A PROPERTY OF LATTICES OF SUBLATTICES CLOSED UNDER TAKING RELATIVE COMPLEMENTS AND ITS CONNECTION TO 2-DISTRIBUTIVITY

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Dedicated to the memory of Professor Béla Csákány, 1932–2022

Abstract. For a lattice L of finite length n, let $\mathrm{RCSub}(L)$ be the collection consisting of the empty set and those sublattices of L that are closed under taking relative complements. That is, a subset X of L belongs to $\mathrm{RCSub}(L)$ if and only if X is join-closed, meet-closed, and whenever $\{a, x, b\} \subseteq S$, $y \in L$, $x \wedge y = a$, and $x \vee y = b$, then $y \in S$. We prove that (1) the poset $\mathrm{RCSub}(L)$ with respect to set inclusion is lattice of length $n + 1$, (2) if $\mathrm{RCSub}(L)$ is a ranked lattice and L is modular, then L is 2-distributive in András P. Huhn's sense, and (3) if L is distributive, then $\mathrm{RCSub}(L)$ is a ranked lattice.

1. Introduction

For a lattice L, $\mathrm{RCSub}(L) = (\mathrm{RCSub}(L), \subseteq)$ will denote the lattice of sublattices *closed under taking relative complements*; $\emptyset \in \mathrm{RCSub}(L)$ by convention. We only deal with lattices L of *finite length*. Our goal is to determine the length of $\mathrm{RCSub}(L)$. Also, for a modular lattice L of finite length, we give a necessary condition and also a sufficient condition that $\mathrm{RCSub}(L)$ is a ranked lattice; a lattice of finite length is *ranked* if it satisfies the Jordan–Hölder chain condition. Finally, we determine the size (that is, the number of elements) of $\mathrm{RCSub}(B_n)$ for the finite Boolean lattice B_n of height n. We present the results in Section 2.

The reader is assumed to be familiar with the rudiments of lattice theory; then the paper is self-contained. The rest of this introductory section is a mini-survey of earlier results that motivate our work.

The importance of taking *complements* of lattice elements is well known; here we only mention three facts to support this opinion. First, complementation plays a crucial role in von Neumann [26], which still belongs to the deepest chapters of lattice theory; see Birkhoff [3] for a laudation of von Neumann's work. Second, Grätzer in [14] surveys the two most famous problems that "shaped a century of lattice theory"; one of these problems, solved by Dilworth [11], is about complementation. (The other problem highlighted in [14] is Dilworth's congruence lattice

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problem, and it was solved by Wehrung [34].) Third, several generalizations of lattices are based on modified versions of complementation; see, for example, Chajda, Halaš and Kühr [4].

There are several papers on the lattices $\mathrm{Sub}(L)$ of sublattices of lattices L; including, for example, Chen, Koh, and Teo [6], Czédli [7, 8], Filippov [12], Lakser [25], and Takách [29, 30, 31, 32]. Some of these papers on $\mathrm{Sub}(L)$ might initiate analogous investigations about $\mathrm{RCSub}(L)$ but this is not targeted in the present paper. On the other hand, Theorem 2.1 (A) and Observation 2.2 are, in some vague sense, counterparts of known results on $\mathrm{Sub}(L)$; see Chen and Koh [5], Koh [24], Ramananda [28] (on convex sublattices) and Stephan [27]. Although 2-distributivity, to be defined later, occurred previously in Czédli [7, 8] and it occurs in this paper again, here the role of 2-distributivity is entirely different from that in [7, 8] and now deeper tools are needed. We recall these tools in Section 3.

Chapter 10 of Grätzer [13] on R-generated sublattices of distributive and, mainly, Boolean lattices also belongs to our motivations. Finally, there are several papers on retracts of lattices and, by a trivial reason, the retracts of distributive lattices are closed under taking relative complements; see Czédli [9].

2. Results

First, we recall some concepts. For elements u, x, v of a lattice L, let

$$\mathrm{rc}_L(u, x, v) := \{y \in L : x \wedge y = u \text{ and } x \vee y = v\}.$$

With this notation, a sublattice S of L is *closed under taking relative complements* or, shortly saying, S is an *RC-closed sublattice* of L if $\mathrm{rc}_L(u, x, v) \subseteq S$ holds for all $u, x, v \in S$. So $\mathrm{RCSub}(L)$ consists of $\emptyset$ and the RC-closed sublattices of L. The poset $\mathrm{RCSub}(L) = \big(\mathrm{RCSub}(L), \subseteq\big)$ is an algebraic lattice, in which the meet is the set theoretic intersection. For $n \in \mathbb{N}^+ := \{1, 2, 3, \dots\}$, a lattice L is *n-distributive* if for all $x, y_0, \dots, y_n \in L$,

$$x \wedge \bigvee\{y_i : 0 \le i \le n\} = \bigvee_{j=0}^{n} \big(x \wedge \bigvee\{y_i : 0 \le i \le n \text{ and } i \ne j\}\big); \tag{2.1}$$

see Huhn [18, 19]. Every distributive (that is, 1-distributive) lattice is 2-distributive but not conversely. The *length* $\mathrm{len}(L)$ of a lattice L is the supremum of the lengths of its finite chains; for an n-element chain C, $\mathrm{len}(C) = n-1$. For a technical reason, we let $\mathrm{len}(\emptyset) := -1$. A *ranked lattice of finite length* is a lattice of finite length in which any two maximal chains are of the same length. E.g., modular lattices of finite length are such. Our aim is to prove the following theorem and some other statements presented in this section; their proofs will be given in Section 4.

