On averaging Frankl's conjecture for large union-closed-sets

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Dedicated to László Lovász, president of IMU, on his sixtieth birthday

Abstract. Let $\mathcal{F}$ be a union-closed family of subsets of an m-element set A. Let $n = |\mathcal{F}| \geq 2$ and for $a \in A$ let $s(a)$ denote the number of sets in $\mathcal{F}$ that contain a. *Frankl's conjecture* from 1979, also known as the *union-closed sets conjecture*, states that there exists an element $a \in A$ with $n - 2s(a) \leq 0$. Strengthening a result of Gao and Yu [7] we verify the conjecture for the particular case when $m \geq 3$ and $n \geq 2^m - 2^{m/2}$. Moreover, for these "large" families $\mathcal{F}$ we prove an even stronger version via averaging. Namely, the sum of the $n - 2s(a)$, for all $a \in A$, is shown to be non-positive. Notice that this stronger version does not hold for all union-closed families; however we conjecture that it holds for a much wider class of families than considered here. Although the proof of the result is based on elementary lattice theory, the paper is self-contained and the reader is not assumed to be familiar with lattices.

1. Introduction and the main theorem

Given an m-element finite set $A = \{a_1, \ldots, a_m\}$, a *family* (or, in other words, a set) $\mathcal{F}$ of subsets of A, i.e. $\mathcal{F} \subseteq P(A)$, is called a *union-closed family* (over A) if $X, Y \in \mathcal{F}$ implies $X \cup Y \in \mathcal{F}$ for all $X, Y \in \mathcal{F}$. We always assume that A is finite with $3 \leq m := |A|$ and $n := |\mathcal{F}| \geq 2$. It was Peter Frankl in 1979 who formulated the following conjecture, now called as *Frankl's conjecture* or *the union-closed sets conjecture*: if $\mathcal{F}$ is as above then there exists an element of A which is contained in at least half of the members of $\mathcal{F}$. In spite of a great number of papers by outstanding authors (only some of them are listed at the end of the present paper but the reader can consult with their bibliographies, too) this conjecture is still open. The known achievements of this field belong to two categories.

The majority of results belong to pure combinatorics, with respect to both the statements and their proofs. They establish the conjecture under some extra stipulations like upper bounds on $m = |A|$ or $\mathcal{F}$ or the presence of certain set(s) in $\mathcal{F}$. For example, Morris [10] resp. Faro [5] settles the case $m \leq 9$ resp. $n \leq 37$, and Roberts [14] improves this for $n \leq 40$. Roberts [14] also verifies the conjecture for

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"small families", i.e. for $n < 4m-1$, while for "large families", i.e. for those with

$$n \geq 2^m - 12\Big(\frac{3}{2}\Big)^{[m/3]} - \frac{1}{2}\binom{m}{3} - \frac{5}{3}m + 44.5\,, \tag{1}$$

this was done by Gao and Yu [7]. For other achievements of combinatorial nature cf., e.g., Norton and Sarvate [11] and Vaughan [16]. One can read more about the problem at http://www.math.uiuc.edu/ west/openp/unionclos.html and, of course, in Frankl [6].

On the other hand, some results together with their proofs belongs to lattice theory. For example, Reinhold [13] proves the lattice theoretic version of the conjecture (to be mentioned later) for lower semimodular lattices; cf. Abe [1] and [2], Abe and Nakano [3] and Herrmann [9] for similar results.

However, there are no real links between the combinatorial and the lattice theoretical approaches, except of course for the statement of their equivalence, cf. Abe and Nakano [3], who gives the credit to Poonen [12] and Stanley [15]. In particular, results that look "combinatorial" are proved by combinatorial methods. One of the novelties of the present work is that although the main result looks combinatorial without mentioning lattices, it is achieved via a purely lattice theoretic method. At this point it is worth assuring the reader from combinatorics that only a very elementary part of lattice theory will be used and the paper is intended to be self-contained.

