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ON THE 2-DISTRIBUTIVITY OF SUBLATTICE LATTICES

By

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I. Introduction

The concept of *n-distributivity* was introduced by HUHN (cf. [4] and [6]). A lattice is said to be n-distributive ($n \geqq 1$) if it satisfies the identity

$$x \wedge \bigvee_{i=0}^{n} y_i \leqq \bigvee_{j=0}^{n} \left(x \wedge \bigvee_{\substack{i=0 \\ i \neq j}}^{n} y_i \right).$$

The n-distributivity of subalgebra lattices of universal algebras proved to be an important property in several cases (cf., e.g., HUHN [4, 5] and NATION [9]). Sublattice lattices were investigated by FILIPPOV [2]. Lattices having modular and (upper) semi-modular sublattice lattices were characterized by KOH [7] and LAKSER [8], respectively. In [1] we have given a structure theorem for distributive lattices having 2-distributive sublattice lattices. In this paper lattices having 2-distributive sublattice lattices will be characterized. A necessary and sufficient condition for distributive lattices to have n-distributive sublattice lattices will be also given. A structure theorem for modular lattices having 2-distributive sublattice lattices will be deduced from the mentioned result of [1].

In what follows, for a lattice L and a subset H of L, let $\mathrm{Su}(L)$ and $[H]$ denote the lattice of sublattices of L and the sublattice generated by H, respectively. ($\mathrm{Su}(L)$ contains the empty set.)

II. Distributive lattices having n-distributive sublattice lattices

We intend to prove the following

THEOREM 1. *For an arbitrary distributive lattice L and integer $n \geqq 1$ the following two conditions are equivalent:*

(i) $\mathrm{Su}(L)$ *is n-distributive;*

(ii) *for any $(n+1)$-element subset H of L we have*

$$[H] = \bigcup_{h \in H} [H \setminus \{h\}].$$

REMARK. Since every finitely generated free distributive lattice is finite, this theorem makes it, at least theoretically, possible to list finitely many finite distributive lattices for each n so that a distributive lattice L has an n-distributive sublattice lattice iff none of the listed lattices is a sublattice of L.

For example, in case $n=1$ only the four-element lattice which is not a chain has to be listed. In case $n=2$, as it follows from Theorem 2 (stated later), S_1 and S_4 (defined in Theorem 2) can be listed.

PROOF. By Lemma 1 in [1] it is enough to show that for any non-negative integer k, the following implication holds:

I_k: for an arbitrary lattice L, if L satisfies (ii) then $[H]=\bigcup_{\substack{G\subset H\\|G|=n}}[G]$ holds for any $(n+k+1)$-element subset H of L. So, the proof goes via induction on k. I_0 is evident. Now suppose $I_0, I_1, \dots, I_{k-1}$ hold for some $k\geqq 1$, but I_k does not hold. Then there exist a lattice L, $H=\{h_1, h_2, \dots, h_{n+k+1}\}\subseteqq L$ and an $(n+k+1)$-ary lattice polynomial p such that $p(h_1, \dots, h_{n+k+1})\notin {}^nH$, where ${}^nH=\bigcup_{\substack{G\subset H\\|G|=n}}[G]$. By the distributivity of L, p can be supposed to be of disjunctive normal form

$$(1)\quad p(x_1, \dots, x_{n+k+1})=p_1(x_1, \dots, x_{n+k+1})\vee p_2(x_1, \dots, x_{n+k+1})\vee\dots\vee p_d(x_1, \dots, x_{n+k+1})$$

where $p_2, \dots, p_d$ are conjunctions of (some of their) variables, p_1 is of disjunctive normal form or is omitted (i.e., $p=p_2\vee\dots\vee p_d$), p_1 does not depend on all the $n+k+1$ variables, and all the $n+k+1$ variables occur in $p_1\vee p_2$ (or in p_2, if p_1 is omitted). Suppose both p and its disjunctive normal form (1) are chosen so that d is minimal. By the induction hypothesis, $p_1(h_1, \dots, h_{n+k+1})=p_0(g_1, \dots, g_n)$ for some $\{g_1, \dots, g_n\}\subset H$ and n-ary polynomial p_0. At least two elements of H, say h_{n+k} and h_{n+k+1}, do not belong to $\{g_1, \dots, g_n\}$. Let $h_0=h_{n+k}\wedge h_{n+k+1}$, $H_0=\{h_0, h_1, \dots, h_{n+k-1}\}$ and observe that