Theorem 2.1. *For every lattice L of finite length, the following three assertions hold.*

(A) *The lattice* $\mathrm{RCSub}(L)$ *is of finite length and* $\mathrm{len}(\mathrm{RCSub}(L)) = \mathrm{len}(L) + 1$.
(B) *If L is modular and* $\mathrm{RCSub}(L)$ *is a ranked lattice, then L is 2-distributive.*
(C) *If L is distributive, then* $\mathrm{RCSub}(L)$ *is a ranked finite lattice.*

In connection with part (C), (3.7) from Section 4 is worth mentioning. For $n \in \mathbb{N}^+$, let B_n denote the Boolean lattice of length n; note that $|B_n| = 2^n$. The n-th *Bell number*, that is, the number of partitions of an n-element set will be denoted by $\mathrm{bell}(n)$. These numbers, named after Bell [1], are well studied; at the time of writing, a MathSciNet search "Title=(Bell number)" returns 170 matches.

n	1	2	3	4	5
r_n	4	11	38	152	675
n	6	7	8	9	10
r_n	3 264	17 008	94 829	562 596	3 535 028
n	11	12	13	14	15
r_n	23 430 841	163 254 886	1 192 059 224	9 097 183 603	72 384 727 658

TABLE 1. $r(n) := |\mathrm{RCSub}(B_n)|$ for $n \in \{1, \dots, 15\}$

Observation 2.2. *For $n \in \mathbb{N}^+$ and the Boolean lattice B_n of length n,* $\mathrm{RCSub}(B_n)$ *is of size*

$$|\mathrm{RCSub}(B_n)| = 1 + \sum_{k=0}^{n} \left(\binom{n}{k} \cdot \sum_{t=0}^{n-k} \binom{n-k}{t} \mathrm{bell}(t) \right). \tag{2.2}$$

For $n \in \{1, 2, \dots, n\}$, $r_n := |\mathrm{RCSub}(B_n)|$ is given in Table 1. We used the computer algebraic program Maple V Release 5 (of Nov. 27, 1997), in which $\mathrm{bell}(n)$ is a built-in function. On a desktop computer with Intel(R) Core(TM) i5-4440 CPU, 3.10 GHz, the computation for Table 1 took less than a millisecond. As n grows, more time is needed; e.g., it took five and a half minutes to obtain that

$$\mathrm{RCSub}(B_{2022})| \approx 9.600\,407\,373\,025\,643\,058\,974\,662\,646\,652\,852\,523 \cdot 10^{4409}.$$

For a lattice L and $X \subseteq L$, let $\mathrm{rcg}_L(X)$ stand for the least RC-closed sublattice of L that includes X as a subset; "g" in the acronym comes from "generated".

Lemma 2.3 (Key Lemma)**.** *If L is a lattice of finite length, Y is an RC-closed sublattice of L, X is a sublattice of Y, and* $\mathrm{len}(X) = \mathrm{len}(Y)$*, then $Y = \mathrm{rcg}_L(X)$.*

For the particular case where L is distributive, Lemma 2.3 could be extracted from Section 10 of Grätzer [13]. Letting $Y := L$, the lemma trivially implies the following.

Corollary 2.4. *If X is a sublattice of a lattice L of finite length such that* $\mathrm{len}(X) = \mathrm{len}(L)$*, then L is RC-generated by X, that is, $L = \mathrm{rcg}_L(X)$.*

3. SOME KNOWN RESULTS ON n-DISTRIBUTIVITY

Two concepts introduced in Huhn [18, 19] have changed a little since their introductions. First, Huhn [18, 19] defined n-distributivity as the conjunction of (2.1) and modularity, but later Huhn himself dropped the assumption that the lattice should be modular; see [22]. Second, the n-diamonds defined by Huhn [18, 19] correspond to the $(n + 1)$-frames of Herrmann and Huhn [17] and even to the $(n + 1)$-diamonds in the terminology used by Day [10]; these concepts are equivalent modulo the theory of modular lattices. Some of the relevant sources appeared in conference proceedings or were written in German. These facts may cause difficulty to some readers. This explains why we collect the results on n-distributivity that are relevant here in this separate section. Note that even though I quote these results from published papers, most of my knowledge goes back to the time when András P. Huhn was my scientific leader.

