Slim patch lattices as absolute retracts and maximal lattices

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Dedicated to the memory of András P. Huhn

Abstract. We prove that *slim patch lattices* are exactly the *absolute retracts* with more than two elements for the category of slim semimodular lattices with length-preserving lattice embeddings as morphisms. Also, slim patch lattices are the same as the *maximal objects* L in this category such that $|L| > 2$. Furthermore, slim patch lattices are characterized as the *algebraically closed lattices* L in this category such that $|L| > 2$. Finally, we prove that if we consider $\{0, 1\}$-preserving lattice homomorphisms rather than length-preserving ones, then the absolute retracts for the class of slim semimodular lattices are the at most 4-element boolean lattices.

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1. Introduction

1.1. Outline

A first look at our goal is presented in Subsection 1.2. In addition to a short historical survey and our motivations, Subsection 1.3 gives most of the definitions that are needed to state our main result, Theorem 1.5, in Subsection 1.4. A related result and two corollaries are also formulated in Subsection 1.4. In Section 2, we prove the results.

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1.2. Goal

We intend to characterize slim patch lattices among slim semimodular lattices as absolute retracts and also as maximal lattices. Theorem 1.5 in Subsection 1.4 indicates that this is possible, provided we turn the class of slim semimodular lattices into a category with appropriately chosen morphisms and we disregard the singleton lattice and the two-element lattice. The present paper continues the "slim semimodular lattices" part of Czédli and Molkhasi [15], where all lattice homomorphisms among these lattices were allowed.

1.3. Definitions and a mini-survey

All lattices in the paper are assumed to be *finite* even where this is not emphasized. For (a finite) lattice L, the set of non-zero *join-irreducible elements* and that of non-unit *meet-irreducible elements* will be denoted by $\mathrm{J}(L)$ and $\mathrm{M}(L)$, respectively. They are posets (that is, partially ordered sets) with respect to the order inherited from L. Following Czédli and Schmidt [16], we say that a lattice L is *slim* if it is finite and $\mathrm{J}(L)$ is the union of two chains. If $x \wedge y \prec x \Rightarrow y \prec x \vee y$ for all $x, y \in L$, then L is *semimodular*. The intensive study of planar semimodular lattices began with Grätzer and Knapp [21, 22]. For *these* lattices, our definition of slimness is equivalent to their original one: a planar semimodular lattice is slim if and only if the five-element modular lattice M_3 with three atoms is not a cover-preserving sublattice of L. Since each planar semimodular lattice is naturally reduced to a slim semimodular lattice by Grätzer and Knapp [21], slim semimodular lattices play a distinguished role among planar semimodular lattices.

By Lemma 2.2 of Czédli and Schmidt [16], slim lattices are *planar*. (Since this is not so if the original definition of slimness from Grätzer and Knapp [21] is used, the term "slim planar semimodular lattice" also occurs in the literature.)

The original importance of slim semimodular lattices in *lattice theory* is explained by their role in studying the congruence lattices of finite lattices; see Grätzer and Knapp [21, 22] together with the book chapter of Czédli and Grätzer [9] and its references. Also, see Czédli [5] for a connection between these lattices and a variant of planarity of bounded posets. Finally, see Czédli [6], Czédli and Grätzer [11], and their references for recent developments.

Notably, slim semimodular lattices have already found applications *outside lattice theory*. First, they played a crucial role in generalizing the classical Jordan–Hölder theorem for *groups* in Grätzer and Nation [23] and Czédli and Schmidt [16]. Second, these lattices led to new results in (combinatorial and convex) *geometry*; see Adaricheva and Bolat [1], Adaricheva and Czédli [2], Czédli [3], Czédli and Kurusa [13], and the references given in [13]. This connection is due to the canonical correspondence between slim semimodular lattices and (combinatorial) convex geometries of convex dimension at most 2; see Propositions 2.1 and 7.3 and Lemma 7.4 in Czédli [4]. Third, these lattices gave rise to interesting *enumerative combinatorial* questions in several papers. For example, even the famous mathematical constant $e = \lim_{n\to\infty}(1 + 1/n)^n \approx 2.718\,2818$ appeared in a lattice theoretical and combinatorial paper; see Czédli, Dékány, Gyenizse, and Kulin [8]. Fourth, some connection between these lattices and finite model theory has recently been found in Czédli [7]. Fifth (and least), a computer game was developed based on these lattices; see Czédli and Makay [14].

Next, we recall the following concept from Czédli and Schmidt [18]; in a slightly modified form that needs less preparation. (The original definition will be given later in (2.3).) An element of a lattice $x \in L$ is said to be *doubly irreducible* if it has exactly one lower cover and exactly one (upper) cover. In other words, if $x \in \mathrm{J}(L) \cap \mathrm{M}(L)$.

Definition 1.1. A slim semimodular lattice is a *slim patch lattice* if it has exactly two doubly irreducible elements, these two elements are coatoms, and their meet is the smallest element of the lattice.

For example, each of the three lattices drawn in Figure 1 is a slim patch lattice. (Their doubly irreducible elements are pentagon-shaped.) By definition, a slim patch lattice consists of at least four elements. We proved in Czédli and Schmidt [18] that every slim semimodular lattice can be obtained from slim patch lattices by gluing them together. Furthermore, [18] also proves that

$$\left.\begin{array}{l}\text{slim patch lattices are characterized as slim}\\ \text{semimodular lattices that are indecomposable}\\ \text{with respect to (Hall–Dilworth) gluing.}\end{array}\right\} \quad (1.1)$$

In this sense, slim patch lattices are the "small building stones" among slim semimodular lattices. On the other hand, slim patch lattices are "large enough" in the sense that each slim semimodular lattice L can be embedded into a slim patch lattice K; for example, such a K is constructed in Czédli and Molkhasi [15, Figure 2]. It will appear from our main result that patch lattices are "maximally large" in some sense.

