ARCHIVUM MATHEMATICUM (BRNO)
Tomus 27 (1991), 149 – 153

NOTES ON CONGRUENCE IMPLICATION

GÁBOR CZÉDLI*

(Received September 8, 1989)

ABSTRACT. Besides the usual implication between lattice identities the classes $\mathrm{Con}\,\mathcal{V} = \{\mathrm{Con}\,A : A \in \mathcal{V}\}$, with $\mathcal{V}$ being closed with respect to some operators, give rise to some new kind(s) of implication. Other kinds of implication arise when $\mathcal{V}$ has a nullary operation e. Then $\mathrm{Con}\,\mathcal{V}$ is said to satisfy a lattice identity $p(x_1,\dots,x_t) \le q(x_1,\dots,x_t)$ at e if the congruence block $[e]\,p(\alpha_1,\dots,\alpha_t)$ is included in $[e]\,q(\alpha_1,\dots,\alpha_t)$ for any $A \in \mathcal{V}$ and arbitrary $\alpha_1,\dots,\alpha_t \in \mathrm{Con}\,A$. This paper shows that some classical results on the implication in congruence varieties (the case when $\mathcal{V}$ is closed with respect to $\boldsymbol{H}$,$\boldsymbol{S}$ and $\boldsymbol{P}$) can be strengthened by using the above-mentioned kinds of implication. An example shows that this strengthening is not always possible.

Let A be an algebra with a nullary operation e and let $\lambda : p(x_1,\dots x_t) \le q(x_1,\dots,x_t)$ be a lattice identity. Following Chajda [1], we say that $\mathrm{Con}\,A$ satisfies λ at e if $[e]\,p(\alpha_1,\dots,\alpha_t) \subseteq [e]\,q(\alpha_1,\dots,\alpha_t)$ for any congruences $\alpha_1,\dots,\alpha_t$ of A. Let $\boldsymbol{P}$, $\boldsymbol{P^s}$, $\boldsymbol{H}$,$\boldsymbol{S}$ and $\boldsymbol{D}$ denote the operators of forming direct products, subdirect products, homomorphic images, subalgebras and directed unions, respectively. The first two, $\boldsymbol{P}$ and $\boldsymbol{P^s}$, may be equipped with subscripts i and/or f to indicate that the factors are identical (i.e., powers are taken) and/or only finitely many factors are allowed, respectively. For a class $\mathcal{V}$ of algebras let $\mathrm{Con}\,\mathcal{V} = \{\mathrm{Con}\,A : A \in \mathcal{V}\}$ and $\mathbf{Con}\ \mathcal{V} = \boldsymbol{HSP}\,\mathrm{Con}\,\mathcal{V}$. If $\mathcal{V}$ is an $\boldsymbol{HSP}$ resp. $\boldsymbol{DHSP_f}$-closed class then $\mathbf{Con}$ $\mathcal{V}$ is called a congruence variety resp. ℓ-congruence variety (cf. Jónsson [7] and [2]). A lattice identity is said to hold in $\mathrm{Con}\,\mathcal{V}$ (at e) if it holds in $\mathrm{Con}\,A$ (at e) for any $A \in \mathcal{V}$. For given operators $X_1,\dots,X_k$ and lattice identities λ and μ we write $\lambda \vDash \mu\ (X_1,\dots,X_k)$ (resp. $\lambda \vDash \mu\ \ (e, X_1,\dots X_k)$) to denote that for any $X_1,\dots,X_k$-closed class $\mathcal{V}$ if λ holds (resp. λ holds at e) in $\mathrm{Con}\,\mathcal{V}$ then so does μ (resp. μ at e). Similarly, $\lambda \vDash \mu$ denotes that all lattices satisfying λ also satisfy μ.

Nation [8] was the first who proved that $\vDash(\boldsymbol{HSP})$ is different from $\vDash$. His results were followed by many similar ones stating that $\lambda \vDash \mu\ (\boldsymbol{HSP})$ without $\lambda \vDash \mu$ for certain λ and μ, cf. Jónsson [7] for a survey. Later $\vDash (\boldsymbol{DHSP_f})$ and even $\vDash (\boldsymbol{P^s_{if}})$

1991 *Mathematics Subject Classification*: 08B05, 08B10.

Key words and phrases: congruence variety, Mal'tsev condition, lattice identity, locally equational class.