Following von Neumann [26] and going also after, say, Herrmann [16], a system

$$F = (a_i, c_{i,j} : i, j \in \{1, \dots, n\}, i \neq j) \tag{3.1}$$

of elements of a modular lattice L is a *non-trivial normalized (von Neumann) n-frame* or, briefly, a *von Neumann n-frame* if, with the notation $0_F := \bigwedge_{i=1}^n a_i$ and $1_F := \bigvee_{i=1}^n a_i$, we have that $0_F \neq 1_F$, $a_j \wedge \bigvee_{t\neq j} a_t = 0_F = a_i \wedge c_{i,j}$, $c_{i,j} = c_{j,i}$, $a_i \vee c_{i,j} = a_i \vee a_j$, and $c_{i,k} = (a_i \vee a_k) \wedge (c_{i,j} \vee c_{j,k})$ for all $\{i, j, k\} \subseteq \{1, \dots, n\}$ with $i \neq j \neq k \neq i$. Here $2 \leq n \in \mathbb{N}^+$. We know from Huhn [21, Proposition 1.2] that

$$\left.\begin{array}{l}\text{for } n \in \mathbb{N}^+\text{, a modular lattice is } n\text{-distributive if and only}\\ \text{if it does not contain a von Neumann } (n+1)\text{-frame.}\end{array}\right\} \tag{3.2}$$

Subsection 1.4 "Reduction of frames" together with Subsection 1.7 of Herrmann and Huhn [17] prove that, for $n \in \mathbb{N}^+$,

$$\left.\begin{array}{l}\text{if } F = (a_i, c_{i,j} : i, j \in \{1, \dots, n\}, i \neq j) \text{ is a von Neumann}\\ n\text{-frame in a modular lattice } L \text{ and } a_1' \in L \text{ such that}\\ 0_F < a_1' < a_1\text{, then } a_1' \text{ belongs to a von Neumann } n\text{-}\\ \text{frame } F = (a_i', c_{i,j}' : i, j \in \{1, \dots, n\}, i \neq j) \text{ such that}\\ 0_F = 0_{F'} < a_i' < a_i \text{ for } i \in \{1, \dots, n\}.\end{array}\right\} \tag{3.3}$$

The von Neumann n-frame F of L from (3.1) is said to be *cover-preserving* if $0_F \prec_L a_i$ for all $i \in \{1, \dots, n\}$. Applying (3.3), repeatedly if necessary, we obtain that, for $2 \leq n \in \mathbb{N}^+$,

$$\begin{array}{l}\text{if a modular lattice of finite length contains a von}\\ \text{Neumann } n\text{-frame, then it also contains a cover-}\\ \text{preserving von Neumann } n\text{-frame.}\end{array} \tag{3.4}$$

Note that for $n = 2$, (3.4) was proved in Jakubík [23].

A projective space is *irreducible* (or, in another terminology, *non-degenerate*) if each of its lines contains at least three points. It is known (and easy to see) that in an irreducible projective plane (which is a projective geometry of dimension 2), each point lies on at least three lines. By Huhn [20, Thm. 1.1],

$$\left.\begin{array}{l}\text{for } n \in \mathbb{N}^+\text{, a modular algebraic lattice is } n\text{-distributive if and}\\ \text{only if none of its sublattices is isomorphic to the subspace}\\ \text{lattice of an irreducible projective geometry of dimension } n.\end{array}\right\} \tag{3.5}$$

The proof of Theorem 2.1 will need two well-known facts, (3.6) and (3.7), about distributive (that is, 1-distributive) lattices. Namely,

$$\text{any atom } q \text{ in a distributive lattice is } \textit{join prime}, \tag{3.6}$$

that is, $q \leq x_1 \vee \cdots \vee x_t$ implies that $q \leq x_i$ for some $i \in \{1, \dots, t\}$, and

$$\text{if } L \text{ is a distributive lattice of finite length, then } L \text{ is finite.} \tag{3.7}$$

For convenience and having no reference at hand, we give a short argument. To verify (3.6), note that $q \leq x_1 \vee \cdots \vee x_t$ gives that $q = q \wedge (x_1 \vee \cdots \vee x_t) = (q \wedge x_i) \vee \cdots \vee (q \wedge x_t)$, whence the join-irreducibility of q applies. To prove (3.7), observe that $L = \{0\} \cup \bigcup\{\uparrow a : 0 \prec a\}$, whereby it suffices to show that L only has at most $\operatorname{len}(L)$ many atoms; indeed, then (3.7) follows by induction on $\operatorname{len}(L)$. For the sake of contradiction, suppose that there exist $t > \operatorname{len}(L)$ pairwise distinct atoms $a_1, a_2, \dots, a_t$ in L. Define $b_0 := 0$ and $b_i := a_1 \vee \cdots \vee a_i$ for $i \in \{1, \dots, t\}$. Clearly, $b_0 < b_1 \leq b_2 \leq \cdots \leq b_t$. If $b_{i-1} = b_i$ for some i, then the join primeness of a_i and the inequality $a_i \leq b_i = b_{i-1}$ would give a $j < i$ with $a_i \leq a_j$, which is impossible since a_i and a_j are distinct atoms. Hence, $b_0 < b_0 < \cdots < b_t$. This contradicts $t > \mathrm{len}(L)$ and proves (3.7).