Next, assume that

$$\left.\begin{array}{l}\mathcal{C}\text{ is a (concrete) category that consists of some}\\ \text{lattices as objects and each morphism of }\mathcal{C}\text{ is a}\\ \text{lattice homomorphism;}\end{array}\right\} \quad (1.2)$$

we do not require that all lattice homomorphisms among the objects of $\mathcal{C}$ are morphisms in $\mathcal{C}$. Using that every singleton subset of a lattice is a sublattice, it follows easily that

$$\left.\begin{array}{l}\text{the monomorphisms of }\mathcal{C}\text{ given in (1.2) are lattice em-}\\ \text{beddings, that is, injective lattice homomorphisms.}\end{array}\right\} \quad (1.3)$$

For lattices $L, K \in \mathcal{C}$, we say that L is a *retract* of K in the category $\mathcal{C}$ if there is a morphism $\iota\colon L \to K$ in $\mathcal{C}$ and a morphism $\rho\colon K \to L$ in $\mathcal{C}$ such that $\rho \circ \iota$ is the identity morphism id_L of L. Here, by (1.2), ι and ρ are lattice homomorphisms; note that we compose them from right to left, that is, $(\rho \circ \iota)(x) = \rho(\iota(x))$. Note also that $\rho \circ \iota = \mathrm{id}_L$ and (1.3) imply that ι is a lattice embedding and it is a monomorphism in $\mathcal{C}$, and ρ is a surjective (in other words, an onto) map. The morphism ρ above is called a *retraction* of ι.

Definition 1.2. Let $\mathcal{C}$ be as in (1.2). A lattice $L \in \mathcal{C}$ is an *absolute retract* for $\mathcal{C}$ if for every $K \in \mathcal{C}$ and every monomorphism $\iota\colon L \to K$, there exists a morphism $\rho\colon K \to L$ in $\mathcal{C}$ such that $\rho \circ \iota = \mathrm{id}_L$. In other words, $L \in \mathcal{C}$ is an absolute retract for $\mathcal{C}$ if every monomorphism of $\mathcal{C}$ with domain L has a retraction in $\mathcal{C}$.

Note that absolute retracts of any category of similar algebras (rather than lattices) were defined in the same way by Reinhold [28] in 1946. Hence, Definition 1.2 is a particular case of Reinhold's well-known definition.

Definition 1.3. Let $\mathcal{C}$ be as in (1.2). We say that a lattice $L \in \mathcal{C}$ is a *maximal object* of $\mathcal{C}$ if every monomorphism $L \to K$ of $\mathcal{C}$ is an isomorphism.

It is quite rare that $\mathcal{C}$ has a maximal object. For a lattice L, an *equation* in L is a formal expression

$$p(a_1, \dots, a_m, x_1, \dots, x_n) \approx q(a_1, \dots, a_m, x_1, \dots, x_n) \tag{1.4}$$

where $m \in \mathbb{N}_0 = \{0, 1, 2, \dots\}$, $n \in \mathbb{N}^+ = \mathbb{N}_0 \setminus \{0\}$, p and q are $(m+n)$-ary lattice terms, the *parameters* (also know as *coefficients*) $a_1, \dots, a_m$ are in L, and $x_1, \dots, x_n$ are the *unknowns* of (1.4). If $\mu\colon L \to K$ is a lattice homomorphism, then the *μ-image* of equation (1.4) is the equation

$$p(\mu(a_1), \dots, \mu(a_m), x_1, \dots, x_n) \approx q(\mu(a_1), \dots, \mu(a_m), x_1, \dots, x_n)$$

in K. For a set Σ of equations in L, we let $\mu(\Sigma) := \{\mu(e) : e \in \Sigma\}$. The following definition is taken from Schmid [29], and it was used later in Molkhasi [25, 26, 27].

Definition 1.4. Let $\mathcal{C}$ be as in (1.2). We say that a lattice $L \in \mathcal{C}$ is *strongly algebraically closed in* $\mathcal{C}$ if for any set Σ of equations in L and any monomorphism $\iota\colon L \to K$ in $\mathcal{C}$, if $\iota(\Sigma)$ has a solution in K, then Σ has a solution in L. If we replace "any set Σ" by "any finite set Σ", then we obtain the concept of an *algebraically closed* lattice $L \in \mathcal{C}$.

From now on,

$$\left.\begin{array}{l}\text{let } \mathcal{S} \text{ denote the category of slim semimodular}\\ \text{lattices with all lattice homomorphisms.}\end{array}\right\} \tag{1.5}$$

In accordance with the general assumption for the paper, every lattice in $\mathcal{S}$ is *finite*. Czédli and Molkhasi [15] proved that for a lattice $L \in \mathcal{S}$, the following four conditions are equivalent: (1) L is algebraically closed in $\mathcal{S}$, (2) L is strongly algebraically closed in $\mathcal{S}$, (3) L is an absolute retract for $\mathcal{S}$, and (4) L is the singleton lattice. In addition to the importance of patch lattices, this result is also one of our motivations here.

A semimodular lattice L is finite by definition, whence it has $0 = 0_L$ and $1 = 1_L$. For lattices L and K with 0 and 1, a lattice homomorphism $\varphi\colon L \to K$ is a *$\{0,1\}$-preserving homomorphism* if $\varphi(0) = 0$ and $\varphi(1) = 1$. If, in addition, φ is injective, then φ is a *$\{0,1\}$-preserving embedding*. If $\varphi\colon L \to K$ is a lattice homomorphism with the property that $\varphi(x) \prec \varphi(y)$ in K whenever $x \prec y$ in L, then φ is *cover-preserving*. Using that the kernel (congruence) of φ is determined by the covering pairs it collapses (and that we only deal with finite lattices), it follows that a cover-preserving lattice homomorphism is necessarily an embedding. This explains partly that

$$\left.\begin{array}{l}\text{by a } \textit{length-preserving embedding} \text{ we mean a lattice homomor-}\\ \text{phism that is both } \{0,1\}\text{-preserving and cover-preserving.}\end{array}\right\} \tag{1.6}$$

The *length* of a finite lattice L is $\text{length}(L) := \max\{|C| - 1 : C \text{ is a chain in } L\}$. Using that a finite semimodular lattice satisfies the Jordan–Hölder chain condition, it is easy to see that for finite semimodular lattices L and K,

$$\left.\begin{array}{l}\text{if there is a length-preserving embedding } L \to K, \text{ then}\\ \text{length}(L) = \text{length}(K), \text{ and}\end{array}\right\} \tag{1.7}$$

$$\left.\begin{array}{l}\text{if } \varphi\colon L \to K \text{ is a lattice embedding and length}(L) =\\ \text{length}(K), \text{ then } \varphi \text{ is a length-preserving embedding.}\end{array}\right\} \tag{1.8}$$

The terminology "length-preserving embedding" is also explained by (1.7). To formulate our results, we define two categories.

$$\left.\begin{array}{l}\text{Let } \mathcal{S}_{\mathbf{01}} \text{ denote the category of slim semimodular lat-}\\ \text{tices with } \{0,1\}\text{-preserving homomorphisms.}\end{array}\right\} \tag{1.9}$$

$$\left.\begin{array}{l}\text{Let } \mathcal{S}_{\mathbf{len}} \text{ denote the category of slim semimodular}\\ \text{lattices with length-preserving embeddings; see (1.6).}\end{array}\right\} \tag{1.10}$$

With more details, (1.10) says that the objects of $\mathcal{S}_{\mathbf{len}}$ are the slim semimodular lattices and the morphisms of $\mathcal{S}_{\mathbf{len}}$ are the length-preserving embeddings among these lattices, and analogously for (1.9).