$$p_1(h_1, \dots, h_{n+k+1})\vee p_2(h_1, \dots, h_{n+k+1}) =$$

$$= p_0(g_1, \dots, g_n)\vee p_2(h_1, \dots, h_{n+k-1}, h_{n+k}, h_{n+k+1}) = q(h_0, h_1, \dots, h_{n+k-1})$$

for some polynomial q. By applying the induction hypothesis (first to H_0 and then once more if necessary) we obtain $q(h_0, \dots, h_{n+k-1})=r(h_1, h_2, \dots, h_{n+k+1})$ where r is a polynomial depending on at most n variables. It can be assumed that r is of disjunctive normal form, whence either all the $n+k+1$ variables occur on the right hand side of the equation $p(h_1, \dots, h_{n+k+1})=r(h_1, \dots, h_{n+k+1})\vee \vee p_3(h_1,\dots, h_{n+k+1})\vee\dots\vee p_d(h_1, \dots, h_{n+k+1})$, which contradicts the minimality of d, or $p(h_1, \dots, h_{n+k+1})\in {}^nH$ is obtained from the induction hypothesis, which is a contradiction again. Q.E.D.

III. Lattices having 2-distributive sublattice lattices

We intend to prove the following

THEOREM 2. *For an arbitrary lattice L the following three conditions are equivalent:*

(i) $\mathrm{Su}(L)$ *is 2-distributive;*
(ii) *None of the lattices* $S_1, S_2, \dots, S_8$ *(see below) is a sublattice of L;*

(iii) *For any three elements* a, b, c *in* L, *if* $a \parallel b$, $a \parallel c$ *and* $b \parallel c$ *then* $[a, b, c]$ *is isomorphic to one of the lattices* $P_1, P_2, \ldots, P_7$, *while if* $a \parallel b$, $a \parallel c$ *and* $b < c$ *then* $[a, b, c]$ *is isomorphic to* R_1 *or* R_2.

We define the lattices occurring in Theorem 2 by their diagrams as follows (S^d stands for the dual of S):

