} \Rrightarrow \mathfrak{q} \text{ iff } \mathrm{con}(\mathfrak{p}) \supseteq \mathrm{con}(\mathfrak{q}), \tag{2.3}$$

see, e.g. G. Grätzer [10, Lemma I.3.6] or [11, Thm. 230], or see also G. Grätzer [9, Sect. III.1] with a different terminology. We shall need the following particular case of (2.3):

$$\text{if } \mathfrak{p}, \mathfrak{q} \in \operatorname{Pri}(M) \text{ are transposed intervals, then } \operatorname{con}_M(\mathfrak{p}) = \operatorname{con}_M(\mathfrak{q}). \tag{2.4}$$

Relations which are reflexive and transitive are called *quasiorderings* (many authors call them *preorderings*). If ν is a quasiordering on a set A, then $(A; \nu)$ is said to be a *quasiordered set*. We recall some basic properties of these sets, see G. Grätzer [11]. Let $\nu^{\cap}$ denote $\nu \cap \nu^{-1}$, the *equivalence induced* by ν. On the quotient set $A/\nu^{\cap}$ we can define $\hat{\nu} = \{([a]\nu^{\cap}, [a]\nu^{\cap}) : (a, b) \in \nu\}$. Then $\hat{\nu}$ and $(A/\nu^{\cap}; \hat{\nu})$ are called the *ordering* and the *order associated with the quasiordering* ν.

For $H \subseteq A^2$, the least quasiordering of A that includes H will be denoted by $\operatorname{quo}_A(H)$, or simply by $\operatorname{quo}(H)$ if there is no danger of confusion. For $H = \{(a, b)\}$, we will of course write $\operatorname{quo}(a, b)$. Quite often, especially if we intend to exploit the transitivity of ν, we write $a \leq_\nu b$ or $b \geq_\nu a$ instead of $(a, b) \in \nu$. The set of all quasiorderings on A form a complete lattice $\operatorname{Quo} A$ under set inclusion. For $\nu, \tau \in \operatorname{Quo} A$, the join $\nu \vee \tau$ is $\operatorname{quo}(\nu \cup \tau)$.

Next, let $(A_1; \nu_1)$ and $(A_2; \nu_2)$ be quasiordered sets. A *homomorphism* $g\colon (A_1; \nu_1) \to (A_2; \nu_2)$ is a map $g\colon A_1 \to A_2$ such that $g(\nu_1) \subseteq \nu_2$, that is, $\big(g(x), g(y)\big) \in \nu_2$ holds for all $(x, y) \in \nu_1$. Following G. Czédli and A. Lenkehegyi [3],

$$\overrightarrow{\operatorname{Ker}}\, g := \big\{(x, y) \in A_1^2 : \big(g(x), g(y)\big) \in \nu_2\big\} \tag{2.5}$$

is called the *directed kernel* of g. Clearly, it is a quasiordering on A_1.

A *quasi-colored lattice* is a lattice M of finite length together with a surjective map γ, called *quasi-coloring*, from $\operatorname{Pri}(M)$ onto a quasiordered set $(H; \nu)$ such that γ satisfies the following two properties:

(C1) if $\gamma(\mathfrak{p}) \geq_\nu \gamma(\mathfrak{q})$, then $\operatorname{con}(\mathfrak{p}) \geq \operatorname{con}(\mathfrak{q})$,
(C2) if $\operatorname{con}(\mathfrak{p}) \geq \operatorname{con}(\mathfrak{q})$, then $\gamma(\mathfrak{p}) \geq_\nu \gamma(\mathfrak{q})$.

The values of γ are called *colors* (rather than quasi-colors). If $\gamma(\mathfrak{p}) = b$, then we say that $\mathfrak{p}$ is colored by b. In figures, colors appear as labels of edges. Usually, not all the edges are labeled. If $(H; \nu)$ is an order, then the above γ is called a *coloring*. This concept of coloring is due to G. Grätzer and E. Knapp [14]. The name “coloring” was used for surjective maps onto antichains satisfying (C2) in G. Grätzer, H. Lakser, and E. T. Schmidt [20], and for surjective maps onto antichains satisfying (C1) in G. Grätzer [10, page 39].

With a quasi-coloring $\gamma\colon \operatorname{Pri}(M) \to (H; \nu)$, we can associate the map

$$\hat{\gamma}\colon \operatorname{Pri}(M) \to (H/\nu^{\cap}; \hat{\nu}), \qquad \mathfrak{p} \mapsto [\gamma(\mathfrak{p})]\nu^{\cap}. \tag{2.6}$$

Clearly, $\hat{\gamma}$ is a coloring. Let D_0 be a finite distributive lattice. Since it is determined by the order $\big((\operatorname{Ji}(D_0); \leq\big)$, we know from G. Grätzer and E. Knapp [14] that

$$D_0 \cong \operatorname{Con} M \text{ iff } M \text{ can be colored by } \big(\operatorname{Ji}(D_0); \leq\big). \tag{2.7}$$

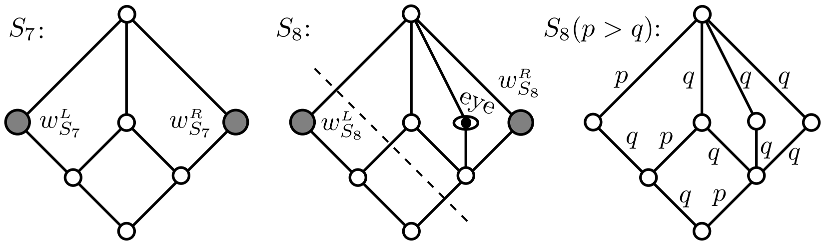

FIGURE 2. S_7 and the semimodular gadget S_8

Furthermore, let $\gamma\colon \mathrm{Pri}(M) \to \big(\mathrm{Ji}(D_0);\leq\big)$ be a coloring. Then

$$\alpha\colon D_0 \to \mathrm{Con}\, M, \text{ where } x \mapsto \bigvee\big\{\mathrm{con}_M(\mathfrak{p}) : \mathfrak{p} \in \mathrm{Pri}(M) \text{ and } \gamma(\mathfrak{p}) \leq x\big\}, \tag{2.8}$$

is an isomorphism. Note that [14] attributes (2.7) and (2.8) to J. Jakubík [25]. Note also that (2.6) and (2.7) imply that *every quasi-coloring of M determines* $\mathrm{Con}\, M$ up to isomorphism. As (2.7) suggests, a part of our job is to find a rectangular lattice K together with a coloring $\gamma\colon \mathrm{Pri}(K) \to \big(\mathrm{Ji}(D);\leq\big)$. While (C1) for γ is fairly easy to achieve, (C2) needs a bit more work.

An example of a coloring is given by our "basic gadget" $S_8 = S_8(p > q)$ in Figure 2, which is due to G. Grätzer, H. Lakser, and E. T. Schmidt [20], see also G. Grätzer [10, Fig. 9.1] and E. T. Schmidt [29]. It is a rectangular lattice colored by $\big(\{p,q\};\leq\big)$ where $q < p$. Since $\big(\{p,q\};\leq\big)$ is the order of all non-zero join-irreducible elements of the three-element chain $\{0 < q < p\}$, $\mathrm{Con}\, S_8$ is (isomorphic to) the three-element lattice. The atom of $\mathrm{Con}\, S_8$ is indicated by a dotted line. Another example, for an arbitrary finite lattice M, is the so-called *natural coloring* $\mathrm{Pri}(M) \to \big(\mathrm{Ji}(M);\leq\big)$, $\mathfrak{p} \mapsto \mathrm{con}_M(\mathfrak{p})$.

A finite lattice M has many quasi-colorings. The following lemma gives a useful way to derive a new quasi-coloring from a given one.

Lemma 2.1. *Let M be a finite lattice, and let $(Q;\nu)$ and $(P;\sigma)$ be quasiordered sets. Let $\gamma_0\colon \mathrm{Pri}(M) \to (Q;\nu)$ be a quasi-coloring. Assume that $g\colon (Q;\nu) \to (P;\sigma)$ is a surjective homomorphism such that $\overrightarrow{\mathrm{Ker}}\, g \subseteq \nu$. Then $g \circ \gamma_0\colon \mathrm{Pri}(M) \to (P;\sigma)$, where $\mathfrak{p} \mapsto g(\gamma_0(\mathfrak{p}))$, is a quasi-coloring.*

Proof. Let $\gamma_1 = g \circ \gamma_0$. Evidently, it is surjective. Assume that $\mathfrak{p}, \mathfrak{q} \in \mathrm{Pri}(M)$ such that $\gamma_1(\mathfrak{p}) \leq_\sigma \gamma_1(\mathfrak{q})$. Then $g\big(\gamma_0(\mathfrak{p})\big) \leq_\sigma g\big(\gamma_0(\mathfrak{q})\big)$ shows that $\big(\gamma_0(\mathfrak{p}), \gamma_0(\mathfrak{q})\big) \in \overrightarrow{\mathrm{Ker}}\, g \subseteq \nu$. Since γ_0 is a quasi-coloring, this implies $\mathrm{con}(\mathfrak{p}) \leq \mathrm{con}(\mathfrak{q})$. Thus γ_1 satisfies (C1).

To show (C2), assume that $\mathrm{con}(\mathfrak{p}) \leq \mathrm{con}(\mathfrak{q})$. Since γ_0 is a quasi-coloring, we infer that $\gamma_0(\mathfrak{p}) \leq_\nu \gamma_0(\mathfrak{q})$. This yields that $\gamma_1(\mathfrak{p}) = g\big(\gamma_0(\mathfrak{p})\big) \leq_\sigma g\big(\gamma_0(\mathfrak{q})\big) = \gamma_1(\mathfrak{q})$ since g is a homomorphism. Thus γ_1 satisfies (C2). $\square$

3. Quasi-colorings versus congruence-preserving extensions

Let M_1 be a sublattice of a lattice M_2. Following the terminology introduced by G. Grätzer and H. Lakser [16], see G. Grätzer [11, I.3.8] and G. Grätzer and E. T. Schmidt [23], M_2 is a *congruence-preserving extension* of M_1 (or M_1 is a *congruence-preserving* sublattice of M_2), if the restriction map $\varrho_{M_1}^{M_2}\colon \operatorname{Con} M_2 \to \operatorname{Con} M_1$ is an isomorphism. Before stating a lemma, which witnesses that quasi-colorings offer a reasonable way to congruence-preserving extensions, we have to introduce another kind of extension.

A prime interval of M_1 (sublattice of M_2) need not be a prime interval of M_2. By $\operatorname{Pri}(M_1)\cap\operatorname{Pri}(M_2)$ we denote the set of $\mathfrak{p}\in\operatorname{Pri}(M_2)$ such that $0_{\mathfrak{p}}, 1_{\mathfrak{p}}\in M_1$. Assume that $\gamma_i\colon \operatorname{Pri}(M_i)\to(Q_i;\nu_i)$ are quasi-colorings for $i=1,2$. We say that γ_2 *extends* γ_1 if $\gamma_2(\mathfrak{p})=\gamma_1(\mathfrak{p})$ for all $\mathfrak{p}\in\operatorname{Pri}(M_1)\cap\operatorname{Pri}(M_2)$.

Lemma 3.1. *Let M_1 be a sublattice of a finite lattice M_2. Let the map $\gamma\colon \operatorname{Pri}(M_1)\to\bigl(\operatorname{Ji}(\operatorname{Con} M_1);\leq\bigr)$ with $\mathfrak{p}\mapsto\operatorname{con}_{M_1}(\mathfrak{p})$ be the natural coloring. Assume that the restriction of γ to $\operatorname{Pri}(M_1)\cap\operatorname{Pri}(M_2)$ is surjective, and that γ can be extended to a coloring $\delta\colon\operatorname{Pri}(M_2)\to\bigl(\operatorname{Ji}(\operatorname{Con} M_1);\leq\bigr)$. Then M_2 is a congruence-preserving extension of M_1.*

Proof. Denote $\operatorname{Pri}(M_1)\cap\operatorname{Pri}(M_2)$ by $\mathfrak{P}$, and the restriction of γ to $\mathfrak{P}$ by $\gamma\rceil_{\mathfrak{P}}$. We know from (2.2) that, for every $y\in\operatorname{Con} M_1$,

$$y=\bigvee\bigl\{\operatorname{con}_{M_1}(\mathfrak{p}) : \mathfrak{p}\in\operatorname{Pri}(M_1)\text{ and }\operatorname{con}_{M_1}(\mathfrak{p})\leq y\bigr\}. \tag{3.1}$$

We claim that, for every $y\in\operatorname{Con} M_1$,

$$y=\bigvee\bigl\{\operatorname{con}_{M_1}(\mathfrak{p}) : \mathfrak{p}\in\mathfrak{P}\text{ and }\operatorname{con}_{M_1}(\mathfrak{p})\leq y\bigr\}. \tag{3.2}$$

The "$\geq$" inequality is trivial. To show the reverse, let $\operatorname{con}_{M_1}(\mathfrak{p})$ be one of the joinands in (3.1). That is, $\mathfrak{p}\in\operatorname{Pri}(M_1)$ and $\operatorname{con}_{M_1}(\mathfrak{p})\leq y$. Since $\operatorname{con}_{M_1}(\mathfrak{p})=\gamma(\mathfrak{p})$ and $\gamma\rceil_{\mathfrak{P}}$ is surjective, there exists a $\mathfrak{q}\in\mathfrak{P}$ such that $\gamma(\mathfrak{p})=\gamma(\mathfrak{q})$. Hence, $\operatorname{con}_{M_1}(\mathfrak{p})=\gamma(\mathfrak{p})=\gamma(\mathfrak{q})=\operatorname{con}_{M_1}(\mathfrak{q})$ shows that $\operatorname{con}_{M_1}(\mathfrak{p})$ equals one of the joinands in (3.2). This proves (3.2).

Applying (2.8) for $(\delta, M_2, \operatorname{Con} M_1)$ instead of (γ, M, D_0), we obtain that

$$\begin{aligned}\alpha\colon\ &\operatorname{Con} M_1\to\operatorname{Con} M_2,\\ &x\mapsto\bigvee\bigl\{\operatorname{con}_{M_2}(\mathfrak{p}) : \mathfrak{p}\in\operatorname{Pri}(M_2),\ \delta(\mathfrak{p})\leq x\bigr\}\end{aligned} \tag{3.3}$$

is an isomorphism. We claim that, for $x\in\operatorname{Con} M_1$,

$$\alpha(x)=\bigvee\bigl\{\operatorname{con}_{M_2}(\mathfrak{p}) : \mathfrak{p}\in\mathfrak{P}\text{ and }\delta(\mathfrak{p})\leq x\bigr\}. \tag{3.4}$$

Since $\operatorname{Pri}(M_2)\supseteq\mathfrak{P}$, the "$\geq$" inequality in (3.4) is obvious. To see the reverse, assume that $\operatorname{con}_{M_2}(\mathfrak{p})$ is a joinand in (3.3). The range of δ is the same as that of γ, which equals the range of $\gamma\rceil_{\mathfrak{P}}=\delta\rceil_{\mathfrak{P}}$. Hence, there is a $\mathfrak{q}\in\mathfrak{P}$ with $\delta(\mathfrak{q})=\delta(\mathfrak{p})$. Since δ is a coloring, (C1) yields that $\operatorname{con}_{M_2}(\mathfrak{p})=\operatorname{con}_{M_2}(\mathfrak{q})$. So, $\operatorname{con}_{M_2}(\mathfrak{p})$ equals one of the joinands in (3.4). Thus (3.4) holds.

Next, we claim that $x=\varrho_{M_1}^{M_2}\bigl(\alpha(x)\bigr)$, for every $x\in\operatorname{Con} M_1$.

First we show that $x \le \varrho^{M_2}_{M_1}\bigl(\alpha(x)\bigr)$. By (3.2), applied to $y := x$, it suffices to show that if $\mathfrak{p} \in \mathfrak{P}$ and $\mathrm{con}_{M_1}(\mathfrak{p}) \le x$, then $\mathrm{con}_{M_1}(\mathfrak{p}) \le \varrho^{M_2}_{M_1}\bigl(\alpha(x)\bigr)$. Assume that $\mathfrak{p} \in \mathfrak{P}$ and $\mathrm{con}_{M_1}(\mathfrak{p}) \le x$. Then $\delta(\mathfrak{p}) = \gamma(\mathfrak{p}) = \mathrm{con}_{M_1}(\mathfrak{p}) \le x$ indicates that $\mathrm{con}_{M_2}(\mathfrak{p})$ is one of the joinands in (3.4). So $\mathrm{con}_{M_2}(\mathfrak{p}) \le \alpha(x)$. Hence, $\alpha(x)$ and, therefore, $\varrho^{M_2}_{M_1}\bigl(\alpha(x)\bigr)$ collapse $\mathfrak{p}$, whence $\mathrm{con}_{M_1}(\mathfrak{p}) \le \varrho^{M_2}_{M_1}\bigl(\alpha(x)\bigr)$.

