Periodica Mathematica Hungarica Vol. 23 (1), (1991), pp. 49–58

ON THE WORD PROBLEM OF LATTICES WITH THE HELP OF GRAPHS

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1. The result

The aim of the present paper is to give a new algorithm for the word problem of finitely presented lattices. Although there are known algorithms solving this problem (cf. Dean [2], Evans [3] and McKinsey [5]), our approach is entirely different and the result looks simpler.

The problem is to give an algorithm which decides whether an arbitrary (universally quantified) lattice Horn sentence holds in all lattices. Let us fix a Horn sentence χ as follows:

$$\chi : (p_0 \le q_0 \,\&\, p_1 \le q_1 \,\&\ldots\&\, p_t \le q_t) \Longrightarrow p \le q$$

where $p_0, \ldots, p_t$, $q_0, \ldots, q_t$, p, q are lattice terms over a finite set X of variables. Without loss of generality (similarly to Evans [3]) we may assume that χ is of a canonical form, i.e.,

$q \in X$ (i.e., q is a variable),

$p, p_0, \ldots, p_t$ are join-free (i.e., they are variables or meets of variables),

$q_0, \ldots, q_t$ are meet-free (i.e., variables or joins of variables).

Indeed, there is an obvious algorithm that associates a canonical Horn sentence with a given Horn sentence such that these two Horn sentences are equivalent modulo lattice theory. All we have to do is introducing new variables for subterms and adding somenew relations to the premise. Instead of going into details this will be illustrated by the well-known

$$\mathrm{SD}_\wedge : (x \wedge y \le x \wedge z \,\&\, x \wedge z \le x \wedge y) \Longrightarrow x \wedge (y \vee z) \le x \wedge y.$$

Now we consider new variables a, b, c corresponding to $x \wedge y$, $x \wedge z$ and $y \vee z$, respectively. Then $\mathrm{SD}_\wedge$ is equivalent to

$$\mathrm{SD}'_\wedge : (a \le b \,\&\, b \le a \,\&\, a \le x \,\&\, a \le y \,\&\, x \wedge y \le a \,\&\, b \le x \,\&\, b \le z \,\&\, x \wedge z \le b \,\&\, c \le y \vee z \,\&\, y \le c \,\&\, z \le c) \Longrightarrow x \wedge c \le a.$$

(Note that this is not the shortest possibility. However, it indicates generality.)

Returning to the canonical χ we have fixed, let X, M, M_i and J_i denote the set of variables occuring in χ, p, p_i and q_i, respectively. We define a mapping $T_j : \mathbf{P}(X) \to \mathbf{P}(X)$ via induction for each non-negative integer j where $\mathbf{P}(X)$ is the

* This work was supported by NSERC, Canada, Operating Grant A8190, when started, and by Hungarian National Foundation for Scientific Research Grant No. 1813, when completed.

Mathematics subject classification numbers, 1980/85. Primary 06B25.

Key words and phrases. Word problem, lattice, Horn sentence.

Akadémiai Kiadó, Budapest
Kluwer Academic Publishers, Dordrecht

power set of X. We will write, for $y \in X$, $T_j(y)$ rather than $T_j(\{y\})$. Let T_0 be the identical $\mathbf{P}(X) \to \mathbf{P}(X)$ mapping. For $A \subseteq X$ let

$$T_{j+1}(A) = T_j(a) \cup \bigcup_{0 \le i \le t, M_i \subseteq T_j(A)} \bigcap_{x \in J_i} T_j(x).$$

(Note that $T_j = T_1^j$.) Since X is finite and, for any $A \subseteq X$, $T_0(A) \subseteq T_1(A) \subseteq T_2(A) \subseteq \ldots$, there is an n such that $T_n = T_{n+1}$. Then $T = T_n = T_{n+1} = T_{n+2} = \cdots = T_{2n} = \ldots$ is a closure operator on X and $T(A)$ can be determined by an easy algorithm. Our main result is the following:

THEOREM.. χ holds in all lattices if and only if $q \in T(M)$.