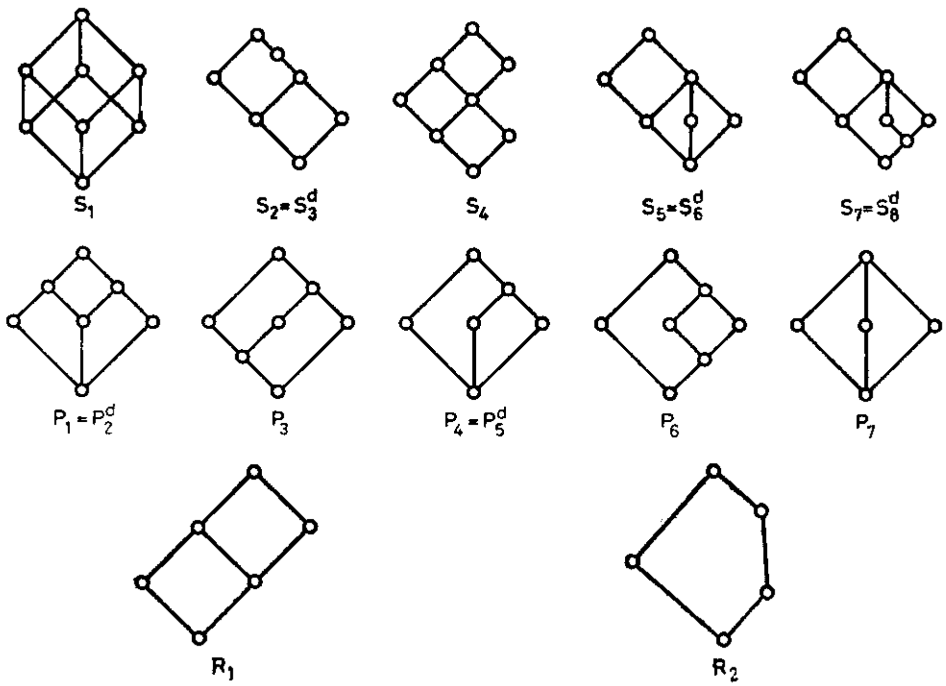

Fig. 1

In order to prove Theorem 2 we need the following

LEMMA 1. *For an arbitrary idempotent algebra* A *with at most binary fundamental operations,* $\mathrm{Su}(A)$ *is 2-distributive if and only if* $[H] = \bigcup_{a \in H} [H \setminus \{a\}]$ *for any three-element subset* H *of* A.

PROOF. Let us write ${}^2H$ instead of $\bigcup_{a, b \in H} [a, b]$. Consider an idempotent algebra A with at most binary fundamental operations and suppose $[H] = {}^2H$ for any three-element subset H of A. By Lemma 1 of [1] it is enough to show that $[H] = {}^2H$ holds for any subset H of A. In other words, it is enough to show that any k-ary polynomial p (in the similarity type of A) has the following property:

For any k-element subset $G = \{a_1, \ldots, a_k\}$ of A, there exist a binary polynomial q and elements b_1, b_2 in G such that $p(a_1, \ldots, a_k) = q(b_1, b_2)$.

Let us assume that this is not true and p is a polynomial of minimal length $|p|$ not having the above property. Then, for any k-element subset $H = \{a_1, \ldots, a_k\}$ of A we have either $p(a_1, \ldots, a_k) = f_0(q_0(a_1, \ldots, a_k))$ or $p(a_1, \ldots, a_k) =$

$=f_1(q_1(a_1, \ldots, a_k), q_2(a_1, \ldots, a_k))$, where q_0, q_1 and q_2 are k-ary polynomials, f_0 and f_1 are fundamental operations, f_1 is binary and f_0 is at most unary. In both cases $|q_i|<|p|$ implies $q_i(a_1, \ldots, a_k)=r_i(b_{2i}, b_{2i+1})$ $(i=0, 1, 2)$ for some binary polynomial r_i and elements b_{2i}, b_{2i+1} in H. Thus in the first case $p(a_1, \ldots, a_k)=$ $=f_0(q_0(a_1, \ldots, a_k))=f_0(r_0(b_0, b_1))\in{}^2H$, a contradiction. If $|\{b_2, b_3, b_4, b_5\}|\leqq 3$ in the second case, then our assumption on A applies and we have $p(a_1, \ldots, a_k)=$ $=f_1(q_1(a_1, \ldots, a_k), q_2(a_1, \ldots, a_k))=f_1(r_1(b_2, b_3), r_2(b_4, b_5))\in{}^2H$, which is a contradiction again. If $|\{b_2, b_3, b_4, b_5\}|=4$, then by applying our assumption on A (first to the set $\{r_1(b_2, b_3), b_4, b_5\}$ and then once more if necessary) we obtain the contradiction $p(a_1, \ldots, a_k)=f_1(r_1(b_2, b_3), r_2(b_4, b_5))\in{}^2H$ again. Q.E.D.

PROOF OF THEOREM 2. It can be checked by Lemma 1 that $\mathrm{Su}(S_i)$ $(i=1, 2, \ldots, 8)$ is not 2-distributive, whence the implication (i)→(ii) follows. An easy calculation by Lemma 1 shows that $\mathrm{Su}(P_i)$ $(i=1, 2, \ldots, 7)$, $\mathrm{Su}(R_1)$ and $\mathrm{Su}(R_2)$ are 2-distributive. Similarly, for any lattice $M=[a, b, c]$ with at most one of $a\parallel b$, $a\parallel c$ and $b\parallel c$, $\mathrm{Su}(M)$ is 2-distributive. Thus the implication (iii)→(i) follows from Lemma 1.