4. Proofs

Proof of Lemma 2.3. We give a proof by contradiction. With $S := \mathrm{rcg}_L(X)$, suppose that $S \neq Y$. Since $X \subseteq S \subset Y$, $\mathrm{len}(Y) = \mathrm{len}(X) \leq \mathrm{len}(S) \leq \mathrm{len}(Y)$ gives that $\mathrm{len}(S) = \mathrm{len}(Y)$. Therefore, we can

$$\text{fix a maximal chain } T \text{ in } S \text{ such that } \mathrm{len}(T) = \mathrm{len}(Y). \tag{4.1}$$

For $a \in L$, the principal ideal and the principal filter generated by a are denoted by $\downarrow_L a := \{x \in L : x \leq a\}$ and $\uparrow_L a := \{x \in L : x \geq a\}$, respectively. We write $\downarrow a$ and $\uparrow a$ if L is understood. If $u \leq v$ in Y, then the *length* of the interval $[u, v]$, understood in Y, will be denoted by $\mathrm{len}_Y([u, v])$. For $u \leq v \in T$, the notation $\mathrm{len}_T([u, v])$ is analogously defined. It follows from (4.1) that for $u \leq v \in T$, we have that $\mathrm{len}_T([u, v]) = \mathrm{len}_Y([u, v])$. For $b \in Y$, we define

$$b_{-T} := \bigvee (T \cap \downarrow_Y b) \quad \text{ and } \quad b^{+T} := \bigwedge (T \cap \uparrow_Y b).$$

Using that any lattice of finite length is complete, T is a sublattice, $0_T = 0_L$, and $0_L = 1_L$, we obtain that both b_{-T} and b^{+T} exist and they belong to T, provided that $b \in Y$. Thus, for $y \in Y \setminus T$, we have that $y_{-T} < y < y^{+T}$, whence $\mathrm{len}_T([b_{-T}, b^{+T}]) = \mathrm{len}_Y([b_{-T}, b^{+T}]) \geq 2$. Next,

$$\text{choose an element } p \in Y \setminus S \text{ such that } \mathrm{len}_Y([p_{-T}, p^{+T}]) \text{ is minimal.} \tag{4.2}$$

Since $p \notin S$ and so $p \notin T$, we have that $\mathrm{len}_T([p_{-T}, p^{+T}]) \geq 2$. This allows us to pick an element $t \in T$ such that $p_{-T} < t < p^{+T}$. We claim that

$$p \vee t \in S \quad \text{ and } \quad p \wedge t \in S. \tag{4.3}$$

By duality, it suffices to deal with the first part of (4.3). For the sake of contradiction, suppose that $p \vee t \notin S$. Since $p, t \in \downarrow p^{+T}$, we have that $r := p \vee t \leq p^{+T}$. Note that r belongs to Y as so do p and t. Using that $T \ni t \leq r \leq p^{+T} \in T$, we obtain that $t \leq r_{-T}$ and $r^{+T} \leq p^{+T}$. So $p_{-T} < t \leq r_{-T} \leq r^{+T} \leq p^{+T}$, which yields that $\mathrm{len}_Y([r_{-T}, r^{+T}]) < \mathrm{len}_Y([p_{-T}, p^{+T}])$. By the choice of p, see (4.2), this inequality rules out that $r \in Y \setminus S$. Hence, $p \vee t = r \in S$, proving (4.3).

Finally, as a consequence of (4.3), $t \in T \subseteq S$, and that S is RC-closed, we obtain that $p \in \mathrm{rcg}_L(p \wedge t, t, p \vee t) \in S$. This contradicts (4.2) and completes the proof of Lemma 2.3. □

Proof of Theorem 2.1. Let $n := \mathrm{len}(L)$. Clearly, we can assume that $n \geq 2$. Note that, as always in lattice theory, "$\subset$" will denote the conjunction of "$\subseteq$" and "$\neq$".

To prove part (A), let $0 = c_0 \prec c_1 \cdots \prec c_n = 1$ be a maximal chain in L. Let $X_{-1} := \emptyset$ and, for $i \in \{0, \dots, n\}$, let $X_i := \downarrow c_i$. Since all these X_i belong to $\mathrm{RCSub}(L)$ and $X_{-1} \subset X_0 \subset X_1 \subset \cdots \subset X_n = L$, we obtain that $\mathrm{len}(\mathrm{RCSub}(L)) \geq n + 1 = \mathrm{len}(L) + 1$.