1.4. The results of the paper

Now, based on Definitions 1.1–1.3 and notation (1.10), we are in the position to formulate the main result of the paper.

Theorem 1.5 (Main Theorem). *For a slim semimodular lattice L, the following three conditions are equivalent.*

(M1) *L is an absolute retract for $\mathcal{S}_{\mathbf{len}}$.*
(M2) *L is a maximal object of $\mathcal{S}_{\mathbf{len}}$.*
(M3) *L is a slim patch lattice or $|L| \le 2$.*

This theorem clearly yields the following corollary, which explains the title of the paper. Let $\mathcal{S}^{\ge 3}_{\mathbf{len}}$ denote the full subcategory of $\mathcal{S}_{\mathbf{len}}$ consisting of at least three-element slim semimodular lattices and all $\mathcal{S}_{\mathbf{len}}$-morphisms among them.

Corollary 1.6. *In $\mathcal{S}^{\ge 3}_{\mathbf{len}}$, slim patch lattices are characterized as absolute retracts. Also, slim patch lattices are characterized as the maximal objects of $\mathcal{S}^{\ge 3}_{\mathbf{len}}$.*

It is not rare that a class of "important objects" in algebra (and in some other fields of mathematics) has a category theoretical characterization. Corollary 1.6 gives two such characterizations of the class of slim patch lattices. Hence, in addition to (1.1) and the original motivation of introducing these lattices in Czédli and Schmidt [18], Corollary 1.6 is another sign that slim patch lattices deserve attention. So is the following corollary, which is based on Definitions 1.1 and 1.4; it will be proved in Section 2.

Corollary 1.7. *For a slim semimodular lattice L, the following three conditions are equivalent.*

(i) *L is a slim patch lattice or $|L| \le 2$.*
(ii) *L is algebraically closed in $\mathcal{S}_{\mathbf{len}}$.*
(iii) *L is strongly algebraically closed in $\mathcal{S}_{\mathbf{len}}$.*

Next, we turn our attention to category $\mathcal{S}_{\mathbf{01}}$; see (1.9). It is not hard to see that there is no maximal object in $\mathcal{S}_{\mathbf{01}}$. (For example, this will prompt follow from (2.1).) The counterpart of the Main Theorem for this category is the following.

Proposition 1.8. *Let L be a slim semimodular lattice. Then L is an absolute retract for $\mathcal{S}_{\mathbf{01}}$ if and only if L is an at most 4-element boolean lattice.*

Similarly to Corollary 1.7, we have the following statement.

Corollary 1.9. *For a slim semimodular lattice L, the following three conditions are equivalent.*

(i) *L is algebraically closed in $\mathcal{S}_{\mathbf{01}}$.*
(ii) *L is strongly algebraically closed in $\mathcal{S}_{\mathbf{01}}$.*
(iii) *L is an at most 4-element boolean lattice.*

2. Proofs

Whenever we deal with a slim semimodular lattice, we always assume that a *planar diagram* of this lattice *is fixed*. Some of the concepts we are going to use depend on how this diagram is chosen but this will not cause any trouble. Below, for the sake of our proofs, we recall some concepts and statements from earlier papers. These concepts are also given in the book chapter of Czédli and Grätzer [9].

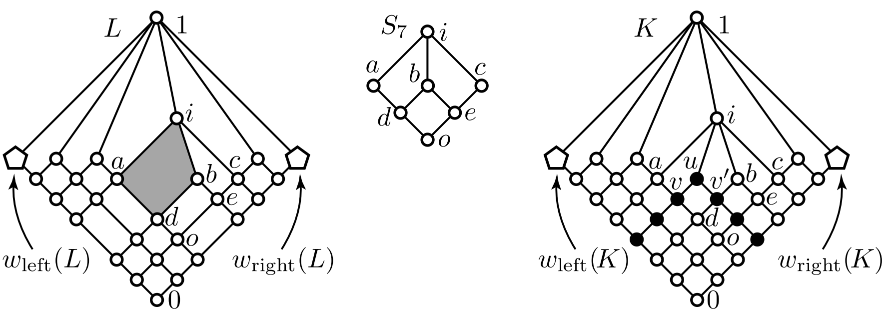

Figure 1. Three slim patch lattices

A cover-preserving four-element boolean sublattice of a slim semimodular lattice L is called a *4-cell*. Given a 4-cell C of this L, we can obtain a so-called *fork extension* of L by *adding a fork* to the 4-cell C of L as it is shown in Figure 5 of Czédli and Schmidt [17]; this is also shown here in Figure 1, where we add a fork to the grey-filled 4-cell of L to obtain K. (The new elements, that is, the elements of $K \setminus L$, are black-filled.) Note that a fork extension K of L is always a *proper extension*, that is, the original lattice is a *proper sublattice* in it; *proper* means that $L \neq K$. We know from Czédli and Schmidt [17, Theorem 11] that

$$\left.\begin{array}{l}\text{a fork extension of a slim semimodular lattice}\\ \text{is again a slim semimodular lattice.}\end{array}\right\} \tag{2.1}$$

For a slim semimodular lattice L, keeping in mind that it is planar and a planar diagram of L is fixed, the *left boundary chain* and the *right boundary chain* of L are denoted by $B_{\text{left}}(L)$ and $B_{\text{right}}(L)$, respectively. The union of these two chains is the *boundary* of L; it is denoted by $\text{Bnd}(L)$. The elements of $\text{Bnd}(L)$ and the edges among these elements form a polygon in the plane, the *boundary polygon* of (the fixed diagram of) L. Following Grätzer and Knapp [22], a slim semimodular lattice L is a *slim rectangular lattice* if $B_{\text{left}}(L)$ has exactly one doubly irreducible element, denoted by $w_{\text{left}}(L)$, $B_{\text{right}}(L)$ has exactly one doubly irreducible element, denoted by $w_{\text{right}}(L)$, and these two elements are complementary, that is,

$$w_{\text{left}}(L) \wedge w_{\text{right}}(L) = 0 \quad \text{and} \quad w_{\text{left}}(L) \vee w_{\text{right}}(L) = 1. \tag{2.2}$$

Note that the original definition of slim patch lattices in Czédli and Schmidt [18] is the following:

$$\left.\begin{array}{l}\text{a lattice } L \text{ is a \textit{slim patch lattice} if it is a slim rectangular}\\ \text{lattice such that } w_{\text{left}}(L) \text{ and } w_{\text{right}}(L) \text{ are coatoms.}\end{array}\right\} \tag{2.3}$$

The doubly irreducible coatoms of each of the three slim patch lattices in Figure 1 are the pentagon-shaped elements.