Let $\mathcal{F}$ be a union-closed family over A and let the notations $n = |\mathcal{F}| \geq 2$, $m = |A| = |\{a_1, \dots, a_m\}|$ be fixed throughout. For $a \in A$ let $s(a) = |\{B \in \mathcal{F} : a \in B\}|$. Then Frankl's conjecture claims the existence of an $a \in A$ with $n - 2s(a) \leq 0$. Let us say that $\mathcal{F}$ satisfies the *averaged Frankl's property* if

$$\sum_{a\in A} \big(n - 2s(a)\big) \leq 0\,.$$

Although this property clearly implies that Frankl's conjecture holds for the given $\mathcal{F}$, there are many union-closed families for which the averaged Frankl's property fails; examples will be given later, in the lattice environment.

For a given $m = |A|$, the maximum value of n is of course 2^m. If n is close to 2^m then we say that the family $\mathcal{F}$ is large. Our main result on large families is the following.

Theorem 1. *If $\mathcal{F}$ is a union-closed family over a nonempty m-element set A, $m \geq 3$, and $\mathcal{F}$ is large in the sense*

$$n := |\mathcal{F}| \geq 2^m - 2^{m/2} = 2^m - \sqrt{2^m} \tag{2}$$

then $\mathcal{F}$ satisfies the averaged Frankl's property $\sum_{a\in A}\big(n - 2s(a)\big) \leq 0$.

This theorem strengthens the afore-mentioned result of Gao and Yu [7] in two ways: it deals with the *averaged* Frankl's property and (2) allows much more families than formula (1). Some more discussion on this theorem will be given at the end of the paper.

2. Lattices and proofs

In order to fix our notations we recall some well known concepts from lattice theory. By a lattice $(L;\leq)$ we mean a partially ordered set such that for any $x, y \in L$ the supremum and infimum of $\{x, y\}$ exist; they are denoted by $x \vee y$ and $x \wedge y$, respectively. We deal only with *finite* lattices; they necessarily have a unique least element 0 and a unique largest element 1. An element z of L is said to be *join-irreducible* if for all $x, y \in L$ the equation $z = x \vee y$ implies $z \in \{x, y\}$. The set of join-irreducible elements distinct from 0 will be denoted by $J(L)$. For $a \leq b \in L$ the subset $\{x \in L : a \leq x \leq b\}$ is denoted by $[a, b]$ and it is called an *interval* of L. When $a = 0$ or $b = 1$ then a particular notation applies: $\uparrow a = [a, 1]$ and $\downarrow b = [0, b]$. The *covering relation* is defined via $a \prec b$ iff $a \leq b$ and $|[a, b]| = 2$. The basic facts on lattices can be found practically in any textbook on algebra like Burris and Sankappanavar [4]. (For the present status of lattice theory, which is not needed here, cf. Grätzer [8].) We will see soon that a union-closed family $\mathcal{F}$ corresponds to a lattice consisting of $|\mathcal{F}|$ elements. An advantage of lattices is that while in case of, say, $|\mathcal{F}| = 12$ usually it is hopeless to visualize $\mathcal{F}$, it is fairly easy and inspirational to depict a twelve element lattice.

It is well-known, cf. Abe and Nakano [3], Poonen [12] or Stanley [15], that Frankl's conjecture is equivalent to its lattice theoretical version, i.e., to the following conjecture: "for each finite lattice L with at least two elements there exists an $a \in J(L)$ with $|\uparrow a| \leq |L|/2$". In particular, the lattice theoretic Frankl's conjecture implies the original one. Since we are interested in the *averaged* property for *large* families, we have to analyze *the proof* of this implication.

