REPRESENTING A MONOTONE MAP BY PRINCIPAL LATTICE CONGRUENCES

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Dedicated to the memory of Gábor Szász

ABSTRACT. For a lattice L, let $\mathrm{Princ}(L)$ denote the ordered set of principal congruences of L. In a pioneering paper, G. Grätzer proved that bounded ordered sets (in other words, posets with 0 and 1) are, up to isomorphism, exactly the $\mathrm{Princ}(L)$ of bounded lattices L. Here we prove that for each 0-separating bound-preserving monotone map ψ between two bounded ordered sets, there are a lattice L and a sublattice K of L such that, in essence, ψ is the map from $\mathrm{Princ}(K)$ to $\mathrm{Princ}(L)$ that sends a principal congruence to the congruence it generates in the larger lattice.

1. Historical background

A classical theorem of Dilworth [4] states that each finite distributive lattice is isomorphic to the congruence lattice of a finite lattice. Since this first result, the *congruence lattice representation problem* has attracted many researchers, and dozens of papers belonging to this topic have been written. The story of this problem were mile-stoned by Huhn [8] and Schmidt [10], reached its summit in Wehrung [11] and Růžička [9], and was summarized in Grätzer [5]; see also Czédli [2] for some additional, recent references. In [6], Grätzer started an analogous new topic of Lattice Theory. Namely, for a lattice L, let $\mathrm{Princ}(L) = \langle \mathrm{Princ}(L), \subseteq \rangle$ denote the ordered set of principal congruences of L. A congruence is *principal* if it is generated by a pair $\langle a, b\rangle$ of elements. Ordered sets (also called partially ordered sets or posets) and lattices with 0 and 1 are called *bounded*. Clearly, if L is a bounded lattice, then $\mathrm{Princ}(L)$ is a bounded ordered set. The pioneering theorem in Grätzer [6] states the converse: each bounded ordered set P is isomorphic to $\mathrm{Princ}(L)$ for an appropriate bounded lattice L of length 5. Up to isomorphism, he also characterized finite bounded ordered sets as the $\mathrm{Princ}(L)$ of finite lattices L. The ordered sets $\mathrm{Princ}(L)$ of countable lattices L were characterized by Czédli [1].

2. Our result

Given two bounded ordered sets, P and Q, a map $\psi\colon P \to Q$ is called a *bound-preserving monotone map* if $\psi(0_P) = 0_Q$, $\psi(1_P) = 1_Q$, and, for all $x, y \in P$, $x \leq_P y$

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implies $\psi(x) \leq_Q \psi(y)$. If, in addition, 0_P is the only preimage of 0_Q, that is, if $\psi^{-1}(0_Q) = \{0_P\}$, then we say that ψ is a 0*-separating bound-preserving monotone map.* For a lattice L and $x, y \in L$, the principal congruence generated by $\langle x, y\rangle$ is denoted by $\mathrm{con}(x, y)$ or $\mathrm{con}_L(x, y)$. If L is bounded, K is a sublattice of L, and $0_L, 1_L \in K$, then K is a $\{0, 1\}$*-sublattice* of L. In this case, the map

$$\zeta_{K,L} \colon \mathrm{Princ}(K) \to \mathrm{Princ}(L) \quad \text{defined by} \quad \mathrm{con}_K(x, y) \mapsto \mathrm{con}_L(x, y)$$

is well-defined since $\zeta_{K,L}(\mathrm{con}_K(x, y))$ is the least congruence of L including $\mathrm{con}_K(x, y)$, and it is a 0-separating bound-preserving monotone map.

Our aim is to prove that each 0-separating bound-preserving monotone map can be represented in this way. We compose maps from right to left, that is, $(\psi_1 \circ \psi_2)(x) = \psi_1(\psi_2(x))$.