To prove the reverse inequality, let $Y_{-1} \subset Y_0 \subset \cdots \subset Y_k = L$ be an arbitrary chain in $\mathrm{RCSub}(L)$. Clearly, $\mathrm{len}(Y_{i-1}) \leq \mathrm{len}(Y_i)$ for all $i \in \{0, 1, \dots, k\}$. We claim that $\mathrm{len}(Y_{i-1}) < \mathrm{len}(Y_i)$ for all meaningful i. Suppose the contrary. Then $Y_{i-1} \subset Y_i$ and $\mathrm{len}(Y_{i-1}) = \mathrm{len}(Y_i)$ for some i. With reference to Lemma 2.3 at "$=^*$", $Y_{i-1} = \mathrm{rcg}_L(Y_{i-1}) =^* Y_i$, which is a contradiction proving that $\mathrm{len}(Y_{i-1}) < \mathrm{len}(Y_i)$ for $i \in \{0, 1, \dots, k\}$. Therefore, since $-1 = \mathrm{len}(\emptyset) \leq \mathrm{len}(Y_{-1})$, we obtain that

$k \le n+1$. Thus, $\mathrm{len}(\mathrm{RCSub}(L)) \le n+1 = \mathrm{len}(L)+1$, completing the proof of part (A).

To prove part (B) by contradiction, suppose that L is a modular lattice of finite length such that $\mathrm{RCSub}(L)$ is a ranked lattice (of finite length by part (A)) but L is not 2-distributive. By (3.2) and (3.4), L contains a cover-preserving von Neumann 3-frame $F = (a_i, c_{i,j} : i \ne j,\ i,j \in \{1,\dots,3\})$. The definition of a 3-frame together with, say, Grätzer [15, Theorem 360] imply that $\{a_1,a_2,a_3\}$ is an independent set of atoms in the filter $\uparrow 0_F$; the independence of $\{a_1,a_2,a_3\}$ means that this three element set generates a Boolean sublattice. This fact together with modularity (in fact, semimodularity) yield that $1_F = a_1 \vee a_2 \vee a_3$ is of height 3 in $\uparrow 0_F$, that is, $\mathrm{len}_L([0_F,1_F]) = 3$. By (3.2), the interval $I := [0_F,1_F]$ is not 2-distributive. Using (3.5), we obtain that the subspace lattice S of a projective plane G is a sublattice of I. Since $\mathrm{len}(I) = 3 = \mathrm{len}(S)$, it follows that $0_S = 0_F = 0_I$, $1_s = 1_I$, and S is a cover-preserving sublattice of I and L, that is, for any $x,y \in S$, we have that $x \prec_S y \iff x \prec_Y y \iff x \prec_L y$. Let $\mathrm{At}(S)$ and $\mathrm{Coat}(S)$ denote the set of atoms and that of coatoms of S, respectively. These two sets are disjoint, $S = \{0_S,1_S\} \cup \mathrm{At}(S) \cup \mathrm{Coat}(S)$, $\mathrm{At}(S) = \{a \in I : 0_I \prec a,\ a \in S\}$, and dually. A trivial geometric argument shows that for $\forall a \in \mathrm{At}(S)$ and $\forall b \in \mathrm{Coat}(S)$,

$$1_S = \bigvee\bigl(\mathrm{At}(S) \setminus \{a\}\bigr) \quad \text{and} \quad 0_S = \bigwedge\bigl(\mathrm{Coat}(S) \setminus \{b\}\bigr). \tag{4.4}$$

Define $Z_{-1} := \emptyset$, $Z_0 := \{0_S\}$, $Z_1 := \{0_S,1_S\}$, and $Z_2 := \mathrm{rcg}_L(S)$. We claim that

$$Z_{-1}, Z_0, Z_1, Z_2 = I \in \mathrm{RCSub}(L) \text{ and } Z_{-1} \prec Z_0 \prec Z_1 \prec Z_2 \text{ in } \mathrm{RCSub}(L). \tag{4.5}$$

Clearly, $I \in \mathrm{RCSub}(L)$, whereby Lemma 2.3 gives that $Z_2 = \mathrm{rcg}_L(S) = I$. Trivially, $Z_{-1} \prec Z_0 \prec Z_1$ and $Z_1 \subset Z_2$. To verify that $Z_1 \prec Z_2 = I$, assume that $Z_1 \subset X \subseteq I$ for some $X \in \mathrm{RCSub}(L)$. Pick an element $u \in X \setminus Z_1$. Then $0_S < u < 1_S$. Since $\mathrm{len}(S) = 3$, either u is of height 1, or it is of height 2. First, assume that u is of height 2, that is, $u \prec 1_S$ in I (and in L). If we had that $|\mathrm{At}(S) \setminus {\downarrow}u| \le 1$, then (4.4) would give that $1_S = \bigvee\bigl(\mathrm{At}(S) \cap {\downarrow}u\bigr) \le u$, contradicting that $u < 1_S$. Hence, $|\mathrm{At}(S) \setminus {\downarrow}u| \ge 2$, and we can pick two distinct elements, v and w, from $\mathrm{At}(S) \setminus {\downarrow}u$. Using that $v \not\le u$, $u,v \in [0_S,1_S]$, $0_S \prec v$, and $u \prec 1_S$, we obtain that $v \in \mathrm{rc}_L(0_S,u,1_S)$. Hence, $v \in X$; we obtain similarly that $w \in X$.