Let $\mathcal{F} \subseteq P(A)$ be a *large* union-closed family with $3 \leq |A| = m$; assuming $\emptyset \in \mathcal{F}$ does not hurt the generality. Then the family $\mathcal{D} := \{A \setminus X : X \in \mathcal{F}\}$ is intersection-closed, in other words, it is a closure system. Therefore $\mathcal{D}$ is a lattice with respect to the set inclusion; the set theoretic intersection serves as the meet while the join is usually different from the set union. Now let us consider an arbitrary $X \in J(\mathcal{D})$, and let $Z = \bigvee\{Y \in \mathcal{D} : Y < X\}$. Then $Z < X$, i.e. $Z \subset X$, for X is join-irreducible. Hence we can choose an element $a_X \in X \setminus Z$. We claim that, for any $Y \in \mathcal{D}$, $a_X \in Y$ iff $X \subseteq Y$. Indeed, if $a_X \in Y$ but $X \nsubseteq Y$ then $X \cap Y = X \wedge Y \neq X$ gives $X \cap Y \subseteq Z$, contradicting $a_X \notin Z$. Notice that a_X is unique; indeed, otherwise we had another element $b \in A$ such that each $Y \in \mathcal{D}$ (and therefore each $Y \in \mathcal{F}$) contained either both a_X and b or none of them, which easily led to $|\mathcal{F}| \leq 2^{m-1}$, a contradiction. Hence the mapping $J(\mathcal{D}) \to A$, $X \mapsto a_X$ is injective. Clearly, $2^m - 2^{m/2} \leq |\mathcal{F}| = |\mathcal{D}| \leq |P(J(\mathcal{D}))|$ gives $|J(D)| \geq m = |A|$. Therefore, the aforementioned mapping is a bijection.

Now, for each $a = a_X \in A$, $|\{Y \in \mathcal{F} : a \in Y\}| = |\{Y \in \mathcal{D} : a_X \notin Y\}| = |\{Y \in \mathcal{D} : X \nsubseteq Y\}| = |\mathcal{D} \setminus (\uparrow X)|$. This gives $|\{Y \in \mathcal{F} : a_X \in Y\}| \geq |\mathcal{F}|/2 = n/2$ iff $|\uparrow X| \leq n/2$, and this makes it clear that Theorem 1 is a consequence of the following, purely lattice theoretic theorem. Before formulating this theorem, we introduce some notations for the rest of the paper. For $a \in J(L)$ let $r(a) = |L| - 2 \cdot |\uparrow a|$, and let $r(L) = \sum\{r(a) : a \in J(L)\}$.

Theorem 2. *Let L be a finite lattice consisting of at least two elements, and let $m = |J(L)| \geq 3$. If $|L| \geq 2^m - 2^{m/2}$ then $r(L) \geq 0$.*

When proving this theorem, L is often treated as a $\{0, \vee\}$ semilattice. This means that we forget about the meet operation $\wedge$ and by a congruence we mean an equivalence relation compatible with the join operation $\vee$ (but not necessarily with $\wedge$). Let $X = \{x_1, \ldots, x_m\}$ be a fixed m-element set and consider its power set $P(X) = \bigl(P(X), \subseteq\bigr)$ as a $\{0, \vee\}$-semilattice; of course 0 is the empty set and $\vee$ stands for the set union $\cup$.

Lemma 1. *There is a congruence Θ of the $\{0, \vee\}$-semilattice $\bigl(P(X), \subseteq\bigr)$ such that L, as a $\{0, \vee\}$-semilattice, is isomorphic to the factor semilattice $P(X)/\Theta$.*

Proof. The lemma is a trivial consequence of the description of free $\{0, \vee\}$-semilattices, which belongs to the folklore of lattice theory and universal algebra, cf. e.g. Exercise 4 of Section \$11 (in page 85) in Burris and Sankappanavar [4]. $\square$