Theorem 2.1. *Let $\langle P; \leq_P\rangle$ and $\langle Q; \leq_Q\rangle$ be bounded ordered sets. If ψ is a 0-separating bound-preserving monotone map from $\langle P; \leq_P\rangle$ to $\langle Q; \leq_Q\rangle$, then there exist a bounded lattice L, a $\{0, 1\}$-sublattice K of L, and order isomorphisms*

$$\xi_1 \colon \langle P; \leq_P\rangle \to \langle \mathrm{Princ}(K); \subseteq\rangle \quad \text{and} \quad \xi_2 \colon \langle Q; \leq_Q\rangle \to \langle \mathrm{Princ}(L); \subseteq\rangle$$

such that $\psi = \xi_2^{-1} \circ \zeta_{K,L} \circ \xi_1$; that is, the diagram

$$\begin{array}{ccc} \langle P; \leq_P\rangle & \xrightarrow{\psi} & \langle Q; \leq_Q\rangle \\ \xi_1 \Big\downarrow & & \xi_2^{-1} \Big\uparrow \\ \langle \mathrm{Princ}(K); \subseteq\rangle & \xrightarrow{\zeta_{K,L}} & \langle \mathrm{Princ}(L); \subseteq\rangle \end{array}$$

is commutative.

Due to the tools developed in Czédli [1], the proof here is short.

3. Lemmas and proofs

A *quasiordered set* is a structure $\langle H; \nu\rangle$ where $H \neq \varnothing$ is a set and $\nu \subseteq H^2$ is a *quasiordering*, that is, a reflexive, transitive relation, on H. Quasiordered sets are also called *preordered sets.* If $g \in H$ and $\langle x, g\rangle \in \nu$ for all $x \in H$, then g is a *greatest element* of H; *least elements* are defined dually. They are not necessarily unique; if they are, then they are denoted by 1_H and 0_H. Given $H \neq \varnothing$, the set of all quasiorderings on H is denoted by $\mathrm{Quord}(H)$. It is a complete lattice with respect to set inclusion. Therefore, for $X \subseteq H^2$, there exists a least quasiordering on H including X; it is denoted by $\mathrm{quo}_H(X)$ or $\mathrm{quo}(X)$. We write $\mathrm{quo}(a, b)$ rather than $\mathrm{quo}(\{\langle a, b\rangle\})$. If $\langle H; \nu\rangle$ is a quasiordered set, then $\Theta_\nu = \nu \cap \nu^{-1}$ is an equivalence relation, and the definition $\langle [x]\Theta_\nu, [y]\Theta_\nu\rangle \in \nu/\Theta_\nu \iff \langle x, y\rangle \in \nu$ turns the quotient set H/Θ_ν into an ordered set $\langle H/\Theta_\nu; \nu/\Theta_\nu\rangle$. For an ordered set H and $x, y \in H$, $\langle x, y\rangle$ is called an *ordered pair* of H if $x \leq y$. This notation is consistent with the one used in previous work on (principal) lattice congruences. The set of ordered pairs of H is denoted by $\mathrm{Pairs}^{\leq}(H)$.

We need the concept of strong auxiliary structures from Czédli [1]; however, we do not need all the details. In particular, the reader does not have to know what the axioms (A1),...,(A13) are. By a *strong auxiliary structure* we mean a structure

$$\mathcal{L} = \langle L; \gamma, H, \nu, \delta, \varepsilon, \mathcal{Z}\rangle \tag{3.1}$$

such that the axioms (A1),...,(A13) from [1] hold. What we only have to know is the following. If $\mathcal{L}$ in (3.1) is a strong auxiliary structure, then L is a bounded lattice, $\langle H;\nu\rangle$ is a quasiordered set, $\gamma\colon \mathrm{Pairs}^{\leq}(L) \to H$ is a map (called quasi-coloring), δ and ε are maps from H to $L\setminus\{0_L,1_L\}$, $\varepsilon(p)$ covers or equals $\delta(p)$ for all $p\in H$, and $\mathcal{Z}$ is a set of certain 9-tuples of L. The following statement follows trivially from (A1), (A4) and the (short) proof of Lemma 2.1 in [1].

Lemma 3.1. *If $\mathcal{L}$ in (3.1) is a strong auxiliary structure, then the map*

$$\xi\colon \langle H/\Theta_\nu;\nu/\Theta_\nu\rangle \to \langle \mathrm{Princ}(L);\subseteq\rangle,\ \textit{defined by } [p]\Theta_\nu \mapsto \mathrm{con}_L(\delta(p),\varepsilon(p)),$$

is an order isomorphism.