To show the converse inequality, $\varrho^{M_2}_{M_1}\bigl(\alpha(x)\bigr) \le x$, we apply (3.2) with $y := \varrho^{M_2}_{M_1}\bigl(\alpha(x)\bigr)$. That is, we assume that $\mathfrak{p} \in \mathfrak{P}$ and $\mathrm{con}_{M_1}(\mathfrak{p}) \le \varrho^{M_2}_{M_1}\bigl(\alpha(x)\bigr)$, and we have to show that $\mathrm{con}_{M_1}(\mathfrak{p}) \le x$. Our assumption yields that $\varrho^{M_2}_{M_1}\bigl(\alpha(x)\bigr)$ collapses $\mathfrak{p}$, and so does $\alpha(x)$. Hence, $\mathrm{con}_{M_2}(\mathfrak{p}) \le \alpha(x)$. Consequently, combining (2.1), (2.2), and (3.4), we get a prime interval $\mathfrak{p}' \in \mathfrak{P}$ such that $\mathrm{con}_{M_2}(\mathfrak{p}) \le \mathrm{con}_{M_2}(\mathfrak{p}')$ and $\delta(\mathfrak{p}') \le x$. (C2) yields that $\delta(\mathfrak{p}) \le \delta(\mathfrak{p}') \le x$. We also have $\mathrm{con}_{M_1}(\mathfrak{p}) = \gamma(\mathfrak{p}) = \delta(\mathfrak{p})$ since $\delta\rceil_{\mathfrak{P}} = \gamma\rceil_{\mathfrak{P}}$. Hence, we conclude the desired $\mathrm{con}_{M_1}(\mathfrak{p}) \le x$ by transitivity. This proves that $\varrho^{M_2}_{M_1}\bigl(\alpha(x)\bigr) = x$.

Finally, we already know that $\varrho^{M_2}_{M_1} \circ \alpha = \mathrm{id}_{\mathrm{Con}\, M_1}$. Multiplying both sides by α^{-1} we get that $\varrho^{M_2}_{M_1} = \alpha^{-1}$ is an isomorphism. ☐

Lemma 3.1 asserts that certain extensions of colorings give rise to congruence-preserving extensions of lattices. The converse statement also holds.

Lemma 3.2. *Assume that M' is a congruence-preserving sublattice of a finite lattice M. Let $\delta'\colon \mathrm{Pri}(M') \to (Q;\nu)$ be a quasi-coloring such that its restriction to $\mathrm{Pri}(M') \cap \mathrm{Pri}(M)$ is surjective. Then*

(i) *δ' can be extended to a quasi-coloring $\delta\colon \mathrm{Pri}(M) \to (Q;\nu)$.*
(ii) *Furthermore, if $\eta\colon \mathrm{Pri}(M) \to \mathrm{Pri}(M')$ is surjective map such that, for all $\mathfrak{p} \in \mathrm{Pri}(M)$, $\varrho^{M}_{M'}\bigl(\mathrm{con}_M(\mathfrak{p})\bigr) = \mathrm{con}_{M'}\bigl(\eta(\mathfrak{p})\bigr)$ and η acts identically on $\mathrm{Pri}(M') \cap \mathrm{Pri}(M)$, then $\delta' \circ \eta\colon \mathrm{Pri}(M) \to (Q;\nu)$ is quasi-coloring extending δ'.*

Proof. Firstly, we show the following easy property of monotone maps. Let T_1 and T_2 be orders, let $f_1 : T_1 \to T_2$ be an order isomorphism, and let $f_2\colon T_2 \to T_1$ be a monotone map. Assume that $x_2 \le f_1\bigl(f_2(x_2)\bigr)$ and $f_2\bigl(f_1(x_1)\bigr) \le x_1$ hold for all $x_1 \in T_1$ and $x_2 \in T_2$. Then

$$f_2 \text{ is also an order isomorphism, and } f_2 = f_1^{-1}. \tag{3.5}$$

Indeed, from the first inequality we obtain that $f_1^{-1}(x_2) \le f_2(x_2)$. But x_2 is of the form $x_2 = f_1(x_1)$, whence the second inequality yields that $f_2(x_2) \le f_1^{-1}(x_2)$. Hence, $f_2 = f_1^{-1}$, proving (3.5).

Let $\mathfrak{P} := \mathrm{Pri}(M') \cap \mathrm{Pri}(M)$. By our assumption, $\varrho^{M}_{M'}\colon \mathrm{Con}\, M \to \mathrm{Con}\, M'$ is an isomorphism. Obviously, $\mathrm{con}_M\colon \mathrm{Con}\, M' \to \mathrm{Con}\, M$ is a monotone map. Clearly, $x_2 \le \varrho^{M}_{M'}\bigl(\mathrm{con}_M(x_2)\bigr)$ and $\mathrm{con}_M\bigl(\varrho^{M}_{M'}(x_1)\bigr) \le x_1$ for all $x_1 \in \mathrm{Con}\, M$ and $x_2 \in \mathrm{Con}\, M'$. So it follows from (3.5) that con_M is the inverse map of $\varrho^{M}_{M'}$. This implies that

$$\text{for all } \mathfrak{p} \in \mathfrak{P},\ \varrho^{M}_{M'}\bigl(\mathrm{con}_M(\mathfrak{p})\bigr) = \mathrm{con}_{M'}(\mathfrak{p}) \tag{3.6}$$

since $\mathrm{con}_M(\mathfrak{p}) = \mathrm{con}_M\bigl(\mathrm{con}_{M'}(\mathfrak{p})\bigr)$.

If $\mathfrak{p} \in \mathrm{Pri}(M)$, then $\varrho^M_{M'}(\mathrm{con}_M(\mathfrak{p})) \in \mathrm{Ji}(\mathrm{Con}\, M')$ since $\varrho^M_{M'}$ is an isomorphism. Hence, we can choose a $\mathfrak{q} \in \mathrm{Pri}(M')$ such that $\varrho^M_{M'}(\mathrm{con}_M(\mathfrak{p})) = \mathrm{con}_{M'}(\mathfrak{q})$. If $\mathfrak{p} \in \mathfrak{P}$, then (3.6) allows us to choose $\mathfrak{q} := \mathfrak{p}$. Consequently, we can fix a surjective map η that satisfies the premise of part (ii). With this η, we define $\delta := \delta' \circ \eta$. Clearly, δ extends δ'. We claim that $\delta\colon \mathrm{Pri}(M) \to (Q;\nu)$ is a quasi-coloring. It is surjective, since so is $\delta'\rceil_{\mathfrak{P}}$. Assume that $\mathfrak{p}, \mathfrak{q} \in \mathrm{Pri}(M)$. Then $\bigl(\delta(\mathfrak{p}), \delta(\mathfrak{q})\bigr) \in \nu$ iff $\bigl(\delta'(\eta(\mathfrak{p})), \delta'(\eta(\mathfrak{q}))\bigr) \in \nu$. Since δ' is a quasi-coloring, this is equivalent to $\mathrm{con}_{M'}(\eta(\mathfrak{p})) \le \mathrm{con}_{M'}(\eta(\mathfrak{q}))$. By the choice of η, this is the same as $\varrho^M_{M'}\bigl(\mathrm{con}_M(\mathfrak{p})\bigr) \le \varrho^M_{M'}\bigl(\mathrm{con}_M(\mathfrak{q})\bigr)$. Since $\varrho^M_{M'}$ is an isomorphism, the last inequality is equivalent to $\mathrm{con}_M(\mathfrak{p}) \le \mathrm{con}_M(\mathfrak{q})$. Thus $\bigl(\delta(\mathfrak{p}), \delta(\mathfrak{q})\bigr) \in \nu$ iff $\mathrm{con}_M(\mathfrak{p}) \le \mathrm{con}_M(\mathfrak{q})$, proving that δ is a quasi-coloring. $\square$

4. Slim semimodular lattices

A *slim lattice* is a finite lattice M such that the order $\mathrm{Ji}(M)$ contains no three-element antichain. Equivalently, see R. P. Dilworth [6], G. Grätzer and E. Knapp [13], or G. Czédli and E. T. Schmidt [5], a finite lattice M is slim iff $\mathrm{Ji}(M)$ is the union of two chains. By G. Czédli and E. T. Schmidt [4, Lemma 6], slim lattices are planar. The study of planar lattices started in D. Kelly and I. Rival [26]. Many properties of these lattices we visually accept are rigorously proved in [26]. When we speak of a planar lattice, always a fixed planar diagram is assumed. If this diagram divides the plane into *covering squares* (i.e., cover-preserving four-element Boolean sublattices), then, following G. Grätzer and E. Knapp [13], we speak of a 4*-cell lattice*. For example, M_3 (the diamond) and all finite chains are 4-cell lattices but the five-element non-modular lattice is not. The cells of a 4-cell lattice are called 4*-cells*. Notice that 4-cells are always covering squares but not conversely; indeed, M_3 has three covering squares but (with respect to a fixed diagram) only two 4-cells. Let us emphasize that a 4-cell lattice is planar by definition. If S is a 4-cell of a lattice, then its largest element, least element, left (weak) corner (element) and right (weak) corner will be denoted by 1_S, 0_S, w^L_S and w^R_S, respectively. The same notation applies if S is a covering square of a not necessarily planar finite lattice; then we have a choice which one of its atoms is denoted by w^L_S.

Lemma 4.1 (G. Grätzer and E. Knapp [13], see also G. Czédli and E. T. Schmidt [5] for the present form)**.** *Let M be finite lattice. Then M is a slim semimodular lattice iff M is a* 4*-cell lattice such that no two distinct* 4*-cells of M have the same bottom.*

For example, none of the semimodular lattices M and M^* in Figure 3 are slim, but each of them would be slim if we deleted the black-filled element.

Lemma 4.2 (G. Grätzer and E. Knapp [13, Lemmas 4 and 5])**.** *Let M be a planar lattice. Then M is semimodular iff it is a* 4*-cell lattice and $0_A = 0_B$ implies $1_A = 1_B$, for any two* 4*-cells A and B of M.*

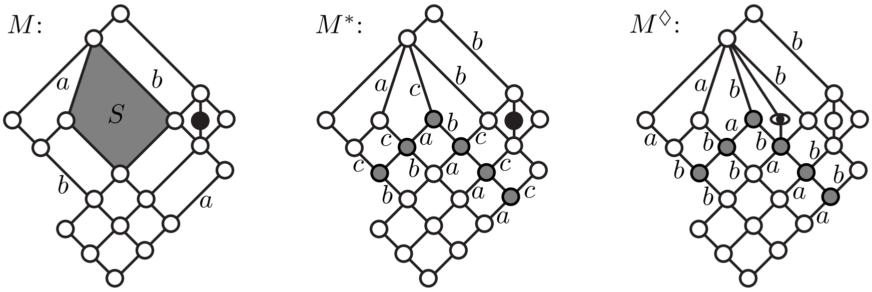

FIGURE 3. Adding a fork and adding an e-fork

Next, consider a planar semimodular lattice M. A 4-cell S of M will be called a *slim* 4-*cell*, if the principal ideal $\downarrow 1_S = \{x \in M : x \leq 1_S\}$ is a slim (and necessarily semimodular) lattice. Assume that S is a slim 4-cell of M. For an example, see the grey-filled 4-cell of M in Figure 3. Replace this 4-cell by a copy of S_7. Then, starting at S and going to the southwest (in the original lattice) from 4-cell (understood in M) to adjacent 4-cells as long as possible, divide these 4-cell into two 4-cells by a northeast-southwest new edge. (Since no two distinct 4-cells of $\downarrow 1_S$ have the same bottom, the next 4-cell is always to the southwest from the previous cell.) In the next stage, do the same sort of steps to the southeast direction. This way we get an extension of M to a new lattice M^*, see Figure 3. The order $F = M^* - M$ of the new elements is called a *fork*; see the grey-filled elements in Figure 3. Notice that F is the disjoint union of its top element, 1_F, its *left chain*, F_{left}, and its *right chain*, F_{right}. Notice that for each $x \in F$, there is a unique (upper) cover x^+ of x in $M = M^* - F$. Similarly, for all $y \in F - \{1_F\}$, the unique lower cover of y outside F will be denoted by y^-. Note that the map $F \to M$ with $x \mapsto x^+$ is an order-embedding. We say that M^* is obtained from M by *adding a fork* (at the 4-cell S). In view of Lemmas 4.1 and 4.2, the following statement is obvious; see G. Czédli and E. T. Schmidt [5] for the particular case when M is a slim semimodular lattice.

Lemma 4.3. *Let S be a slim 4-cell of a planar semimodular lattice M, and let u be an element of M such that $\downarrow u$ is slim. Then M^*, defined above, is semimodular. Moreover, $\downarrow u = \{x \in M^* : x \leq u\}$ is a slim semimodular lattice.*

Concerning M^*, we will need the following two lemmas.

Lemma 4.4. *Assume that $\mathfrak{p}, \mathfrak{q} \in \mathrm{Pri}(M^*)$, $\mathfrak{p} \overset{\text{up}}{\twoheadrightarrow}_{M^*} \mathfrak{q}$, $0_{\mathfrak{q}} \in F$ and $1_{\mathfrak{q}} \notin F$. Then $0_{\mathfrak{p}} \in F$ and $1_{\mathfrak{p}} \notin F$.*

Proof. Observe that $\mathfrak{q}$ is of the form $\mathfrak{q} = [x, x^+]$ where $x \in F$. Let $\mathfrak{p} = [a, b]$. Since $a \prec b$, it follows easily that $\mathfrak{p}$ and $\mathfrak{q}$ are transposed intervals. Hence, $x^+ = b \vee x$ and $a = b \wedge x$ holds in M^*. We can assume that $\mathfrak{p} \neq \mathfrak{q}$. Then $b \parallel x$. Fix a maximal chain C in $\uparrow 1_F$. Then $C \cup F_{\text{left}}$, which is a maximal chain of M^*, divides M^* into two parts, a left side and a right side. The intersection of this two sides is $C \cup F_{\text{left}}$. Similarly, $C \cup F_{\text{right}}$ also divides M^* into two sides. Combining these sides, M^* is divided into three parts: the left side of $C \cup F_{\text{left}}$, the right side of $C \cup F_{\text{right}}$ and, finally, $\downarrow 1_F$. Note that F is a subset of the boundary of $\downarrow 1_F$.

Since F is an order-filter in $\downarrow 1_F$, $x \in F$, and $x^+ = x \vee b \notin F$, it follows that $b = 1_{\mathfrak{p}} \notin \downarrow 1_F$. So, modulo left-right symmetry, we can assume that b is on the left of $C \cup F_{\text{left}}$. Moreover, $b \notin C \cup F_{\text{left}}$ since $b \parallel x$. Hence, b is strictly on the left side of $C \cup F_{\text{left}}$.

Assume by way of contradiction that $a = 0_{\mathfrak{p}} \notin F$. Then $a = b \wedge x \in \downarrow x \subseteq \downarrow 1_F$ implies that a is strictly on the right side of $C \cup F_{\text{left}}$. Therefore D. Kelly and I. Rival [26, Lemma 1.2] yields an element $z \in C \cup F_{\text{left}}$ such that $a < z < b$. However, this contradicts that $a \prec b$. $\square$

Lemma 4.5. *Consider the retraction map*

$$\psi\colon M^* \to M, \quad \textit{where} \quad x \mapsto \begin{cases} x, & \textit{if } x \in M; \\ x^+, & \textit{if } x \in F. \end{cases}$$

Then ψ is a lattice homomorphism.

Proof. An easy result (see [1, Thm. IV.20] and compare with (2.4)) from the folklore of lattice theory says that an equivalence $\boldsymbol{\alpha}$ is a congruence iff

(i) the $\boldsymbol{\alpha}$-blocks are convex sublattices, and
(ii) for any pair $(\mathfrak{p}, \mathfrak{q})$ of transposed intervals, if $\boldsymbol{\alpha}$ collapses $\mathfrak{p}$, then it collapses $\mathfrak{q}$.

First we show that $\boldsymbol{\alpha} := \operatorname{Ker}(\psi)$ is a lattice congruence. The $\boldsymbol{\alpha}$-blocks are the prime intervals $[x, x^+]$, for $x \in F$, and the singletons $\{y\}$, for $y \in M$. Hence, (i) holds. Assume that $\mathfrak{p}, \mathfrak{q} \in \operatorname{Pri}(M^*)$ are transposed prime intervals and $\mathfrak{p} \overset{\text{up}}{\twoheadrightarrow}_{M^*} \mathfrak{q}$. If $\mathfrak{q}$ is collapsed by $\boldsymbol{\alpha}$, then so is $\mathfrak{p}$ by Lemma 4.4. Assume that $\mathfrak{p} = [x, x^+]$ is collapsed by $\boldsymbol{\alpha}$. Then $x \in F$. If $0_{\mathfrak{q}} \notin F$, then $x \leq 0_{\mathfrak{q}}$ implies that $x^+ \leq 0_{\mathfrak{q}}$, and $1_{\mathfrak{q}} = x^+ \vee 0_{\mathfrak{q}} = 0_{\mathfrak{q}}$ is a contradiction. Hence, $0_q \in F$. Since $x^+ \notin \downarrow 1_F \supseteq F$ and $x^+ \leq 1_{\mathfrak{q}}$, we get that $1_{\mathfrak{q}} \notin F$ and $1_{\mathfrak{q}} = 0_{\mathfrak{q}}^+$. Hence, $\mathfrak{q}$ is collapsed by $\boldsymbol{\alpha}$. Thus (ii) holds and $\operatorname{Ker}(\psi) = \boldsymbol{\alpha}$ is a congruence. Since the $\operatorname{Ker}(\psi)$-blocks intersect M in singletons, we conclude easily that ψ is a homomorphism. $\square$

5. Elementary extensions

The statements of this section are intuitively clear. Nevertheless, their proofs need some work.

