On the lattice of congruence varieties of locally equational classes

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1. Introduction

For a class $\mathcal{K}$ of algebras, let $\mathbf{Con}(\mathcal{K})$ denote the lattice variety generated by the class of congruence lattices of all members of $\mathcal{K}$. A lattice variety $\mathcal{U}$ will be called an *l-congruence variety* if $\mathcal{U}=\mathbf{Con}(\mathcal{K})$ for some locally equational class $\mathcal{K}$ of algebras. In particular, every congruence variety is an l-congruence variety. Our aim is to show that l-congruence varieties form a complete lattice, which is a join-subsemilattice of the lattice of all lattice varieties (while meet is not preserved). We also show that the minimal modular congruence varieties described by FREESE [1] and the minimal modular l-congruence varieties are the same.

The notion of locally equational class has been introduced by HU [2]. For the definition, let F be a subset of an algebra A of type τ and let t_1, t_2 be n-ary τ-terms. The identity $t_1=t_2$ is said to be valid in F if for all $(a_1, a_2, \ldots, a_n)\in F^n$ we have $t_1(a_1, a_2, \ldots, a_n)=t_2(a_1, a_2, \ldots, a_n)$. Suppose $\mathcal{K}$ is a class of algebras of type τ and denote by $\mathbf{L}(\mathcal{K})$ the class of all algebras A of type τ having the following property:

> for each finite subset G of A there is a finite family $\{B_i: i\in I\}$ in $\mathcal{K}$ and there is for each $i\in I$ a finite subset $F_i\subseteq B_i$ such that every identity valid in F_i for all $i\in I$ is also valid in G.

Now, $\mathbf{L}$ is a closure operator on classes of similar algebras. $\mathbf{L}(\mathcal{K})$ is called the *locally equational class* (or, briefly, *local variety*) generated by $\mathcal{K}$, and $\mathcal{K}$ is said to be a local variety if $\mathbf{L}(\mathcal{K})=\mathcal{K}$. We often write $\mathbf{L}(A)$ instead of $\mathbf{L}(\{A\})$.

Denote by $\mathbf{H}, \mathbf{S}, \mathbf{P}_f, \mathbf{D}$ the operators of forming homomorphic images, subalgebras, direct products of finite families and directed unions, respectively, and let us recall

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Theorem 1.1. (Hu [2]) (a) *Every variety is a local variety. The converse does not hold, e.g. all torsion groups form a local variety.*

(b) *For a class $\mathscr{K}$ of similar algebras* $\mathbf{L}(\mathscr{K})=\mathbf{DHSP}_f(\mathscr{K})$; *consequently,*

(c) *$\mathscr{K}$ is locally equational if and only if it is closed under* $\mathbf{D}, \mathbf{H}, \mathbf{S}, \mathbf{P}_f$.

Our main tool is the following

Theorem 1.2. (Pixley [11]) *There is an algorithm which, for each lattice identity λ and pair of integers $n, k \geqq 2$, determines a strong Mal'cev condition (i.e., a finite set of equations of polynomial symbols of unspecified type) $U_{n,k}=U_{n,k}(\lambda)$ such that for an arbitrary algebra A of type τ the following three conditions are equivalent:*

(i) *λ is satisfied throughout* $\mathbf{Con}(\mathbf{L}(A))$;

(ii) *for each finite subset F of A and integer $n \geqq 2$ there is an integer $k=k(n, F, \lambda)$ and a τ-realization $U^{\tau}_{n,k}$ of $U_{n,k}$ such that $U^{\tau}_{n,k}$ is valid in F;*

(iii) *for each finite subset F of A and integer $n \geqq 2$ there is a $k_0=k_0(n, F, \lambda)$ such that for each $k \geqq k_0$ there is a τ-realization $U^{\tau}_{n,k}$ of $U_{n,k}$ which is valid in F.*

We have supplemented Pixley's theorem with condition (iii) which is implicit in the proof in [11] of the theorem. We shall make essential use of

Proposition 1.3. *In the above theorem each polynomial of $U^{\tau}_{n,k}$ is idempotent in F.*

This follows easily from the construction of $U_{n,k}$ described in [11].