By modularity (in fact, by semimodularity), $v \prec v \vee w \in X$. Since $\mathrm{len}(I) = 3$, $0_S = 0_I \prec v \prec v \vee w < 1_S = 1_I$. This chain, being in X, shows that $3 \le \mathrm{len}(X)$. On the other hand, $\mathrm{len}(X) \le \mathrm{len}(I) = 3$. Using Lemma 2.3 at "$=^*$" and that $X \in \mathrm{RCSub}(L)$, we have that $X = \mathrm{rcg}_L(X) =^* I = Z_2$. Thus, $Z_1 \prec Z_2$, completing the proof of (4.5).

By (4.5), $Z_{-1} \prec \dots \prec Z_2$ extends to a maximal chain $\vec{Z} : Z_{-1} \prec \dots \prec Z_k$ of $\mathrm{RCSub}(L)$. Like in the proof of part (A), Lemma 2.3 applies and we obtain that $-1 \le \mathrm{len}(Z_{-1}) < \mathrm{len}(Z_0) < \dots < \mathrm{len}(Z_k) = \mathrm{len}(L)$. But now we know more: $\mathrm{len}(Z_1) + 1 = 2 < 3 = \mathrm{len}(Z_2)$. Thus, the maximal chain $\vec{Z}$ consists of at most $\mathrm{len}(L)+1$ elements, whence $\mathrm{len}(\vec{Z}) \le \mathrm{len}(L)$. But $\mathrm{RCSub}(L)$ is a ranked lattice of finite length, whereby $\mathrm{len}(\mathrm{RCSub}(L)) = \mathrm{len}(\vec{Z}) \le \mathrm{len}(L)$, contradicting part (A) of the theorem. This proves part (B).

To prove part (C), let L be a distributive lattice of finite length. By (3.7), L is finite, whence so is $\mathrm{RCSub}(L)$. Since L is a ranked lattice, it suffices to show that for any $U,V \in \mathrm{RCSub}(L)$,

$$U \prec V \text{ in } \mathrm{RCSub}(L) \iff \bigl(U \subset V \text{ and } \mathrm{len}(V) = \mathrm{len}(U)+1\bigr). \tag{4.6}$$

To prove the $\Rightarrow$ direction, assume that $U \prec V$. Clearly, $U \subset V$ and $\mathrm{len}(U) \leq \mathrm{len}(V)$. If we had that $\mathrm{len}(U) = \mathrm{len}(V)$, then Lemma 2.3 would give that $U = \mathrm{rcg}_L(U) = V$, a contradiction. Hence, $\mathrm{len}(U) < \mathrm{len}(V)$. We are going to show by way of contradiction that $\mathrm{len}(V) = \mathrm{len}(U) + 1$. Suppose the contrary; then $k := \mathrm{len}(U) \leq \mathrm{len}(V) - 2$. Take a maximal chain C_0 in U. Since $\mathrm{len}(C_0) = k$ and $\mathrm{len}(V) \geq k + 2$, we can extend C_0 to a chain C of V such that $\mathrm{len}(C) = k + 1$. Let $W := \mathrm{rcg}_L(C)$. It follows from Lemma 2.3 that $U = \mathrm{rcg}_L(C_0)$. Hence, $U = \mathrm{rcg}_L(C_0) \subseteq \mathrm{rcg}_L(C) = W$. Since $\mathrm{len}(W) \geq \mathrm{len}(C) > \mathrm{len}(U)$, $W \neq U$. Thus, $U \subset W$. The inclusion $C \subseteq V$ gives that $W = \mathrm{rcg}_L(C) \subseteq \mathrm{rcg}_L(V) = V$. Combining $U \subset W \subseteq V$ and $U \prec V$, we obtain that $W = V$.

Next, we write C in the form $C = \{c_0 < c_1 < \cdots < c_{k+1}\}$. (Note that, say, $c_0 \prec_L c_1$ need not hold.) By Birkhoff [2], we can fix a finite Boolean lattice D such that L is a sublattice of D. We define the elements $b_i \in D$ for $i \in \{1, \dots, k+1\}$ by $b_i \in \mathrm{rc}_D(c_0, c_{i-1}, c_i)$. Since D is a Boolean lattice, b_i exists and it is uniquely determined. We claim that, for $i = 2, 3, \dots, k+1$,

$$c_0 \notin \{b_1, \dots, b_i\},\ c_{i-1} = b_1 \vee \cdots \vee b_{i-1},\ \text{and } (b_1 \vee \cdots \vee b_{i-1}) \wedge b_i = c_0. \tag{4.7}$$