For example, let us consider $\mathrm{SD}'_\wedge$. Then $X = \{x, y, z, a, b, c\}$. We do not have to calculate $T(A)$ for every $A \subseteq X$ and do not need to known n either. To determine $T(M) = T(\{x, c\})$ we need $T(y)$ and $T(z)$, which require $T(c)$. Considering the first few values of j we can see that

$$T_j(y) \subseteq \{y, c\}, \quad T_j(z) \subseteq \{z, c\}, \quad T_j(c) \subseteq \{c\},$$

which follows for all j via induction. This implies $T_j(\{x, c\}) \subseteq \{x, c\}$, whence $T(\{x, c\}) \subseteq \{x, c\}$. Consequently $a \notin T(\{x, c\})$ and we obtain the well-known fact: $\mathrm{SD}_\wedge$ does not hold in all lattices.

2. The proof of the theorem

The original idea of the proof is to apply a slightly modified version of the method described in [1] to the *variety* of sets when all the joins $u \vee v$ in $q_0, \ldots, q_t$ are replaced by $u \circ v \circ u \circ v$, and to refer to the type 3 representability of lattices by equivalences (Jónsson [4]). Yet, the present proof is much shorter and easier to understand even if it does not reveal anything about the motivation.

Let P_χ denote the premise of χ and let $\mathbf{P}^+(X) = \mathbf{P}(X) \backslash \{0\}$. We will prove the following stronger version of the theorem.

Let $x \in X$ and $A \in \mathbf{P}^+(X)$. Then the (canonical) Horn sentence

$$P_\chi \Longrightarrow \bigwedge_{y \in A} y \le x$$

holds in all lattices if and only if $x \in T(A)$.

To prove the sufficiency it sufficies to show that if certain elements of a lattice satisfy P_χ then, for every j,

$$(\forall A \in \mathbf{P}^+(X)) (\forall x \in T_j(A)) (\bigwedge_{y \in A} y \le x).$$

Since this is obvious for $j = 0$ assume it for some j. Let $x \in T_{j+1}(A) \backslash T_j(A)$. Then there is an $i, 0 \le i \le t$, such that $M_i \subseteq T_j(A)$ and

$$x \in \bigcap_{z \in J_i} T_j(z).$$

By the induction hypothesis, $z \leq x$ for $z \in J_i$ and $\bigwedge_{y \in A} y \leq u$ for $u \in T_j(A)$. Thus

$$\bigwedge_{y \in A} y = \bigwedge_{y \in T_j(A)} y \leq \bigwedge_{y \in M_i} y = p_i \leq q_i = \bigvee_{z \in J_i} z \leq x.$$

We have obtained $\bigwedge_{y \in A} y \leq x$ for all $x \in T_{j+1}(A)$, proving the sufficiency of the theorem.

The neccesity part of the proof needs some preliminaries.

We fix an ordering on X, say $X = \{x_1, x_2, \ldots, x_w\}$, and adopt the notation $h(x : x \in X)$ for any lattice term $h(x_1, x_2, \ldots, x_w)$.