Assume that S is a slim 4-cell of a planar semimodular lattice M. Let $\gamma\colon \operatorname{Pri}(M) \to (Q; \beta)$ be a quasi-coloring. Let a and b be the colors of the northwest edge and the northeast edge of S, respectively. Then we say that the 4-cell S is *colored by* (a, b). On the set $Q^* := Q \cup \{c\}$, where $c \notin Q$, let

$\beta^* = \mathrm{quo}_{Q^*}\big(\beta \cup \{(c,a),(c,b)\}\big)$. Note that here the "elementary quasiordering" mentioned at the end of Section 1 is $\mathrm{quo}_{Q^*}\big((c,a),(c,b)\big)$. Note also that if β happens to be an ordering, not just a quasiordering, then β^* is an ordering as well. Let M^* denote the lattice we obtain by adding a fork F to M at our (a,b)-colored 4-cell, S, see Figure 3. Define a map γ^* as follows:

$$\gamma^* \colon \mathrm{Pri}(M^*) \to (Q^*;\beta^*), \qquad \text{where}$$

$$\mathfrak{p} \mapsto \begin{cases} c, & \text{if } \mathfrak{p} = [x, x^+] \text{ with } x \in F; \\ a, & \text{if } \mathfrak{p} = [x^-, x] \text{ with } x \in F_{\text{right}}; \\ b, & \text{if } \mathfrak{p} = [x^-, x] \text{ with } x \in F_{\text{left}}; \\ \gamma([x^+, y^+]), & \text{if } \mathfrak{p} = [x, y] \text{ with } x, y \in F; \\ \gamma(\mathfrak{p}), & \text{if } 0_\mathfrak{p}, 1_\mathfrak{p} \in M. \end{cases}$$

Note that γ^* extends γ.

Lemma 5.1 (Fork Lemma)**.** *Assume that M, M^*, γ and γ^* are given as above. Then $\gamma^* \colon \mathrm{Pri}(M^*) \to (Q^*;\beta^*)$ is a quasi-coloring.*

Proof of Lemma 5.1. It is fairly evident by the construction that γ^* satisfies (C1). To show that (C2) also holds, let us assume that $\mathfrak{p}_1, \mathfrak{p}_2 \in \mathrm{Pri}(M^*)$ such that $\mathrm{con}_{M^*}(\mathfrak{p}_1) \geq \mathrm{con}_{M^*}(\mathfrak{p}_2)$. We want to show that $\gamma^*(\mathfrak{p}_1) \geq_{\beta^*} \gamma^*(\mathfrak{p}_2)$. We have to deal with two cases.

Case 1. We assume that $\big\{\gamma^*(\mathfrak{p}_1), \gamma^*(\mathfrak{p}_2)\big\} \subseteq Q$. By (2.3), $\mathfrak{p}_1 \Rrightarrow_{M^*} \mathfrak{p}_2$. Hence, there are intervals $\mathfrak{r}_i = [x_i, y_i] \in \mathrm{Int}(M^*)$ that form a sequence

$$\mathfrak{p}_1 = \mathfrak{r}_0 \twoheadrightarrow_{M^*} \mathfrak{r}_1 \twoheadrightarrow_{M^*} \cdots \twoheadrightarrow_{M^*} \mathfrak{r}_k = \mathfrak{p}_2. \tag{5.1}$$

Observe that the retraction homomorphism ψ provided by Lemma 4.5 sends (5.1) to a congruence-perspectivity sequence

$$\psi(\mathfrak{p}_1) = \psi(\mathfrak{r}_0) \twoheadrightarrow_M \psi(\mathfrak{r}_1) \twoheadrightarrow_M \cdots \twoheadrightarrow_M \psi(\mathfrak{r}_k) = \psi(\mathfrak{p}_2)$$

in M. Indeed, since all prime intervals collapsed by $\mathrm{Ker}(\psi)$ are c-colored and $c \notin Q$, $\psi(\mathfrak{p})$ and $\psi(\mathfrak{q})$ remain prime intervals, and therefore all the $\psi(\mathfrak{r}_i)$ are intervals (rather than singletons). By (2.3) again, the new sequence gives that $\mathrm{con}_M\big(\psi(\mathfrak{p}_1)\big) \geq \mathrm{con}_M\big(\psi(\mathfrak{p}_2)\big)$. Since γ is a quasi-coloring, we conclude that $\gamma\big(\psi(\mathfrak{p}_1)\big) \geq_\beta \gamma\big(\psi(\mathfrak{p}_2)\big)$. By $\beta \subseteq \beta^*$, this entails that $\gamma\big(\psi(\mathfrak{p}_1)\big) \geq_{\beta^*} \gamma\big(\psi(\mathfrak{p}_2)\big)$. The assumption $\gamma^*(\mathfrak{p}_i) \notin Q$ and the definition of γ^* imply that $\gamma\big(\psi(\mathfrak{p}_i)\big) = \gamma^*(\mathfrak{p}_i)$, for $i = 1, 2$. Hence, $\gamma^*(\mathfrak{p}_1) \geq_{\beta^*} \gamma^*(\mathfrak{p}_2)$, as intended.

Case 2. We assume that $\{\gamma^*(\mathfrak{p}_1), \gamma^*(\mathfrak{p}_2)\}$ is not a subset of Q. We want to show that $\gamma^*(\mathfrak{p}_1) \geq_{\beta^*} \gamma^*(\mathfrak{p}_2)$. This is obvious if $\gamma^*(\mathfrak{p}_1) = c = \gamma^*(\mathfrak{p}_2)$. The case $\gamma^*(\mathfrak{p}_1) = c \neq \gamma^*(\mathfrak{p}_2)$ would contradict that $\mathrm{con}_{M^*}(\mathfrak{p}_1) \geq \mathrm{con}_{M^*}(\mathfrak{p}_2)$ since $\mathrm{con}_{M^*}(\mathfrak{p}_1) = \mathrm{Ker}(\psi)$ from Lemma 4.5 would be an atom in $\mathrm{Con}\, M^*$. Therefore, we assume that $\gamma^*(\mathfrak{p}_1) = d \in Q$ and $\gamma^*(\mathfrak{p}_2) = c$. It suffices to find an $\mathfrak{s} \in \mathrm{Pri}(M^*)$ such that $\mathrm{con}_{M^*}(\mathfrak{p}_1) \geq \mathrm{con}_{M^*}(\mathfrak{s})$ and $\gamma^*(\mathfrak{s}) \in \{a, b\}$. Indeed, then the previous case will clearly imply that $\gamma^*(\mathfrak{p}_1) \geq_{\beta^*} \gamma^*(\mathfrak{s}) \in \{a, b\}$, yielding that $\gamma^*(\mathfrak{p}_1) \geq_{\beta^*} c = \gamma^*(\mathfrak{p}_2)$.

If $\mathfrak{p}_1 \notin \mathrm{Pri}(M)$, then there exists a $\mathfrak{p}_1' \in \mathrm{Pri}(M)$ such that $\gamma^*(\mathfrak{p}_1) = \gamma^*(\mathfrak{p}_1')$. Then $\gamma^*(\mathfrak{p}_1) \leq_{\beta^*} \gamma^*(\mathfrak{p}_1')$, $\gamma^*(\mathfrak{p}_1) \geq_{\beta^*} \gamma^*(\mathfrak{p}_1')$, and (C1) yield that $\mathrm{con}_{M^*}(\mathfrak{p}_1) = \mathrm{con}_{M^*}(\mathfrak{p}_1')$. Therefore, we can assume that $\mathfrak{p}_1 \in \mathrm{Pri}(M)$.

By the assumption $\mathrm{con}_{M^*}(\mathfrak{p}_1) \geq \mathrm{con}_{M^*}(\mathfrak{p}_2)$, there is a sequence (5.1) such that $\gamma^*(\mathfrak{p}_1) = d$ and $\gamma^*(\mathfrak{p}_2) = c$. It is sufficient to find an $\mathfrak{r}_i$ that has a prime subinterval $\mathfrak{s}$ such that $\gamma^*(\mathfrak{s}) \in \{a, b\}$. Indeed, then we would get $\mathrm{con}_{M^*}(\mathfrak{p}_1) \geq \mathrm{con}_{M^*}(\mathfrak{r}_i) \geq \mathrm{con}_{M^*}(\mathfrak{s})$, which would do the job.

To harmonize with Figure 3, we say that the intervals of M are *white*. The intervals of the order F are *grey*. The rest of the intervals, such as $[x, x^+]$ for $x \in F$, are *bicolored*. (This terminology has nothing to do with the quasi-colorings γ and γ^*.) We know that $\mathfrak{p}_1$ is white and $\mathfrak{p}_2$ is bicolored. Let $\mathfrak{r}_i$ be the first interval in the sequence (5.1) that is not white.

Let us assume that $\mathfrak{r}_i$ is grey. We can assume that y_i is not the top of F, since otherwise, with respect to γ^*, an a-colored or b-colored prime subinterval would be clearly at our disposal. Let j be the largest number such that $i \leq j \leq k$ and $\mathfrak{r}_i, \mathfrak{r}_{i+1}, \dots, \mathfrak{r}_j$ are all grey. Obviously, $\mathfrak{r}_j$ is a subinterval of $\mathfrak{r}_i$. Hence, $[x_j^+, y_j^+]$ is a subinterval of $[x_i^+, y_i^+]$. Therefore,

$$[x_i^{\sigma_1}, y_i^{\sigma_1}] \Rrightarrow_M [x_j^{\sigma_2}, y_j^{\sigma_2}] \tag{5.2}$$

for any choice of the "signs" $\sigma_1, \sigma_2 \in \{+, -\}$. Since $y_j \neq 1_F$, it follows easily that $\mathfrak{r}_{j+1}$ is white. Hence, (5.2) clearly allows us to replace the "grey segment" $\mathfrak{r}_i \twoheadrightarrow_{M^*} \mathfrak{r}_{i+1} \twoheadrightarrow_{M^*} \cdots \twoheadrightarrow_{M^*} \mathfrak{r}_j$ of (5.1) by a white part. For example, if $\mathfrak{r}_{i-1} \twoheadrightarrow_{M^*} \mathfrak{r}_i$ is an $\mathfrak{r}_{i-1} \overset{\mathrm{up}}{\twoheadrightarrow}_{M^*} \mathfrak{r}_i$, then we replace $\mathfrak{r}_i$ by $[x_i^+, y_i^+]$. If, say, $\mathfrak{r}_j \twoheadrightarrow_{M^*} \mathfrak{r}_{j+1}$ is an $\mathfrak{r}_j \overset{\mathrm{up}}{\twoheadrightarrow}_{M^*} \mathfrak{r}_{j+1}$, then we replace $\mathfrak{r}_j$ by $[x_j^-, y_j^-]$. After these two replacements, (5.2) allows us to get rid of the grey segment in question. We can do the same for all grey segments of (5.1).

Based on these considerations, we can assume that no grey interval occurs in (5.1). Therefore $\mathfrak{r}_i$ is bicolored. Let us assume that $y_i \in F$; that is, we assume that $\mathfrak{r}_i$ is a [white, grey] interval. If it is not the top of F, then $[y_i^-, y_i]$ is an a- or b-colored subinterval of $\mathfrak{r}_i$ and we are ready. If y_i is the top of F, then it has exactly two lower covers, whence $\mathfrak{r}_i$ still contains an a- or b-colored subinterval, and we are ready again. The subscript i plays no special role. Hence, it suffices to show that there is an $\ell \in \{0, \dots, k\}$ such that

$$\mathfrak{r}_\ell \text{ has a [white, grey] subinterval.} \tag{5.3}$$

Using (5.3), we assume that $\mathfrak{r}_i$ is a [grey, white] interval, that is, $y_i \notin F$ and $x_i \in F$. Then $\mathfrak{r}_{i-1} \twoheadrightarrow_{M^*} \mathfrak{r}_i$ is an up congruence-perspectivity, since otherwise $x_{i-1}, y_i \in M$ would imply that $x_i \in M$. Let $z_0 = y_{i-1} \wedge x_i$. If z_0 is grey, then $[x_{i-1}, z_0]$ is a [white, grey] subinterval of $\mathfrak{r}_{i-1}$, and we are ready by (5.3). So, we assume that z_0 is white.

Next, take a maximal chain $z_0 \prec z_1 \prec \cdots \prec z_m = y_{i-1}$ in $[z_0, y_{i-1}]$. Let $z_\ell' := z_\ell \vee x_i$ for $\ell = 0, \dots, m$. By semimodularity, $x_i = z_0' \preceq z_1' \preceq \cdots \preceq z_m' = y_i$. Since $z_0' = x_i$ is grey and $z_m' = y_i$ is white, there is a subscript $s \in \{1, \dots, m\}$ such that z_{s-1}' is grey, z_s' is white and $z_{s-1}' \prec z_s'$. Since

$z_s \vee z'_{s-1} = z_s \vee z'_{s-1} \vee x_i = z_s \vee x_i \vee z'_{s-1} = z'_s \vee z'_{s-1} = z'_s$, we see that $[z_{s-1}, z_s] \overset{\text{up}}{\twoheadrightarrow}_{M^*} [z'_{s-1}, z'_s]$. We infer from Lemma 4.4 that z_{s-1} is grey and z_s is white. Consequently, $[x_{i-1}, z_{s-1}]$ is a [white, grey] subinterval of $\mathfrak{r}_{i-1}$, and we are ready by (5.3). $\square$

The next construction is well-known and quite easy. Let S be a 4-cell of a planar lattice M. (Semimodularity is not assumed here.) Replace this 4-cell by a copy of M_3, often called *diamond*, the five-element nondistributive modular lattice. This means that we insert a new element, which is called the *eye*, see G. Grätzer and E. Knapp [13]. We say that the new lattice $M^{\circledcirc}$ is obtained from M by *adding an eye* to the 4-cell S. For example, this is the way how S_8 is obtained from S_7, see Figure 2. This example indicates that this construction destroys slimness. However, by G. Grätzer and E. Knapp [13, Lemma 2],

$$M \text{ is semimodular iff } M^{\circledcirc} \text{ is semimodular.} \tag{5.4}$$

Assume that $\gamma\colon \mathrm{Pri}(M) \to (Q;\nu)$ is a quasi-coloring of a planar lattice M. Let S be a 4-cell of M colored by (a, b). Define $\tau = \mathrm{quo}_Q\big(\nu \cup \{(a, b), (b, a)\}\big) \in \mathrm{Quo}\, Q$. Note that this time the "elementary quasiordering" mentioned at the end of Section 1 is $\mathrm{quo}_Q\big(\{(a, b), (b, a)\}\big)$. We add an eye to the covering square S; the lattice we obtain is denoted by $M^{\circledcirc}$. We extend γ to a map

$$\gamma^{\circledcirc}\colon \mathrm{Pri}(M^{\circledcirc}) \to (Q;\tau), \text{ where } \mathfrak{p} \mapsto \begin{cases} \gamma(\mathfrak{p}), & \text{if } \mathfrak{p} \in \mathrm{Pri}(M);\\ a, & \text{otherwise.}\end{cases}$$

Note that $\mathrm{Pri}(M) \subseteq \mathrm{Pri}(M^{\circledcirc})$ and $\gamma^{\circledcirc}$ is an extension of γ.

Lemma 5.2 (Eye Lemma)**.** *For a planar lattice M, the map $\gamma^{\circledcirc}$ is a quasi-coloring.*

Proof. If $\mathfrak{s} \in \mathrm{Pri}(M^{\circledcirc}) - \mathrm{Pri}(M)$, then there is a $\mathfrak{s}' \in \mathrm{Pri}(M)$ such that $\gamma^{\circledcirc}(\mathfrak{s}') = \gamma^{\circledcirc}(\mathfrak{s})$, $\mathfrak{s}'$ is transposed to $\mathfrak{s}$ and, therefore, $\mathrm{con}_{M^{\circledcirc}}(\mathfrak{s}') = \mathrm{con}_{M^{\circledcirc}}(\mathfrak{s})$. Thus to verify (C1) and (C2) for $\gamma^{\circledcirc}$, is suffices to deal with prime intervals of M.