We show this by induction on i. Clearly, $b_1 = c_1 \neq c_0$. From $b_2 \in \mathrm{rc}_D(c_0, c_1, c_2)$, we obtain that $b_2 \neq c_0$ since otherwise $c_2 = c_1 \vee b_2 = c_1$ would be a contradiction. Also, $b_2 \in \mathrm{rc}_D(c_0, c_1, c_2)$ gives that $b_1 \wedge b_2 = c_1 \wedge b_2 = c_0$. Hence, the base of the induction holds, that is, (4.7) is satisfied for $i = 2$. Assume that $2 \leq i < k+1$ and (4.7) holds for this i. As before, $b_{i+1} \neq c_0$ since otherwise $b_{i+1} \in \mathrm{rc}_D(c_0, c_i, c_{i+1})$ would lead to $c_{i+1} = b_{i+1} \vee c_i = c_i$, which is a contradiction. Using the definition of b_i and the induction hypothesis, we have that $c_i = c_{i-1} \vee b_i = b_1 \vee \cdots \vee b_{i-1} \vee b_i$, that is, the second equality of (4.7) holds for $i+1$. Using this equality and the definition of b_{i+1}, we have that $(b_1 \vee \cdots \vee b_i) \wedge b_{i+1} = c_i \wedge b_{i+1} = c_0$. This completes the induction step and proves that (4.7) holds for $i = 2, 3, \dots, k+1$.

By Grätzer [15, Theorem 360] and (4.7), $\{b_1, b_2, \dots, b_{k+1}\}$ is a $(k+1)$-element independent set in the filter $\uparrow_D c_0$, whereby this set generates a 2^{k+1}-element Boolean sublattice E. Using that any element in an interval of a distributive lattice has at most one relative complement with respect to the interval in question and E as a Boolean lattice is closed under taking relative complements, we obtain that E is RC-closed. It is clear from (4.7) and $b_{k+1} \in \mathrm{rc}_D(c_0, c_k, c_{k+1})$ that $C \subseteq E$. Hence, $\mathrm{rcg}_D(C) \subseteq \mathrm{rcg}_D(E) = E$. Now let F be a maximal chain in W. Since $W = \mathrm{rcg}_L(C) \subseteq \mathrm{rcg}_D(C) \subseteq E$, we have that F is a chain in E. But $\mathrm{len}(E) = k+1$, implying that $\mathrm{len}(F) \leq k+1$. So $\mathrm{len}(W) = \mathrm{len}(F) \leq k+1$. On the other hand, $k+1 = \mathrm{len}(C) \leq \mathrm{len}(W)$. Thus, $\mathrm{len}(W) = k+1$, which is a contradiction since $W = V$ and $\mathrm{len}(V) = k+2$. This proves the $\Rightarrow$ direction of (4.6).

To prove the $\Leftarrow$ direction, assume that $U \subset V$ and $\mathrm{len}(V) = \mathrm{len}(U) + 1$. Assume also that $H \in \mathrm{RCSub}(L)$ such that $U \subseteq H \subseteq V$. Clearly, $\mathrm{len}(U) \leq \mathrm{len}(H) \leq \mathrm{len}(V)$, whence $\mathrm{len}(H) \in \{\mathrm{len}(U), \mathrm{len}(V)\}$. If $\mathrm{len}(H) = \mathrm{len}(U)$, then Lemma 2.3 gives that $H = \mathrm{rcg}_L(U) = U$. Similarly, if $\mathrm{len}(H) = \mathrm{len}(V)$, then the same lemma yields that $H = \mathrm{rcg}_L(H) = V$. Therefore, $U \prec V$, completing the proof of part (C) and that of the theorem. $\square$

Proof of Observation 2.2. Apart from the empty set, we classify the members S of $\mathrm{RCSub}(B_n)$ according to the height $h(0_S)$ of their bottoms, 0_S. This justifies the outer $\sum$ in (2.2). Since B_n is isomorphic to the powerset lattice over an n-element set, 0_S of height k can be chosen in $\binom{n}{k}$ ways; this is where the first binomial coefficient in (2.2) comes from. After choosing 0_S, we choose $t := \mathrm{len}_L([0_S, 1_S])$ from $\{0, 1, \dots, n-k\}$; this explains the second $\sum$ in (2.2). (Note that t can be larger than $\mathrm{len}(S)$ since S need not be a cover-preserving sublattice of B_n.) The filter $\uparrow_L 0_S$ is a Boolean sublattice of B_n and this filter is of length $n-k$. Hence, $\uparrow_L 0_S$ has exactly $n-k$ atoms, and there is a bijective correspondence between the set of elements of height t in $\uparrow_L 0_S$ and the set of t-element subsets $\mathrm{At}(\uparrow_L 0_S)$. Thus, to obtain 1_S such that $\mathrm{len}_L([0_S, 1_S]) = t$, we select t atoms of the Boolean lattice $\uparrow_L 0_S$ and then we form their join to obtain 1_S. The second binomial coefficient in (2.2) tells us how many ways these t atoms, denoted by $p_1, \dots, p_t$, of $\uparrow_L 0_S$ can be chosen.