Now only the implication (ii)→(iii) has to be proved, so suppose L satisfies (ii) and $\{a, b, c\}$ is a three-element subset of L. Several cases have to be dealt with. If $a\parallel b$, $a\parallel c$ and $b<c$ then $[a, b, c]$ is isomorphic to the factor lattice K/Θ for some congruence relation Θ (cf. GRÄTZER [3, p. 11]) where K denotes the lattice

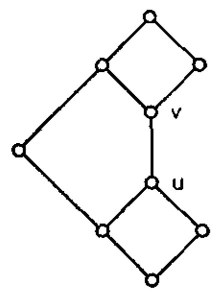

Fig. 2

Now it is not hard to check that $[a, b, c]\cong R_1$ in case $(u, v)\in\Theta$, and $[a, b, c]\cong R_2$ otherwise.

So we have proved that in case $a\parallel b$, $a\parallel c$ and $b<c$ $[a, b, c]$ is isomorphic to R_1 or R_2. This has an important consequence *that will be often used in the proof*:

Suppose $a\parallel b$, $a\parallel c$ and $b<c$ $(a, b, c\in L)$. Then $a\vee b\neq a\vee c$ implies $a\wedge b=a\wedge c$ and $(a\vee b)\wedge c=b$. Similarly, $a\wedge b\neq a\wedge c$ implies $a\vee b=a\vee c$ and $(a\wedge c)\vee b=c$.

Having three elements a, b, c in L with $a\parallel b$, $a\parallel c$ and $b<c$, *we shall refer to this consequence by* $R_{a,b,c}$.

In what follows let us assume $a\parallel b$, $a\parallel c$ and $b\parallel c$. We have

$$a\vee b\vee c\in\{a\vee b, a\vee c, b\vee c\} \quad\text{and}\quad a\wedge b\wedge c\in\{a\wedge b, a\wedge c, b\wedge c\} \tag{2}$$

since otherwise $[a\vee b, a\vee c, b\vee c]$ or $[a\wedge b, a\wedge c, b\wedge c]$ would be isomorphic to S_1 by Lemma 9 in GRÄTZER [3, p. 38]. Let $j=|\{\{x, y\}: \{x, y\}\subseteq\{a, b, c\}$ and $x\vee y=$ $=a\vee b\vee c\}|$ and $m=|\{\{x, y\}: \{x, y\}\subseteq\{a, b, c\}$ and $x\wedge y=a\wedge b\wedge c\}|$. Then (2) states that $j\geqq 1$ and $m\geqq 1$.

Claim 1. If $j=1$ then $[a, b, c]\cong P_1$.

To prove this claim suppose $j=1$ and $a\vee c=a\vee b\vee c$. We obtain $a\wedge(b\vee c)=$ $=a\wedge b$ and $(a\vee b)\wedge c=b\wedge c$ from $R_{a,b,b\vee c}$ and $R_{c,b,a\vee b}$, respectively. If we had $(a\vee b)\wedge(b\vee c)>b$, then $\{a, b, a\wedge b, a\vee b, a\vee c, b\vee c, (a\vee b)\wedge(b\vee c)\}$ could be easily shown to be a sublattice of L isomorphic to S_3. Thus $(a\vee b)\wedge(b\vee c)=b$.

Case 1.1: $a\wedge b\parallel c$. We obtain $a\wedge b\wedge c=(a\vee b)\wedge c$ from $R_{c,a\wedge b,a\vee b}$. We have $(a\wedge b)\vee c<b\vee c$ since otherwise $\{a\wedge b, b, a\vee b, a\vee c, b\vee c, c, a\wedge b\wedge c\}$ would be a sublattice of L isomorphic to S_3. From $a\wedge b=a\wedge(a\wedge b)\leqq a\wedge((a\wedge b)\vee c)\leqq a\wedge(b\vee c)=$ $=a\wedge b$ we obtain $a\wedge((a\wedge b)\vee c)=a\wedge b$. Hence $\{a, c, b\vee c, a\wedge b, (a\wedge b)\vee c, a\wedge c,$ $a\vee c\}\cong S_2$, which is a contradiction. Therefore Case 1.1 is impossible.