For any $A \in \mathbf{P}^+(X)$ and $0 \leq j \leq \omega$, where ω is the least infinite ordinal, we will define a graph $G_j(A)$. Before starting the definition of the these graphs it is reasonable to make some agreements. $G_j(A)$ will have neither loop edges nor parallel edges. It will have two distinguished vertices, the so-called left and right endpoints. They will be figured on the left-hand side and right-hand side, respectively. The edges of $G_j(A)$ will be coloured by the elements of $\mathbf{P}^+(X)$, but we often write x instead of $\{x\} \in \mathbf{P}^+(X)$. The B-coloured edge connecting the vertices a and b will be denoted by (a, B, b) or $(a, ., b)$ if B is irrelevant. The edges $(a, ., b)$ and $(b, ., a)$ are considered equal. Let G be one of the graphs occuring in the sequel. Put $E(G) = \{(a, B, b) : (a, B, b) \text{ is an edge of } G\}$. For technical reasons, we will need the notation $E'(G) = \{(a, B, b) : (a, B, b) \in E(G) \text{ and } a < b\}$. Here $<$ means a fixed well-ordering on the vertex set $V(G)$ of G. To avoid extra technicalities this well-ordering will never be defined but we assume that left endpoint < right endpoint. The edge (left endpoint,.,right endpoint) will have a particular role and will be called the *initial edge* of G. For $B \in \mathbf{P}^+(X)$ the smallest equivalence relation on $V(G)$ that includes $\{(a, b) : (a, C, b) \in E(G) \text{ and } B \subseteq C\}$ is denoted by $e(B, G)$; we will write $e(x, G)$ instead of $e(\{x\}, G)$. If G and H are graphs, $V(G) \subseteq V(H)$, $\{(a, b) : (a, ., b) \in E(G)\} \subseteq \{(a, b) : (a, ., b) \in E(H)\}$ and $(a, B, b) \in E(G)$, $(a, C, b) \in E(H)$ imply $B \subseteq C$ then G is called a *weak subgraph* of H. If, in addition, $\{(a, b) : (a, ., b) \in E(G)\} = \{(a, b) : (a, ., b) \in E(H), a, b \in V(G)\}$ then G is called a *subgraph* of H. If G is a subgraph of H and each edge of G has the same colour in H as in G then G is called a *strong subgraph* of H. If G is a weak subgraph of $G_\omega(A)$ (to be defined later) then G^* will denote the strong subgraph of $G_\omega(A)$ for which $V(G) = V(G^*)$. By isomorphism we will mean a colour-preserving graph isomorphism.

If $x_1 \in X$ then let $H(x_1)$ be the graph consisting of an x-coloured initial edge only (cf. Figure 1).

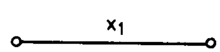

Fig. 1

If $x_1, x_2, \ldots, x_n \in X$ are pairwise distinct then $H(x_1, \ldots, x_n)$ is obtained via a "serial connection" of two copies of $H(x_1, \ldots, x_{n-1})$ alternating with two copies of $H(x_n)$, cf. Figure 2.

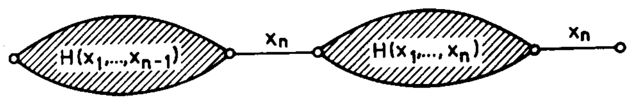

Fig. 2

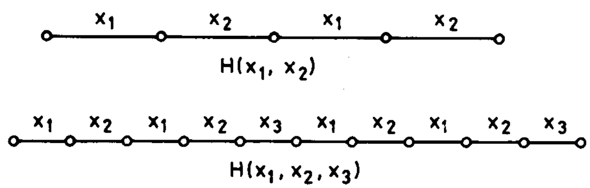

Fig. 3

E.g., $H(x_1, x_2)$ and $H(x_1, x_2, x_3)$ are depicted on Figure 3. This defines $H(x_1, \ldots, x_n)$ for all positive integers n.

If

$$q_i = x_{\ell_1} \vee \cdots \vee x_{\ell_n} \qquad (\text{i.e., } J_i = \{x_{\ell_1}, \ldots, x_{\ell_n}\})$$

then let $H(q_i)$ stand for

$$H(x_{\ell_1}, \ldots, x_{\ell_n}).$$

(To make $H(q_i)$ unique we suppose $\ell_1 < \cdots < \ell_n$.)

Now let $G_0(A)$ be the graph consisting of its A-coloured initial edge only. I.e., $V(G_0(A)) = \{$left endpoint, right endpoint$\}$ and $E'(G_0(A)) = \{$(left endpoint, A, right endpoint)$\}$.

Suppose $G_j(A)$ is alredy defined for some $j < \omega$. Then we obtain $G_{j+1}(A)$ from $G_j(A)$ in three steps. (These steps will be depicted on Figures 4-6 where the special case $t = 4$, $A = \{x\}$,

$$\mathrm{P}_\chi : x \le y \,\&\, x \wedge y \le z \vee u \,\&\, z \le v \,\&\, u \le v \,\&\, x \wedge y \wedge v \le u$$

is considered).