Let $\widetilde{X}$ stand for $\bigl\{\{x\} : x \in X\bigr\} = J(P(X))$. The Θ-class of an element $u \in P(X)$ will be denoted by $[u]\Theta$ or simply by $[u]$. In virtue of Lemma 1 we will assume that L equals $P(X)/\Theta$ and $A = \{a_1, \ldots, a_m\} = J(L)$ such that $a_i = [\{x_i\}]$ for $i \in \{1, \ldots, m\}$. For a Θ-class $[u] \in P(X)/\Theta = L$ let $e([u]) = e_\Theta([u]) = |[u]\Theta \setminus \{u\}| = |[u]| - 1$, the *excess* of $[u] \in L$. Sometimes we use the notation $e_\Psi([u]) = |[u]\Psi \setminus \{u\}|$ for another equivalence Ψ (not necessarily a congruence) on $P(X)$; then the subscript Ψ is never dropped. Since the isomorphism between $P(X)/\Theta$ and L is considered fixed, we can use the notation $e(b)$ for any $b \in L$. Clearly, we have

$$2^{m/2} \geq 2^m - n = |P(X)| - |L| = \sum_{b \in L} e(b). \tag{3}$$

An element $b \in L$ will be called an *abundant element* if $e(b) > 0$. In accordance with the terminology of lattices, for $u \in P(X)$ the *height* of u, denoted by $h(u)$ is defined as $|u| = |{\downarrow} u \cap \widetilde{X}|$.

Lemma 2. *If $[u] \in L$ is abundant then $h(u) \geq m/2 - 1$.*

Proof. It belongs to the folklore (or it can trivially be extracted from the proof of Lemma I.3.7 in Grätzer [8]) that the Θ-classes of $P(X)$ are convex subsemilattices. This means that for every $[u] \in P(X)/\Theta$, $[u]$ is closed with respect to join and $v_1 \leq v_2 \leq v_3 \in P(X)$ together with $v_1, v_3 \in [u]$ imply $v_2 \in [u]$. Hence, without loss of generality, we may assume that u is a minimal element in its abundant Θ-class $[u]$ and there is an element $v \in [u]$ such that $u \prec v$. Since $P(X)$ and therefore any of its interval can also be considered as a Boolean algebra, we may take the unique (relative) complement v' of v in ${\uparrow} u$. We have $v \wedge v' = u$ and $v \vee v' = 1$, $h(v) = h(u) + 1$ and $h(v') = m - 1$.

Let Ψ denote the smallest equivalence (not a congruence!) including $\{(t, t \vee v) : t \in [u, v']\}$. Observe that for every $t \in [u, v']$, $|[t]\Psi| = 2$ and $e_\Psi([t]\Psi) = 1$. Indeed, otherwise $t_1 \vee v = t_2 \vee v$ would hold for some distinct $t_1, t_2 \in [u, v']$ and distributivity would easily lead to a contradiction: $t_1 = (t_1 \wedge v') \vee u = (t_1 \wedge v') \vee (v \wedge v') =$

$(t_1 \vee v) \wedge v' = (t_2 \vee v) \wedge v' = \ldots = t_2$. Since for each $t \in [u, v']$ we have $(t, t \vee v) = (t \vee u, t \vee v) \in \Theta$, we obtain that $\Psi \subseteq \Theta$. Now, for a Θ-class $b \in L$, assume that $b \cap [u, v'] = \{t_1, \ldots, t_\ell\}$ with $\ell \geq 1$ and $t_i \neq t_j$ for $i \neq j$. Then $\Psi \subseteq \Theta$ yields that the $t_i \vee v$ belong to b, whence $e(b) \geq 2\ell - 1 \geq l = e_\Psi([t_1]\Psi) + \cdots + e_\Psi([t_\ell]\Psi)$. Hence we conclude

$$\sum_{b \in L} e(b) \geq \sum_{t \in [u,v']} e_\Psi([t]\Psi) \geq |[u, v']| = 2^{h(v')-h(u)} = 2^{m-1-h(u)}. \tag{4}$$

Now (3) and (4) entail $m/2 \geq m - 1 - h(u)$, implying the lemma. $\square$

Lemma 3. *There is at most one $u \in P(X)$ such that $h(u) < m/2$ and $[u]$ is abundant.*