Case 1.2: $a\wedge b\not\parallel c$. We obtain $a\wedge b\wedge c=a\wedge b=a\wedge c$ from $a\wedge b<c$ and $a\wedge(b\vee c)=a\wedge b$.

Case 1.2.1: $b\wedge c\neq a\wedge b\wedge c$. If we had $a\vee b\neq a\vee(b\wedge c)$ then $\{a, b\wedge c, a\vee(b\wedge c),$ $a\vee b, a\wedge b, c, a\vee c\}$ would be isomorphic to S_2. Therefore $a\vee b=a\vee(b\wedge c)$ and so $\{a, b, a\vee b, b\wedge c, a\wedge b, b\vee c, a\vee c\}$ is isomorphic to S_3. This contradiction shows that Case 1.2.1 is impossible.

Case 1.2.2: $b\wedge c=a\wedge b\wedge c$. Then the earlier equalities yield $[a, b, c]\cong P_1$. This completes the proof of Claim 1.

Claim 2. If $j=2$ and $m=3$ then $[a, b, c]\cong P_4$.

To prove this claim suppose $b\vee c\neq a\vee b\vee c$.

Case 2.1: $a\wedge(b\vee c)\neq a\wedge b\wedge c$. Since $[a, b, c]\not\cong S_5$, we have either $(a\wedge(b\vee c))\vee$ $\vee b\neq b\vee c$ or $(a\wedge(b\vee c))\vee c\neq b\vee c$. Thus, e.g., $(a\wedge(b\vee c))\vee b\neq b\vee c$ can be assumed. But then $\{a, b, b\vee c, a\vee b, a\wedge b, a\wedge(b\vee c), (a\wedge(b\vee c))\vee b\}\cong S_2$, a contradiction, showing that Case 2.1 is impossible.

Case 2.2: $a\wedge(b\vee c)=a\wedge b\wedge c$. Then $[a, b, c]$ is isomorphic to P_4, which completes the proof of Claim 2.

For $j=m=2$ the only essentially different cases are

$$a\wedge b\neq a\wedge b\wedge c \quad\text{and}\quad b\vee c\neq a\vee b\vee c;$$

$$b\wedge c\neq a\wedge b\wedge c \quad\text{and}\quad b\vee c\neq a\vee b\vee c.$$

Claim 3. If $j=m=2, a\wedge b\neq a\wedge b\wedge c$ and $b\vee c\neq a\vee b\vee c$ then $[a, b, c]\cong P_3$.