In order to show that $\gamma^{\circledcirc}$ satisfies (C1), assume that $\mathfrak{p}, \mathfrak{q} \in \mathrm{Pri}(M)$ such that $\big(\gamma^{\circledcirc}(\mathfrak{p}), \gamma^{\circledcirc}(\mathfrak{q})\big) \in \tau$. We have to show that $\mathrm{con}_{M^{\circledcirc}}(\mathfrak{p}) \le \mathrm{con}_{M^{\circledcirc}}(\mathfrak{q})$. By transitivity, we can assume that $\big(\gamma^{\circledcirc}(\mathfrak{p}), \gamma^{\circledcirc}(\mathfrak{q})\big) = \big(\gamma(\mathfrak{p}), \gamma(\mathfrak{q})\big) \in \nu \cup \{(a, b), (b, a)\}$. If $\big(\gamma(\mathfrak{p}), \gamma(\mathfrak{q})\big) \in \nu$, then the fact that γ is a quasi-coloring yields that $\mathrm{con}_M(\mathfrak{p}) \le \mathrm{con}_M(\mathfrak{q})$. Then $\mathfrak{q} \Rrightarrow_M \mathfrak{p}$ implies that $\mathfrak{q} \Rrightarrow_{M^{\circledcirc}} \mathfrak{p}$, whence $\mathrm{con}_{M^{\circledcirc}}(\mathfrak{p}) \le \mathrm{con}_{M^{\circledcirc}}(\mathfrak{q})$. If $\big(\gamma(\mathfrak{p}), \gamma(\mathfrak{q})\big) \in \{(a, b), (b, a)\}$, then since γ is a quasi-coloring, both $\mathfrak{p}$ and $\mathfrak{q}$ are congruence-equivalent to appropriate edges of the just inserted diamond. Consequently, $\mathrm{con}_{M^{\circledcirc}}(\mathfrak{p}) \le \mathrm{con}_{M^{\circledcirc}}(\mathfrak{q})$ follows again. Thus $\gamma^{\circledcirc}$ satisfies (C1).

Next, to show that $\gamma^{\circledcirc}$ satisfies (C2), assume that $\mathfrak{p}, \mathfrak{q} \in \mathrm{Pri}(M)$ such that $\mathrm{con}_{M^{\circledcirc}}(\mathfrak{p}) \ge \mathrm{con}_{M^{\circledcirc}}(\mathfrak{q})$. Using (2.3), we obtain that there are $\mathfrak{r}_i = [x_i, y_i] \in \mathrm{Pri}(M^{\circledcirc})$, $i = 0, \dots, k$, such that

$$\mathfrak{p} = \mathfrak{r}_0 \twoheadrightarrow_{M^{\circledcirc}} \mathfrak{r}_1 \twoheadrightarrow_{M^{\circledcirc}} \cdots \twoheadrightarrow_{M^{\circledcirc}} \mathfrak{r}_k = \mathfrak{q}. \tag{5.5}$$

Obviously, $x_i < y_i$, for $i = 0, \dots, k$. Let e denote the unique element of $M^{\circledcirc} - M$, the "eye" inserted into S. Intervals whose bottom or top is e will be called *e-critical.* Let $\mathfrak{r}_i$ be the first e-critical interval. (The case when (5.5) contains no e-critical intervals will be considered later.) Let $j \in \{i, \dots, k\}$ be the largest subscript such that $\mathfrak{r}_i, \dots, \mathfrak{r}_j$ are e-critical. Clearly,

$$\mathfrak{p} \Rrightarrow_M \mathfrak{r}_{i-1}. \tag{5.6}$$

By duality, we can assume that $\mathfrak{r}_{i-1} \overset{\text{up}}{\twoheadrightarrow} \mathfrak{r}_i$. Since e is a doubly irreducible element, it cannot be y_i. Hence, $x_i = e$, $y_i = y_{i-1} \vee e$ and $x_{i-1} \le e$. Since $e \notin \{x_{i-1}, y_i\}$, we see that $x_{i-1} \le 0_S$ and $1_S \le y_i$. Extend the chain $\{x_{i-1}, 0_S, e, 1_S, y_i\}$ to a maximal chain C of $M^{\circledcirc}$. This chain cuts $M^{\circledcirc}$ into a left side and a right side, see D. Kelly and I. Rival [26, Lemma 1.2]. The intersection of these sides is C. Suppose $y_{i-1} \parallel e$. Then we choose the corner w_i of S opposite to y_{i-1}. That is, if y_{i-1} is on the left of C, then let $w_i = w^R_S$, and let $w_i = w^L_S$ if y_{i-1} is on the right of C. It is easy to see (based on D. Kelly and I. Rival [26, Lemma 1.2]) that $y_i = y_{i-1} \vee e = y_{i-1} \vee w^R_S$. That is, $\mathfrak{r}_{i-1} \overset{\text{up}}{\twoheadrightarrow}_M [w_i, y_i]$. Next, assume that y_{i-1} is comparable with e. Then we can choose $w_i \in \{w^L_S, w^R_S\}$ arbitrarily. Since $y_{i-1} = y_i$, and $x_{i-1} \le 0_S$, we have $\mathfrak{r}_{i-1} \overset{\text{up}}{\twoheadrightarrow}_M [w_i, y_i]$ again.

By the definition of j, all the congruence-perspectivities (if there are any) between $\mathfrak{r}_i$ and $\mathfrak{r}_j$ are down congruence-perspectivities. Hence, $y_j \le y_i$ and $x_j = e$. Let $w_j = w_i$, and observe that $\mathfrak{r}_{i-1} \Rrightarrow_M [w_j, y_j]$. This and (5.6) imply that

$$\mathfrak{p} \Rrightarrow_M [w_j, y_j]. \tag{5.7}$$

Assume that $\mathfrak{r}_j \twoheadrightarrow_{M^{\circledcirc}} \mathfrak{r}_{j+1}$ is an up congruence-perspectivity. Then the meet-irreducibility of e yields that $y_j \wedge x_{j+1} \ge 1_S$. Hence, $[w_j, y_j] \overset{\text{up}}{\twoheadrightarrow}_M r_{j+1}$, which together with (5.7) yields that $\mathfrak{p} \Rrightarrow_M r_{j+1}$. Let us replace the initial segment of (5.5) preceding $\mathfrak{r}_{j+1}$ by a sequence witnessing $\mathfrak{p} \Rrightarrow_M \mathfrak{r}_{j+1}$ in M. So, in case of $\mathfrak{r}_j \overset{\text{up}}{\twoheadrightarrow}_{M^{\circledcirc}} \mathfrak{r}_{j+1}$, we can reduce the number of e-critical intervals in (5.5).

Next, assume that $\mathfrak{r}_j \overset{\text{dn}}{\twoheadrightarrow}_{M^{\circledcirc}} \mathfrak{r}_{j+1}$. Let us take a maximal chain that extends the chain $\{x_{j+1}, 0_S, e, 1_S, y_j\}$, and fix a $w'_j \in \{w^L_S, w^R_S\}$ such that y_{j+1} and w'_j are on opposite sides of this maximal chain. Keeping $e \wedge y_{j+1} = x_{j+1}$ in mind and using D. Kelly and I. Rival [26, Lemma 1.2] again, we derive that $w'_j \wedge y_{j+1} = x_{j+1}$. Then $[w'_j, y_j] \overset{\text{dn}}{\twoheadrightarrow}_M \mathfrak{r}_{j+1}$. If $w'_j = w_j$, then (5.7) implies $\mathfrak{p} \Rrightarrow_M \mathfrak{r}_{j+1}$ again, and we reduce the number of e-critical intervals in (5.5) by improving the initial segment of (5.5) the same way as previously.

We are left with the most complex case when $w'_j \ne w_j$. Then we consider the sequence

$$[w'_j, y_j] \twoheadrightarrow_{M^{\circledcirc}} \mathfrak{r}_{j+1} \twoheadrightarrow_{M^{\circledcirc}} \cdots \twoheadrightarrow_{M^{\circledcirc}} \mathfrak{r}_k = \mathfrak{q}. \tag{5.8}$$

This sequence has fewer e-critical intervals than the original (5.5).

For $I, J \in \operatorname{Int}(M^{\circledcirc})$, let us say that the interval I $(\gamma^{\circledcirc}, M)$-*majorizes* the interval J if we can fix a maximal chain $C(I)$ in $I \cap M$ and a maximal chain

$C(J)$ in $J \cap M$ such that for each $\mathfrak{r} \in \operatorname{Pri}\bigl(C(J)\bigr)$ there exists an $\mathfrak{r}' \in \operatorname{Pri}\bigl(C(I)\bigr)$ with $\gamma^{\circledcirc}(\mathfrak{r}') \geq_\tau \gamma^{\circledcirc}(\mathfrak{r})$.

We claim that $[w_j, y_j]$, the second interval in (5.7), $(\gamma^{\circledcirc}, M)$-majorizes the first interval of (5.8), $[w_j', y_j]$. To see this, let C be a maximal chain in $[1_S, y_j]$. Define $C\bigl([w_j, y_j]\bigr) := C \cup \{w_j\}$ and $C\bigl([w_j', y_j]\bigr) := C \cup \{w_j'\}$. The only difference between $\operatorname{Pri}\bigl(C([w_j', y_j])\bigr)$ and $\operatorname{Pri}\bigl(C([w_j, y_j])\bigr)$ is that $[w_j, 1_S]$ is replaced by $[w_j', 1_S]$. The pair of their $\gamma^{\circledcirc}$-colors, (a, b) or (b, a), belongs to τ. Hence, $[w_j, y_j]$ $(\gamma^{\circledcirc}, M)$-majorizes $[w_j', y_j]$.

Thus, instead of the original sequence (5.5), now we have a congruence-projectivity in M, (5.7), and a new sequence, (5.8), such that the last member of (5.7) $(\gamma^{\circledcirc}, M)$-majorizes the first member of the new sequence, and the new sequence has fewer e-critical intervals than the original one. Iterating the above procedure to the new sequence, finally we receive a finite number $t \in \mathbb{N}_0$ of congruence-projectivities

$$\mathfrak{p} = I_0' \Rrightarrow_M I_1,\ I_1' \Rrightarrow_M I_2,\ I_2' \Rrightarrow_M I_3,\ \ldots,\ I_t' \Rrightarrow_M I_{t+1} = \mathfrak{q} \tag{5.9}$$

such that I_i $(\gamma^{\circledcirc}, M)$-majorizes I_i' for $i = 1, \ldots, t$. Observe that this assumption works (with $t = 0$) also for the case when (5.5) contains no e-critical interval. The fixed chains for $(\gamma^{\circledcirc}, M)$-majorizations will be denoted by $C(I_i)$ and $C(I_i')$, respectively.

Next, we define intervals $\mathfrak{s}_{t+1} = \mathfrak{q}, \mathfrak{s}_t', \mathfrak{s}_t, \mathfrak{s}_{t-1}', \mathfrak{s}_{t-1}, \ldots, \mathfrak{s}_1, \mathfrak{s}_0'$ such that $\mathfrak{s}_i \in \operatorname{Pri}\bigl(C(I_i)\bigr)$ and $\mathfrak{s}_i' \in \operatorname{Pri}\bigl(C(I_i')\bigr)$ for all meaningful subscripts. We know that $\mathfrak{s}_{t+1} = \mathfrak{q}$. Let $0 \leq i \leq t$, and suppose that $\mathfrak{s}_{i+1}$ is already defined. We get from $I_i' \Rrightarrow_M I_{i+1}$ that

$$\bigvee_{\mathfrak{r} \in \operatorname{Pri}(C(I_i'))} \operatorname{con}_M(\mathfrak{r}) = \operatorname{con}_M(I_i') \geq \operatorname{con}_M(I_i) \geq \operatorname{con}_M(\mathfrak{s}_{i+1})$$

holds in $\operatorname{Con} M$. Hence, (2.1) together with (2.2) yield the existence of an $\mathfrak{s}_i' \in \operatorname{Pri}\bigl(C(I_i')\bigr)$ such that $\operatorname{con}_M(\mathfrak{s}_i') \geq \operatorname{con}_M(\mathfrak{s}_{i+1})$. Since γ is a quasi-coloring, we have that $\gamma(\mathfrak{s}_i') \geq_\nu \gamma(\mathfrak{s}_{i+1})$. But $\gamma^{\circledcirc}$ extends γ and $\tau \supseteq \nu$, so we get that

$$\gamma^{\circledcirc}(\mathfrak{s}_i') \geq_\tau \gamma^{\circledcirc}(\mathfrak{s}_{i+1}). \tag{5.10}$$

Since I_i $(\gamma^{\circledcirc}, M)$-majorizes I_i', we can choose an $\mathfrak{s}_i \in \operatorname{Pri}\bigl(C(I_i)\bigr)$ such that

$$\gamma^{\circledcirc}(\mathfrak{s}_i) \geq_\tau \gamma^{\circledcirc}(\mathfrak{s}_i'). \tag{5.11}$$

Finally, $\mathfrak{s}_0' = \mathfrak{p}$ since $I_0' = \mathfrak{p}$ has no other prime subinterval. Taking $\mathfrak{s}_0' = \mathfrak{p}$, $\mathfrak{s}_{t+1} = \mathfrak{q}$, and (5.10) and (5.11) for all meaningful subscripts into account, the desired inequality $\gamma^{\circledcirc}(\mathfrak{p}) \geq_\tau \gamma^{\circledcirc}(\mathfrak{q})$ follows by transitivity. Thus $\gamma^{\circledcirc}$ satisfies (C2). $\square$

Next, assume that M is a planar semimodular lattice, and S is a slim 4-cell of M. Add a fork to M at S; we get M^*. Then, in the place of S, a copy of S_7 appears. For example, see Figure 3. By adding an eye to the right upper 4-cell of this S_7 we obtain the lattice $M^\diamond := (M^*)^{\circledcirc}$. See Figure 3 for an illustration. We will say that $M^\diamond$ is obtained from M by *adding a (right) e-fork at S*. (Here

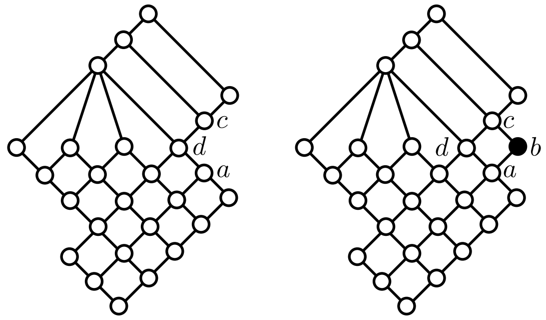

FIGURE 4. Adding a strong corner

"e" comes from "eye".) Note that the original S of M has changed to an S_8, whose left-right (west-east) orientation is relevant.

Assume that $\gamma\colon \operatorname{Pri}(M) \to (Q;\delta)$ is a quasi-coloring of M and, in addition to the previous paragraph, S is (a,b)-colored. In $\operatorname{Quo} Q$, let $\delta^\diamondsuit = \operatorname{quo}\bigl(\delta\cup\{(b,a)\}\bigr)$. Consider the map

$$\gamma^\diamondsuit\colon \operatorname{Pri}(M^\diamondsuit) \to (Q;\delta^\diamondsuit),\quad \mathfrak{p} \mapsto \begin{cases} \gamma^*(\mathfrak{p}), & \text{if } \mathfrak{p}\in \operatorname{Pri}(M^*) \text{ and } \gamma^*(\mathfrak{p})\in Q;\\ b, & \text{otherwise,}\end{cases}$$

see Figure 3. Lemmas 5.2 and 5.1 obviously imply the following statement.

Lemma 5.3 (E-fork Lemma)**.** *With the above assumptions and notations, $\gamma^\diamondsuit$ is a quasi-coloring.*

Let $M^\triangleright$ be an arbitrary lattice of finite length, and let b be a doubly irreducible element of $M^\triangleright$, that is, $b \in \operatorname{Ji}(M^\triangleright)\cap \operatorname{Mi}(M^\triangleright)$. Then $M = M^\triangleright - \{b\}$ is a sublattice of $M^\triangleright$. Since $b \notin \{0,1\}$ by definition, it has a unique lower cover a and a unique upper cover c. Assume that there is a unique element $d \in M$ such that $a \prec d \prec c$ in M. Assume also that $a \in \operatorname{Mi}(M)$ and $c \in \operatorname{Ji}(M)$. Then b is a *quasi-corner* of $M^\triangleright$. We say that $M^\triangleright$ is obtained from M by *adding a quasi-corner* to the "short chain" $a \prec d \prec c$. Clearly, we can add a quasi-corner to a short chain $a \prec d \prec c$ of M iff $a \in \operatorname{Mi}(M)$ and $c \in \operatorname{Ji}(M)$.

We are usually interested in the case when b is on the boundary of $M^\triangleright$, see Figure 4. Then a and c are also on the boundary. In this situation, b is called a *strong corner* of $M^\triangleright$ and we speak of *adding a strong corner* to a *short subchain $a \prec d \prec c$ of the boundary*, and we say that the short chain $a \prec d \prec c$ *admits a strong corner*. The notion of strong corners (but not using the adjective "strong") for planar semimodular lattices was introduced in G. Czédli and E. T. Schmidt [5]. While strong corners are necessarily weak corners and quasi-corners, the converse is not true. For example, $w^L_{S_7}$ in Figure 2 witnesses that, even in a rectangular lattice, weak corners are not necessarily strong corners. The notion of quasi-corners and that of weak corners are independent.