*Research partially supported by Hungarian National Foundation for Scientific Research grant no.1813

were found to be different from $\models$, cf. [2] and Freese, Herrmann and Huhn [4]. Although it is still not known whenther $\models$ $(\boldsymbol{HSP})$, $\models(\boldsymbol{DHSP}_f)$ and $\models$ $(\boldsymbol{P}^s_{if})$ are all the same, we have

Proposition 1. *$\models(\boldsymbol{DHSP}_f)$ and $\models(\boldsymbol{SP}_{if})$ coincide. In other words, the ℓ-congruence varieties are just the* **Con** *$\mathcal{V}$ with $\mathcal{V}$ being $\boldsymbol{SP}_{if}$-closed.*

Proof. Assume that $\mathcal{V}$ is $\boldsymbol{SP}_{if}$-closed. We need to show that **Con** $\mathcal{V}$ is an ℓ-congruence variety. It was remarked in [2, page 45] that for any $\boldsymbol{SP}_f$-closed class $\mathcal{U}$, **Con** $\mathcal{U}$ is an ℓ-congruence variety. (Indeed, **Con** $\mathcal{U}$ = **Con** $(\boldsymbol{H}\,\mathcal{U})$ =**Con** $(\boldsymbol{DH}\,\mathcal{U})$ and $\boldsymbol{DH}\,\mathcal{U}$ is $\boldsymbol{DHSP}_f$-closed already.) Therefore, for any $A \in \mathcal{V}$, **Con** $(\boldsymbol{SP}_f\{A\})$ is an ℓ-congruence variety, which is included in **Con** $\mathcal{V}$ as $\boldsymbol{SP}_f\{A\} = \boldsymbol{SP}_{if}\{A\}$. Clearly, **Con** $\mathcal{V} = \bigvee_{A\in\mathcal{V}} \textbf{Con}\,(\boldsymbol{SP}_f\{A\})$ where the join is taken in the lattice of all lattice varieties. But the join of ℓ-congruence varieties is an ℓ-congruence variety by [2, Thm. 2.1], completing the proof. □

Now let us revive one of Chajda's motivations for dealing with congruence identities at e. The proof and even the more explicit formulation of the folowing proposition would not only wander too far from the subject of this paper but it would be quite straightforward for those being familiar with Wille [10] or Pixley [9], and Chajda [1].

Proposition 2. *Given a lattice identity λ, let $(\forall\ m)\,(\exists\ n)\,(U_{mn})$ denote the weak Mal'tsev condition associated with λ by the Wille – Pixley algorithm. Denote U'_{mn} the (strong Mal'tsev) condition obtained from U_{mn} via substituting e for the first variables of term symbols everywhere. (E.g., we write $t_3(x_2,\dots,x_k)$ resp. $e = t_2(x_2,e,x_2,e)$ instead of $t_3(x_1,x_2,\dots,x_k)$ resp. $x_1 = t_2(x_1,x_2,x_1,x_2,x_1)$).) Then, for any variety $\mathcal{V}$ with a nullary operation e, $\mathrm{Con}\,\mathcal{V}$ satisfies λ at e iff $(\forall\ m)\,(\exists\ n)\,(U'_{mn})$ holds in $\mathcal{V}$.*

Now if $\mathcal{V}$ is a variety of modules then the satisfaction of λ in Con $\mathcal{V}$ can already be characterized by $U_{2,2}$. This was quite useful e.g. in [5]. However, $U'_{2,2}$, having one variable less, gives a better characterization since module congruences are determined by their blocks containing $e = 0$.

It is almost trivial that if $\lambda \models \mu\ (e, X_1 \dots X_k)$ then $\lambda \models \mu\ (X_1 \dots X_k)$. (The clue to this assertion is to equip each algebra in $\mathcal{V}$ with a new nullary operation e in all possible ways.) Therefore the following theorem strengthens its classical counterpart about $\models$ $(\boldsymbol{HSP})$, which was found by Nation [8]. Denoting the distributive law $x\,(y+z) \le xy + xz$ by dist and calling a lattice identity nontrivial if it does not hold in all lattices we have

Theorem. *Suppose ε is a nontrivial lattice identity of the form*

$$\sigma_0\, w \le \sum_{i=1}^{n} \sigma_0 \sigma_i$$

where the lattice terms $\sigma_0, \sigma_1, \ldots, \sigma_n$ are joins of variables and σ_0 and w have no variable in common. Then $\varepsilon \models$ dist $(e,$ $\boldsymbol{SP}_{if})$.