Proof. By way of contradiction we suppose that u_1 and u_2 are distinct abundant elements of $P(X)$ and $h(u_i) < m/2$ for $i = 1, 2$. It follows from Lemma 2 that $h(u_1) = h(u_2) = \lfloor (m-1)/2 \rfloor$ and u_i is a minimal element in $[u_i]$ for $i \in \{1, 2\}$. Like in the previous proof, for $i \in \{1, 2\}$ there is a $v_i \in P(X)$ such that $v_i \in [u_i]$, $u_i \prec v_i$ and v_i has a unique (relative) complement $v_i' \in \uparrow u_i$. Let $\alpha_i = \{(t, t \vee v_i) : t \in [u_i, v_i']\}$, and let $\Psi = \alpha_1 \cup \alpha_2$. (In general, they are equivalences, not necessarily semilattice congruences.) The proof of Lemma 2 shows that each of the α_i classes has at most two elements, $|[t]\alpha_i| = 2$ for all $t \in [u_i, v_i']\}$, and $\Psi \subseteq \Theta$. Hence for $i = 1, 2$, like in case of (4),

$$\sum_{[t]\alpha_i \in L/\alpha_i} e_{\alpha_i}([t]\alpha_i) = |[u_i, v_i']| = 2^{m-1-\lfloor (m-1)/2 \rfloor} = 2^{\lfloor m/2 \rfloor}. \tag{5}$$

This may give the feeling that

$$\sum_{b \in L} e(b) \geq \sum_{[t]\Psi \in L/\Psi} e_\Psi([t]\Psi) \geq^* 2^{\lfloor m/2 \rfloor} + 2^{\lfloor m/2 \rfloor} = 2^{\lfloor m/2 \rfloor + 1}. \tag{6}$$

However, the above estimation for the *total excess* $\sum_{b \in L} e(b)$ is not correct at $\geq^*$ since the contribution of (5) for $i = 1$ and that for $i = 2$ are not necessarily "disjoint", so the "common contribution" has to be subtracted from $2^{\lfloor m/2 \rfloor + 1}$.

Now consider a Ψ-class H as a graph of $\Psi|_H$. We disregard from loop edges. Then this graph is a connected one, and each of its edges has a unique color from the color set $\{\alpha_1, \alpha_2\}$. Two parallel edges with distinct colors are possible. Since the α_i-classes have at most two elements, the degree of each vertex of this graph is at most two. If this graph contains no circle then, in connection with H, nothing has to be subtracted from $2^{\lfloor m/2 \rfloor + 1}$. This is exemplified by, say, $H = \{w_1, \ldots, w_6\}$ with $(w_1, w_2), (w_3, w_4), (w_5, w_6) \in \alpha_1$ and $(w_2, w_3), (w_4, w_5) \in \alpha_2$, then $e_\Psi(H) = 5$, and this is the same as the sum $e_{\alpha_1}([w_1]\alpha_1) + e_{\alpha_1}([w_3]\alpha_1) + e_{\alpha_1}([w_5]\alpha_1) + e_{\alpha_2}([w_2]\alpha_2) + e_{\alpha_2}([w_4]\alpha_2)$.

So $2^{\lfloor m/2 \rfloor + 1}$ needs correction only for those H that contain a circle. Since H is connected with vertex degrees ≤ 2, this means that H is a circle, and the colors α_1 and α_2 alternate on this circle. Since both α_1 and α_2 are included in the $<$ relation of $P(X)$, we can consider H as an oriented graph such that the start point of each edge should be less then its endpoint. In fact, we imagine H as a regular $|H|$-gon in the plain. Since the relation $<$ is irreflexive, it is impossible that all edges are oriented clock-wise or they are all oriented anti-clockwise. Therefore there are consecutive elements t_1, t, t_2 of H such that $t_1 > t < t_2$ and $(t, t_1) \in \alpha_i$ and $(t, t_2) \in \alpha_{1-i}$. (The possibility $t_1 = t_2$ is allowed.) From $(t, t_1) \in \alpha_i$ we conclude that $u_i \le t \le v_i'$ while $(t, t_2) \in \alpha_{1-i}$ entails $u_{1-i} \le t \le v_{1-i}'$. Hence t belongs to the interval $[u_1 \vee u_2, v_1' \wedge v_2']$. Since the "$\Psi$-excess" $e_\Psi([t]\Psi) = e(H)$ is one less than the sum of the "α_i-excesses" (with alternating i) of its edges, we have to subtract one from $2^{\lfloor m/2\rfloor+1}$ according to H. We can associate the above $t \in [u_1 \vee u_2, v_1' \wedge v_2']$ with this subtraction; t is not necessarily unique but distinct circles H give rise to distinct elements t. Hence the total subtraction is at most $|[u_1 \vee u_2, v_1' \wedge v_2']|$.