First, we assume that $t > 0$, the case $t = 0$ will be discussed later. We know that $1_S = p_1 \vee \dots \vee p_t$. For an atom u of S, in notation for $u \in \mathrm{At}(S)$, we let $H_u := \{i : 1 \le i \le t \text{ and } p_i \le u\}$. Let $\mathcal{E}_u := \{H_u : u \in \mathrm{At}(S)\}$. We claim that $\mathcal{E}_u$ is a partition of $\{1, \dots, t\}$. Clearly, $H_u \neq \emptyset$ if $u \in \mathrm{At}(S)$. For distinct $u, u' \in \mathrm{At}(S)$, $u \wedge u' = 0_S$ yields that $H_u \cap H_{u'} = \emptyset$. For $i \in \{1, \dots, t\}$, $p_i \le 1_S = \bigvee \mathrm{At}(S)$. Hence (3.6) gives that $i \in H_u$ for some $u \in \mathrm{At}(S)$. Thus, $\mathcal{E}_u := \{H_u : u \in \mathrm{At}(S)\}$ is a partition of $\{1, \dots, t\}$. We claim that

$$\text{for each } u \in \mathrm{At}(S), \quad u = \bigvee \{p_i : i \in H_u\}. \tag{4.8}$$

To show this, observe that u is certainly the join of some atoms of the Boolean lattice $\uparrow_L 0_S$, whence it suffices to show that for every atom v of $\uparrow_L 0_S$ such that $v \le u$, we have that $v \in \{p_1, \dots, p_t\}$. But if $v \in \mathrm{At}(\uparrow_L 0_S) \cap \downarrow u$ then $v \le 1_S = p_1 \vee \dots \vee p_t$ yields the required $v \in \{p_1, \dots, p_t\}$ by (3.6) since any two comparable atoms of $\uparrow_L 0_S$ coincide. Thus, (4.8) holds.

It follows from (4.8) that the partitions of $\{1, \dots, t\}$ and the Boolean sublattices S with fixed 0_S and 1_S such that 0_S is of height k and $\mathrm{len}_L([0_S, 1_S]) = t$ mutually determine each other. Thus, the number of these S is $\mathrm{bell}(t)$. This is also true for $t = 0$, when there is only $1 = \mathrm{bell}(0)$ such S. In this way, we have explained $\mathrm{bell}(t)$ in (2.2), completing the proof of Observation 2.2. $\square$

5. Odds and ends

We do not know whether parts (B) and (C) of Theorem 2.1 can be strengthened in a reasonable way; this section only mentions some easy facts.

Remark 5.1. For a lattice L, $\mathrm{RCSub}(L)$ is Boolean if and only if L is a chain. If L is not a chain, then the lattice $\mathrm{RCSub}(L)$ is not even semimodular.

Proof. For a chain L, $\mathrm{RCSub}(L)$ is the powerset lattice of L, whence it is Boolean. Assume that L is not a chain, and pick $a, b \in L$ such that these two elements are incomparable. Let $u := a \wedge b$. Then $\emptyset, \{a\}, \{b\}, \{b, u\} \in \mathrm{RCSub}(L)$ and $\{a\} \vee \{b\} = \mathrm{rcg}_L(\{a, b\})$ contains a, b, and u. Since $\emptyset \prec \{a\}$ but $\emptyset \vee \{b\} < \{b, u\} < \{a\} \vee \{b\}$, $\mathrm{RCSub}(L)$ is not semimodular. $\square$

Remark 5.2. For the four element Boolean lattice B_2, the lattice $\mathrm{RCSub}(B_2)$ is not lower semimodular.

Proof. With the notation $B_2 = \{0, a, b, 1\}$, both $\{0, a\}$ and $\{b, 1\}$ are coatoms of $\mathrm{RCSub}(B_2)$ but their meet is $\emptyset$, the bottom element. Thus, if $\mathrm{RCSub}(B_2)$ was lower semimodular, then it would be of length 2, contradicting Theorem 2.1(A). $\square$

L	$\mathrm{len}(\mathrm{RCSub}(L)) = \mathrm{len}(L)+1$	is $\mathrm{RCSub}(L)$ ranked?	is L ranked?	is L modular?
N_5	4	no	no	no
N_6	4	yes	yes	no
M_3	3	yes	yes	yes
F	4	no	yes	yes

Table 2. Examples

We conclude the paper with four examples given by Figure 1 and Table 2; note that F is the subspace lattice of the Fano plane and the table is justified by a straightforward argument and (4.5) (applied to $S = L := F$).

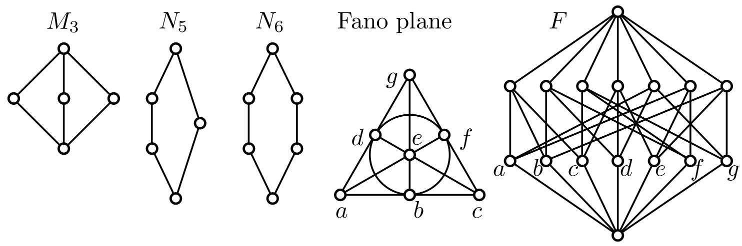

Figure 1. Examples

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