The proof we are going to present is almost the same as that of Nation's Theorem 3.7 in Jónsson [7]. (We make [7] our main reference because following the original proof in [8] the reader would have to consider several preliminary statements as well.) Therefore instead of repeating whole pages from [7] mutatis mutandis we only point out the necessary changes so that the original proof settle the lion's share of our theorem.

Proof. Let $\mathcal{V}$ be an $\boldsymbol{SP}_{if}$-closed class such that ε holds in Con $\mathcal{V}$ at e. Then so does ε', which was defined in [7, page 379]. Now the original proof starts to use $\mathcal{V}$-free algebras, which we do not have in general. Thus we need to consider an appropriate substitute. If F and B are algebras, $X \subseteq F$ and $Y \subseteq B$ are finite subsets, X generates F and each mapping $X \to Y$ extends a homomorphism $F \to B$ then $F = F(X; Y \subseteq B)$ is called a locally free algebra (with respect to X, Y and B). We claim that for each $B \in \mathcal{V}$, each finite subset $Y \subseteq B$ with more than one element and each finite set X there is a locally free algebra $F = F(X; Y \subseteq B)$ in $\mathcal{V}$. Really, consider the direct power $\prod_{f \in M} B$ where M is the set of all $X \to Y$ mappings. Denoting $(f(x) : f \in M)$ by $\hat{x}$ the subset $\hat{X} = \{\hat{x} : x \in X\}$ of $\prod_{f \in M} B$ can be identified with X. Let F be the subalgebra of $\prod_{f \in M} B$ generated by $\hat{X}$. Clearly, any map $g : \hat{X} \to Y$ extends to the homomorphism $F \to B$, $s \mapsto s(g)$.

Returning to the original proof, let $S = \{b, c\}$ with $b \neq c$ and let $F(S)$ denote an arbitrary (not necessarily locally free) algebra in $\mathcal{V}$ such that $S \subseteq F(S)$. Put $T = \{e, u_1, \ldots, u_n\}$ and let $F(T)$ denote a locally free algebra $F(\{u_1, \ldots, u_n\}; S \subseteq F(S))$ in $\mathcal{V}$. If both u_0 and a are understood as e then the original proof (apart form its last sentence) remains valid and yields

(1) $(e, b) \in \alpha\beta + \alpha\gamma$ where $\alpha =$ con $(e, b), \beta =$ con (e, c) and $\gamma =$ con (b, c).

Although from (1) we cannot deduce identities as Chajda [1], following Jónsson [6], did the ideas from [6] still can be used. Assume that $A \in \mathcal{V}$, $\alpha', \beta', \gamma' \in$ Con A, $b' \in A \setminus \{e\}$ and $(e, b') \in \alpha'(\beta' + \gamma')$. We need to show that (e, b') belongs to $\delta' = \alpha'\beta' + \alpha'\gamma'$. By the assumption there are elements $u_0 = e$, $u_1, u_2 \ldots, u_m = b'$ in A such that $u_i \beta' u_{i+1}$ for i even while $u_i \gamma' u_{i+1}$ for i odd ($0 \leq i < m$). From now on let $F(S)$ be a locally free $F(S; \{u_0, u_1, \ldots, u_m\} \subseteq A)$ algebra in $\mathcal{V}$. Since $(e, b) \in \alpha\beta + \alpha\gamma$, there are elements $t_0 = e$, $t_1, t_2, \ldots, t_r = b$ in $F(S)$ such that $t_0 \alpha t_1 \alpha t_2 \alpha \ldots \alpha t_r$, $t_i \beta t_{i+1}$ for i even and $t_i \gamma t_{i+1}$ for i odd ($0 \leq i < r$). Further, there exist binary terms q_i such that $t_i = q_i(c, b)$, $0 \leq i \leq r$. Let φ_i denote the unique $F(S) \to A$ homomorphism for which $\varphi_i(b) = b'$ and $\varphi_i(c) = u_i$, $0 \leq i \leq m$. Observe that for any $x, y \in F(S)$ and $\mu \in \{\alpha, \beta, \gamma\}$

(2) if μ' collapses the φ_i-image of the generic pair of μ and $(x, y) \in \mu$ then $(\varphi_i(x), \varphi_i(y)) \in \mu'$.