Step (a). For each $(a, B, b) \in E'(G_j(A))$ we replace the colour B of this edge by

$$B \cup \bigcup_{|J_i|=1, M_i \subseteq B} J_i.$$

The graph we obtain this way is denoted by $G'_j(A)$.

Step (b). For each $(a, B, b) \in E'(G'_j(A))$ and for each i such that $M_i \subseteq B, |J_i| \ge 2$ and $(a, b) \notin q_i(e(x, G'_j(A)) : x \in X)$ we take a copy of $H(q_i)$

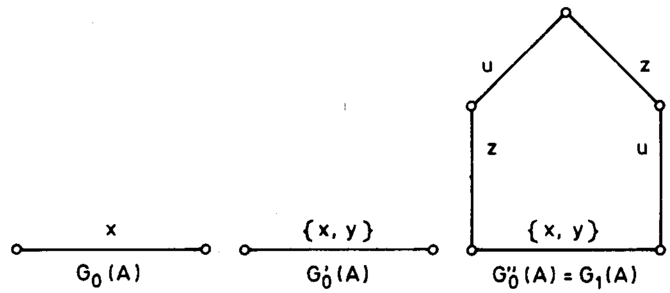

Fig. 4

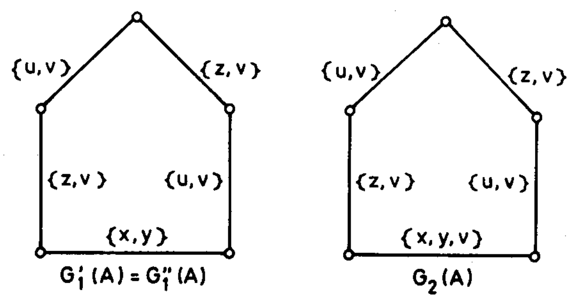

Fig. 5

(all these copies are pairwise distinct and they are distinct from $G_j'(A)$, too) and identify the left resp. right endpoint of $H(q_i)$ with a and b respectively. Motivated by Pudlák and Tuma [6], we call this step as "fermentation of $G_j'(A)$ at $(a, ., b)$ by $H(q_i)$". Performing all the fermentations at the same time we obtain $G_j''(A)$.

Step (c). Changing the colour B of each (a, B, b) to $\{x : x \in X$ and $(a, b) \in e(x, G_j''(A))\}$ on each edge $(a, B, b) \in E'(G_j''(A))$ we obtain $G_{j+1}(A)$.

Now $G_j(A)$ is defined for all $j < \omega$, and $G_j(A)$ is a subgraph of $G_{j+1}(A)$. Let $G_\omega(A)$ be the union of $G_j(A), j < \omega$. I.e., $V(G_\omega(A)) = \cup_{j<\omega} V(G_j(A)), (a, ., b) \in E(G_\omega(A)) \iff (\exists j < \omega)((a, ., b) \in E(G_j(A)))$, and the colour of $(a, ., b) \in E(G_\omega(A))$ is the union of the colours of this edge in $G_j(A)$, $j < \omega$.

Let G be a graph and $(a, ., b) \in E'(G)$. We define a graph $S(a, b; G)$, called the strong subgraph of G *spanned by the edge* $(a, ., b)$, as follows: let $S(a, b; G)$ consist of all vertices c of G such that every path in G connecting c and either endpoint

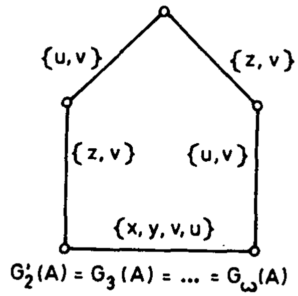

Fig. 6

of G goes through at least one of a and b. (By definition, a path goes through its endpoints.) The left and right endpoints of $S(a,b;G)$ are a and b, respectively. Note that S (left endpoint, right endpoint; G)=G.