To prove this claim suppose $[a, b, c]\not\cong P_3$. Then either $a\wedge(b\vee c)\neq a\wedge b$ or $(a\wedge b)\vee c\neq b\vee c$. By the lattice theoretical Duality Principle $a\wedge(b\vee c)\neq a\wedge b$ can be assumed. Then we have $(a\wedge b)\vee c\neq b\vee c$ as well, since otherwise $\{a, c, b\vee c,$ $a\wedge b, a\wedge(b\vee c), a\wedge c, a\vee c\}$ would be isomorphic to S_3. Similarly, we have $a\wedge(b\vee c)\not\equiv$ $\not\equiv(a\wedge b)\vee c$ since otherwise $\{a, c, b\vee c, (a\wedge b)\vee c, a\wedge(b\vee c), a\wedge c, a\vee c\}$ would be isomorphic to S_2. Therefore $\{a\wedge(b\vee c), b, (a\wedge b)\vee c\}$ is an antichain (i.e., a set of pairwise incomparable elements). Clearly, we have $a\wedge((a\wedge b)\vee c)=$ $=(a\wedge(b\vee c))\wedge((a\wedge b)\vee c)$. Therefore $(a\wedge(b\vee c))\vee((a\wedge b)\vee c)=b\vee c$ since otherwise $\{a, b\vee c, (a\wedge b)\vee c, a\wedge(b\vee c), a\vee c, a\wedge((a\wedge b)\vee c), (a\wedge(b\vee c))\vee((a\wedge b)\vee c)\}$ would be isomorphic to S_2. The dual argument shows $(a\wedge(b\vee c))\wedge((a\wedge b)\vee c)=a\wedge b$. From $R_{a,b,b\vee c}$ and $R_{c,a\wedge b,b}$ we get $(a\wedge(b\vee c))\vee b=b\vee c$ and $((a\wedge b)\vee c)\wedge b=a\wedge b$. On the other hand, it is clear that $(a\wedge(b\vee c))\wedge b=a\wedge b$ and $((a\wedge b)\vee c)\vee b=b\vee c$. It follows from the equations we have obtained that $\{a, b, a\wedge b, b\vee c, a\vee c, a\wedge(b\vee c), (a\wedge b)\vee c\}$ is isomorphic to S_5. This contradiction completes the proof of Claim 3.

Claim 4. If $j=m=2, b\wedge c\neq a\wedge b\wedge c$ and $b\vee c\neq a\vee b\vee c$ then $[a, b, c]$ is isomorphic to P_6.

Suppose $[a, b, c]\not\cong P_6$. Then we have $a\wedge(b\vee c)\neq a\wedge c$ or $a\vee(b\wedge c)\neq a\vee c$, so $a\wedge(b\vee c)\neq a\wedge c$ can be assumed by the Duality Principle. From $R_{a,b\wedge c,b\vee c}$ we obtain $a\vee(b\wedge c)=a\vee c$. We have $(a\wedge(b\vee c))\vee(b\wedge c)=b\vee c$ since otherwise $\{a, a\wedge(b\vee c), a\wedge c, a\vee c, b\vee c, b\wedge c, (a\wedge(b\vee c))\vee(b\wedge c)\}$ would be isomorphic to S_2. Therefore $[a, b, c]$ is isomorphic to S_7, which is a contradiction, completing the proof of Claim 4.

Clearly, if $j=m=3$, then $[a, b, c]\cong P_7$. Hence (2) together with Claims 1, 2, 3, 4 and their dual statements complete the proof of Theorem 2.

IV. Structure theorem for modular lattices having 2-distributive sublattice lattices

First we recall the notion of the *special sum of lattices* from [1]. Let a set of indices I which is a chain and lattices L_i $(i\in I)$ be given. Define the following binary relation ϑ on the ordinal sum $\sum_{i\in I} L_i$ of the lattices L_i:

$(a, b)\in\vartheta$ iff there exist $i, j\in I$ such that j covers i, a is the greatest element of L_i and b is the lowest element of L_j.

Let Θ be the equivalence relation generated by ϑ. Then Θ is a congruence relation. The factor lattice $\sum_{i\in I} L_i/\Theta$ will be called the special sum of the lattices L_i and will be denoted by $\sum'_{i\in I} L_i$.

For example, if $I=\{1, 3, 7\}$ with $1<3<7$, then $\sum'_{i\in I} S_i$ is the following lattice:

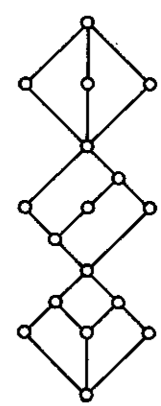

Fig. 3

Now we can state the following theorem, which generalizes the main result of [1].