Lemma 5.4 (Corner Lemma). *Assume that $M^\triangleright$ is obtained from M by adding a quasi-corner b to a short chain $a \prec d \prec c$. Then*

(i) *M is a congruence-preserving sublattice of $M^\triangleright$;*

(ii) *if $\delta\colon \mathrm{Pri}(M) \to (Q;\nu)$ is a quasi-coloring, then its "natural extension"*

$$\delta^\triangleright\colon \mathrm{Pri}(M^\triangleright) \to (Q;\nu),\ \textit{where}\ \mathfrak{p} \mapsto \begin{cases} \delta(\mathfrak{p}), & \textit{for}\ \mathfrak{p} \in \mathrm{Pri}(M);\\ \delta\bigl([d,c]\bigr), & \textit{for}\ \mathfrak{p} = [a,b];\\ \delta\bigl([a,d]\bigr)], & \textit{for}\ \mathfrak{p} = [b,c], \end{cases}$$

is also a quasi-coloring.

Proof. In order to prove (i), we define

$$\gamma^\triangleright : \mathrm{Pri}(M^\triangleright) \to \bigl(\mathrm{Ji}(\mathrm{Con}\, M);\le\bigr), \quad \mathfrak{p} \mapsto \begin{cases} \mathrm{con}_M(\mathfrak{p}), & \text{if } \mathfrak{p} \in \mathrm{Pri}(M),\\ \mathrm{con}_M(d,c), & \text{if } \mathfrak{p} = [a,b],\\ \mathrm{con}_M(a,d), & \text{if } \mathfrak{p} = [b,c]. \end{cases}$$

It suffices to show that $\gamma^\triangleright$ is a coloring. Indeed, then Lemma 3.1 together with $\mathrm{Pri}(M) \subseteq \mathrm{Pri}(M^\triangleright)$ imply the present statement. It is easy to see that $\gamma^\triangleright$ satisfies (C1). To prove that it satisfies (C2), assume that $\mathfrak{p}, \mathfrak{q} \in \mathrm{Pri}(M^\triangleright)$ such that $\mathrm{con}_{M^\triangleright}(\mathfrak{p}) \ge \mathrm{con}_{M^\triangleright}(\mathfrak{q})$. Like in case of (Eye) Lemma 5.2 (see the first paragraph of its proof), we can assume that $\mathfrak{p}, \mathfrak{q} \in \mathrm{Pri}(M)$. Using (2.3) we obtain that there are a $k \in \mathbb{N}_0$ and $\mathfrak{r}_i = [x_i, y_i] \in \mathrm{Pri}(M^\triangleright)$, $i = 0, \dots, k$, such that

$$\mathfrak{p} = \mathfrak{r}_0 \twoheadrightarrow_{M^\triangleright} \mathfrak{r}_1 \twoheadrightarrow_{M^\triangleright} \cdots \twoheadrightarrow_{M^\triangleright} \mathfrak{r}_k = \mathfrak{q}. \tag{5.12}$$

For $\mathfrak{r} \in \mathrm{Int}(M^\triangleright)$, let $\mathrm{Int}_M(\mathfrak{r})$ stand for the collection of all those subintervals $\mathfrak{s}$ of $\mathfrak{r}$ for which $0_\mathfrak{s}, 1_\mathfrak{s} \in M$. We claim that, for $i = 0, 1 \dots, k$,

$$\mathrm{con}_M(\mathfrak{p}) \ge \mathrm{con}_M(\mathfrak{r}_i') \text{ for all } \mathfrak{r}_i' \in \mathrm{Int}_M(\mathfrak{r}_i). \tag{5.13}$$

Since $\mathrm{Int}_M(\mathfrak{q}) = \{\mathfrak{q}\}$, (5.13) will imply $\gamma^\triangleright(\mathfrak{p}) = \mathrm{con}_M(\mathfrak{p}) \ge \mathrm{con}_M(\mathfrak{r}_k) = \mathrm{con}_M(\mathfrak{q}) = \gamma^\triangleright(\mathfrak{q})$ and, therefore, (C2) for $\gamma^\triangleright$. Hence, it suffices to prove (5.13).

The validity of (5.13) for $i = 0$ needs no proof. Assume that $i < k$ and (5.13) holds for $0, \dots, i$; we want to show that it holds for $i+1$. We can assume that $\mathfrak{r}_i \twoheadrightarrow_{M^\triangleright} \mathfrak{r}_{i+1}$ is a parallel perspectivity, since otherwise $\mathfrak{r}_{i+1}$ is a subinterval of $\mathfrak{r}_i$ and (5.13) for $i+1$ follows evidently from $\mathrm{Int}_M(\mathfrak{r}_i) \supseteq \mathrm{Int}_M(\mathfrak{r}_{i+1})$. By duality, we can assume that $\mathfrak{r}_i \twoheadrightarrow_{M^\triangleright} \mathfrak{r}_{i+1}$ is a parallel up-perspectivity $\mathfrak{r}_i \overset{\mathrm{up}}{\twoheadrightarrow}_{M^\triangleright} \mathfrak{r}_{i+1}$. That is,

$$y_i \parallel x_{i+1},\ y_{i+1} = y_i \vee x_{i+1} \ \text{ and } \ x_i \le x_{i+1}. \tag{5.14}$$

There are four cases to consider.

Case 1. We assume that $\{x_i, y_i\} \subseteq M$ and $\{x_{i+1}, y_{i+1}\} \subseteq M$. Then we have $\mathrm{con}_M(\mathfrak{p}) \ge \mathrm{con}_M(\mathfrak{r}_i)$ by the induction hypothesis, $\mathrm{con}_M(\mathfrak{r}_i) \ge \mathrm{con}_M(\mathfrak{r}_{i+1})$ since $\mathfrak{r}_i \overset{\mathrm{up}}{\twoheadrightarrow}_M \mathfrak{r}_{i+1}$, and we conclude $\mathrm{con}_M(\mathfrak{p}) \ge \mathrm{con}_M(\mathfrak{r}_{i+1})$ by transitivity. Thus the inequality $\mathrm{con}_M(\mathfrak{p}) \ge \mathrm{con}_M(\mathfrak{r}_{i+1}')$ for all $\mathfrak{r}_{i+1}' \in \mathrm{Int}_M(\mathfrak{r}_{i+1})$ is clear.

Case 2. We assume that $\{x_i, y_i\} \subseteq M$ but $\{x_{i+1}, y_{i+1}\} \not\subseteq M$. Since b is join-irreducible in $M^{\triangleright}$, (5.14) gives that $b \neq y_{i+1}$. Hence, $b = x_{i+1}$. If $\mathfrak{r}'_{i+1} \in \mathrm{Int}_M(\mathfrak{r}_{i+1})$, then $\mathfrak{r}'_{i+1} \in \mathrm{Int}_M\bigl([c, y_{i+1}]\bigr)$ since c is the only upper cover of b in $M^{\triangleright}$. By (5.14) we have $\mathfrak{r}_i \overset{\mathrm{up}}{\twoheadrightarrow}_M [c, y_{i+1}]$. Hence, $\mathrm{con}_M(\mathfrak{p}) \geq \mathrm{con}_M\bigl([c, y_{i+1}]\bigr) \geq \mathrm{con}_M(\mathfrak{r}'_{i+1})$ follows easily like in the previous case.

Case 3. We assume that $\{x_i, y_i\} \not\subseteq M$ and $\{x_{i+1}, y_{i+1}\} \not\subseteq M$. Since $\mathfrak{r}_i \twoheadrightarrow_{M^{\triangleright}} \mathfrak{r}_{i+1}$ is a parallel congruence-perspectivity, $b \in \{x_i, y_i\}$ simultaneously with $b \in \{x_{i+1}, y_{i+1}\}$ is impossible. So this case cannot occur.

Case 4. We assume that $\{x_i, y_i\} \not\subseteq M$ but $\{x_{i+1}, y_{i+1}\} \subseteq M$. If $b = x_i$, then $c \leq y_i \wedge x_{i+1}$ follows from (5.14) and $b \in \mathrm{Mi}(M^{\triangleright})$. Hence, $[c, y_i] \overset{\mathrm{up}}{\twoheadrightarrow}_M \mathfrak{r}_{i+1}$, and $\mathrm{con}_M(\mathfrak{p}) \geq \mathrm{con}_M(\mathfrak{r}'_{i+1})$ follows easily like in the previous cases. So, we can assume that $y_i = b$. Let $t := x_{i+1} \vee a$. Since $a \in \mathrm{Mi}(M)$ and $x_{i+1} \not\leq a$ by (5.14), $t > a$. This and $y_{i+1} = b \vee x_{i+1} = c \vee x_{i+1}$ imply that

$$t = x_{i+1} \vee a = x_{i+1} \vee d \text{ and } y_{i+1} = t \vee c. \tag{5.15}$$

Let j be the smallest subscript such that $y_j = b$. Then $1 \leq j \leq i + 1$ since $y_{i+1} = b$. Since $y_{j-1} \neq b$ and $b \in \mathrm{Ji}(M^{\triangleright})$, $\mathfrak{r}_{j-1} \twoheadrightarrow_{M^{\triangleright}} \mathfrak{r}_j$ cannot be an up congruence-perspectivity. So $\mathfrak{r}_{j-1} \overset{\mathrm{dn}}{\twoheadrightarrow}_{M^{\triangleright}} \mathfrak{r}_j$. Assume first that this is a comparable congruence-perspectivity. Then $\mathfrak{r}_j$ is a subinterval of $\mathfrak{r}_{j-1}$, whence $a, c \in \mathfrak{r}_{j-1}$. So $[d, c] \in \mathrm{Int}_M(\mathfrak{r}_{j-1})$ and the induction hypothesis yields that $\mathrm{con}_M(\mathfrak{p}) \geq \mathrm{con}_M(d, c)$. Secondly, assume that $\mathfrak{r}_{j-1} \overset{\mathrm{dn}}{\twoheadrightarrow}_{M^{\triangleright}} \mathfrak{r}_j$ is a parallel congruence-perspectivity. Then $y_{j-1} > b$ gives that $y_{j-1} \geq c$. On the other hand, $x_{j-1} \not\geq b$ yields that $x_{j-1} \not\geq c$, whence $x_{j-1} \wedge c < c$ together with $c \in \mathrm{Mi}(M)$ implies $x_{j-1} \wedge c \leq d$. Therefore, we have that $\mathfrak{r}_{j-1} \overset{\mathrm{dn}}{\twoheadrightarrow}_M [x_{j-1} \wedge c, c] \overset{\mathrm{up}}{\twoheadrightarrow}_M [d, c]$. Since $\mathrm{con}_M(\mathfrak{p}) \geq \mathrm{con}_M(\mathfrak{r}_{j-1})$ by the induction hypothesis, $\mathrm{con}_M(\mathfrak{p}) \geq \mathrm{con}_M(d, c)$ follows again. So, $\mathrm{con}_M(\mathfrak{p}) \geq \mathrm{con}_M(d, c)$ in both cases. This implies by $[d, c] \overset{\mathrm{up}}{\twoheadrightarrow}_M [t, t \vee c] = [t, y_{i+1}]$ that $\mathrm{con}_M(\mathfrak{p}) \geq \mathrm{con}_M(t, y_{i+1})$.

Next, $[x_i, a] \in \mathrm{Int}_M(\mathfrak{r}_i)$, and $\mathrm{con}_M(\mathfrak{p}) \geq \mathrm{con}_M(x_i, a)$ by the induction hypothesis. Since $[x_i, a] \overset{\mathrm{up}}{\twoheadrightarrow}_M [x_{i+1}, t]$ by (5.15), we obtain that $\mathrm{con}_M(\mathfrak{p}) \geq \mathrm{con}_M(x_{i+1}, t)$. Thus $\mathrm{con}_M(\mathfrak{p}) \geq \mathrm{con}_M(x_{i+1}, t) \vee \mathrm{con}_M(t, y_{i+1}) = \mathrm{con}_M(\mathfrak{r}_{i+1}) \geq \mathrm{con}_M(\mathfrak{r}'_{i+1})$. This completes Case 4, the induction for (5.13), and the proof of part (i).

Finally, after letting $\eta\bigl([a, b]\bigr) := [d, c]$ and $\eta\bigl([b, c]\bigr) := [a, d]$, we conclude part (ii) from part (i) and Lemma 3.2. $\square$

Lemma 5.5 (Zero Lemma)**.** *Let M be a finite lattice. Let us assume that $\gamma\colon \mathrm{Pri}(M) \to (Q; \nu)$ is a quasi-coloring. Let M° be the lattice that we get from M by adding a new least element. With $e \notin Q$, we define $Q^\circ = Q \cup \{e\}$ and $\nu^\circ = \mathrm{quo}_{Q^\circ}(\nu) \in \mathrm{Quo}\, Q^\circ$. Then*

$$\gamma^\circ\colon \mathrm{Pri}(M^\circ) \to Q^\circ, \text{ where } \mathfrak{p} \mapsto \begin{cases} \gamma(\mathfrak{p}), & \text{if } \mathfrak{p} \in \mathrm{Pri}(M); \\ e, & \text{if } \mathfrak{p} = [0_{M^\circ}, 0_M], \end{cases}$$

is a quasi-coloring.

Clearly, e is incomparable with all elements of Q. Hence, if γ is a coloring, then so is γ°. We notice (but will not use) the following consequence of the above lemma: $\mathrm{Con}\, M^\circ$ is (isomorphic to) the direct product of $\mathrm{Con}\, M$ and the two-element chain.

Proof of Lemma 5.5. Since $\mathrm{con}_{M^\circ}(0_{M^\circ}, 0_M)$ collapses no old edge, and for any old edge $\mathfrak{p} \in \mathrm{Pri}(M)$, $\mathrm{con}_{M^\circ}(\mathfrak{p})$ does not collapse the new edge $[0_{M^\circ}, 0_M]$, the lemma follows trivially. □

6. The structure of rectangular lattices

Assume that M_1 and M_2 are lattices, and that $S := M_1 \cap M_2$ is a filter of M_1 and an ideal of M_2. Then we can form the classical gluing $M := M_1 \cup_S M_2$ of M_1 and M_2 over S, see G. Grätzer [10, I.2.4], G. Grätzer [9, Exercise I.4.20], or G. Grätzer [11, IV.2.1], for example. We know that M_1 is an ideal and M_2 is a filter of M, and $M = M_1 \cup M_2$. The following lemma is easy.

Lemma 6.1. *Gluing preserves semimodularity.*

Having no reference at hand, we give a proof.

Proof. With the previous notation, it is well-known that for $x \in M_1$ and $y \in M_2$,

$$x \le y \text{ iff there is a } z \in S \text{ with } x \le z \le y. \tag{6.1}$$

Assume that $a, b, c \in M$ with $a \prec b$; we have to show that $a \vee c \preceq b \vee c$. We get from (6.1) that $\{a, b\} \subseteq M_1$ or $\{a, b\} \subseteq M_2$.

Assume first that $\{a, b\} \subseteq M_2$. Assume that $c \notin M_2$ since otherwise the semimodularity of M_2 does the job. Then (6.1) yields a $d \in S$ such that $c \le d \le a \vee c$. Hence, the semimodularity of M_2 implies $a \vee c = a \vee d \preceq b \vee d = b \vee c$.

Secondly, assume that $\{a, b\} \subseteq M_1$. Assume $c \notin M_1$ since otherwise the semimodularity of M_1 does the job. Then (6.1) yields a $d \in S$ such that $a \le d \le a \vee c$. The semimodularity of M_1 yields that $d = a \vee d \preceq b \vee d$. Then the semimodularity of M_2 gives the desired $a \vee c = d \vee c \preceq b \vee d \vee c = b \vee a \vee c = b \vee c$. □

By G. Grätzer and E. Knapp [14, Lemma 4], the principal filters and the principal ideals that the weak corners of a rectangular lattice generate are chains; these chains will be called the *northwest boundary (chain)*, the *northeast boundary (chain)*, etc. For example, the northwest boundary of S_8 is a two-element chain while the southeast boundary is a three-element chain, see Figure 2. The following lemma offers an easy way to recognize rectangular lattices. For another characterization see Lemma 6.4 to follow.

Lemma 6.2 (G. Grätzer and E. Knapp [14, Lemma 6])**.** *A planar semimodular lattice with exactly on left weak corner and exactly one right weak corner is rectangular iff* 1 *is join-reducible and* 0 *is meet-reducible.*

By Lemma 4.1, all 4-cells of a slim lattice are slim.

Lemma 6.3. *Let M be a slim rectangular lattice. Assume that M^* is obtained from M by adding a fork to a 4-cell. Then M^* is a slim rectangular lattice.*

Proof. We know from Lemma 4.3 that M^* is a slim semimodular lattice. Clearly, M is a $\{0,1\}$-sublattice of M^*. Let $F = M^* - M$ denote the fork in question. Then the boundary of M^* is the union of the boundary of M and $\{0_{F_{\rm left}}, 0_{F_{\rm right}}\}$. Since $0_{F_{\rm left}}$ and $0_{F_{\rm right}}$ are meet-reducible, the weak corners of M^* and M are the same. Therefore, the statement follows from Lemma 6.2. □

The distinguished role of rectangularity is partly explained by the following two lemmas.