This lemma shows the importance of strong auxiliary structures, and it explains why we are going to construct a quasiordered set from a monotone map. In the rest of the paper, ψ denotes a bound-preserving monotone map from a bounded ordered set $\langle P;\leq_P\rangle$ to another one, $\langle Q;\leq_Q\rangle$. Since there will be several orderings and quasiorderings and since they are needed in various contexts, we will write ν_1 instead of $\leq_P$ and ν_2 instead of $\leq_Q$. Without loss of generality, we can assume that $0_P=0_Q$, $1_P=1_Q$, and $P\cap Q=\{0_P,1_P\}=\{0_Q,1_Q\}$. Let $R=P\cup Q$, $0_R=0_P$ (that is, $0_R=0_Q$), $1_R=1_P=1_Q$, and note that $\nu_1\subseteq R^2$, $\nu_2\subseteq R^2$, $\psi\subseteq R^2$, and $\psi^{-1}=\{\langle x,y\rangle : x=\psi(y)\}\subseteq R^2$. Hence, we can define $\nu_3=\mathrm{quo}_R(\nu_1\cup\nu_2\cup\psi\cup\psi^{-1})$. To make our notation easier, Θ_i will stand for $\Theta_{\nu_i}=\nu_i\cap\nu_i^{-1}$, for $i\in\{1,2,3\}$.

Lemma 3.2. *There exists a unique map $\kappa\colon \langle R/\Theta_3;\nu_3/\Theta_3\rangle \to \langle Q;\nu_2\rangle$ such that*

$$\kappa([x]\Theta_3)=\begin{cases} x, & \textit{if } x\in Q,\\ \psi(x), & \textit{if } x\in P,\end{cases}\tag{3.2}$$

and this map is an order isomorphism.

Proof. Consider the map $\kappa_0\colon \langle R;\nu_3\rangle\to\langle Q;\nu_2\rangle$, defined by

$$\kappa_0(x)=\begin{cases} x, & \text{if } x\in Q,\\ \psi(x), & \text{if } x\in P.\end{cases}\tag{3.3}$$

Since ψ is the identity map on $P\cap Q=\{0_P,1_P\}$, Definitions (3.2) and (3.3) are nonambiguous. First, we show that κ_0 is monotone, that is, for all $x,y\in R$,

$$\text{if}\quad \langle x,y\rangle\in\nu_3,\quad\text{then}\quad \langle\kappa_0(x),\kappa_0(y)\rangle\in\nu_2.\tag{3.4}$$

By the transitivity of ν_3 and that of ν_2, it is sufficient to consider the case where $\langle x,y\rangle\in\nu_1\cup\nu_2\cup\psi\cup\psi^{-1}$. If $\langle x,y\rangle\in\nu_1$, then $\langle\kappa_0(x),\kappa_0(y)\rangle=\langle\psi(x),\psi(y)\rangle\in\nu_2$, since ψ is monotone. If $\langle x,y\rangle\in\nu_2$, then $\langle\kappa_0(x),\kappa_0(y)\rangle=\langle x,y\rangle\in\nu_2$. If $\langle x,y\rangle\in\psi$, that is $\psi(x)=y$, then $\langle\kappa_0(x),\kappa_0(y)\rangle=\langle y,y\rangle\in\nu_2$ by reflexivity. Similarly, if $\langle x,y\rangle\in\psi^{-1}$, that is $\psi(y)=x$, then $\langle\kappa_0(x),\kappa_0(y)\rangle=\langle x,x\rangle\in\nu_2$. This proves (3.4).

Next, if $[x]\Theta_3=[y]\Theta_3$, then $\langle x,y\rangle,\langle y,x\rangle\in\nu_3$. So, (3.4) and the antisymmetry of ν_2 yield that $\kappa_0(x)=\kappa_0(y)$. Hence, the map κ is well-defined. Note the rule $\kappa([x]\Theta_3)=\kappa_0(x)$. This, together with (3.4), implies that κ is monotone. Since κ_0 is surjective, so is κ. Hence, to complete the proof, it suffices to show that

$$\text{if}\quad \langle\kappa([x]\Theta_3),\kappa([y]\Theta_3)\rangle\in\nu_2,\quad\text{then}\quad \langle[x]\Theta_3,[y]\Theta_3\rangle\in\nu_3/\Theta_3.\tag{3.5}$$

Assume that $\langle\kappa([x]\Theta_3),\kappa([y]\Theta_3)\rangle\in\nu_2$. This means that $\langle\kappa_0(x),\kappa_0(y)\rangle\in\nu_2$, and we have to show that $\langle x,y\rangle\in\nu_3$. There are four cases to consider.