Indeed, denoting the canonical $A \to A/\mu'$ homomorphism by κ we have $\mu \subseteq$ Ker $(\kappa \circ \varphi_i)$, which implies (2). Form (2) we infer that all the $\varphi_i(t_j)$, $0 \leq i \leq m$

and $0 \le j \le r$, belong to the same α'-block. Thus whenever two of them is found to be congruent modulo β' or γ' then they will be congruent modulo δ' as well. Observe that $\varphi_i(t_j) = \varphi_i(q_j(c,b)) = q_j(u_i, b')$. Therefore, for any j, $\varphi_0(t_j) = q_j(u_0,b')\,\beta' q_j\,(u_1,b')\,\gamma' q_j\,(u_2,b')\,\beta' q_j(u_3,b')\,\gamma' \dots q_j\,(u_m,b') = \varphi_m\,(t_j)$. This yields $\varphi_0\,(t_j)\,\delta'\varphi_m\,(t_j)$. By (2), $\varphi_0\,(t_j)\,\beta'\varphi_0\,(t_{j+1})$ for j even and $\varphi_m\,(t_j)\,\gamma'\varphi_m\,(t_{j+1})$ for j odd. Thus $\varphi_0\,(t_j)\,\delta'\varphi_0\,(t_{j+1})$ for j even and $\varphi_m\,(t_j)\delta'\varphi_m\,(t_{j+1})$ for j odd. Now $e = \varphi_0\,(t_0)\,\delta'\varphi_0\,(t_1)\,\delta'\varphi_m\,(t_1)\,\delta'\varphi_m\,(t_2)\,\delta'\varphi_0\,(t_2)\,\delta'\varphi_0\,(t_3)\,\delta' \dots \delta'\varphi_m\,(t_r) = \varphi_m\,(b) = b'$, completing the proof. □

One might think that other classical results on $\models$ $(\boldsymbol{HSP})$ can be strengthened similarly. Sometimes, e.g. in case of Day [3], this is true, but far from always. The following problem, which may look surprising at the first sight, would be completely trivial without e. Let dist* denote the identity $(x+y)\,(x+z) \le x+yz$, the dual of dist.

Problem. Is it true that dist* $\models$ dist $(e, \boldsymbol{SP}_{if})$ or at least dist* $\models$ dist $(e, \boldsymbol{HSP})$?

Example. dist$\models$ dist* $(e, \boldsymbol{HSP})$ is false.

Proof. If $\mathcal{V}$ is the variety of meet semilattices with 0 then dist holds in Con $\mathcal{V}$ at 0 by Chajda [1, Example 3]. On the other hand, consider the seven element semilattice $A = \{a, b, c, ab, ac, bc, abc = 0\}$. Then the congruences α, β, γ corresponding to the respective partitions

$\{\{a,b,ab,ac,bc,0\}, \{c\}\}$, $\{\{a,c,ac\}, \{b\}, \{ab,bc,0\}\}$
$\{\{a\}, \{ac,ab,0\}, \{b,c,bc\}\}$ witness that dist* fails in Con A at 0. □

References

[1] I. Chajda, *Congruence distributivity in varieties with constants*, Archivum Mathematicum (Brno), 22, (1986), 121-124.

[2] G. Czédli, *On the lattice of congruence varieties of locally equational classes*, Acta Sci. Math. (Szeged) 41, (1979), 39-45.

[3] A. Day, *p-modularity implies modularity equational classes*, Algebra Universalis 3 (1973), 398-399.

[4] R. Freese, G. Herrmann and A.P. Huhn, *On some identities valid in modular congruence varieties*, Algebra Universalis 12 (1981), 322-334.

[5] G. Hutchinson and G. Czédli, *A test for identities satisfied in lattices of submodules*, Algebra Universalis 8 (1978), 269-309.

[6] B. Jónsson, *Algebras whose congruence lattices are distributive*, Math. Scandinavica 21 (1967) 110-121.

[7] B. Jónsson, *Congruence varieties*, Algebra Universalis 10 (1980) 355-394.

[8] J.B. Nation, *Varieties whose congruences satisfy certain lattice identities*, Algebra Universalis 4 (1974) 78-88.

[9] A.F. Pixley, *Local Mal'cev conditions*, Canad. Math. Bull. 15 (1972) 559-568.
[10] R. Wille, *Kongruenzklassengeometrien*, Springer-Verlag Lecture Notes in Math. 113, Berlin-Heidelberg-New York 1970.

GÁBOR CZÉDLI
JATE BOLYAI INSTITUTE
SZEGED, ARADI VÉRTANUK TERE 1
HUNGARY H-6720