For $u \in L$, the *ideal* $\{x \in L : x \leq u\}$ and the filter $\{x \in L : x \geq u\}$ are denoted by $\downarrow u$ and $\uparrow u$, respectively. We know from Lemmas 3 and 4 of Grätzer and Knapp [22] that for any rectangular lattice L,

$$\downarrow w_{\text{left}}(L),\ \downarrow w_{\text{right}}(L),\ \uparrow w_{\text{left}}(L), \text{ and } \uparrow w_{\text{right}}(L) \text{ are chains,} \tag{2.4}$$

$$\uparrow w_{\text{left}}(L) \setminus \{1\} \subseteq \text{M}(L), \qquad \uparrow w_{\text{right}}(L) \setminus \{1\} \subseteq \text{M}(L), \tag{2.5}$$

$$\downarrow w_{\text{left}}(L) \setminus \{0\} \subseteq \text{J}(L), \text{ and } \downarrow w_{\text{right}}(L) \setminus \{0\} \subseteq \text{J}(L). \tag{2.6}$$

The direct product of two non-singleton chains is a *grid*. Clearly, grids are distributive slim rectangular lattices. We know from (the last sentence of) Theorem 11 and Lemma 22 in [17] that

$$\left.\begin{array}{l}L \text{ is a \textit{slim rectangular} lattice if and only if it can be ob-}\\ \text{tained from a grid by adding forks, one by one, in a finite}\\ \text{(possibly zero) number of steps.}\end{array}\right\} \tag{2.7}$$

Without needing it later, we mention that grids and distributive slim rectangular lattices are the same. (To derive this from (2.7), observe that if we add a fork to a 4-cell C, then C turns into S_7 from Figure 1 and S_7 is not distributive.)

For $x \in \mathrm{J}(L)$ and $y \in \mathrm{M}(L)$, the unique lower cover of x and the unique cover of y are denoted by x^- and y^+, respectively. Following Czédli and Schmidt [17] again, a *corner* of a slim semimodular lattice L is a doubly irreducible element u (necessarily on the boundary of L) such that u^+ covers exactly two elements and u^- is covered by exactly two elements. Note that $w_{\text{left}}(L)$ and $w_{\text{right}}(L)$ in (2.2) and (2.3) are not corners in general and in Figure 1; these two elements are *weak corners* in the sense of in Czédli and Schmidt [17]. We know from Lemma 21 of [17] that

$$\left.\begin{array}{l}\text{a lattice } L \text{ is a slim semimodular lattice if and only if } |L| \le 2\\ \text{or } L \text{ can be obtained from a slim rectangular lattice by}\\ \text{removing finitely many corners, one by one.}\end{array}\right\} \quad (2.8)$$

Let u be a corner of a slim rectangular lattice K, and denote by v the unique cover of u^- such that $v \neq u$. Then $u \parallel v$. So if we extend $\{v\}$ to a maximal chain C of K, then $u \notin C$ and C is a maximal chain of $K \setminus L$. Hence, the lattices K and $K \setminus \{u\}$ are of the same length, and we obtain from (1.8) that

$$\left.\begin{array}{l}\text{if } L, K \in \mathcal{S}_{\mathbf{len}} \text{ and } L \text{ is obtained from } K \text{ by removing}\\ \text{a corner of } K\text{, then the embedding } \iota\colon L \to K \text{ defined}\\ \text{by } \iota(x) = x \text{ for all } x \in L \text{ is a monomorphism in } \mathcal{S}_{\mathbf{len}}.\end{array}\right\} \quad (2.9)$$

The *congruence lattice* of a lattice L will be denoted by $\mathrm{Con}(L)$, and Δ_L will stand for the *equality relation* $\{(x,x) : x \in L\}$, which is the least element of $\mathrm{Con}(L)$. For $\Theta \in \mathrm{Con}(L)$ and $H \subseteq L$, the *restriction* $\{(x,y) \in H^2 : (x,y) \in \Theta\}$ of Θ to H will be denoted by $\Theta\rceil_H$. For $\Theta \in \mathrm{Con}(L)$ and $u \in L$, we denote by u/Θ the Θ-*block* $\{x \in L : (u,x) \in \Theta\}$ of u. We often write $u \mathrel{\Theta} v$ instead of $(u,v) \in \Theta$.

For $a, b \in L$, the least congruence of L containing the pair (a,b) will be denoted by $\mathrm{con}(a,b)$. If $[a,b]$ is a *prime interval*, which means that b covers a, in notation $a \prec b$, then the prime intervals collapsed by $\mathrm{con}(a,b)$ are efficiently described by G. Grätzer's Swing Lemma. Below, for the reader's convenience, we recall this lemma from Grätzer [19]; alternatively, see Czédli, Grätzer, and Lakser [12] or Czédli and Makay [14] for secondary sources. To state the lemma, we need some definitions and notations. For prime intervals $[p_0,p_1]$ and $[q_0,q_1]$ of a slim semimodular lattice L, we say that $[p_0,p_1]$ is *up-perspective* to $[q_0,q_1]$ or, equivalently, $[q_0,q_1]$ is *down-perspective* to $[p_0,p_1]$ if $p_1 \wedge q_0 = p_0$ and $p_1 \vee q_0 = q_1$. For these two prime intervals, we say that $[p_0,p_1]$ *swings* to $[q_0,q_1]$ if $q_1 = p_1$, q_1 has at least three lower covers in L, and q_0 is join-reducible in the interval $[q_{1*}, q_1]$ where q_{1*} stands for the meet of all lover covers of q_1. For example, in the lattice K given in Figure 1, $[u,i]$ and $[b,i]$ mutually swing to each other. However, $[a,i]$ swings to $[u,i]$ but $[u,i]$ does not swing to $[a,i]$.