- If $x, y \in P$, then $\langle x, \psi(x)\rangle \in \psi \subseteq \nu_3$, $\langle \psi(x), \psi(y)\rangle = \langle \kappa_0(x), \kappa_0(y)\rangle \in \nu_2 \subseteq \nu_3$, and $\langle \psi(y), y\rangle \in \psi^{-1} \subseteq \nu_3$ imply $\langle x, y\rangle \in \nu_3$.
- If $x, y \in Q$, then $\langle x, y\rangle = \langle \kappa_0(x), \kappa_0(y)\rangle \in \nu_2 \subseteq \nu_3$.
- If $x \in P$ and $y \in Q$, then $\langle x, \psi(x)\rangle \in \psi \subseteq \nu_3$ and $\langle \psi(x), y\rangle = \langle \kappa_0(x), \kappa_0(y)\rangle \in \nu_2 \subseteq \nu_3$ yield that $\langle x, y\rangle \in \nu_3$.
- Finally, if $x \in Q$ and $y \in P$, then we conclude $\langle x, y\rangle \in \nu_3$ from $\langle x, \psi(y)\rangle = \langle \kappa_0(x), \kappa_0(y)\rangle \in \nu_2 \subseteq \nu_3$ and $\langle \psi(y), y\rangle \in \psi^{-1} \subseteq \nu_3$. □

Proof of Theorem 2.1. Assume that $|P| = 1$. Since ψ is bound-preserving, we have $|Q| = 1$, and $|K| = |L|$ witness the validity of the theorem.

Assume that $|P| = 2$. Since ψ is bound-preserving, we have $|Q| \geq 2$. If $|Q| = 2$, then we can pick a simple lattice $K = L$ of length 5. Hence, we also assume that $|Q| \geq 3$. By Grätzer [6] (or by the rest of the proof), there exists a bounded lattice L such that $\mathrm{Princ}(L) \cong \langle Q; \leq_Q\rangle = \langle Q; \leq_2\rangle$, and we can let $K = \{0_L, 1_L\}$.

Therefore, in the rest of the proof, we assume that $|P| \geq 3$. Since ψ is bound-preserving, we have $|Q| \geq 2$. Let $\nu_0 = (\{0_P\} \times P) \cup (P \times \{1_P\})$; note that $\langle P; \nu_0\rangle$ is a modular lattice of length 2. Let $\mathcal{L}_0 = \langle L_0; \gamma_0, P, \nu_0, \delta_0, \varepsilon_0, \mathcal{Z}_0\rangle$ denote the strong auxiliary structure defined in Example 2.2 (and Figure 5) of [1], with $\langle P; \nu_0\rangle$ playing the role of $\langle H; \nu\rangle$. Similarly, let $\nu'_0 = (\{0_R\} \times R) \cup (R \times \{1_R\})$, and let $\mathcal{L}'_0 = \langle L'_0; \gamma'_0, R, \nu'_0, \delta'_0, \varepsilon'_0, \mathcal{Z}'_0\rangle$ denote the strong auxiliary structure defined in Example 2.2 (and Figure 5) of [1], with $\langle R; \nu'_0\rangle$ playing the role of $\langle H; \nu\rangle$. It follows trivially from the construction, described in [1], that L_0 is a $\{0, 1\}$-sublattice of L'_0, δ_0 is the restriction $\delta'_0\rceil_P$ of δ'_0 to P, and $\varepsilon'_0 = \varepsilon'_0\rceil_P$.

Clearly, $\nu_0 \subseteq \nu_1$. Hence, we can apply [1, Lemma 5.3] so that $\langle P, \nu_0, P, \nu_1\rangle$ plays the role of $\langle H, \nu, H^{\blacktriangleright}, \nu^{\blacktriangleright}\rangle$. In this way, we obtain a strong auxiliary structure $\mathcal{L}_1 = \langle L_1; \gamma_1, P, \nu_1, \delta_1, \varepsilon_1, \mathcal{Z}_1\rangle$. Note that L_0 is a $\{0, 1\}$-sublattice of L_1, $\delta_1 = \delta_0$, and $\varepsilon_1 = \varepsilon_0$. Let $\nu'_1 = \mathrm{quo}_R(\nu_1) = \nu_1 \cup \nu'_0$. Giving the role of $\langle H, \nu, H^{\blacktriangleright}, \nu^{\blacktriangleright}\rangle$ to $\langle R, \nu'_0, R, \nu'_1\rangle$, [1, Lemma 5.3] yields a strong auxiliary structure $\mathcal{L}'_1 = \langle L'_1; \gamma'_1, R, \nu'_1, \delta'_1, \varepsilon'_1, \mathcal{Z}'_1\rangle$ such that L_1 is a $\{0, 1\}$-sublattice of L'_1, $\delta'_1 = \delta'_0$, and $\varepsilon'_1 = \varepsilon'_0$.