Lemma 6.4. *Let M be planar semimodular lattice.*

(i) *M is a rectangular lattice iff we cannot add a strong corner to it.*
(ii) *We can get a rectangular lattice M' from M by adding strong corners, one by one, in a finite number of steps.*

Proof. We fix a planar diagram of M (in the Euclidean plane). Omit all the eyes (middle elements of cover-preserving diamonds) from M. That is, we perform a (total) slimming in the sense of G. Grätzer and E. Knapp [13]. This way we obtain M', which is a slim semimodular lattice by (5.4) and Lemmas 4.1 and 4.2.

Since M' is derived from M by omitting "inner" elements (the eyes of cover-preserving diamonds), the boundary of M' is the same as the boundary of M. Let $x \notin \{0,1\}$ be an element of this common boundary. Observe that if we add an eye to a planar lattice, then the join-irreducible elements remain join-irreducible, and the join-reducible elements remain join-reducible. This fact and its dual yield that, for any x on the common boundary,

$$x \in \mathrm{Ji}(M) \text{ iff } x \in \mathrm{Ji}(M'), \text{ and } x \in \mathrm{Mi}(M) \text{ iff } x \in \mathrm{Mi}(M'). \tag{6.2}$$

Therefore, if $a \prec d \prec c$ is a short chain on the common boundary, then

$$a \prec d \prec c \text{ in } M \text{ admits a corner iff } a \prec d \prec c \text{ in } M' \text{ admits a corner.} \tag{6.3}$$

On the other hand, M' is a $\{0,1\}$-sublattice of M. Hence, it follows easily from (6.2) and Lemma 6.2 that

$$M \text{ is rectangular iff } M' \text{ is rectangular.} \tag{6.4}$$

Combining (6.3) and (6.4), we conclude that it is sufficient to prove the lemma for slim semimodular lattices. So from now on in the proof, let M be a slim semimodular lattice.

Firstly, we assume that M is rectangular. Assume by way of contradiction that a short subchain $a \prec d \prec c$ of the boundary of M admits a strong corner. Let, say, d be on the left boundary. Then so are a and c, the only lower cover and the only upper cover of d, since the left boundary is a maximal chain. As usual, the left weak corner of M is denoted by w_M^L. Recall from G. Grätzer and E. Knapp [14, Lemmas 3 and 4] that

$$\uparrow w_M^L \text{ and } \downarrow w_M^L \text{ are subchains of the left boundary,} \tag{6.5}$$

$$\uparrow w_M^L - \{1\} \subseteq \mathrm{Mi}(M) \text{ and } \downarrow w_M^L - \{0\} \subseteq \mathrm{Ji}(M). \tag{6.6}$$

Therefore, since w_F^L is the only doubly irreducible element on the left boundary and $a \in \mathrm{Mi}(M)$, $a \notin \downarrow w_M^L - \{0, w_M^L\}$. Hence, $a = 0$ or $a \geq w_M^L$. But $0 = w_M^L \wedge w_M^R$ is meet-reducible, which excludes $a = 0$. Hence, $a \geq w_M^L$. The dual argument yields $c \leq w_M^L$. So we get $c \leq w_M^L \leq a$, which contradicts $a < c$. Thus, if M is rectangular, then we cannot add a strong corner to it.

Conversely, assume that M is not rectangular. Combining G. Czédli and E. T. Schmidt [5, Thm. 12] with Lemma 6.3, we conclude that M can be obtained from an appropriate slim rectangular lattice M' by removing strong corners, one by one. In other words, starting from M and adding strong corners, one by one, we can get a slim rectangular lattice. This proves part (ii). Since M is a slim semimodular non-rectangular lattice, it follows that at least one strong corner can be added to it. This proves part (i). ☐

Lemma 6.5. *Let M_1 be a planar semimodular lattice with a filter F, and assume that F is a rectangular lattice. Then M_1 is a congruence-preserving sublattice of an appropriate rectangular lattice M_2 such that F is a filter of M_2 and* $\mathrm{Pri}(M_1) \subseteq \mathrm{Pri}(M_2)$.

Proof. We infer from Lemma 6.4 that by adding strong corners to M_1, one by one, we can get a rectangular lattice M_2. Obviously, $\mathrm{Pri}(M_1) \subseteq \mathrm{Pri}(M_2)$. (Corner) Lemma 5.4 yields that M_1 is a congruence-preserving sublattice of M_2. So, all we have to show is that F remains a filter by adding just one strong corner.

Let $F = \uparrow u$ in M_1. Assume by way of contradiction that we can add a strong corner b to a short subchain $a \prec d \prec c$ of the boundary of M_1 such that, in the enlarged lattice, $b \in \uparrow u$. Then $u \leq a \prec d \prec c$. Since the intersection of F with the boundary of M_1 is clearly a subset of the boundary of $F = \uparrow u$, we obtain that we can add a strong corner to the rectangular lattice F. But this contradicts Lemma 6.4 since F is a rectangular lattice. Hence, $F = \uparrow u$ in M_2, so F remains a filter. ☐

7. Congruence-preserving rectangular filters

Given a subset S of a lattice M, S is said to be a *congruence-determining subset*, if each congruence of M is determined by its restriction to S. If S is a sublattice, not just a subset, then we speak of a *congruence-determining sublattice*, see G. Grätzer [11, III.1.6]. Clearly, a sublattice S is congruence-determining iff $\varrho_S^M \colon \mathrm{Con}\, M \to \mathrm{Con}\, S$ is injective. For example,

Lemma 7.1 (G. Grätzer and J. B. Nation [21])**.** *Every maximal chain of a finite length semimodular lattice is a congruence-determining sublattice.*

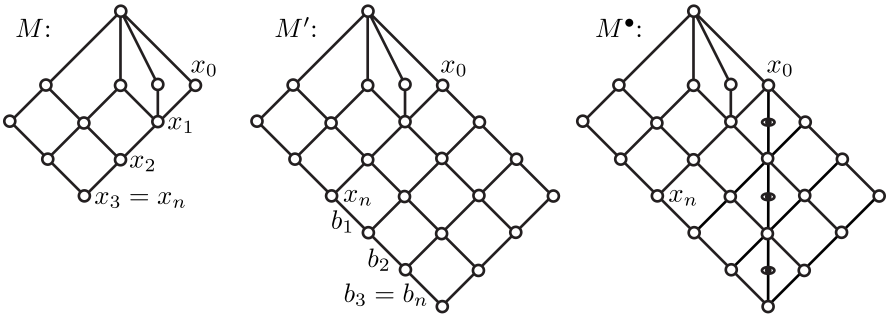

FIGURE 5. The construction of $M^\bullet$ for Lemma 7.2

By a *chain ideal*, we mean an ideal that is a chain.

Lemma 7.2. *Each rectangular lattice M is a congruence-preserving filter of an appropriate rectangular lattice $M^\bullet$ such that the southwest boundary of $M^\bullet$ is a congruence-determining chain ideal.*

Proof. Let M be a rectangular lattice. We claim that

$$\downarrow w_M^L \cup \downarrow w_M^R \text{ is a congruence-determining subset of } M. \tag{7.1}$$

Here $\downarrow w_M^L$ and $\downarrow w_M^R$ are the southwest boundary chain and the southeast boundary chain by (6.5). Let G denote the (total) slimming of M in the sense of G. Grätzer and E. Knapp [13]. That is, we obtain G from M by deleting eyes (the middle elements of cover-preserving M_3 sublattices) as long as possible. Since deleting an eye results in a congruence-determining (but not necessarily congruence-preserving) sublattice, we conclude that G is a congruence-determining sublattice of M. Furthermore, $\downarrow w_G^L = \downarrow w_M^L$, and $\downarrow w_G^R = \downarrow w_M^R$. Hence, to prove (7.1), it suffices to prove that

$$\downarrow w_G^L \cup \downarrow w_G^R \text{ is a congruence-determining subset of } G; \tag{7.2}$$

the progress is that G, as opposed to M, is a *slim* rectangular lattice. We know from (6.5) and Lemma 7.1 that the left boundary chain is $\uparrow w_G^L \cup \downarrow w_G^L$, and it is a congruence-determining sublattice of G. Hence, it suffices to show that for each $\mathfrak{p} \in \mathrm{Pri}(\uparrow w_G^L \cup \downarrow w_G^L)$ there is a $\mathfrak{q} \in \mathrm{Pri}(\downarrow w_G^L \cup \downarrow w_G^R)$ such that $\mathfrak{p} \twoheadleftarrow\!\!\twoheadrightarrow \mathfrak{q}$. For $\mathfrak{p} \in \mathrm{Pri}(\downarrow w_G^L)$, we can let $\mathfrak{q} := \mathfrak{p}$.

So assume that $\mathfrak{p}$ belongs to the northwest boundary chain, that is, $\mathfrak{p} \in \mathrm{Pri}(\uparrow w_G^L)$. By G. Czédli and E. T. Schmidt [4, Lemma 12], there is a sequence of prime intervals $\mathfrak{r}_0 = \mathfrak{p}, \mathfrak{r}_1, \dots, \mathfrak{r}_k$ such that $\mathfrak{r}_{i-1} \cup \mathfrak{r}_i$ is a covering square (that is, a 4-cell) for $i = 1, \dots, k$, and $\mathfrak{r}_k$ belongs to $\in \mathrm{Pri}(\uparrow w_G^R \cup \downarrow w_G^R)$, the right boundary chain. It follows that $\mathfrak{r}_{i-1} \twoheadrightarrow_G \mathfrak{r}_i$ for $i = 1, \dots, k$. Let $\mathfrak{q} := \mathfrak{r}_k$. Since $0_\mathfrak{p} \in \uparrow w_G^L$, $0_{\mathfrak{r}_0} = 0_\mathfrak{p} \in \mathrm{Mi}(G)$ by (6.6). Hence, $\mathfrak{r}_0 \twoheadrightarrow_G \mathfrak{r}_1$ actually is $\mathfrak{r}_0 \overset{\mathrm{dn}}{\twoheadrightarrow}_G \mathfrak{r}_1$. Hence, it follows from [4, Lemma 12] that all the $\mathfrak{r}_{i-1} \twoheadrightarrow_G \mathfrak{r}_i$ are down congruence-perspectivities. In particular, $\mathfrak{r}_{k-1} \overset{\mathrm{dn}}{\twoheadrightarrow}_G \mathfrak{r}_k = \mathfrak{q}$, whence $0_\mathfrak{q}$ is meet-reducible. Hence, the left-right dual of (6.6) implies that $0_\mathfrak{q} \notin \uparrow w_G^R$.

Consequently, $\mathfrak{q} \in \mathrm{Pri}(\downarrow w^{R}_{G})$, as requested. This proves (7.2) and, therefore, (7.1).

Utilizing (7.1), we are now close to constructing $M^\bullet$, see Figure 5. Let $w^{R}_{M} = x_0 \succ x_1 \succ \cdots \succ x_n = 0_M$ be the southeast boundary chain of M. Take its direct square, T. Apply the gluing to T and M along the southeast boundary chain of M. This way we get M', see Figure 5. Lemma 6.2 yields easily that M' is a rectangular lattice.

Let us denote the order $\bigl(\mathrm{Ji}(\mathrm{Con}(M)); \leq\bigr)$ by $(Q;\nu)$. Consider the natural coloring $\gamma\colon \mathrm{Pri}(M) \to (Q;\nu)$, defined by $\mathfrak{p} \mapsto \mathrm{con}_M(\mathfrak{p})$. Let $Q' = Q \cup \{b_n,\dots,b_1\}$ where the set $\{b_n,\dots,b_1\}$ is disjoint from Q. Let $\nu' = \mathrm{quo}_{Q'}(\nu) \in \mathrm{Quo}\, Q'$. Note that, modulo ν', every b_i is incomparable with all elements of Q. Clearly, M' is obtained from M by adding a new 0 (n times) and then adding a strong corner n^2 times. Hence, it follows easily from (Zero) Lemma 5.5 and (Corner) Lemma 5.4 that γ extends to a coloring $\gamma'\colon \mathrm{Pri}(M') \to (Q';\nu')$.

In the next step, add an eye to each diagonal square of T, see Figure 5. This way we get $M^\bullet$. It is rectangular, since so is M'. Hence, the southwest boundary of $M^\bullet$ is a chain ideal by (6.5). Let

$$\nu^\bullet := \mathrm{quo}_{Q'}\Bigl(\nu' \cup \bigcup_{1\leq i\leq n} \{(a_i,b_i),(b_i,a_i)\}\Bigr) = \mathrm{quo}_{Q'}\Bigl(\nu \cup \bigcup_{1\leq i\leq n} \{(a_i,b_i),(b_i,a_i)\}\Bigr).$$

A repeated use of (Eye) Lemma 5.2 yields a quasi-coloring $\gamma^\bullet\colon \mathrm{Pri}(M^\bullet) \to (Q;\nu^\bullet)$ that extends γ'. Note that $\gamma^\bullet$ also extends γ. Next, a straightforward application of Lemma 2.1 gives a coloring $\delta\colon \mathrm{Pri}(M^\bullet) \to (Q;\nu)$ such that δ is an extension of γ. Since $\mathrm{Pri}(M) \cap \mathrm{Pri}(M^\bullet) = \mathrm{Pri}(M)$, we conclude from Lemma 3.1 that $M^\bullet$ is a congruence-preserving extension of M.

Finally, the construction of $M^\bullet$ together with (6.5) and (6.6) makes it clear that the southwest boundary chain of $M^\bullet$ is the union of the chain intervals $[0_M, w^{L}_{M}]$ and $[0_{M^\bullet}, 0_M]$. These intervals are understood in $M^\bullet$. Observe that $[0_M, w^{L}_{M}]$ is $\downarrow w^{L}_{M}$ in M. Observe also that, due to the eyes added to T, each prime interval of $\downarrow w^{R}_{M}$ is congruence-equivalent to a prime interval of $[0_{M^\bullet}, 0_M]$. These facts, together with (7.1) and the fact that M is a congruence-determining (since congruence-preserving) sublattice of $M^\bullet$, imply that the southwest boundary chain of $M^\bullet$ is a congruence-determining subset of $M^\bullet$. $\square$

8. The Main Lemma

Given two quasiordered sets, $(A_1;\nu_1)$ and $(A_2;\nu_2)$, we say that $(A_2;\nu_2)$ is an *extension* of $(A_1;\nu_1)$ if $A_1 \subseteq A_2$ and $\nu_1 \subseteq \nu_2$. (Note that ν_1 may be a proper subset of the restriction of ν_2 to A_1.) The key construction of the paper is provided by the following lemma.

Lemma 8.1 (Main Lemma). *Let L be a rectangular lattice. Let $\gamma\colon \mathrm{Pri}(L) \to (Q;\nu)$ be a quasi-coloring. Assume that $(R;\zeta)$ is a quasiordered set that is an*

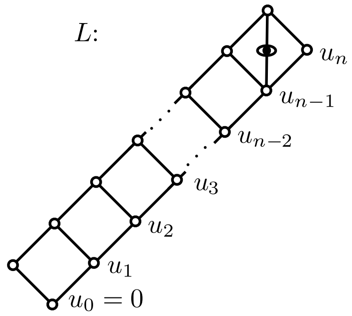

FIGURE 6. The "one-eyed ladder" of length $n+1$

extension of the quasiordered set $(Q;\nu)$. *Then there exist a rectangular lattice* K *and a quasi-coloring* $\delta\colon \mathrm{Pri}(K) \to (R;\zeta)$ *such that* L *is a filter of* K *and* δ *extends* γ.

Note that in this situation, $\mathrm{Pri}(L) \subseteq \mathrm{Pri}(K)$ since L is a filter. Hence, the restriction $\delta\rceil_{\mathrm{Pri}(L)}$ equals γ. Before proving Lemma 8.1, we illustrate its power with the following proof.

