PERMUTING 2-UNIFORM TOLERANCES ON LATTICES

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Dedicated to the memory of Ivo G. Rosenberg

Abstract. A 2-*uniform tolerance* on a lattice is a compatible tolerance relation such that all of its blocks are 2-element. We characterize permuting pairs of 2-uniform *tolerances* on lattices of finite length. In particular, any two 2-uniform *congruences* on such a lattice permute.

1. Introduction and result

In addition to his famous theorem on functional completeness over finite sets, the words "tolerance" and "lattice" also remind me of Ivo G. Rosenberg, since both are common in the title of the present paper and that of our joint lattice theoretical paper [5] (co-authored also by I. Chajda). A part of my motivation is to keep his memory alive.

This short paper is structured as follows. First, after few necessary definitions, we formulate our main result, Theorem 1.1. Then, still in this section, we present the rest of our motivation and we point out how the present theorem supersedes its precursor on 2-uniform congruences. Section 2 is devoted to the proof of Theorem 1.1.

Definitions and the result. By a *tolerance* T on a lattice L we mean a reflexive, symmetric, and compatible relation on L. Following the monograph Chajda [3], several papers referenced in [3], and other papers like Bandelt [1], Chajda, Czédli and Halaš [4, 5], Czédli [8, 11], Czédli and Grätzer [12], and Grygiel and Radeleczki [14], the maximal subsets X of L such that $X^2 \subseteq T$ are called the *blocks* of T. So a block of T is a maximal subset X of L with the property that $(a, b) \in T$ for all $a, b \in X$. Note that tolerance blocks occur in mathematics outside lattice theory often under different names including cliques in graphs and simplexes in geometry.

If T is a tolerance such that each of its blocks consists of exactly two elements, then we call it a 2-*uniform tolerance* on L. As usual, for tolerances T and S on L, the *product* $T \circ S$ is defined to be $\{(x, z) :$ there exists a $y \in L$ such that $(x, y) \in T$ and $(y, z) \in S\}$. We say that T and S *permute* if $T \circ S = S \circ T$. Note by Chajda and Zelinka [7] that T and S permute if and only if $T \circ S$ is also a tolerance.

Tolerance blocks are known to be convex sublattices by, say, Czédli [8]. Hence, each block of a 2-uniform tolerance is a two-element interval, which is represented

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by an edge in the diagram if the lattice happens to be finite. Let T be a 2-uniform tolerance on a lattice L, and let $u \in L$. Since every T-block containing u is a two-element interval and there exists such a block, at least one of the following two possibilities holds:

(i) there exists a lower cover v of u (in notation, $v \prec u$) such that $\{v, u\}$ is a T-block; then u is called a *T-top* (element) and v is the *lower T-neighbour* of u; or

(ii) there exists an upper cover w of u such that $\{u, w\}$ is a T-block; then u is called a T-bottom (element) and w is the *upper T-neighbour* of u.

Observe that v in (i) is unique and so is w in (ii) (this explains the definite articles preceding them). Indeed, if v_1 and v_2 were distinct lower covers of u with $\{v_1, u\}$ and $\{v_2, u\}$ being T-blocks, then $(v_1, u) \in T$ and $(v_2, u) \in T$ would lead to $(v_1 \wedge v_2, u \wedge u) = (v_1 \wedge v_2, u) \in T$, we could extend $\{v_1 \wedge v_2, u\}$ to a T-block, this T-block would contain $v_1 \wedge v_2$, v_1, v_2, and u by its convexity, but this would contradict the assumption that all T-blocks are two-element. This proves that v in (i) is unique, and the uniqueness of w in (ii) follows by duality. We have mentioned that at least one of (i) and (ii) holds. Note that they can simultaneously hold, that is, both v and w can exist; see the atoms on the left of Figure 1 for examples.

Next, assume that T and S are 2-uniform tolerances on a lattice L, and let $u \in L$. If u is both a T-bottom and an S-bottom, then we call it a *two-fold (T, S)-bottom*, or a *two-fold* bottom if T and S are understood. *Two-fold (T, S)-tops* are defined dually as elements that are simultaneously T-tops and S-tops. Finally, we say that T and S are *amicable* if the following two conditions hold for every u in L.

(A1) If u is a two-fold (T, S)-top, $u \prec v$ and $(u, v) \in T \cup S$, then v is also a two-fold (T, S)-top.

(A2) If u is a two-fold (T, S)-bottom, $v \prec u$ and $(v, u) \in T \cup S$, then v is also a two-fold (T, S)-bottom.