Lemma 2.1 (Swing Lemma, Grätzer [19]). *Let $[a,b]$ and $[x,y]$ be prime intervals of a slim semimodular lattice L. Then $(x,y) \in \mathrm{con}(a,b)$ if and only if there is a finite sequence $[u_0,v_0]$, $[u_1,v_1]$, ..., $[u_n,v_n]$ of not necessarily distinct prime intervals of L such that $[u_n,v_n]=[x,y]$, $[a,b]$ is up-perspective to $[u_0,v_0]$, and for all $i \in \{1,\dots,n\}$, $[u_{i-1},v_{i-1}]$ is either down-perspective or swings to $[u_i,v_i]$.*

Armed with the tools and notations listed so far in this section, we are prepared for the proof of our main result.

Proof of Theorem 1.5. First, to show the implication (M2) ⇒ (M1), assume that L is a maximal object for $\mathcal{S}_{\mathbf{len}}$. Now if $\iota\colon L \to K$ is a monomorphism in $\mathcal{S}_{\mathbf{len}}$, then ι is an isomorphism since L is a maximal object. Hence, we can let $\rho := \iota^{-1}\colon K \to L$, and $\rho \circ \iota = \mathrm{id}_L$ is clear. Thus, L is an absolute retract for $\mathcal{S}_{\mathbf{len}}$, proving the implication (M2) ⇒ (M1).

We recall from Grätzer and Nation [23] that

$$\left.\begin{array}{l}\text{if } C \text{ is a maximal chain of a finite semimodular}\\ \text{lattice } L\text{, then every congruence } \Theta \text{ of } L \text{ is deter-}\\ \text{mined by its restriction } \Theta\rceil_C \text{ to } C.\end{array}\right\} \tag{2.10}$$

Note that Grätzer and Nation [23] proved a more general result by allowing "finite length" instead of "finite".

Next, to show the implication (M1) ⇒ (M2), assume that $L \in \mathcal{S}_{\mathbf{len}}$ is an absolute retract for $\mathcal{S}_{\mathbf{len}}$, and let $\iota\colon L \to K$ be a monomorphism in $\mathcal{S}_{\mathbf{len}}$. By (1.3), ι is injective. Take a maximal chain C in L. Then $\iota(C) := \{\iota(x) : x \in C\}$ is a maximal chain in K since ι is a length-preserving embedding. Since L is an absolute retract for $\mathcal{S}_{\mathbf{len}}$, ι has a retraction $\rho\colon K \to L$ in $\mathcal{S}_{\mathbf{len}}$. For later reference, let us mention that in the rest of our argument proving the implication (M1) ⇒ (M2),

$$\left.\begin{array}{l}\text{to prove that our length-preserving embedding } \iota\\ \text{is an isomorphism, we only use that } \rho \text{ is a lattice}\\ \text{homomorphism such that } \rho\circ\iota = \mathrm{id}_L\text{, but we do}\\ \text{not use that } \rho \text{ is a morphism in } \mathcal{S}_{\mathbf{len}}.\end{array}\right\} \tag{2.11}$$

Let $\Theta \in \mathrm{Con}(K)$ denote the kernel of ρ. Since $\rho \circ \iota = \mathrm{id}_L$ gives that, for every $\iota(x) \in \iota(C)$, $\rho(\iota(x)) = x$, the restriction of ρ to $\iota(C)$ is injective. Hence, $\Theta\rceil_{\iota(C)} = \Delta_{\iota(C)}$. Applying (2.10), we obtain that $\Theta = \Delta_K$. Hence, ρ is injective. But it is also surjective since $\rho \circ \iota = \mathrm{id}_L$. Thus, ρ is a lattice isomorphism, whereby it has an inverse, $\rho^{-1}\colon L \to K$, which is also a lattice isomorphism. Using that $\rho\circ\iota = \mathrm{id}_L$, we obtain that $\iota = \mathrm{id}_K \circ \iota = (\rho^{-1}\circ\rho)\circ\iota = \rho^{-1}\circ(\rho\circ\iota) = \rho^{-1}\circ \mathrm{id}_L = \rho^{-1}$, showing that ι is a lattice isomorphism. Thus, L is a maximal object of $\mathcal{S}_{\mathbf{len}}$, and we have proved the implication (M1) ⇒ (M2).

Next, to prove the implication (M2) ⇒ (M3), assume that L is a maximal object of $\mathcal{S}_{\mathbf{len}}$ such that $|L| \geq 3$. It follows from (2.8) and (2.9) that L is a slim rectangular lattice. Hence, $w_{\mathrm{left}}(L)$ and $w_{\mathrm{right}}(L)$ make sense. We claim that

$$w_{\mathrm{left}}(L) \text{ and } w_{\mathrm{right}}(L) \text{ are coatoms.} \tag{2.12}$$

For the sake of contradiction, suppose that (2.12) fails and, say, $w_{\mathrm{left}}(L)$ is not a coatom. Then $\uparrow w_{\mathrm{left}}(L)$, which is a chain by (2.4), has at least three elements. Hence, there are unique elements $u, v \in \uparrow w_{\mathrm{left}}(L)$ such that $u \prec v \prec 1$. Extend L to a poset $K := L \cup \{d\}$ so that $d \notin L$ and $u \prec d \prec 1$. In the diagram of K, we position d to the left of v. Since $u \in \mathrm{M}(L)$ by (2.5), we have that $u = d^-$ has exactly two covers in K. Hence, it follows from Proposition 10(i) of Czédli and Schmidt [17] that $K \in \mathcal{S}_{\mathbf{len}}$. Since d is doubly irreducible in K, $L = K \setminus \{d\}$ is a sublattice of K. Clearly, $\mathrm{length}(L) = \mathrm{length}(K)$ since a maximal chain of K that extends $\{v\}$ is also a maximal chain of L. It follows from (2.9) that $\iota\colon L \to K$ defined by $\iota(x) = x$ for all $x \in L$ is a length-preserving embedding, that is, a monomorphism of $\mathcal{S}_{\mathbf{len}}$. However, ι is not an isomorphism since $|K| = |L \cup \{d\}| = |L| + 1 > |L|$. This contradict the assumption that L is a maximal object of $\mathcal{S}_{\mathbf{len}}$ and proves the implication (M2) $\Rightarrow$ (M3).

Finally, to prove the validity of (M3) $\Rightarrow$ (M2), observe that the one-element lattice and the two-element lattice are trivially maximal objects of $\mathcal{S}_{\mathbf{len}}$. So we assume that L is a slim patch lattice, and we need to show that it is a maximal object of $\mathcal{S}_{\mathbf{len}}$. In fact, it suffices to show that an isomorphic copy of L is a maximal object in $\mathcal{S}_{\mathbf{len}}$; this is why we can take the map $x \mapsto x$ instead of a more involved embedding below.