Second proof of Lemma 1.4. Let $(R;\zeta)$ stand for $\big(\mathrm{Ji}(D);\leq\big)$. Denoting the equality relation by ω, let $(Q;\nu)$ be $\big(\mathrm{Ji}(D);\omega_{\mathrm{Ji}(D)}\big)$. Denote $|\mathrm{Ji}(D)|$ by n. Define L, which is a "one-eyed ladder" of length $n+1$, by Figure 6. It is trivial (and follows easily from Lemmas 5.2 and 5.4) that $\mathrm{Ji}(\mathrm{Con}\,L)$ is an n-element antichain. Hence, it can be colored by $(Q;\nu)$. Applying Lemma 8.1, we get a rectangular lattice colored by the order $(R;\zeta)$, and $\mathrm{Con}\,K \cong D$ follows from (2.7). $\square$

Proof of Lemma 8.1. First we assume that the southwest boundary chain of L, denoted by C, is a congruence-determining sublattice. By (6.5), C is an ideal of L. Its top element will be denoted by 1_C. Let $\{a_1,\dots,a_n\}$ be the γ-colors of the prime intervals $\mathfrak{s}_1,\dots,\mathfrak{s}_n$ of C, listing them downwards, see (a part of) Figure 7. (Since only the southwest boundary of L is relevant, the interior of L is not drawn.) Note the a_i are not necessarily distinct, and $\{a_1,\dots,a_n\} \subset Q$ (proper subset) is possible. Let $\{c_1,\dots,c_m\} := Q - \{a_1,\dots,a_n\}$; possibly an empty set with $m=0$. We claim that

$$(\forall i \in \{1,\dots,m\})\ (\exists j \in \{1,\dots,n\})\ \big((c_i,a_j)\in\nu \text{ and } (a_j,c_i)\in\nu\big). \tag{8.1}$$

Let $i \in \{1,\dots,m\}$. Since γ is surjective by definition, there exists a $\mathfrak{p} \in \mathrm{Pri}(L)$ with $\gamma(\mathfrak{p}) = c_i$. Since C is congruence-determining, each congruence of L is the join of some $\mathrm{con}_L(\mathfrak{s}_j)$. Applying this fact to the join-irreducible $\mathrm{con}_L(\mathfrak{p})$, we get a $j \in \{1,\dots,n\}$ such that $\mathrm{con}_L(\mathfrak{p}) = \mathrm{con}_L(\mathfrak{s}_j)$. That is, $\mathrm{con}_L(\mathfrak{p}) \leq \mathrm{con}_L(\mathfrak{s}_j)$ and $\mathrm{con}_L(\mathfrak{s}_j) = \mathrm{con}_L(\mathfrak{p})$. Since $\gamma(\mathfrak{p}) = c_i$ and $\gamma(\mathfrak{s}_j) = a_j$, so (C2) implies (8.1).

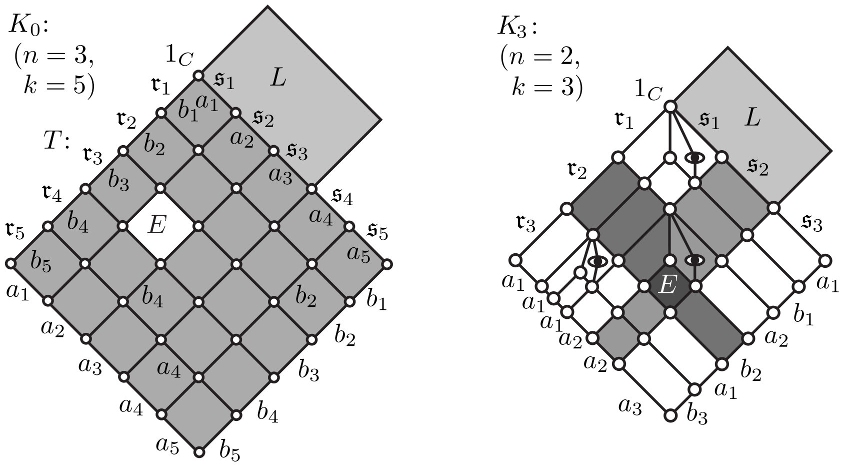

FIGURE 7. Examples for K_0 and K_3

Let $\{a_{n+1}, \dots, a_k\} := R - Q$; if it is an empty set then $k = n$. Let C_0 be a chain of length $n - k$ with prime intervals $\mathfrak{s}_{n+1}, \dots, \mathfrak{s}_k$, enumerating them downwards.

Let L' be the Hall-Dilworth gluing of C_0 and L over the singleton $\{0_L\} = \{1_{C_0}\}$. Let $C' = C \cup C_0$; it is a chain ideal of L' with prime intervals $\mathfrak{s}_1, \dots, \mathfrak{s}_k$. Define $\nu' = \mathrm{quo}_R(\nu) \in \mathrm{Quo}\, R$. Note that $\nu \subseteq \zeta$ implies that $\nu' \subseteq \zeta$. Extend the map γ from $\mathrm{Pri}(L)$ to $\mathrm{Pri}(L')$ as follows:

$$\gamma' \colon \mathrm{Pri}(L') \to (R; \nu'), \text{ where } \mathfrak{p} \mapsto \begin{cases} \gamma(\mathfrak{p}), & \text{if } \mathfrak{p} \in \mathrm{Pri}(L); \\ a_i, & \text{if } \mathfrak{p} = \mathfrak{s}_i \in \mathrm{Pri}(C_0). \end{cases}$$

Since L' is obtained from L by adding a new zero, one by one, $k - n$ times, we infer from (Zero) Lemma 5.5 that γ' is a quasi-coloring.

Next, let $S = R \cup \{b_1, \dots, b_k\}$ where the b_i are distinct new elements outside R. That is, our sets are as follows:

$$\begin{aligned} Q &= \{a_1, \dots, a_n, c_1, \dots, c_m\}, \\ R &= \{a_1, \dots, a_n, \dots, a_k, c_1, \dots, c_m\}, \\ S &= \{a_1, \dots, a_n, \dots, a_k, c_1, \dots, c_m, b_1, \dots, b_k\}. \end{aligned}$$

Take the direct square T of C'. It is a rectangular lattice. We identify the northeast boundary of T with C'. So C' becomes a filter of T. The prime intervals of the northwest boundary of T will be denoted by $\mathfrak{r}_1, \dots, \mathfrak{r}_k$ (enumerating them downwards). Let K_0 be the gluing of T and L' over C', see Figure 7. Note that $\downarrow 1_C$ in K_0 is T.

Let $\xi_0 := \mathrm{quo}_S(\nu) = \mathrm{quo}_S(\nu') \in \mathrm{Quo}\, S$. Extend γ' as follows:

$$\beta_0\colon \mathrm{Pri}(K_0) \to (S;\xi_0), \text{ where } \mathfrak{p} \mapsto \begin{cases} \gamma'(\mathfrak{p}), & \text{if } \mathfrak{p} \in \mathrm{Pri}(L'),\\ a_i, & \text{if } \mathfrak{p} \in \mathrm{Pri}(T) \text{ and } \mathfrak{p} \Lleftarrow\!\!\!\Rrightarrow_T \mathfrak{s}_i,\\ b_i, & \text{if } \mathfrak{p} \in \mathrm{Pri}(T) \text{ and } \mathfrak{p} \Lleftarrow\!\!\!\Rrightarrow_T \mathfrak{r}_i. \end{cases}$$

See Figure 7, where the colors are given only for some of the prime intervals. Instead of gluing T to L', we can get K_0 also in the following way. First, by adding new zeros k times, we add the southeast boundary of T. Then we add a left strong corner k^2 times. Thus, using (Zero) Lemma 5.5 k times and (Corner) Lemma 5.4 k^2 times, we infer that β_0 is a quasi-coloring.

Next, we linearly order $\{1,\dots,k\} \times \{1,\dots,k\}$: $(1,1) \lessdot (2,1) \lessdot (1,2) \lessdot (3,1) \lessdot (2,2) \lessdot (1,3) \lessdot (4,1) \lessdot \cdots \lessdot (k,k)$. That is, $(i_1,j_1) \lessdot (i_2,j_2)$ iff either $i_1+j_1 < i_2+j_2$, or $i_1+j_1 = i_2+j_2$ and $j_1 < j_2$. Let

$$\zeta \cap \{a_1,\dots,a_k\}^2 \text{ be listed as } \{(a_{i_1},a_{j_1}),\dots,(a_{i_t},a_{j_t})\} \tag{8.2}$$

such that $(i_1,j_1) \lessdot (i_2,j_2) \lessdot \cdots \lessdot (i_t,j_t)$.

Now we are ready to define rectangular lattices K_f, quasiorderings ξ_f and quasi-colorings $\beta_f\colon \mathrm{Pri}(K_f) \to (S;\xi_f)$ for $f = 0,1,\dots,t$ by induction as follows. The case $f = 0$ is already done. We will obtain K_f from K_{f-1} by adding an e-fork. The eyes of the previously added e-forks will play a special role. Notice that only T will change but L' will not, and L and L' will remain filters. Furthermore, $\downarrow 1_C$ will remain a rectangular lattice. Note also that β_f will be an extension of β_{f-1}, and K_f will be an extension of K_{f-1}.

Let $0 < f \le t$, and assume that the desired objects are already defined for $f-1$. Utilizing the quasi-coloring β_{f-1}, we select a (b_{j_f}, a_{i_f})-colored slim 4-cell E in K_{f-1} as follows. For $f = 1$, $j_f = 3$, and $i_f = 2$, see it in Figure 7. For $f = 4$, $j_f = 2$, and $i_f = 2$, see the dark grey 4-cell of K_3 in Figure 7. (Note that because of space considerations, K_3 in the figure originates from *another* K_0, not from the previous one.)

The exact definition of E is the following. Remove the eyes of the previously added e-forks. (There are $f-1$ eyes.) Then, starting from $\mathfrak{s}_{i_f}$ and proceeding to the southwest, there is a northeast-southwest row of adjacent 4-cells in K_{f-1} that reaches the southwest boundary. (See the light grey cells in Figure 7.) Similarly, starting from $\mathfrak{r}_{j_f}$ and going to the southeast there is a northwest-southeast row of adjacent 4-cells. (See the medium grey cells in Figure 7.) The intersection of these two rows determines a unique 4-cell E. At present, $\downarrow 1_C$ is slim; this follows easily from Lemma 4.3. Now we put back the $f-1$ eyes we omitted. Thanks to the $\lessdot$ arrangement, none of these eyes is in $\downarrow 1_E$. Hence, it follows from Lemma 4.1 that E is a slim 4-cell of K_{f-1}.

We obtain K_f from K_{f-1} by adding a right e-fork at E. We let $\xi_f = \mathrm{quo}_S\big(\xi_{f-1} \cup \{(a_{i_f}, b_{j_f})\}\big)$. Extend β_{f-1} to β_f according to (E-fork) Lemma 5.3, that is, let $\beta_f = \beta_{f-1}^{\diamondsuit}$. Then $\beta_f\colon \mathrm{Pri}(K_f) \to (S;\xi_f)$ is a quasi-coloring.

After t steps, we arrive at a lattice K_t and a quasi-coloring $\beta_t\colon \mathrm{Pri}(K_t) \to (S;\xi_t)$, where

$$\xi_t = \mathrm{quo}\big(\nu' \cup \{(a_i, b_j) : (a_i, a_j) \in \zeta\}\big).$$

We claim that, for $i = 1, \dots, k$ and $f = 0, \dots, t$ and with respect to β_f,

$$\text{there is a } (b_i, a_i)\text{-colored 4-cell in } K_f. \tag{8.3}$$

This is obvious for $f = 0$. To see that K_f inherits the validity of (8.3) from K_{f-1}, let F be a (b_i, a_i) colored 4-cell of K_{f-1}, with respect to β_{f-1}. If F is also a 4-cell of K_f, then it is a (b_i, a_i) colored 4-cell of K_f with respect to β_f since β_f extends β_{f-1}. Otherwise F in K_f is divided into two or four new 4-cells. Then there is a unique 4-cell F' of these two or four new cells such that $0_{F'} = 0_F$, and F' is (b_i, a_i) colored with respect to β_f. This shows (8.3).

Notice that the 4-cell provided by (8.3) is not slim in general. This explains that instead of using Lemma 5.3 together with its left-right dual, we are going to resort to Lemma 5.2.

It follows from (8.3) that we can fix an (a_i, b_i)-colored 4-cell F_i in K_t, for $i = 1, \dots, k$. Then we add an eye to F_i one by one, for $i = 1, \dots, k$. This way we get K'. Observe that L remains a filter in K'. The successive use of (Eye) Lemma 5.2 yields a quasi-coloring $\delta^o\colon \mathrm{Pri}(K') \to (S;\chi)$ where

$$\chi = \mathrm{quo}_S\Big(\underbrace{\nu' \cup \{(a_i, b_j) : (a_i, a_j) \in \zeta\}}_{\text{generates } \xi_t} \cup \underbrace{\bigcup_{1\le i\le k} \{(a_i, b_i), (b_i, a_i)\}}_{\text{generates } \mu}\Big). \tag{8.4}$$

Note that δ^o is still an extension of γ. Let $\mu = \mathrm{quo}_S\Big(\bigcup_{i=1}^k \{(a_i, b_i), (b_i, a_i)\}\Big)$. It is an equivalence relation on S. We claim that

$$\chi = \mathrm{quo}_S(\zeta \cup \mu). \tag{8.5}$$

To prove the "$\supseteq$" inclusion, let $(x, y) \in \zeta$. By (8.1) and $\nu \subseteq \nu'$, there are $i, j \in \{1, \dots, k\}$ such that

$$(x, a_i), (a_i, x), (y, a_j), (a_j, y) \in \nu'. \tag{8.6}$$

Transitivity, $\nu' \subseteq \zeta$ and (8.6) yield that $(a_i, a_j) \in \zeta$. Hence, $(a_i, b_j) \in \chi$. So using transitivity, $(b_j, a_j) \in \chi$, and (8.6) together with $\nu' \subseteq \chi$, we obtain that $(x, y) \in \chi$. This together with $\chi \supseteq \mu$ proves that $\chi \supseteq \mathrm{quo}_S(\zeta\cup\mu)$. To prove the reverse inclusion of (8.5), observe that if $(a_i, a_j) \in \zeta$, then $(a_i, b_j) \in \mathrm{quo}_S(\zeta\cup\mu)$ by transitivity. Hence, each pair occurring in (8.4) belongs to $\mathrm{quo}_S(\zeta\cup\mu)$. This fact, together with $\nu' \subseteq \zeta$, yields that $\chi \subseteq \mathrm{quo}_S(\zeta \cup \mu)$, proving (8.5).

Next, motivated by a more general result of G. Czédli and A. Lenkehegyi [3, Thms. 1.3 and 1.6], we define a surjective map

$$\alpha\colon (S;\chi) \to (R;\zeta), \text{ where } x \mapsto \begin{cases} x, & \text{if } x \in R,\\ a_i, & \text{if } x = b_i. \end{cases}$$

Note that the equivalence kernel of α is μ. We claim that

$$\text{for all } x, y \in S,\ \ (x, y) \in \chi \text{ iff } \big(\alpha(x), \alpha(y)\big) \in \zeta. \tag{8.7}$$

As the first step in proving (8.7), observe that $\big(z,\alpha(z)\big)\in\mu=\mu^{-1}$, for all $z\in S$. Assume that $\big(\alpha(x),\alpha(y)\big)\in\zeta$. Then $\big(x,\alpha(x)\big),\big(\alpha(y),y\big)\in\mu\subseteq\chi$, $\zeta\subseteq\chi$, and the transitivity of χ imply $(x,y)\in\chi$.

Conversely, assume that $(x,y)\in\chi$. We can assume that $x\neq y$. By (8.5), there is a sequence $z_0=x,z_1,\dots,z_h=y$ in S such that $(z_{i-1},z_i)\in\zeta\cup\mu$ for $i=1,\dots,h$. If $(z_{i-1},z_i)\in\zeta$, then $z_{i-1},z\in R$ and $\big(\alpha(z_{i-1}),\alpha(z_i)\big)=(z_{i-1},z_i)$ belongs to ζ. Otherwise $(z_{i-1},z_i)\in\mu$ and $\alpha(z_{i-1})=\alpha(z_i)$ yield that $\big(\alpha(z_{i-1}),\alpha(z_i)\big)\in\zeta$. Hence, $\big(\alpha(x),\alpha(y)\big)=\big(\alpha(z_0),\alpha(z_h)\big)\in\zeta$ by transitivity. This proves (8.7).

In the next step, we define $\delta':=\alpha\circ\delta^o$. Since δ^o extends γ and α acts identically in $Q\subseteq R$, the map δ' extends γ. From (8.7) and the fact that δ^o is a quasi-coloring we conclude in a straightforward way that the map $\delta'\colon \mathrm{Pri}(K')\to(R,\zeta)$, defined by $\mathfrak{p}\mapsto\alpha(\delta^o(\mathfrak{p}))$, is a quasi-coloring.

Next, Lemma 6.5 gives us a rectangular lattice K such that L is a filter of K, K is a congruence-preserving extension of K', and $\mathrm{Pri}(K')\subseteq\mathrm{Pri}(K)$. Using Lemma 3.2, we can extend δ' to a quasi-coloring $\delta\colon\mathrm{Pri}(K)\to(R,\zeta)$. Clearly, δ extends γ.