Since $u_1 \ne u_2$, $h(u_1 \vee u_2) \ge h(u_1) = h(u_2)$, so $h(u_1 \vee u_2) \ge h(u_1) + 1 = 1 + \lfloor (m-1)/2 \rfloor = \lfloor (m+1)/2 \rfloor$. We cannot say $v_1' \ne v_2'$, we have only $h(v_1' \wedge v_2') \le h(v_1') = m - 1$. So we obtain

$$|[u_1 \vee u_2, v_1' \wedge v_2']| \le 2^{m-1-\lfloor (m+1)/2 \rfloor} = 2^{\lfloor m/2 \rfloor - 1}\,. \tag{7}$$

Now subtracting (7) from the right hand side of (6) we obtain that the total excess is at least

$$\sum_{b\in L} e(b) \ge 2^{\lfloor m/2\rfloor+1} - 2^{\lfloor m/2\rfloor-1} = (3/2)\cdot 2^{\lfloor m/2\rfloor}. \tag{8}$$

Finally, after inspecting even and odd values of m separately, we see that (3) contradicts (8), completing the proof of Lemma 3. □

Now we are in the position of proving Theorem 2:

Proof of Theorem 2. Let $H_1, \ldots, H_\ell$ be a complete list of abundant (i.e., non-singleton) Θ-classes. It has already been mentioned that the H_i are (convex) subsemilattices. Therefore each H_i has a unique largest element w_i. Then $H_i = [w_i]$. Denote $H_i \setminus \{w_i\}$ by G_i and let $G = G_1 \cup \cdots \cup G_\ell$. Clearly, $|G| = \sum_{b\in L} e(b)$, the total excess of L. We claim that

$$\sum_{g\in G} |{\downarrow} g \cap \widetilde{X}| = \sum_{g\in G} h(g) \ge \frac{m}{2}\cdot |G|. \tag{9}$$

The equality is trivial by definitions. The inequality is almost clear by Lemmas 2 and 3, for all but at most one summands satisfy $h(g) \ge m/2$. Suppose there is a $g \in G$ with $h(g) < m/2$. Then this g is unique and $h(g) = \lfloor (m-1)/2 \rfloor$ by Lemmas 2 and 3. Since $[g]$ is a convex subsemilattice and g is not its largest element, there is an element $v \in [g] \setminus \{g\}$ such that $g \prec v$. Let v' be the complement of v in $P(X)$. Then $h(g \vee v') = m - 1$. Joining $(g, v) \in \Theta$ and $(v', v') \in \Theta$ we have $(g \vee v', 1) \in \Theta$, which yields that $g \vee v' \in G$. Now, exploiting $m \ge 3$ the first time, $\lfloor (m-1)/2 \rfloor < m - 1$ gives $g \ne g \vee v'$ and $h(g) + h(g \vee v') = \lfloor (m-1)/2 \rfloor + m - 1 \ge 2 \cdot m/2$ proves (9).