For the sake of contradiction, suppose that L is not a maximal object and take a slim semimodular lattice K such that L is a proper sublattice of K and the map $L \to K$ defined by $x \mapsto x$ is a length-preserving embedding. In particular, we know from (1.6) and (1.7) that $\mathrm{length}(L) = \mathrm{length}(K)$, $0 := 0_K = 0_L$, $1 := 1_K = 1_L$, and L is a cover-preserving sublattice of K; that is, for all $x, y \in L$, if $x \prec_L y$ in L, then $x \prec_K y$ in K. Since φ is injective, we also know that

$$\text{for } x, y \in L, \qquad x \prec_L y \iff x \prec_K y. \tag{2.13}$$

Fix a planar diagram of K, and pick an element $p \in K \setminus L$. It follows from the Jordan–Hölder chain condition and $\mathrm{length}(L) = \mathrm{length}(K)$ that p is not on any edge of L, and similarly for any other element of $K \setminus L$. By (2.13), the edges of L in the fixed diagram of K are among the edges of K. Therefore, if we remove the elements of $K \setminus L$ with all edges adjacent to them, then we get a planar diagram of L; let this diagram be what we fix for L. We know that $L \subset K$. (As opposed to some other branches of mathematics, "$\subset$" is the conjunction of "$\subseteq$" and "$\neq$".)

By the classical Jordan curve theorem, the boundary polygon of L divides the plane into three pairwise disjoint subsets: an interior region, an exterior region, and the (set of geometrical points on the) boundary polygon. The first two subsets are topologically open while the third one is closed. Since p is not on any edge of L, it is not on the boundary polygon. Therefore,

p is either in the interior region of the boundary polygon, or it is in the exterior region of the boundary polygon; these two possibilities need separate treatments.

First, assume that p is in the interior region of the boundary polygon. Since this region is divided into 4-cells by Lemma 4 of Grätzer and Knapp [21] and p is not on any edge of L, the element p is inside the topologically open region determined by a 4-cell $H = \{b = u \wedge v, u, v, t = u \vee v\}$ of L. By Kelly and Rival [24, Proposition 1.4], we obtain that $b < p < t$. Using the Jordan–Hölder chain condition and $b \prec u \prec t$, it follows that $b \prec p \prec t$. Thus, u, v, and p are three different covers of b, which contradicts Lemma 8 of Grätzer and Knapp [21].

Second, assume that p is in the exterior region of the boundary polygon. Note that $0 < p < 1$. Take a maximal chain C in K that contains p. Since p is in the (topologically open) exterior region of the boundary polygon but $0 \in C$ is not, there are consecutive elements $r \prec s$ of C such that s is in the exterior region of the boundary polygon but r is not. Using planarity or, to be more precise, Kelly and Rival [24, Lemma 1.2], we obtain that $r \in \mathrm{Bnd}(L)$. In particular, $r \in L$. By left–right symmetry, we can assume that $r \in B_{\mathrm{left}}(L)$. By the already mentioned Lemma 8 of Grätzer and Knapp [21], r has at most two covers in K. Since $s \in K$ is a cover of r and $r \neq 1$ yields that r also has at least one cover in L, we obtain that r has exactly one cover in L. That is, $r \in \mathrm{M}(L)$. Since L is a slim patch lattice, $B_{\mathrm{left}}(L)$ is the disjoint union of $\downarrow w_{\mathrm{left}}(L) \setminus \{w_{\mathrm{left}}(L)\}$, $\{w_{\mathrm{left}}(L)\}$, and $\{1\}$.

If we had that $r \in \downarrow w_{\mathrm{left}}(L) \setminus \{w_{\mathrm{left}}(L)\}$, then $r \in \mathrm{M}(L)$ would belong to $\mathrm{J}(L)$ by (2.6), so r would be doubly irreducible, we would have that $r = w_{\mathrm{right}}(L)$ by Definition 1.1, and $w_{\mathrm{right}}(L) = r = w_{\mathrm{left}}(L) \wedge r = w_{\mathrm{left}}(L) \wedge w_{\mathrm{right}}(L) = 0$ would be a contradiction. Hence, taking also $r \neq 1$ into account, we have that $r = w_{\mathrm{left}}(L)$. Thus, $w_{\mathrm{left}}(L) \prec_K s \leq p < 1$, which gives that 1 does not cover $w_{\mathrm{left}}(L)$ in K. This contradicts the facts that $w_{\mathrm{left}}(L)$ is a coatom in L and L is a cover-preserving sublattice of K.

Regardless the position of p, we have obtained a contradiction. This yields the implication (M3) $\Rightarrow$ (M2) and completes the proof of Theorem 1.5. □

Proof of Proposition 1.8. To prove the "only if" part, assume that $L \in \mathcal{S}_{\mathbf{01}}$ is an absolute retract for $\mathcal{S}_{\mathbf{01}}$. We are going to show that L is a maximal object of $\mathcal{S}_{\mathbf{len}}$. Let $\iota\colon L \to K$ be a monomorphism in $\mathcal{S}_{\mathbf{len}}$. By (1.3) and (1.6), ι is also a monomorphism in $\mathcal{S}_{\mathbf{01}}$. Since we have assumed that L is an absolute retract for $\mathcal{S}_{\mathbf{01}}$, there exists a $\{0,1\}$-preserving homomorphism $\rho\colon K \to L$ such that $\rho \circ \iota = \mathrm{id}_L$. Applying (2.11), it follows that ι is an isomorphism. This shows that L is a maximal object of $\mathcal{S}_{\mathbf{len}}$, as required. Thus, we obtain from Theorem 1.5 that $|L| \leq 2$ or L is a slim patch lattice. We can assume that L is a slim patch lattice since lattices with at most two elements are boolean. Then $|L| \geq 4$.

For the sake of contradiction, suppose that $|L| \geq 5$. We know from (2.7) that L can be obtained from a grid G by adding forks. When we add a fork to a slim rectangular lattice R, then $w_{\mathrm{left}}(R)$ and $w_{\mathrm{right}}(R)$ remain doubly irreducible and the lengths of the intervals $[w_{\mathrm{left}}(R),1]$ and $[w_{\mathrm{right}}(R),1]$ do not change. Since L is not only rectangular but it is a patch lattice, $w_{\mathrm{left}}(G)$ and $w_{\mathrm{right}}(G)$ are coatoms of G. This means that G is the 4-element boolean lattice. Then, since $|G| = 4 < 5 \le |R|$, it follows that at least one fork has been added to G to obtain L. Thus, thinking of the last fork added, we obtain that the lattice S_7 given in the middle of Figure 1 is a cover-preserving sublattice of L.