Note that (A1) is the dual of (A2). The conjunction of (A1) and (A2) is easy to imagine as follows: in every component of the graph $(L; T \cup S)$, covers of two-fold tops are two-fold tops and lower covers of two-fold bottoms are two-fold bottoms. Note that for permuting 2-uniform tolerances T and S of a finite lattice L, the graph $(L; T \cup S)$ can have several components; this is exemplified by Figure 1. Now, we are in the position to formulate our result.

Theorem 1.1. *Let T and S be 2-uniform tolerances on a lattice L that contains no infinite chain. Then T and S permute if and only if they are amicable.*

History and further motivation. Beginning with Chajda and Zelinka [6], several papers deal with tolerances on lattices and lattice-like structures. Without seeking completeness, this is exemplified by Bandelt [1, 2], Chajda [3], Chajda, Czédli, and Rosenberg [4], Czédli [8], Czédli and Grätzer [12], Grygiel and Radeleczki [14], and Kindermann [16]. However, the history of the research leading to the present paper began with a problem raised by Grätzer, Quackenbush, and E. T. Schmidt [13]. They asked whether a finite lattice L is necessarily congruence permutable if any two blocks of each congruence are isomorphic (sub)lattices. Soon thereafter, Kaarli [15] gave an affirmative answer; in fact, he proved even more: if any two blocks of each congruence are of the same size, then the finite lattice in question is congruence permutable. This result was followed by Czédli [9] and [10], which state that in certain finite algebras (including finite lattices), any two

2-uniform congruences permute; a 2-*uniform congruence* is, of course, a 2-uniform tolerance that happens to be a congruence. Recently, Czédli [11] has applied 2-uniform (and even more general) tolerances in a new construction of modular lattices.

Clearly, any two 2-uniform congruences are amicable. Hence, Theorem 1.1 immediately implies the following corollary.

Corollary 1.2. *If all chains of a lattice L are finite, then any two 2-uniform congruences of L permute.*

Although this statement is formulated only for lattices, it supersedes [9] and [10] in the sense that the lattice in Corollary 1.2 need not be finite. For $n \in \mathbb{N} := \{1, 2, 3, \dots\}$, let C_n denote the n-element chain. In order to show an infinite example that belongs to the scope of Corollary 1.2, let K be an arbitrary infinite lattice without infinite chains. (For example, we can take all the C_n, $3 \le n \in \mathbb{N}$, and glue their bottoms into a common bottom and glue their tops into a common top.) Define $L := \mathsf{C}_2 \times \mathsf{C}_2 \times K$, and let α and β be the kernel of the first projection and that of the second projection, respectively; then α and β are 2-uniform congruences and L has no infinite chain. Two examples of amicable pairs of 2-uniform tolerances are shown if Figure 1, where the T-blocks are given by solid grey ovals while the S blocks by dotted black ones. Finally, note that neither Theorem 1.1, nor Corollary 1.2 can be extended to an arbitrary lattice. This is exemplified by the lattice of all integer numbers with the usual ordering; this lattice has exactly two 2-uniform congruences but they do not permute.

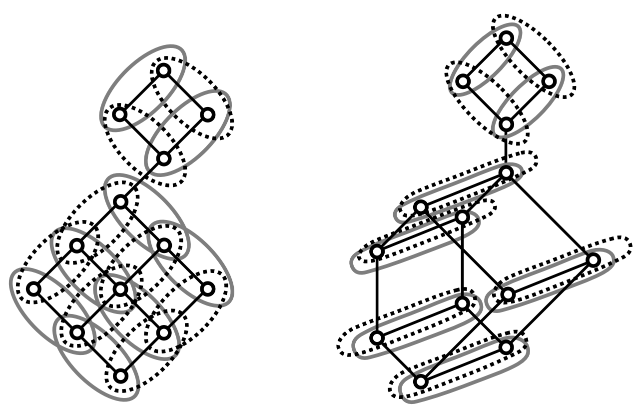

FIGURE 1. Two examples

2. The proof of the result

Lemma 2.1. *Let L be a lattice without infinite chains, and let R, T, and S be 2-uniform tolerances on L. Then, for any $x, y, z, a, b, u \in L$, the following assertions hold.*

(i) *If x and y are lower R-neighbours of z, then $x = y$.*
(ii) *If $(x, y) \in R$, then $x = y$, or $x \prec y$, or $y \prec x$.*
(iii) *If $a \neq b$, a is the lower T-neighbour of u, and b is the lower S-neighbour of u, then $a \wedge b$ is the lower S-neighbour of a and the lower T-neighbour of b.*

Although this lemma is a trivial folkloric consequence of definitions, we give a short proof for convenience.