We still need to define another kind of subgraphs. Assume that $(a,.,b) \in E'(G_{j-1}(A))$ $(j \geq 1)$ and when we obtained $G_j(A)$ from $G_{j-1}(A)$ then $G'_{j-1}(A)$ was fermented by $H(q_i)$ at $(a,.,b)$. The copy of this $H(q_i)$ whose endpoints where identified with a and b constitutes a weak subgraph of $G_j(A)$. We denote this weak subgraph by $Q(a,b;q_i;G_j(A))$. If we add the edge $(a,.,b) \in E'(G_j(A))$ to this weak subgraph then we obtain a strong subgraph of $G_j(A)$, which will be denoted by $Q(a,.,b;q_i;G_j(A))$.

The following lemma states that, roughly saying, certain subgraphs develop independently from their neighbourhood when $j \to \infty$.

LEMMA 1.. *Assume that* $A \in \mathbf{P}^+(X)$, $0 < j < \omega$, *and* $(a,.,b) \in E'(G_j(A)) \setminus E(G_{j-1}(A))$. *Let* B *denote the colour of* $(a,.,b)$ *in* $G_j(A)$. *Then, for any* $k \leq \omega$, $S(a,b;G_{j+k}(A))$ *is isomorphic to* $G_k(B)$.

PROOF.. We may assume that $k < \omega$. The case $k = 0$ being trivial suppose Lemma 1 is true for some $k < \omega$. Then we have $S(a,b;G'_{j+k}(A)) \cong G'_k(B)$. Let $\varphi : S(a,b;G'_{j+k}(A)) \to G'_k(B)$ be an isomorphism. To prove that $S(a,b;G''_{j+k}(A))$ and $G''_k(B)$ are isomorphic it sufficies to show that, for any $i \leq t$ and every $(u,.,v) \in E(S(a,b;G'_{j+k}(A)))$, $(u,v) \in q_i(e(x,G'_{j+k}(A)) : x \in X)$ iff $(u\varphi, v\varphi) \in q_i(e(x,G'_k(B)) : x \in X)$. The "if" part being evident assume that the "only if" part is false. I.e., suppose $(u,v) \in q_i(e(x,G'_{j+k}(A)) : x \in X)$ but $(u\varphi, v\varphi) \notin q_i(e(x,G'_k(B)) : x \in X)$. Let us say that a path is a q_i-path if the colour of any of its edges is not disjoint from J_i. Now there is a *shortest* q_i-path α in $G'_{j+k}(A)$ connecting u and v. This α cannot be entirely in $S(a,b;G'_{j+k}(A)) \cong G'_k(A)$.

By the assumptions, there is a $(g,.,h) \in E'(G_{j-1}(A))$ and an $m \leq t$ such that $|J_m| \geq 2$ and $(a,B,b) \in E(Q(g,h;q_m;G_j(A)))$. (Note that $|B| = 1$.) Since any $x \in J_m$ occurs, as a colour, in $H(q_m)$ at least twice, there exists a $(c, B, d) \in E(Q(g, h; q_m; G_j(A)))$ such that $\{a, b\} \cap \{c, d\} = \emptyset$ (cf. Figure 7, where S'_a and S'_c stand for $S(a, b; G'_{j+k}(A))$ and $S(c, d; G'_{j+k}(A))$, respectively).

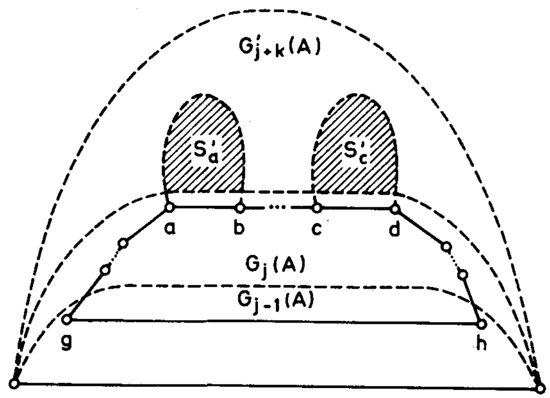

Fig. 7

Since $S(a, b; G_{j+k}(A)) \cong G_k(B) \cong S(c, d; G_{j+k}(A))$ by the induction hypothesis, we obtain $S'_a \cong S'_c$. In particular, $(a, ., b)$ and $(c, ., d)$ have the same colour D in $G'_{j+k}(A)$.