The elements of this S_7 will be denoted as in Figure 1. Take the upper left 4-cell of this S_7; it is grey-filled on the left of Figure 1. Add a fork to L to obtain a new lattice denoted by K; see on the right of the figure. The new meet-irreducible element is denoted by u, its lower covers are v and v', as it is shown in the figure. By (2.1), $K \in \mathcal{S}_{\mathbf{01}}$. Clearly, the embedding $\iota\colon L \to K$ defined by $x \mapsto x$ is a morphism in $\mathcal{S}_{\mathbf{01}}$. Since we have assumed that L is an absolute retract for $\mathcal{S}_{\mathbf{01}}$, ι has a retraction $\rho\colon K \to L$ in $\mathcal{S}_{\mathbf{01}}$. That is, ρ is a $\{0,1\}$-preserving-homomorphism such that $\rho\circ\iota = \mathrm{id}_L$. In particular, $\rho(x) = x$ for all $x \in L$. As in the previous proof, we let $\Theta = \ker\rho$. Observe that since $\rho(x) = x$ for all $x \in L$,

$$\text{the restriction } \Theta \text{ to } L \text{ is } \Delta_L, \text{ that is, } \Theta\rceil_L = \Delta_L. \tag{2.14}$$

Since $\rho(u) = (\rho\circ\iota)(\rho(u)) = \rho((\iota\circ\rho)(u)) = \rho(\iota(\rho(u))) = \rho(\rho(u))$, we have that $u \mathrel{\Theta} \rho(u)$. Also, $u \neq \rho(u)$ since $\rho(u) \in L$ but $u \notin L$. Depending on whether $u \nleq \rho(u)$ or $u \ngeq \rho(u)$, we have that $u > u\wedge\rho(u)$ or $u < u\vee\rho(u)$. Since $(u,\rho(u)) \in \Theta$ gives that $\{u\wedge\rho(u), u\vee\rho(u)\} \subseteq u/\Theta$, it follows that u is not a minimal element or not a maximal element of u/Θ. Using that u/Θ is a convex subset of K and taking into account that u covers or is covered by exactly three elements, i, v, and v', we obtain that at least one of (u,i), (v,u), and (v',u) belongs to Θ. This gives us three cases to consider; each of them leads to contradiction in a different way.

If $(u,i) \in \Theta$, then it follows from (the Swing) Lemma 2.1 that $(b,i) \in \Theta$, contradicting (2.14). If $(v,u) \in \Theta$, then $(a,i) = (a\vee v, a\vee u) \in \Theta$ contradicts (2.14). If $(v',u) \in \Theta$, then $(b,i) = (b\vee v', b\vee u) \in \Theta$, contradicting (2.14) again. Thus, $|L| \ge 5$ leads to a contradiction and it follows that $|L| = 4$. Finally, a four-element patch lattice is boolean, and we have shown the "only if" part of Proposition 1.8.

Next, in order to prove the "if" part, assume that L is a boolean lattice with at most four elements, $L \in \mathcal{S}_{\mathbf{01}}$, and $\iota\colon L \to L'$ is a monomorphism in $\mathcal{S}_{\mathbf{01}}$. That is, ι is a lattice embedding preserving 0 and 1. We need to find a morphism ρ in $\mathcal{S}_{\mathbf{01}}$ such that $\rho\circ\iota = \mathrm{id}_L$. If $|L| = 1$, then the preservation of 0 and 1 gives that $|L'| = 1$, ι is an isomorphism, and we can let $\rho := \iota^{-1}$. Hence, in the rest of the proof, it suffices to deal with the cases $|L| = 2$ and $|L| = 4$. Before doing so, let us recall from Grätzer [20, Corollary 14] that

$$\left.\begin{array}{l}\text{if } 1 \neq p \in {\uparrow}w_{\mathrm{left}}(K) \cup {\uparrow}w_{\mathrm{right}}(K) \text{ in a slim rectan-}\\ \text{gular lattice } K, \text{ then } {\downarrow}p \text{ is a prime ideal of } K.\end{array}\right\} \tag{2.15}$$

First, assume that $|L| = 2$. Then $L = \{0,1\}$. We can assume that $|L'| > 2$ since otherwise ι is an isomorphism and $\rho := \iota^{-1}$ does the job. It follows from (2.8) and (2.9) that there exists a slim rectangular lattice K and a monomorphism $\iota'\colon L' \to K$ in $\mathcal{S}_{\mathbf{01}}$. (In fact, ι' belongs even to $\mathcal{S}_{\mathbf{len}}$.) By (1.3), ι' is a lattice embedding. We know from (2.15) that $I := \mathord{\downarrow} w_{\text{left}}(K)$ is a prime ideal of K. Hence, the map

$$\rho'\colon K \to L, \text{ defined by } x \mapsto \begin{cases} 0, & \text{if } x \in I,\\ 1, & \text{if } x \in K \setminus I,\end{cases}$$

is a lattice homomorphism. In fact, ρ' is a morphism in $\mathcal{S}_{\mathbf{01}}$. Let $\rho := \rho' \circ \iota'$. Then ρ is a map from L' to L. Since both ρ' and ι' are morphisms in $\mathcal{S}_{\mathbf{01}}$, so is their product, ρ. By the same reason, $\rho \circ \iota\colon L \to L$ is again a morphism in $\mathcal{S}_{\mathbf{01}}$. Since $L = \{0,1\}$, id_L is the only $L \to L$ morphism belonging to $\mathcal{S}_{\mathbf{01}}$. Hence, $\rho \circ \iota = \mathrm{id}_L$, showing that ρ is a retraction of ι. Therefore, L is an absolute retract for $\mathcal{S}_{\mathbf{01}}$.