THEOREM 3. *For an arbitrary modular lattice L the following four conditions are equivalent:*

(i) $\mathrm{Su}(L)$ *is 2-distributive;*

(ii) *None of* S_1, S_4, S_5 *and* S_8 *is a sublattice of* L*;*

(iii) S_4 *is not a sublattice of* L *and any three-element antichain in* L *generates a sublattice isomorphic to* P_7*;*

(iv) L *is isomorphic to a special sum* $\sum_{i\in I}' L_i$ *where* I *is a chain, and for all* $i\in I$ *one of the following three conditions is satisfied:*

(a) L_i *is a chain;*

(b) L_i *is the direct product of a chain and the two-element lattice;*

(c) L_i *has lowest and greatest elements (in notation* 0_i *and* 1_i*), and* $L_i\setminus\{0_i, 1_i\}$ *is an antichain.*

PROOF. The equivalence of (i), (ii) and (iii) immediately follows from Theorem 2 and the well-known criterion of modularity (cf. GRÄTZER [3, page 59]). The implication (iv)→(iii) is straightforward. So, only the implication (iii)→(iv) has to be shown. For a lattice M let us define $C'(M)=\{x\in M: x\neq 0_M, x\neq 1_M$ and $x \not\parallel y$ for any $y\in M\}$. By [1, Lemma 2] and [1, Theorem] it is enough to show that whenever M is a modular, non-distributive lattic efor which (iii) and $C'(M)=\varnothing$ hold then $M\setminus\{0_M, 1_M\}$ is an antichain. Suppose M is a modular, non-distributive lattice for which (iii) and $C'(M)=\varnothing$ hold. By the well-known criterion of distributivity (cf. GRÄTZER [3, page 59]), M contains a three-element antichain. Thus, by Zorn's Lemma, M contains an at least three-element *maximal* antichain A. Set $B=A\cup\{a\vee b, a\wedge b\}$ where a and b are distinct elements in A. By (iii) B is a sublattice of M, so it is enough to show that $B=M$. Suppose $a\vee b$ is not the greatest element of M. Then $x\parallel a\vee b$ for some $x\in M$. Choosing two distinct elements y, z from A such that $y\parallel x$ and $z\parallel x$, $x\vee y\vee z=x\vee a\vee b\neq a\vee b=y\vee z$ contradicts (iii). Therefore $a\vee b=1=1_M$ and, similarly, $a\wedge b=0=0_M$. Suppose x is an element in $M\setminus B$. Since $A\cup\{x\}$ is not an antichain, $x\not\parallel y$, say $x<y$, for some y in A. Choose two distinct elements d, e in $A\setminus\{y\}$. Since $\{x, d, e\}$ is an antichain, by (iii) we have $x\vee d=d\vee e=1$, $x\wedge d=d\wedge e=0$. Hence $\{0, 1, x, y, d\}$ is isomorphic to R_2, which contradicts the modularity of M. Therefore $M=B$. Q.E.D.

References

[1] G. CZÉDLI, Which distributive lattices have 2-distributive sublattice lattices, *Acta Math. Acad. Sci. Hungar.*, **35** (1980), 455—463.

[2] N. D. FILIPPOV, Probjectivity of lattices (in Russian), *Mat. Sbornik*, **70** (1966), 36—54.

[3] G. GRÄTZER, *General lattice theory*, Akademie-Verlag (Berlin, 1978).

[4] A. P. HUHN, Schwach distributive Verbände, *Acta F.R.N. Univ. Comen.—Mathematica*, Nim. cislo (1971), 51—56. (Summary of the author's lecture at the Summer School in 1970, Czechoslovakia.)

[5] A. P. HUHN, Schwach distributive Verbände. I, *Acta Sci. Math. Szeged*, **33** (1972), 297—305.

[6] A. P. HUHN, *n*-distributivity and some questions of the equational theory of lattices, *Colloquia Math. Soc. János Bolyai*, **17**. *Contributions to Universal Algebra*, Szeged (Hungary, 1975).

[7] K. M. KOH, On sublattices of a lattice, *Nanta Math.*, **6** (1973), 68—79.

[8] H. LAKSER, A note on the lattice of sublattices of a finite lattice, *Nanta Math.*, **6** (1973), 55—57.

[9] J. B. NATION, Varieties whose congruences satisfy certain lattice identities, *Algebra Univ.*, **4** (1974), 78—88.

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