2. Lattice of *l*-congruence varieties

A lattice variety $\mathscr{U}$ is called a *congruence variety* (Jónsson [8]) if $\mathscr{U}=\mathbf{Con}(\mathscr{K})$ for some variety $\mathscr{K}$, and $\mathscr{U}$ will be called an *l-congruence variety* if $\mathscr{U}=\mathbf{Con}(\mathscr{V})$ for some local variety $\mathscr{V}$. Let $\mathfrak{C}$ and $\mathfrak{C}^*$ denote the "sets" consisting of all *l*-congruence varieties and all *l*-congruence varieties of the form $\mathbf{Con}(\mathbf{L}(A))$, respectively. Let $\mathfrak{C}$ and $\mathfrak{C}^*$ be partially ordered by inclusion. Our main result is

Theorem 2.1. *$\mathfrak{C}$ is a complete lattice. The (infinitary) join of arbitrary l-congruence varieties in $\mathfrak{C}$ and their join taken in the lattice of all lattice varieties coincide.*

Although there exists a local variety which cannot be generated by a single algebra (Hu [2]), we have

Theorem 2.2. *For any local variety $\mathscr{V}$ there is an algebra A (*not necessarily *of the same type as $\mathscr{V}$) such that* $\mathbf{Con}(\mathscr{V})=\mathbf{Con}(\mathbf{L}(A))$. *Thus $\mathfrak{C}=\mathfrak{C}^*$.*

Proof of Theorems 2.1 and 2.2. First we show the following statement:

(1) For any algebra A of type τ there exists an algebra B such that $\mathbf{Con}(\mathbf{L}(A))=$ $=\mathbf{Con}(\mathbf{L}(B))$ and B has a one-element subalgebra.

Let $b_0\in A$, $\Phi=\{\lambda:\lambda$ is a lattice identity satisfied throughout $\mathbf{Con}(\mathbf{L}(A))\}$ and $H=\{F: F$ is a finite subset of A containing $b_0\}$. By Thm. 1.2 choose a $k=k(n, F, \lambda)$ and a τ-realization $U^{\tau}_{n,k}(F,\lambda)$ of $U_{n,k}(\lambda)$ for all $\lambda\in\Phi$, $F\in H$ and $n\geqq 2$ such that $U^{\tau}_{n,k}(F,\lambda)$ is valid in F. Denote by $P(n, F, \lambda)$ the set of τ-polynomials occuring in $U^{\tau}_{n,k}(F,\lambda)$ and define an algebra B as follows: B has the same carrier as A and the set of its operations is $\cup\{P(n, F, \lambda): n\geqq 2, F\in H, \lambda\in\Phi\}$ (i.e. B is a reduct of A). Since $U^{\tau}_{n,k}$ is also valid in $F\setminus\{b_0\}$, $\mathbf{Con}(\mathbf{L}(A))=\mathbf{Con}(\mathbf{L}(B))$ follows from Thm. 1.2. By Prop. 1.3, $\{b_0\}$ is a subalgebra of B, which completes the proof of (1).

Now we prove that

(2) For an arbitrary set Γ of indices and for any algebras $A_\gamma(\gamma\in\Gamma)$ there is an algebra A' such that $\bigvee_{\gamma\in\Gamma}\mathbf{Con}(\mathbf{L}(A_\gamma))=\mathbf{Con}(\mathbf{L}(A'))$ in the lattice of all lattice varieties.

We can assume $\Gamma\neq\emptyset$ (otherwise the statement is trivial) and

— $\{a_\gamma\}$ is a one-element subalgebra of A_γ for each $\gamma\in\Gamma$,

— all the algebras $A_\gamma(\gamma\in\Gamma)$ are of the same similarity type τ (otherwise the set of operations of A_γ can be supplemented with projections since for polynomially equivalent algebras B_1 and B_2 over the same carrier, $\mathbf{Con}(\mathbf{L}(B_1))=\mathbf{Con}(\mathbf{L}(B_2))$ by Thm. 1.2), and

— for each $\gamma\in\Gamma$, every τ-polynomial is equal to some τ-operation over A_γ.

Denote by τ_i the set of i-ary operation symbols in τ and regard $\tau_i'=\tau_i^{\Gamma}$ as a set of i-ary operation symbols $(i=0, 1, 2, \ldots)$. Now, $\tau=\bigcup_{i=0}^{\infty}\tau_i$ and set $\tau'=\bigcup_{i=0}^{\infty}\tau_i'$. For each $\gamma\in\Gamma$, A_γ can be regarded as an algebra A_γ' of type τ' if we define, for $q\in\tau'$, the operation q by $q=q(\gamma)$ $(q(\gamma)\in\tau$, A_γ and A_γ' have the same carrier). Evidently, $\mathbf{Con}(\mathbf{L}(A_\gamma'))=\mathbf{Con}(\mathbf{L}(A_\gamma))$ by Thm. 1.2. Let A' be a weak direct product of the algebras A_γ' defined by

$A'=\{f\in\prod_{\gamma\in\Gamma}A_\gamma'$: for all but finitely many $\gamma\in\Gamma$, $f(\gamma)=a_\gamma\}$.