Second, assume that $|L|$ is the four-element boolean lattice and $\iota\colon L \to L'$ is a monomorphism in $\mathcal{S}_{\mathbf{01}}$. As in the previous case, (1.3), (2.8), and (2.9) yield that there exists a slim rectangular lattice K and a lattice embedding $\iota'\colon L' \to K$ such that ι' is a monomorphism in $\mathcal{S}_{\mathbf{01}}$. As previously, $I := \mathord{\downarrow} w_{\text{left}}(K)$ is a prime ideal of K, and so is $J := \mathord{\downarrow} w_{\text{right}}(K)$. The atoms of L will be denoted by u and v, and we let $\widehat{u} := (\iota' \circ \iota)(u) = \iota'(\iota(u))$ and $\widehat{v} := (\iota' \circ \iota)(v)$. Of course, we have that $(\iota' \circ \iota)(0) = 0$ and $(\iota' \circ \iota)(1) = 1$ since we are in $\mathcal{S}_{\mathbf{01}}$. Since $\iota' \circ \iota$ is a lattice embedding, $\widehat{u} \wedge \widehat{v} = 0$, $\widehat{u} \vee \widehat{v} = 1$, and $|\{0, \widehat{u}, \widehat{v}, 1\}| = 4$. Since I and J are prime ideals, we can observe that $\{\widehat{u}, \widehat{v}\} \nsubseteq I$ and $\{\widehat{u}, \widehat{v}\} \nsubseteq K \setminus I$, and analogously for J, since otherwise $\widehat{u} \vee \widehat{v} = 1$ or $\widehat{u} \wedge \widehat{v} = 0$ would fail. That is

$$|\{\widehat{u}, \widehat{v}\} \cap I| = 1 \quad \text{and} \quad |\{\widehat{u}, \widehat{v}\} \cap J| = 1. \tag{2.16}$$

Thus, using that u and v play symmetrical roles, we can assume that $\widehat{u} \in I$ but $\widehat{v} \notin I$. It follows from (2.2) that $I \cap J = \{0\}$. This fact and $0 \neq \widehat{u} \in I$ give that $\widehat{u} \notin J$. Combining this with (2.16), we obtain that $\widehat{v} \in J$. So (2.16) gives that

$$\widehat{u} \in I, \quad \widehat{u} \notin J, \quad \widehat{v} \in J, \quad \text{and} \quad \widehat{v} \notin I. \tag{2.17}$$

Applying (2.2) again, we conclude easily that

$$w_{\text{left}}(K) \in I,\ w_{\text{left}}(K) \notin J,\ w_{\text{right}}(K) \in J, \text{ and } w_{\text{right}}(K) \notin I. \tag{2.18}$$

Since I is a prime ideal, the equivalence α with blocks I and $K \setminus I$ is a congruence of K. Similarly, let β be the congruence with blocks J and $K \setminus J$. Let $\gamma := \alpha \wedge \beta = \alpha \cap \beta \in \mathrm{Con}(K)$. Since each of α and β has only two blocks, γ has at most four blocks. It follows from (2.18) that 0, 1, $w_{\text{left}}(K)$, and $w_{\text{right}}(K)$ are in different γ-blocks. Hence γ has exactly four blocks. Therefore, using that $w_{\text{left}}(K)$ and $w_{\text{right}}(K)$ are complementary and that 0, 1, $w_{\text{left}}(K)$, and $w_{\text{right}}(K)$ are in different γ-blocks, it follows that K/γ is isomorphic to L and the map

$$\rho' \colon K \to L, \text{ defined by } x \mapsto \begin{cases} u, & \text{if } (x, w_{\text{left}}(K)) \in \gamma, \\ v, & \text{if } (x, w_{\text{right}}(K)) \in \gamma, \\ 0, & \text{if } (x, 0) \in \gamma, \\ 1, & \text{if } (x, 1) \in \gamma \end{cases} \tag{2.19}$$

is a lattice homomorphism and it is a morphism belonging to $\mathcal{S}_{\mathbf{01}}$. Comparing (2.17) and (2.18), we obtain that $(\widehat{u}, w_{\text{left}}(K)) \in \gamma$ and $(\widehat{v}, w_{\text{right}}(K)) \in \gamma$. Thus, it follows from (2.19) that $\rho'(\widehat{u}) = u$, $\rho'(\widehat{v}) = v$. These two equalities and the fact that all the homomorphisms occur in the proof are morphisms in $\mathcal{S}_{\mathbf{01}}$ imply that $\rho' \circ (\iota' \circ \iota) = \mathrm{id}_L$. In other words, $(\rho' \circ \iota') \circ \iota = \mathrm{id}_L$. Thus, with $\rho := \rho' \circ \iota'$, which also belongs to $\mathcal{S}_{\mathbf{01}}$, we have that $\rho \circ \iota = \mathrm{id}_L$. Since ρ is an $L' \to L$ morphism in $\mathcal{S}_{\mathbf{01}}$, L is an absolute retract for $\mathcal{S}_{\mathbf{01}}$, as required. Hence, the "if" part holds and the proof of Proposition 1.8 is complete. □

Finally, we give a joint proof of two corollaries.

Proof of Corollaries 1.7 and 1.9. Let $\mathcal{Y}$ be $\mathcal{S}_{\mathbf{len}}$ or $\mathcal{S}_{\mathbf{01}}$. A lattice $L \in \mathcal{Y}$ is called an *absolute* **H**-*retract* for $\mathcal{Y}$ if for any monomorphism $\iota \colon L \to K$ of the category $\mathcal{Y}$ there exists a lattice homomorphism $\rho \colon K \to L$ such that $\rho \circ \iota = \mathrm{id}_L$. (We do not require here that ρ is a morphism of $\mathcal{Y}$.) We claim that

$$\left.\begin{array}{l}\text{for } L \in \mathcal{Y}, L \text{ is an absolute retract for } \mathcal{Y} \text{ if and} \\ \text{only if } L \text{ is an absolute } \mathbf{H}\text{-retract for } \mathcal{Y}.\end{array}\right\} \tag{2.20}$$

This is trivial for $\mathcal{Y} = \mathcal{S}_{\mathbf{01}}$ since for a $\{0, 1\}$-preserving homomorphism $\iota \colon L \to K$ and a lattice homomorphism $\rho \colon K \to L$, the equality $\rho \circ \iota = \mathrm{id}_L$ implies that ρ is also $\{0, 1\}$-preserving. For $\mathcal{Y} = \mathcal{S}_{\mathbf{len}}$, (2.20) follows from (2.11).

For $L \in \mathcal{Y}$, we know from Czédli and Molkhasi [15, Proposition 1.1] that X is an absolute **H**-retract for $\mathcal{Y}$ if and only if X is strongly algebraically closed in $\mathcal{Y}$, and the same holds if the adverb "strongly" is removed. This fact, (2.20), Theorem 1.5, and Proposition 1.8 imply Corollaries 1.7 and 1.9. □

3. Declarations

Data availability

Data sharing is not applicable to this article as datasets were neither generated nor analyzed.

Compliance with ethical standards

The author declares that he has no conflict of interest.

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