Now the path α, as it is the shortest one and leaves S'_a, must go through both a and b. Further, D is disjoint from J_i as otherwise α could be shortened. Since α leaves S'_a at one of its endpoints and returns to the other of its endpoint, α must go through c and d. The segment of α from c to d is a q_i-path in S'_c. Since $S'_c \cong S'_a$, there is a q_i-path in S'_a connecting a and b. Replacing the segment of α going outside S'_a by a q_i-path in S'_a, we obtain a q_i-path in S'_a from u to v. Since $S'_a \cong G'_k(B)$, $u\varphi$ and $v\varphi$ can be connected by a q_i-path in $G'_k(B)$, which contradicts $(u\varphi, v\varphi) \notin q_i(e(x, G'_k(B)) : x \in X)$. We have seen that $S''_a := S(a, b; G''_{j+k}(A))$ is isomorphic to $G''_k(B)$.

Let us call a path an x-path, $x \in X$, if the colour of any of its edges contains x. An argument similar to the above shows that if $g, h \in V(S''_a)$ can be connected by an x-path in $G''_{j+k}(A)$ then they can be connected by an x-path alredy in S''_a. Since $S''_a \cong G''_k(B)$, $S(a, b; G_{j+k+1}(A)) \cong G_{k+1}(B)$ can be concluded. This proves Lemma 1. ■

LEMMA 2.. For any $A, C_0, C_1, \ldots, C_k \in \mathbf{P}^+(X)$ we have

$$\bigcap_{\ell \leq k} e(C_\ell, G_\omega(A)) = e\left(\bigcup_{\ell \leq k} C_\ell, G_\omega(A)\right).$$

PROOF.. For $B \in \mathbf{P}^+(X)$, call a path a B-path if B is a subset of every colour occuring on its edges. Let $C = \cup_{\ell \leq k} C_\ell, e_\ell = e(C_\ell, G_\omega(A))$ and $e = e(C, G_\omega(A))$. Since every C-path is a C_ℓ-path, $\ell \leq k$, the " $\supseteq$ " part of Lemma 2 is obvious.

Before showing the converse inclusion let us observe the following property of $H(q_i)$ when $|J_i| \geq 2$:

(*) If $a_1, a_2 \in V(H(q_i))$ then either there exists an x-coloured edge between a_1 and a_2 for each $x \in J_i$ or for each $x \in J_i$ there is an X-coloured edge in $H(q_i)$ which is not between a_1 and a_2.

(This property follows via an easy induction on $|J_i|$.)

Now, to show the reverse inclusion, we have to prove that for all $j < \omega$

$$(H_j) \qquad \left(c, d \in V(G_j(A)) \,\&\, (c,d) \in \bigcap_{\ell \leq k} e_\ell \right) \Longrightarrow (c,d) \in e.$$

IF $c = d$ or $(c,.,d) \in E(G_\omega(A))$ then H_j is evident. In particular, (H_0) is true. Assume now that the induction hypothesis (H_{j-1}) holds for some $j, 0 < j < \omega$. To show (H_j), suppose $(c,d) \in \cap_{\ell \leq k} e_\ell$, $c, d \in V(G_j(A))$. We may assume that $c \notin V(G_{j-1}(A))$. Then there is an edge $(a,.,b) \in E'(G_{j-1}(A))$ and an $i, i \leq t$, such that $|J_i| \geq 2$ and c is an inner ($\neq$ endpoint) vertex of $Q(a,b;q_i;G_j(A))$.