By Thm. 1.1 $\mathbf{L}(A_\gamma')\subseteqq\mathbf{L}(A')$, therefore

$$\bigvee_{\gamma\in\Gamma}\mathbf{Con}(\mathbf{L}(A))=\bigvee_{\gamma\in\Gamma}\mathbf{Con}(\mathbf{L}(A_\gamma'))\subseteqq\bigvee_{\gamma\in\Gamma}\mathbf{Con}(\mathbf{L}(A'))=\mathbf{Con}(\mathbf{L}(A')).$$

In order to prove the converse inclusion by means of Thm. 1.2, suppose a lattice identity λ is satisfied throughout each $\mathbf{Con}(\mathbf{L}(A_\gamma))$. Fix an arbitrary finite subset F of A' and $n\geqq 2$. For each $\gamma\in\Gamma$ set $F_\gamma=\{f(\gamma): f\in F\}\subseteqq A_\gamma'$ and choose a non-empty finite $\Delta\subseteqq\Gamma$ such that $\gamma\in\Gamma\setminus\Delta$ implies $F_\gamma=\{a_\gamma\}$. Since λ holds in each $\mathbf{Con}(\mathbf{L}(A_\gamma))$, by Thm 1.2 for each $\gamma\in\Gamma$ there exist $k_\gamma\geqq 2$ and for all $k\geqq k_\gamma$ a τ-realization $U^{\tau}_{n,k}(\gamma)$ of $U_{n,k}$ such that $U^{\tau}_{n,k}(\gamma)$ is valid in F_γ. We can suppose $k_\gamma=2$

if $\gamma\in\Gamma\setminus\Delta$, because F_γ is a subalgebra consisting of a single element. Set $k=\max\{k_\gamma: \gamma\in\Gamma\}$. Then for each $\gamma\in\Gamma$ there exists a realization $U^\tau_{n,k}(\gamma)$ of $U_{n,k}$ which is valid in F_γ. Let $U^\tau_{n,k}(\gamma)$ consist of τ-operations $q_{1,\gamma}, q_{2,\gamma}, \ldots, q_{s,\gamma}$. For $i=1, 2, \ldots, s$ define $q_i\in\tau'$ by $q_i(\gamma)=q_{i,\gamma}$ over $A_\gamma(\gamma\in\Gamma)$. Then the operations $q_1, q_2, \ldots, q_s$ yield a τ'-realization of $U_{n,k}$ which is valid in F. This completes the proof of (2).

Now, let $\mathscr{V}$ be an arbitrary local variety and let Φ consist of all lattice identities which are not satisfied throughout $\mathbf{Con}(\mathscr{V})$. For each $\lambda\in\Phi$ we can choose $A_\lambda\in\mathscr{V}$ such that λ is not satisfied in the congruence lattice of A_λ. Since $\mathbf{L}(A_\lambda)\subseteq\mathscr{V}$ and λ is not satisfied throughout $\mathbf{Con}(\mathbf{L}(A_\lambda))$, it can be easily seen that $\mathbf{Con}(\mathscr{V})= =\bigvee_{\lambda\in\Phi}\mathbf{Con}(\mathbf{L}(A_\lambda))$. Hence Thm. 2.2 follows from (2). Since any complete join-semilattice having a 0-element is a complete lattice, Thm. 2.1 follows from (2) and Thm. 2.2. Q.E.D.