Finally, we drop the initial assumption that the southwest boundary of L is a congruence-determining sublattice. Then we start the proof by invoking Lemma 7.2, to get a rectangular lattice L^* such that L is a congruence-preserving filter of L^* and the southwest boundary of L^* is congruence-determining. Furthermore, $\mathrm{Pri}(L)\subseteq\mathrm{Pri}(L^*)$ since L is a filter of L^*. By Lemma 3.2, we can extend γ to a quasi-coloring $\gamma^*\colon\mathrm{Pri}(L^*)\to(Q;\nu)$. We already know that γ^* extends to an appropriate $\delta\colon\mathrm{Pri}(K)\to(R;\zeta)$. Clearly, δ extends γ and L is a filter of K. □

9. A left adjoint at work

In this section, D and E are finite distributive lattices, and $\varphi\colon D\to E$ is a $\{0,1\}$-lattice homomorphism. For $e\in E$, we let

$$A_e:=\{x\colon x\in D \text{ and } e\leq\varphi(x)\}=\varphi^{-1}(\mathord{\uparrow}e).$$

Since $1_D\in A_e$, A_e is never empty. We define

$$\eta\colon E\to D,\ \text{where}\ x\mapsto\bigwedge A_e. \tag{9.1}$$

Although the following lemma follows from the (Freyd) Adjoint Functor Theorem, see S. Mac Lane [27, p. 117] or P. A. Grillet [24, XVI.7.3], its proof for our particular case is quite short. So, for the reader's convenience, we will present a proof.

Lemma 9.1. *η is a left adjoint of φ. That is, for all $e\in E$ and $d\in D$, $\eta(e)\leq d$ iff $e\leq\varphi(d)$. For each $e\in E$, $\eta(e)$ is the smallest element of A_e. Furthermore, η is monotone, that is, if $e_1\leq e_2\in E$, then $\eta(e_1)\leq\eta(e_2)$.*

Proof. If $x, y \in A_e$, then $e \leq \varphi(x) \wedge \varphi(y) = \varphi(x \wedge y)$ yields that $x \wedge y \in A_e$. Hence, A_e is meet-closed, and $\eta(e) = \bigwedge A_e$ is indeed its smallest element. Assume that $e \leq \varphi(d)$. Then $d \in A_e$. But $\eta(e)$ is the smallest element of A_e, whence $\eta(e) \leq d$. Conversely, assume that $\eta(e) \leq d$. Then

$$\begin{aligned} e &\leq \bigwedge \bigl\{\varphi(x) : x \in D \text{ and } e \leq \varphi(x)\bigr\} \leq \varphi\Bigl(\bigwedge \bigl\{x : x \in D \text{ and } e \leq \varphi(x)\bigr\}\Bigr) \\ &= \varphi\bigl(\bigwedge A_e\bigr) = \varphi\bigl(\eta(e)\bigr) \leq \varphi(d). \end{aligned}$$

Hence, η is a left adjoint of φ. Assume that $e_1 \leq e_2 \in E$. Then $A_{e_1} \supseteq A_{e_2}$, so we get that $\eta(e_1) = \bigwedge A_{e_1} \leq \bigwedge A_{e_2} = \eta(e_2)$. This shows that η is monotone. □

Lemma 9.2. $\eta(\mathrm{Ji}(E)) \subseteq \mathrm{Ji}(D)$.

Proof. Assume that $e \in \mathrm{Ji}(E)$. We have to show that $d := \eta(e) \in \mathrm{Ji}(D)$. If $d = 0$, then $\eta(e) \leq 0$ together with Lemma 9.1 implies $e \leq \varphi(0) = 0$, contradicting $e \in \mathrm{Ji}(E)$. Hence, $d \neq 0$. Next, assume that $d_1, d_2 \in D$ such that $d = d_1 \vee d_2$. Since $\eta(e) \leq \eta(e) = d$, Lemma 9.1 yields that $e \leq \varphi(d) = \varphi(d_1 \vee d_2) = \varphi(d_1) \vee \varphi(d_2)$. Hence, $e \leq \varphi(d_i)$ for some $i \in \{1, 2\}$ by (2.1). So $d_i \in A_e$. Therefore, $d = \eta(e) \leq d_i$ by Lemma 9.1, and we obtain that $d = d_i$. Thus, $\eta(e) = d \in \mathrm{Ji}(D)$. □

To avoid confusion, the ordering of $\mathrm{Ji}(E)$ will often be denoted by $\leq_{\mathrm{Ji}(E)}$ or $\vec{\varpi}_{\mathrm{Ji}(E)}$. Similarly, $\leq_{\mathrm{Ji}(D)}$ or $\vec{\varpi}_{\mathrm{Ji}(D)}$ will denote the ordering of $\mathrm{Ji}(D)$. The *graph* of η, that is, the relation $\bigl\{(x, \eta(x)) : x \in E\bigr\}$, is denoted by $\dot{\eta}$. Then $\dot{\eta}^{-1} = \bigl\{(\eta(x), x) : x \in E\bigr\}$. We can assume, without loss of generality, that $\mathrm{Ji}(D)$ and $\mathrm{Ji}(E)$ are *disjoint*.

We define a quasiordered set $(R; \zeta)$ as follows:

$$R = \mathrm{Ji}(D) \cup \mathrm{Ji}(E), \quad \zeta = \mathrm{quo}_R(\dot{\eta} \cup \dot{\eta}^{-1} \cup \vec{\varpi}_{\mathrm{Ji}(D)} \cup \vec{\varpi}_{\mathrm{Ji}(E)}). \tag{9.2}$$

Clearly, $(R; \zeta)$ is an extension of $(\mathrm{Ji}(E), \leq) = (\mathrm{Ji}(E), \vec{\varpi}_{\mathrm{Ji}(E)})$.

We will need the following property of $(R; \zeta)$. Remember the $\mathrm{Ji}(E)$ and $\mathrm{Ji}(D)$ are *always* assumed to be disjoint.

Lemma 9.3. *Assume that $e \in \mathrm{Ji}(E)$ and $d \in \mathrm{Ji}(D)$. Then $(e, d) \in \zeta$ iff there exists an $e' \in \mathrm{Ji}(E)$ such that $e \leq_{\mathrm{Ji}(E)} e'$ and $\eta(e') \leq_{\mathrm{Ji}(D)} d$.*

Proof. Since $(e', \eta(e')) \in \dot{\eta} \subseteq \zeta$, $\vec{\varpi}_{\mathrm{Ji}(E)} \subseteq \zeta$, and $\vec{\varpi}_{\mathrm{Ji}(D)} \subseteq \zeta$, the "if" part trivially follows by transitivity.

To prove the converse direction, assume that $(e, d) \in \zeta$. By the definition or ζ, there are an $n \in \mathbb{N}_0$ and, for $i = 0, \ldots, n$, elements $x_i, y_i \in \mathrm{Ji}(E)$ and $z_i, t_i \in \mathrm{Ji}(D)$ such that

$$\begin{gathered} x_0 = e, \quad t_n = d, \\ x_i \leq_{\mathrm{Ji}(E)} y_i, \quad \eta(y_i) = z_i, \text{ and } \quad z_i \leq_{\mathrm{Ji}(D)} t_i \quad \text{ for } i = 0, \ldots, n, \text{ and} \\ \eta(x_{j+1}) = t_j \quad \text{for } j = 0, \ldots, n-1. \end{gathered}$$

By Lemma 9.1, η is monotone. Hence, $t_i = \eta(x_{i+1}) \leq_{\mathrm{Ji}(D)} \eta(y_{i+1}) = z_{i+1}$ for $i = 0, \dots, n-1$. Let $e' = y_0$. Then $e = x_0 \leq_{\mathrm{Ji}(E)} y_0 = e'$. Since $\eta(e') = z_0 \leq_{\mathrm{Ji}(D)} t_0 \leq_{\mathrm{Ji}(D)} z_1 \leq_{\mathrm{Ji}(D)} t_1 \leq_{\mathrm{Ji}(D)} z_2 \leq_{\mathrm{Ji}(D)} \cdots \leq_{\mathrm{Ji}(D)} t_n = d$, transitivity yields that $\eta(e') \leq_{\mathrm{Ji}(D)} d$. $\square$

For the terminology of the next statement, see the paragraph ending with (2.5).

Lemma 9.4. *The retraction map*

$$g\colon (R;\zeta) \to (\mathrm{Ji}(D); \leq_{\mathrm{Ji}(D)}), \text{ where } x \mapsto \begin{cases} x, & \text{if } x \in \mathrm{Ji}(D); \\ \eta(x), & \text{if } x \in \mathrm{Ji}(E), \end{cases}$$

is a homomorphism, and $\vec{\mathrm{Ker}}\, g \subseteq \zeta$.

Proof. By Lemma 9.2, $g(R) \subseteq \mathrm{Ji}(D)$. To show that g is a homomorphism, assume that $(x, y) \in \zeta$. Then there are $n \in \mathbb{N}_0$ and $z_0, z_1, \dots, z_n \in R$ such that $z_0 = x$, $z_n = y$, and $(z_i, z_{i+1}) \in \dot\eta \cup \dot\eta^{-1} \cup \vec\varpi_{\mathrm{Ji}(D)} \cup \vec\varpi_{\mathrm{Ji}(E)}$ for $i = 0, \dots, n-1$. It suffices to show that $g(z_i) \leq_{\mathrm{Ji}(D)} g(z_{i+1})$ for all these i since then the desired inequality $g(x) \leq_{\mathrm{Ji}(D)} g(y)$ follows by transitivity. If $(z_i, z_{i+1}) \in \dot\eta \cup \dot\eta^{-1}$, then $g(z_i) = g(z_{i+1})$ gives that $g(z_i) \leq_{\mathrm{Ji}(D)} g(z_{i+1})$. If $(z_i, z_{i+1}) \in \vec\varpi_{\mathrm{Ji}(D)}$, then $g(z_i) = z_i \leq_{\mathrm{Ji}(D)} z_{i+1} = g(z_{i+1})$. If $(z_i, z_{i+1}) \in \vec\varpi_{\mathrm{Ji}(E)}$, then $g(z_i) = \eta(z_i) \leq_{\mathrm{Ji}(D)} \eta(z_{i+1}) = g(z_{i+1})$ since η is monotone. Thus g is a homomorphism.

Next, assume that $(x, y) \in \vec{\mathrm{Ker}}\, g$, that is, $g(x) \leq_{\mathrm{Ji}(D)} g(y)$. If $x, y \in \mathrm{Ji}(D)$, then $(x, y) = \big(g(x), g(y)\big) \in \vec\varpi_{\mathrm{Ji}(D)} \subseteq \zeta$. If $x, y \in \mathrm{Ji}(E)$, then $\big(x, g(x)\big) = \big(x, \eta(x)\big) \in \dot\eta \subseteq \zeta$, $\big(g(x), g(y)\big) \in \vec\varpi_{\mathrm{Ji}(D)} \subseteq \zeta$, $\big(g(y), y\big) = \big(\eta(y), y\big) \in \dot\eta^{-1} \subseteq \zeta$, and transitivity imply $(x, y) \in \zeta$. If $x \in \mathrm{Ji}(E)$ and $y \in \mathrm{Ji}(D)$, then $\big(x, g(x)\big) = \big(x, \eta(x)\big) \in \dot\eta \subseteq \zeta$ and $\big(g(x), y\big) = \big(g(x), g(y)\big) \in \vec\varpi_{\mathrm{Ji}(D)} \subseteq \zeta$ yield that $(x, y) \in \zeta$. Finally, if $x \in \mathrm{Ji}(D)$ and $y \in \mathrm{Ji}(E)$, then $\big(x, g(y)\big) = \big(g(x), g(y)\big) \in \vec\varpi_{\mathrm{Ji}(D)} \subseteq \zeta$ and $\big(g(y), y\big) = \big(\eta(y), y\big) \in \dot\eta^{-1} \subseteq \zeta$ entail that $(x, y) \in \zeta$. Thus $\vec{\mathrm{Ker}}\, g \subseteq \zeta$. $\square$

Now, we are in the position to put the pieces together.

Proof of Theorem 1.3. Let $E = \mathrm{Con}\, L$, and consider η and $(R;\zeta)$ defined in this section, see (9.1) and (9.2). Let us consider the natural coloring $\gamma\colon \mathrm{Pri}(L) \to \big(\mathrm{Ji}(E); \leq_{\mathrm{Ji}(E)}\big)$, defined by $\mathfrak{p} \mapsto \mathrm{con}_L(\mathfrak{p})$. Since $(R;\zeta)$ is an extension of $\big(\mathrm{Ji}(E); \leq_{\mathrm{Ji}(E)}\big)$, Lemma 8.1 yields a rectangular lattice K and a quasi-coloring $\delta\colon \mathrm{Pri}(K) \to (R;\zeta)$ such that L is a filter of K and γ is the restriction of δ to $\mathrm{Pri}(L)$. Consider the homomorphism $g\colon (R;\zeta) \to \big(\mathrm{Ji}(D); \leq_{\mathrm{Ji}(D)}\big)$ provided by Lemma 9.4. Let $\delta' := g \circ \delta$. Since $\vec{\mathrm{Ker}}\, g \subseteq \zeta$ by Lemma 9.4, Lemma 2.1 yields that $\delta'\colon \mathrm{Pri}(K) \to \big(\mathrm{Ji}(D); \leq_{\mathrm{Ji}(D)}\big)$, defined by $\mathfrak{p} \mapsto g\big(\delta(\mathfrak{p})\big)$ is a quasi-coloring. In fact, it is a coloring since its range is an order. Hence, we conclude from (2.8) that

$$\alpha\colon D \to \mathrm{Con}\, K, \text{ where } d \mapsto \bigvee\big\{\mathrm{con}_K(\mathfrak{p}) : \mathfrak{p} \in \mathrm{Pri}(K) \text{ and } \delta'(\mathfrak{p}) \leq d\big\}, \quad (9.3)$$

is an isomorphism. In order to show that $\varphi = \varrho_L^K \circ \alpha$, it suffices to show that $\varphi(d) = \varrho_L^K\big(\alpha(d)\big)$ holds for every $d \in \mathrm{Ji}(D)$. To prove this equation, we fix a $d \in \mathrm{Ji}(D)$. We claim that

$$\varrho_L^K\big(\alpha(d)\big) = \bigvee\big\{\mathrm{con}_L(\mathfrak{q}) : \mathfrak{q} \in A_1\big\}, \tag{9.4}$$

where $A_1 = \big\{\mathfrak{q} \in \mathrm{Pri}(L) : \mathrm{con}_K(\mathfrak{q}) \leq \alpha(d)\big\}$. Indeed, a prime interval $\mathfrak{q} \in \mathrm{Pri}(L)$ is collapsed by $\varrho_L^K\big(\alpha(d)\big)$ iff $(0_{\mathfrak{q}}, 1_{\mathfrak{q}}) \in \alpha(d)$ iff $\mathrm{con}_K(\mathfrak{q}) \leq \alpha(d)$ iff $\mathfrak{q} \in A_1$. Since each congruence of L is determined by the prime intervals it collapses, (9.4) follows. We also claim that

$$\varphi(d) = \bigvee\big\{\mathrm{con}_L(\mathfrak{q}) : \mathfrak{q} \in A_2\big\}, \tag{9.5}$$

where $A_2 := \big\{\mathfrak{q} \in \mathrm{Pri}(L) : \gamma(\mathfrak{q}) \leq_E \varphi(d)\big\}$. Indeed, the $\gamma(\mathfrak{q}) = \mathrm{con}_L(\mathfrak{q})$ are the join-irreducible elements of $E = \mathrm{Con}\, L$, whence we get (9.5).

By virtue of (9.4) and (9.5), it suffices to show that $A_1 = A_2$. Let $\mathfrak{q} \in \mathrm{Pri}(L)$.

Assume that $\mathfrak{q} \in A_1$. Then (2.1) together with (9.3) yield that $\mathrm{con}_K(\mathfrak{q}) \leq \mathrm{con}_K(\mathfrak{p})$ for some $p \in \mathrm{Pri}(K)$ such that $\delta'(\mathfrak{p}) \leq d$. But $\mathrm{con}_K(\mathfrak{q}) \leq \mathrm{con}_K(\mathfrak{p})$ is equivalent to $\delta'(\mathfrak{q}) \leq \delta'(\mathfrak{p})$ since δ' is a coloring. Consequently, $\delta'(\mathfrak{q}) \leq d$. Taking $\delta'(\mathfrak{q}) = g\big(\delta(\mathfrak{q})\big) = g\big(\gamma(\mathfrak{q})\big) = \eta\big(\gamma(\mathfrak{q})\big)$ into account, we get that $\eta\big(\gamma(\mathfrak{q})\big) \leq d$. Since η is a left adjoint of φ by Lemma 9.1, we conclude that $\gamma(\mathfrak{q}) \leq \varphi(d)$, that is, $\mathfrak{q} \in A_2$. Thus $A_1 \subseteq A_2$.

To show the reverse inclusion, assume that $\mathfrak{q} \in A_2$. Then, using Lemma 9.1 again, we get that $\eta\big(\gamma(\mathfrak{q})\big) \leq d$. But $\eta\big(\gamma(\mathfrak{q})\big) = g\big(\gamma(\mathfrak{q})\big) = g\big(\delta(\mathfrak{q})\big) = \delta'(\mathfrak{q})$, whence we get that $\delta'(\mathfrak{q}) \leq d$. Hence, $\mathrm{con}_K(\mathfrak{q})$ is one of the joinands in (9.3), which implies $\mathrm{con}_K(\mathfrak{q}) \leq \alpha(d)$. Thus $\mathfrak{q} \in A_1$, and $A_2 \subseteq A_1$. $\square$

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