Case 1: $d \notin V(Q(a,b;q_i;G_j(A)))$. Then, by (*) and the $a-b$ symmetry, we may assume that for each $y \in J_i$ the weak subgraph $Q(a,b;q_j;G_j(A))$ has a y-coloured edge between c and b. For $\ell \leq k$, consider the shortest C_ℓ-path α in $G_\omega(A)$ that connects c and d. Being the shortest, this path is in $G_j(A)^*$ and, by Step (c) of the construction, any edge of this path in $S(a,b;G_\omega(A))$ is in $Q(a,.,b;q_i;G_j(A))^*$.

Consider the case α goes through b and avoids a. Since any two edges that have the same colour in $Q(a,b;q_i;G_j(A))$ have the same colour in $Q(a,b;q_i;G_j(A))^*$ by Lemma 1, C_ℓ is a subset of the colour of every edge in $Q(a,b;q_i;G_j(A))^*$. Therefore $(a,b) \in e_\ell$. Further, and this the observation we need in the sequel, the path between a and c in $Q(a,b;q_i;G_j(A))^*$ is an e_ℓ-path and, by transitivity, $(d,a) \in e_\ell$. The same observation needs no proof in the other case when α goes through a and avoids b. Since ℓ was arbitrary, we have $(a,c) \in e$. Similarly, giving the role of (c,d) to (d,a), we can obtain a vertex a' in $G_{j-1}(A)$ such that $(a',d) \in e$ and $(a',a) \in \cap_{\ell \leq k} e_\ell$. Now $(a',a) \in e$ follows from the induction hypothesis (H_{j-1}), and the transitivity of e yields $(c,d) \in e$.

Case 2: $d \in V(Q(a,b;q_i;G_j(A)))$. This case can be handled similarly to Case 1.

The proof of Lemma 2 is complete. ∎

LEMMA 3.. For any $A \in \mathbf{P}^+(X)$, the equivalence relations $e(x, G_\omega(A))$ satisfy P_X in the lattice of equivalences of $V(G_\omega(A))$.

PROOF.. Consider $p_i \leq q_i$ from P_X. From Lemma 2 we obtain that

$$p_i(e(x,G_\omega(A)) : x \in X) = \bigcap_{x \in M_i} e(x, G_\omega(A)) = e(\bigcup_{x \in M_i} \{x\}, G_\omega(A)) = e(M_i, G_\omega(A)).$$

Suppose $(a,C,b) \in E(G_\omega(A))$ and $(a,b) \in e(M_i, G_\omega(A))$. Then, by Step (c) of the construction, $M_i \subseteq C$. Further, there is a $j < \omega$ such that $(a,C,b) \in E(G_j(A))$. We may assume that $(a,C,b) \in E'(G_j(A))$. Then, by step (a) or (b) of the construction, $(a,b) \in q_i(e(x,G_{j+1}(A)) : x \in X) \subseteq q_i(e(x,G_\omega(A)) : x \in X)$.

We have shown that $q_i(e(x, G_\omega(A)) : x \in X)$ includes $\{(a,b) : (a,C,b) \in E(G_\omega(A))$ and $M_i \subseteq C\}$. Therefore $e(M_i, G_\omega(A)) \subseteq q_i(e(x, G_\omega(A)) : x \in X)$, proving the lemma. ■

For a graph G let $C(G)$ denote the colour of its initial edge. Then we have

LEMMA 4.. If $j \le \omega$ and $A \in \mathbf{P}^+(X)$ then $C(G_j(A)) \subseteq T(A)$.

PROOF.. It sufficies to consider finite j only. If $j = 0$ then the statement is trivial, for $C(G_0(A)) = A$. To start an induction, let j be fixed and assume the following hypothesis:

$$(\forall u \le j)(\forall B \in \mathbf{P}^+(X))(C(G_u(B)) \subseteq T(B)).$$

Let $A \in \mathbf{P}^+(X)$, $T = T_n$ and compute:

$$C(G_j''(A)) = C(G_j'(A)) = C(G_j(A)) \cup \bigcup_{M_i \subseteq C(G_j(A)), |J_i|=1} J_i \subseteq$$

$$T_n(A) \cup \bigcup_{M_i \subseteq C(G_j(A))} \bigcap_{x \in J_i} \{x\} \subseteq T_n(A) \cup \bigcup_{M_i \subseteq T_n(A)} \bigcap_{x \in J_i} T_n(x) = T_{n+1}(A) = T(A),$$

i.e., $C(G_j''(A)) \subseteq T(A)$.