Now, for $a_i = [\{x_i\}] \in J(L)$, $|{\uparrow} a_i|$, computed in L, equals $|{\uparrow}\{x_i\} \setminus G| = |{\uparrow}\{x_i\} \setminus (G \cap {\uparrow}\{x_i\})| = 2^{m-1} - |(G \cap {\uparrow}\{x_i\})|$. Notice also that $|L| = n$ and for any $y \in P(X)$, $h(y) = |\{x_i \in X : \{x_i\} \le y\}|$. Hence

$$r(L) = \sum_{i=1}^{m} \bigl(|L| - 2\cdot |{\uparrow} a_i|\bigr) = mn - 2\sum_{i=1}^{m} |{\uparrow} a_i| =$$

$$mn - 2\sum_{i=1}^{m}\bigl(2^{m-1} - |(G \cap \uparrow\{x_i\})|\bigr) = mn - m\cdot 2^m + 2\sum_{i=1}^{m}|(G \cap \uparrow\{x_i\})| =$$
$$m(n-2^m) + 2\cdot\bigl|\{(g,x) : x \in X,\ g \in G,\ \text{and } \{x\} \le g\}\bigr| =$$
$$m(n-2^m) + 2\sum_{g\in G}|{\downarrow g} \cap \widetilde{X}| \ge^{(9)} m(n-2^m) + m\cdot|G|$$
$$= m\bigl(n - (2^m - |G|)\bigr) = m\Bigl(n - \bigl(2^m - \sum_{e\in L} e(b)\bigr)\Bigr) =^{(3)} m(n-n) = 0,$$

proving Theorem 2. □

The above proof reveals that condition (2) is far from being optimal for large m. However, we do not see how far we could go with our method, and therefore we have decided not to spoil the simplicity of condition (2) by making the proof much more complicated without reaching the optimal condition. We conjecture that Theorem 1 remains true if $2^m - 2^{m/2}$ is replaced by something even smaller than $2^m - 2^{m-2}$. We also conjecture that $r(L) > 0$ (equivalently, $\sum_{a\in A}\bigl(n - 2s(a)\bigr) < 0$) when $2^m > n > 2^m - 2^{m-2}$ and $m \ge 3$. These conjectures come from a great number of examples examined by computer, and also from the following example.

Let L be the direct product of a Boolean algebra with $m-2$ atoms and the three element chain. (For $m = 4$, L is given in Figure 1; $J(L)$ consists of the black-filled elements.) We omit the details of showing that this lattice L has the properties $J(L) = m$, $|L| = 2^m - 2^{m-2} = 3\cdot 2^{m-2}$ and $r(L) = 0$.

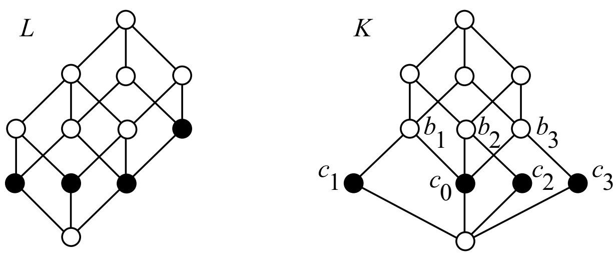

FIGURE 1

We conclude the paper with another example which shows that the averaged Frankl's property does not hold for all lattices or, equivalently, for all union-closed families. Take a Boolean algebra B with k atoms. (The case $k = 3$ is depicted in Figure 1.) Let $b_1, \dots, b_k$ be the atoms of B. Rename the 0 of B as c_0, add a new 0 and for $i = 1, \dots, k$, add a new atom c_i such that $0 \prec c_i \prec b_i$. This way we obtain a lattice K consisting of $2^k + k + 1$ elements. Now $r(c_0) = 2^k + k + 1 - 2\cdot 2^k = k + 1 - 2^k$ and, for $1 \le i \le k$, $r(c_i) = 2^k + k + 1 - 2(1 + 2^{k-1}) = k - 1$. Hence

$$r(K) = r(c_0) + r(c_1) + \cdots + r(c_k) = k^2 + 1 - 2^k < 0$$

when $k \ge 5$.

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