Proof. By Zorn's Lemma, any $X \subseteq L$ with $X^2 \subseteq R$ extends to a block of R, whereby $X^2 \subseteq R$ implies that $|X| \leq 2$. We know from, say, Czédli [8] that the blocks of R are convex sublattices. If x and y were distinct lower R-neighbours of z, then we would have $(u,z) := (x \wedge y, z \wedge z) \in R$, we could pick a block B of R such that $\{u,z\} \subseteq B$, so $[u,z] \subseteq B$, contradicting $\{u,z,x,y\} \subseteq [u,v]$ and $|B| = 2$. This shows (i). Part (ii) follows from the trivial fact that it describes the only possibilities where x and y belong to an interval of size at most 2. Finally, to prove (iii), assume its premise. Then a and b are incomparable (in notation, $a \parallel b$), since both are lower covers of u by (ii). Hence, $\{a,b\} \cap \{a \wedge b\} = \emptyset$. Since $(a \wedge b, a) = (a \wedge b, a \wedge u) \in S$, we have that $\{a \wedge b, a\}^2 \subseteq S$. Hence, $\{a \wedge b, a\}$ is a block of S, and $a \wedge b$ is the lower S-neighbour of a. Since (a,S) and (b,T) play a symmetric role, (iii) follows. □

If u is a *two-fold* (T,S)*-bottom*, then there are two possibilities. Namely, either the upper T-neighbour and the upper S-neighbour of u are different and we say that u is a *split* (T,S)*-bottom*, or these two upper neighbours are the same and we call u an *adherent* (T,S)*-bottom*. Dually, if u is a *two-fold* (T,S)*-top*, then it is either a *split* (T,S)*-top*, or an *adherent* (T,S)*-top*, depending on whether its lower neighbours are distinct or equal, respectively. Armed with these concepts, we are going to prove the following lemma, which is a bit more than what the necessity part of Theorem 1.1 would require.

Lemma 2.2. *If T and S are permuting 2-uniform tolerances on a lattice L without infinite chains, then the following four conditions are satisfied for every $u \in L$.*

- (i) *If u is a split (T,S)-top, $u \prec v$, and $(u,v) \in T \cup S$, then v is a split (T,S)-top.*
- (ii) *If u is an adherent (T,S)-top, $u \prec v$, and $(u,v) \in T \cup S$, then v is an adherent (T,S)-top.*
- (iii) *If u is a split (T,S)-bottom, $v \prec u$, and $(v,u) \in T \cup S$, then v is a split (T,S)-bottom.*
- (iv) *If u is an adherent (T,S)-bottom, $v \prec u$, and $(v,u) \in T \cup S$, then v is an adherent (T,S)-bottom.*

Proof. With the assumptions of the lemma, in order to prove (i), let u be a split (T,S)-top, $u \prec v$ and $(u,v) \in T \cup S$. Since T and S play a symmetric role, we can assume that $(u,v) \in T$. The lower T-neighbour and the lower S-neighbour of u will be denoted by a and b, respectively; note that $a \parallel b$, since a and b are distinct lower covers of u by Lemma 2.1(ii). Since $(b,v) \in S \circ T$ and $S \circ T = T \circ S$, there exists an element c such that $(b,c) \in T$ and $(c,v) \in S$. Observe that $v \nleq c$, because otherwise $b < u < v \leq c$ together with $(b,c) \in T$ would violate Lemma 2.1(ii). Hence, again by 2.1(ii), $c \prec v$ and c is a lower S-neighbour of v. If $c \neq u$, then v is a split (T,S)-top, as required. Hence, it suffices to exclude that $c = u$. For the sake of contradiction, suppose that $c = u$. Then $(b,u) = (b,c) \in T$ indicates that a and b are distinct lower T-neighbours of u, contradicting Lemma 2.1(i). This contradiction completes the argument proving (i). By duality, we conclude the validity of (iii).

Next, to prove (ii), let u be an adherent (T,S)-top, $u \prec v$ and $(u,v) \in T \cup S$. Again, we can assume that $(u,v) \in T$. Denote the common lower T-neighbour and

S-neighbour of u by a. Since $(a,v) \in S \circ T = T \circ S$, there is an element c such that $(a,c) \in T$ and $(c,v) \in S$. Since both $c \le a < u < v$ and $a < u < v \le c$ are excluded by Lemma 2.1(ii), we obtain from Lemma 2.1(ii) that $a \prec c \prec v$. As two upper T-neighbours of a, the elements u and c are the same by the dual of Lemma 2.1(i). Hence, $(u,v) = (c,v) \in S$ shows that v is an adherent (T,S)-top, as required. This shows the validity of (ii), and (iv) follows also by duality. □