Now assume that $x \in C(G_{j+1}(A)) \setminus C(G_j''(A))$. Then, by Step (c) of the construction, there is an x-path α of minimal length $|\alpha|$ in $G_j''(A)$ connecting the endpoints of $G_j''(A)$. Since $|\alpha| > 1$, α goes through all the vertices of an appropriate weak subgraph Q(left endpoint, right endpoint; q_i; $G_k(A)$) where $0 < k \le j$. (As the notations indicate, $G_{k-1}'(A)$ was fermented at its initial edge by $H(q_i)$, and α goes through the vertices of this copy of $H(q_i)$.)

Let a_0 =left endpoint, $a_1, a_2, \ldots, a_s$ =right endpoint be the vertices of Q(left endpoint, right endpoint; q_i; $G_k(A)$) such that $\{(a_m, \cdot, a_{m+1}) : m < s\} = E(Q$(left endpoint, right endpoint; q_i; $G_k(A)$)), and let $\{y_m\}$ be the colour of $(a_m, \cdot, a_{m+1})$ in $G_k''(A)$. Then $\{y_m : m < s\} = J_i$. Since $|J_i| \ge 2$, $\{y_m\}$ is the colour of $(a_m, \cdot, a_{m+1})$ in $G_k(A)$ as well.

From the fact that α is an x-path through $a_0, a_1, a_2, \ldots, a_s$ we infer that $k < j$. Further, it follows by Step (c) of the construction that x belongs to the colour of $(a_m, \cdot, a_{m+1})$ in $G_{j+1}(A)$ for any $m < s$. In other words, $x \in C(S(a_m, a_{m+1}; G_{j+1}(A)))$. But $S(a_m, a_{m+1}; G_{j+1}(A)) \cong G_{j+1-k}(\{y_m\})$ by Lemma 1. Therefore $x \in C(G_{j+1-k}(\{y_m\}))$, and the induction hypothesis yields $x \in T(y_m)$. Since m was arbitrary, we obtain

$$x \in \bigcap_{m<s} T(y_m) = \bigcap_{y \in J_i} T(y) = \bigcap_{y \in J_i} T_n(y).$$

To achieve $x \in T(A) = T_{n+1}(A)$ it remains to show that $M_i \subseteq T_n(A) = T(A)$. Since $G_{k-1}'(A)$ was fermented at its initial edge by q_i, $M_i \subseteq C(G_{k-1}'(A)) \subseteq C(G_k(A))$, whence the induction hypothesis yields $M_i \subseteq T(A)$. Therefore $C(G_{j+1}(A)) \subseteq T(A)$, which completes the proof of Lemma 4. ■

Now the necessity part of the theorem follows easily. Assume that $P_X \Longrightarrow \bigwedge_{y \in A} y \le x$ holds in all lattices. In particular, $\bigwedge_{y \in A} y \le x$ holds for the $e(x, G_\omega(A))$, $x \in X$, by Lemma 3. Lemma 2 yields that (left endpoint, right endpoint) $\in$

$e(A, G_\omega(A)) = e(\cup_{y\in A}\{y\}, G_\omega(A)) = \cap_{y\in A} e(y, G_\omega(A)) \subseteq e(x, G_\omega(A))$. Hence there is an x-path α in $G_\omega(A)$ connecting the endpoints. But α is an x-path in $G_j(A)$ for som sufficiently large $j < \omega$. Therefore $x \in C(G_{j+1}(A))$, by Step (c) of the construction. Hence $x \in T(A)$ follows from Lemma 4. ■

Acknowledgement. A former unpublished version of this paper, containing a weaker result, was written during the author's stay at Lakehead University. The author express his thanks to Professor Alan Day for the invitation and providing excellent circumstances in which to work there.

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(Received August 28, 1989)

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