Finally, using [1, Lemma 5.3] with $\langle R, \nu'_1, R, \nu_3\rangle$ in place of $\langle H, \nu, H^{\blacktriangleright}, \nu^{\blacktriangleright}\rangle$, we obtain a strong auxiliary structure $\mathcal{L}_3 = \langle L_3; \gamma_3, R, \nu_3, \delta_3, \varepsilon_3, \mathcal{Z}_3\rangle$. Again, the construction yields that L'_1 is a $\{0, 1\}$-sublattice of L_3, $\delta_3 = \delta'_1$, and $\varepsilon_3 = \varepsilon'_1$. Hence, L_1 is a $\{0, 1\}$-sublattice of L_3, $\delta_1 = \delta_3\rceil_P$, and $\varepsilon_1 = \varepsilon_3\rceil_P$. For $i \in \{1, 3\}$, the order isomorphism provided by Lemma 3.1 will be denoted by ξ_i. Since ν_1 is an ordering, $\Theta_1 = \Theta_{\nu_1}$ is the equality relation, and so we can disregard it when applying Lemma 3.1. Hence, for $p \in P$, $\xi_1(p) = \mathrm{con}_{L_1}(\delta_1(p), \varepsilon_1(p))$. Consider the following diagram:

$$\begin{array}{ccccc}
\langle P; \nu_1\rangle & \xrightarrow{\kappa^{-1} \circ \psi} & \langle R/\Theta_3; \nu_3/\Theta_3\rangle & \xrightarrow{\kappa} & \langle Q; \nu_2\rangle \\
\xi_1 \big\downarrow & & \xi_3 \big\downarrow & & \xi_3 \circ \kappa^{-1} \big\downarrow \\
\langle \mathrm{Princ}(L_1); \subseteq\rangle & \xrightarrow{\zeta_{L_1, L_3}} & \langle \mathrm{Princ}(L_3); \subseteq\rangle & = & \langle \mathrm{Princ}(L_3); \subseteq\rangle
\end{array} \tag{3.6}$$

The first row of (3.6) makes sense by Lemma 3.2. Obviously, the square on the right commutes. Since the first two vertical arrows are order isomorphisms by Lemma 3.1 and so is κ by Lemma 3.2, we obtain that all the three vertical arrows are order isomorphisms. Next, to show that the square on the left of (3.6) is commutative, consider an arbitrary element $p \in P$. Using $\delta_1 = \delta_3\rceil_P$ and $\varepsilon_1 = \varepsilon_3\rceil_P$, we have that

$$\zeta_{L_1,L_3}(\xi_1(p)) = \zeta_{L_1,L_3}\big(\mathrm{con}_{L_1}(\delta_1(p), \varepsilon_1(p))\big) = \mathrm{con}_{L_3}(\delta_3(p), \varepsilon_3(p)). \tag{3.7}$$

Since (3.2) yields $\kappa^{-1}(\psi(p)) = [p]\Theta_3$, we also have that

$$\xi_3\big((\kappa^{-1} \circ \psi)(p)\big) = \xi_3\big(\kappa^{-1}(\psi(p))\big) = \xi_3([p]\Theta_3) = \mathrm{con}_{L_3}(\delta_3(p), \varepsilon_3(p)). \tag{3.8}$$

Thus, we conclude from (3.7) and (3.8) that the left square of (3.6) commutes. Hence, (3.6) is a commutative diagram. Finally, letting $K = L_1$, $L = L_3$, and $\xi_2 = \xi_3 \circ \kappa^{-1}$, the commutativity of (3.6) proves the theorem. $\square$

Added on March 31, 2015. In a *short* paper based on [1], I cannot prove more. In a subsequent long paper not relying on [1], we generalize Theorem 2.1 from a single 0-separating bound-preserving monotone map to a family of such maps and from bounded lattices to selfdual lattices of length 5; see [3]. Because of selfduality, [3] strengthens the result of Grätzer [6].

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