3. Minimal modular l-congruence varieties

Let P be the set of all prime numbers and set $P_0=P\cup\{0\}$. For $p\in P_0$ denote by Q_p the prime field of characteristic p and by $\mathscr{V}_p$ the variety of all vector spaces over Q_p. The following theorem was announced by FREESE [1]:

Theorem 3.1. *For any modular but not distributive congruence variety $\mathscr{U}$ there is a $p\in P_0$ such that $\mathbf{Con}(\mathscr{V}_p)\subseteq\mathscr{U}$. Consequently, congruence varieties do not form a sublattice in the lattice of all lattice varieties.*

Christian Herrmann has also proved the above theorem. We shall slightly modify his (unpublished) proof to obtain the following

Theorem 3.2. *For any modular but not distributive l-congruence variety $\mathscr{U}$ there is a $p\in P_0$ such that $\mathbf{Con}(\mathscr{V}_p)\subseteq\mathscr{U}$. Consequently, l-congruence varieties do not form a sublattice in the lattice of all lattice varieties.*

The proof is based on the following theorem (which is presented here in a weakened form):

Theorem 3.3. (HUHN [4]) *For an arbitrary modular lattice M and $n\geqq 3$ the following two conditions are equivalent:*

(i) *M is not n-distributive, i.e., the n-distributivity law*

$$x\wedge\bigvee_{i=0}^{n} y_i=\bigvee_{j=0}^{n}\Big(x\wedge\bigvee_{\substack{i=0\\ i\neq j}}^{n} y_i\Big)$$

(cf. HUHN [3] *and* [5]) *is not satisfied in M.*

(ii) *The lattice variety generated by M contains $L_{n+1}(Q_p)$ for some $p\in P_0$ where $L_{n+1}(Q_p)$ denotes the congruence lattice of the $(n+1)$-dimensional vector space over Q_p.*

For a pair of non-negative integers m, k let us define the divisibility condition $D(m, k)$ by the formula $(\exists x)\,(m\cdot x=k\cdot 1)$ where $m\cdot x$ and $k\cdot 1$ mean $x+x+\ldots+x$ (m times) and $1+1+1+\ldots+1$ (k times), respectively. We need the following

Proposition 3.4. *For any lattice identity λ there exist non-negative integers n_0, m, k such that for each $p\in P_0$ the following three conditions are equivalent:*
(i) *λ is satisfied throughout* $\mathbf{Con}(\mathscr{V}_p)$,
(ii) *there exists $n\geqq n_0$ such that λ is satisfied in $L_n(Q_p)$,*
(iii) *the divisibility condition $D(m, k)$ holds in Q_p.*

Proof. The equivalence of (i) and (iii) is a special case of [6, Thm. 3]. As for (ii)$\to$(i), we can argue as follows: Let us construct the identity $\hat{\lambda}$ from λ by replacing the operation symbols $\wedge$ and $\vee$ by $\cap$ and $\circ$ (composition of relations), respectively. By congruence permutability, (i) holds iff $\hat{\lambda}$ is satisfied by arbitrary congruences of any algebra in $\mathscr{V}_p$. Now, Wille's theorem [12] (see also Pixley [11, Thm. 2.2]) involves implicitly that if $\hat{\lambda}$ is satisfied by certain congruences of the free $\mathscr{V}_p$-algebra of rank n_0, for some n_0 depending on $\hat{\lambda}$, then $\hat{\lambda}$ is satisfied by arbitrary congruences of any algebra in $\mathscr{V}_p$. Finally, the congruence lattice of the free $\mathscr{V}_p$-algebra of rank n_0 is a sublattice of $L_n(Q_p)$ whence $\hat{\lambda}$ is satisfied by arbitrary congruences of the free $\mathscr{V}_p$-algebra of rank n_0. Q.E.D.

It follows from a more general result of Nation [10, Thm. 2] that any n-distributive congruence variety is distributive $(n\geqq 1)$. Now we need the following generalization of this fact:

Proposition 3.5. *Let $n\geqq 1$ and $\mathscr{U}$ be an arbitrary l-congruence variety. If $\mathscr{U}$ is n-distributive, then $\mathscr{U}$ is distributive.*

Proof. Certain arguments using Mal'cev conditions for congruence varieties can easily be reformulated for l-congruence varieties. Pixley [11] has pointed out that Jónsson's criterion for congruence distributivity [7] remains valid for l-congruence varieties. Similarly, Mederly's ciiterion for n-distributivity [9, Theorem 2.1] also remains valid. Thus the have:

Proposition 3.6. *For an arbitrary algebra of type τ and $n\geqq 1$ the following two conditions are equivalent:*
(i) $\mathbf{Con}(\mathbf{L}(A))$ *is n-distributive,*
(ii) *For each finite $F\subseteq A$ there exist $k\geqq 2$ and $(n+2)$-ary τ-polynomials*