Proof of Theorem 1.1. The necessity part follows from Lemma 2.2. In order to prove the sufficiency part, assume that T and S are amicable. Since T and S play a symmetric role, it suffices to show that $T \circ S \subseteq S \circ T$. So let $(a,b) \in T \circ S$; we need to show that $(a,b) \in S \circ T$. We can assume that $(a,b) \notin T \cup S$, since otherwise the task is trivial. By the definition of $T \circ S$, there exists an element u such that $(a,u) \in T$ and $(u,b) \in S$. Apart from duality, Lemma 2.1(ii) allows only two cases: either $a \prec u \succ b$, or $a \prec u \prec b$. Since Lemma 2.1(iii) implies immediately that $(a,b) \in S \circ T$ in the first case, it suffices to deal only with the second case. That is, $a \prec u \prec b$. Let $x_0 := a$, $x_1 := u$, $x_2 := b$, and define a sequence $x_3, x_4, \dots$ of further elements as follows. If i is even and x_i is a T-bottom, then let x_{i+1} be the unique upper T-neighbour of x_i. If i is odd and x_i is an S-bottom, then let x_{i+1} be the unique upper S-neighbour of x_i. Note that, in addition to the elements x_i, $i > 2$, the elements $x_1 = u$ and $x_2 = b$ also obey these rules. Since $x_2 \prec x_3 \prec x_4 \prec \dots$ but L has no infinite chain, there is a unique $2 \le n \in \mathbb{N}$ such that $x_2, x_3, \dots, x_n$ are defined but x_{n+1} is not. There are two (similar) cases depending on the parity of n. First, assume that n is even. Since the sequence has terminated with x_n, the element x_{n+1} does not exists, that is, x_n is not a T-bottom. Hence, x_n is a T-top. But x_n is also an S-top, whereby x_n is a two-fold (T,S)-top. The same argument, with the roles of T and S interchanged, shows that x_n is a two-fold (T,S)-top also in the second case where n is odd. So, x_n is a two-fold (T,S)-top no matter what the parity of n is. We claim that

$$x_{n-2} \text{ is a two-fold } (T,S)\text{-bottom.} \tag{2.1}$$

If x_n is an adherent (T,S)-top, then we obtain from Lemma 2.1(i) that x_{n-1} is an adherent (T,S)-bottom, whence x_{n-2} is a two-fold (T,S)-bottom by (A2), as required. If the two-fold (T,S)-top x_n is not an adherent one, then it is a split (T,S)-top, and there are two cases. If n is even, then x_{n-1} is a lower S-neighbour of x_n, and x_n has a unique lower T-neighbour c, which is distinct from x_{n-1}. By Lemma 2.1(iii), $x_{n-1} \wedge c$ is a lower T-neighbour of x_{n-1} and a lower S-neighbour of c. But x_{n-2} is also a lower T-neighbour of x_{n-1}, whence Lemma 2.1(i) gives that $x_{n-1} \wedge c = x_{n-2}$, and so x_{n-2} is a two-fold (split) (T,S)-bottom, as required. The same argument works, with T and S interchanged, if n is odd. Thus, (2.1) has been verified.

Next, we obtain from (A2) and (2.1) that $a = x_0$ is also a two-fold (T,S)-bottom. There are two cases to consider. First, assume that a is a split (T,S)-bottom. Then, in addition that u is an upper T-neighbour of a, the element a has an upper S-neighbour d such that $d \neq u$. By the dual of Lemma 2.1(iii), $u \vee d$ is an upper S-neighbour of u and an upper T-neighbour of d. Since b is also an upper S-neighbour of u, the dual of Lemma 2.1(i) gives that $u \vee d = b$. Hence, $(a,d) \in S$ and $(d,b) = (d, u \vee d) \in T$ yield that $(a,b) \in S \circ T$, as required.

Second, assume that a is an adherent (T,S)-bottom. Then $u = x_1$ is an (adherent) two-fold (T,S)-top. Applying (A1), we have that $b = x_2$ is also a two-fold

(T,S)-top. Hence, b has a unique lower T-neighbour e. We claim that $e = u$; for the sake of contradiction, suppose that $u \neq e$. Applying Lemma 2.1(iii), it follows that $u \wedge e$ is a lower T-neighbour of u. But a is also a lower T-neighbour of u, whereby Lemma 2.1(i) give that $u \wedge e = a$. On the other hand, Lemma 2.1(iii) also gives that $u \wedge e = a$ is a lower S-neighbour of e. Hence, a has two distinct upper S-neighbours, u and e, which contradicts the dual of Lemma 2.1(i). This contradiction shows that $e = u$. Armed with this equality, $(a, u) \in S$ and $(u, b) = (e, b) \in T$, and the required $(a, b) \in S \circ T$ follows. We have shown that $T \circ S \subseteq S \circ T$, and the proof of Theorem 1.1 is complete. □

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