$t_0, t_1, \ldots, t_k$ *on* A *such that the identities*

$$t_0(x_0, x_1, \ldots, x_{n+1}) = x_0, \quad t_k(x_0, x_1, \ldots, x_{n+1}) = x_{n+1},$$

$$t_i(x_0, x_1, \ldots, x_n, x_0) = x_0 \quad (i = 0, 1, \ldots, k),$$

$$t_i(\underbrace{x, x, \ldots, x}_{j+1}, y, y, \ldots, y) = t_{i+1}(\underbrace{x, x, \ldots, x}_{j+1}, y, y, \ldots, y)$$

$(0 \leqq i < k,\ 0 \leqq j \leqq n$ *and* $i \equiv j \pmod{n+1})$ *are valid in* F.

Now, suppose $\mathbf{Con}(\mathbf{L}(A))$ is n-distributive for some $n \geqq 1$. Fix a finite $F \subseteqq A$. Then, by Prop. 3.6, there are $k \geqq 2$ and τ-polynomials $t_0, t_1, \ldots, t_k$ satisfying the required identities in F. Define $j(-1)=0$ and for $i=0, 1, \ldots, k$, $j(i) \equiv i \pmod{n+1}$, $0 \leqq j(i) \leqq n$. Define ternary τ-polynomials $q_0, q_1, \ldots, q_{2k+2}$ as follows: $q_0(x, y, z)=x$ and for $i=0, 1, \ldots, k$

$$q_{2i+1}(x, y, z) = t_i(\underbrace{x, x, \ldots, x}_{j(i-1)+1}, y, y, \ldots, y, z)$$

and

$$q_{2i+2}(x, y, z) = t_i(\underbrace{x, x, \ldots, x}_{j(i)+1}, y, y, \ldots, y, z).$$

It is easy to check that the polynomials $q_0, q_1, \ldots, q_{2k+2}$ satisfy the equations of Prop. 3.6 (ii) in F for $(1, 2k+2)$ instead of (n, k). Hence, by Prop. 3.6, 1-distributivity — which is the usual distributivity — holds throughout $\mathbf{Con}(\mathbf{L}(A))$. Thus Thm. 2.2 completes the proof.

Proof of Theorem 3.2. Let $\mathcal{U}$ be an l-congruence variety as in the theorem. By Prop. 3.5, $\mathcal{U}$ is not distributive for $n=1, 2, 3, \ldots$. Hence, by Thm. 3.3, for each $n>2$ we can choose $p_n \in P_0$ such that $L_{n+1}(Q_{p_n}) \in \mathcal{U}$. Set $S=\{p_n : n>2\}$. If the set $\{n : n>2$ and $p_n=p_t\}$ is infinite for some t, then $\{L_{n+1}(Q_{p_n}) : p_n=p_t\}$ generates $\mathbf{Con}(\mathcal{V}_{p_t})$ by Prop. 3.4 (i, ii). Hence $\mathbf{Con}(\mathcal{V}_{p_t}) \subseteqq \mathcal{U}$. Suppose $\{n : n>2$ and $p_n=p_t\}$ is finite for all $t>2$. Then it suffices to show that $\mathbf{Con}(\mathcal{V}_0)$ is a subvariety of the variety generated by $\{L_{n+1}(Q_{p_n}) : n>2\}$. Suppose λ holds in $L_{n+1}(Q_{p_n})$ for each $n>2$. For a sufficiently large t, λ holds throughout $\mathbf{Con}(\mathcal{V}_{p_n})$ for any $n \geqq t$ by Prop 3.4 (i, ii). Hence there exists an infinite $S' \subseteqq S \setminus \{0\}$ such that λ holds in $\mathbf{Con}(\mathcal{V}_p)$ for each $p \in S'$. Then, by Prop. 3.4, the divisibility condition $D(m, k)$ associated with λ holds in Q_p for each $p \in S'$. Therefore, $D(m, k)$ holds in Q_0 (otherwise $m=0$ and $k \neq 0$, so each $p \in S'$ divides k). Hence, by Prop. 3.4, λ holds throughout $\mathbf{Con}(\mathcal{V}_0)$. Q.E.D.

Remark. If $\mathscr{K}$ is a class of similar algebras closed under S and $\mathbf{P}_f$ then $\mathbf{Con}(\mathscr{K})$ is an l-congruence variety, namely $\mathbf{Con}(\mathscr{K})=\mathbf{Con}(\mathbf{L}(\mathscr{K}))$.

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