NATURAL EQUIVALENCES FROM LATTICE QUASIORDERS TO INVOLUTION LATTICES OF CONGRUENCES

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Abstract. The involution lattice $\mathrm{Quord}(A)$ of quasiorders of a lattice A is known to be isomorphic to the involution lattices $\mathrm{Con}^2(A)$ consisting of pairs of congruences of A. Moreover, the isomorphism described in [9] is supplied by a natural equivalence between the functors Quord and Con^2. The aim of the present paper is to describe and count the possible Quord $\to$ Con^2 natural equivalences. The answer depends on the domain category $\mathcal{L}$, always a prevariety of lattices with the surjective homomorphisms, of the functors Quord and Con^2; and the problem is solved only for very small prevarieties $\mathcal{L}$. An overview on the most recent developments in the theory of involution lattices and quasiorders is also presented.

To the memory of Milan Kolibiar

1. Introduction

The primary purpose of the present paper is to describe all possible natural equivalences from the functor Quord to the functor Con^2. Some new results on this problem will be proved in the following section. This introductory section surveys some related recent developments in the topic of involution lattices.

A quadruplet $L = \langle L; \vee, \wedge, ^* \rangle$ is called an *involution lattice* if $L = \langle L; \vee, \wedge \rangle$ is a lattice and $^* : L \to L$ is a lattice automorphism such that $(x^*)^* = x$ holds for all $x \in L$. To present a natural example, let us consider an algebra A. A binary relation $\rho \subseteq A^2$ is called a *quasiorder* of A if ρ is reflexive, transitive and compatible. (Sometimes we consider a set A rather than an algebra, then all relations are compatible.) Defining $\rho^* = \{\langle x, y\rangle : \langle y, x\rangle \in \rho\}$, the set $\mathrm{Quord}(A)$ of quasiorders of A becomes an involution lattice $\mathrm{Quord}(A) = \langle \mathrm{Quord}(A); \vee, \wedge, ^*\rangle$, where $\wedge$ is the intersection and $\vee$ is the transitive closure of the union. These involution lattices were studied in [3, 6, 9] and Chajda and Pinus [4]. For an involution lattice I, the subalgebra $\{x \in I : x^* = x\}$ is a lattice if we forget about the (trivial) involution operation. In particular, $\{\rho \in \mathrm{Quord}(A) : \rho^* = \rho\}$ is just the congruence lattice of A. For a lattice L, the direct square L^2 of L becomes an involution lattice if we define $\langle x, y\rangle^* = \langle y, x\rangle$ for $\langle x, y\rangle \in L^2$. The involution lattice arising from the congruence lattice $\mathrm{Con}(A)$ of A this way will be denoted by $\mathrm{Con}^2(A)$. There are

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many more examples for involution lattices as related structures, e.g., the ideal lattice of a ring with involution, the lattice of all semigroup varieties, the lattice of clones over a two-element set (the so-called Post lattice), etc., but only $\mathrm{Con}^2(A)$ and $\mathrm{Quord}(A)$ of them will be studied in the present note.

Motivated by the classical Grätzer—Schmidt Theorem [10], Chajda and Pinus [4] asked which involution lattices I are isomorphic to $\mathrm{Quord}(A)$. Some partial answer to this question is given in the following four theorems. Note that an obvious necessary condition on I is that it has to be algebraic as a lattice. The simplest case, when the involution is trivial (i.e. $x^* = x$ for all x), is settled in

Theorem A. *([3] and Pinus [14], independently.) Let I be an algebraic involution lattice such that $x^* = x$ for all x. Then there exists an algebra A such that $I \cong \mathrm{Quord}(A)$.*

When the involution is not assumed to be trivial, much less is known. The quasiorders of an algebra A are called 3-permutable if $\alpha \circ \beta \circ \alpha = \beta \circ \alpha \circ \beta$ holds for any $\alpha, \beta \in \mathrm{Quord}(A)$.

Theorem B. *([3]) For any finite distributive involution lattice I there exists a finite algebra A such that $I \cong \mathrm{Quord}(A)$ and, in addition, the quasiorders of A are 3-permutable.*

We remark that if the quasiorders of all algebras in a given variety V are 3-permutable then $\mathrm{Con}(A) = \mathrm{Quord}(A)$ for all $A \in V$, cf. Chajda and Rachůnek [2]. Sharpening Whitman's result in [18], Jónsson [11] has shown that each modular lattice L has a type 2 representation. We say that an involution lattice I has a type 2 representation if for some set A the involution lattice $\mathrm{Quord}(A)$ has a subalgebra S isomorphic to I such that $\alpha \circ \beta \circ \alpha = \beta \circ \alpha \circ \beta$ holds for any $\alpha, \beta \in S$.

Theorem C. *Each distributive involution lattice L has a type 2 representation.*

For a partial algebra $A = \langle A, F\rangle$, a reflexive and symmetric relation $\rho \subseteq A^2$ is called a quasiorder of A provided for any $f \in F$, say n-ary, and $\langle a_1, b_1\rangle, \dots, \langle a_n, b_n\rangle \in \rho$ if both $f(a_1, \dots, a_n)$ and $f(b_1, \dots, b_n)$ are defined then $\langle f(a_1, \dots, a_n), f(b_1, \dots, b_n)\rangle \in \rho$. The quasiorders of a partial algebra A still constitute an algebraic involution lattice $\mathrm{Quord}(A)$ under the set-theoretic inclusion and $\rho^* = \{\langle x, y\rangle : \langle y, x\rangle \in \rho\}$, but the join is not the transitive closure of the union in general.

Theorem D. *([3]) For any algebraic involution lattice I there is a partial algebra A such that I is isomorphic to $\mathrm{Quord}(A)$.*

The proofs of the above four theorems are not very difficult, for we can borrow a lot of ideas from their classical counterparts for congruences or equivalences. E.g., the yeast graph construction to prove Theorem A in [3] is taken from Pudlák and Tůma [15].

The previous four theorems naturally lead to the question whether every algebraic involution lattice is isomorphic to $\mathrm{Quord}(A)$ for some algebra A. The affirmative answer would imply that any involution lattice I could be embedded in $\mathrm{Quord}(A)$ for some set A, for I is embedded in the (algebraic) involution lattice of its lattice ideals. Unfortunately, as the next few lines witness, this is not the case.

On the set $\{x, y, z, t, u, v, w\}$ of variables let us define the following involution lattice terms

$$\begin{aligned} s_1 &= (z \vee u) \wedge (u^* \vee x \vee z^* \vee t^*), \\ s_2 &= (y \vee w) \wedge (y^* \vee x \vee v^* \vee w^*), \\ s_3 &= (y \vee s_1) \wedge (u^* \vee x \vee z^* \vee t^*), \\ s_4 &= (u \vee s_2) \wedge (y^* \vee x \vee v^* \vee w^*). \end{aligned}$$

Theorem E. *([6]) The Horn sentence*

$$x \leq y \vee u \;\&\; y \leq z \vee t \;\&\; u \leq v \vee w \Longrightarrow x \leq s_3 \vee s_4 \vee z^* \vee w^*$$

holds in Quord(A) *for any set A but does not hold in all involution lattices.*

The proof of Theorem E needs a computer implementation of an algorithm to solve the word problem for involution lattices (and also for lattices) This computer program is based on [5] and is available from the author upon request.

The description of quasiorders of a lattice L is due to Szabó [16]. Later, in [9], this description was deduced from the following theorem, which made the proof substantially easier. Let I denote an involution lattice and let $L = \{x \in I : x^* = x\}$ be regarded as a lattice. As previously, L^2 is an involution lattice.

Theorem F. *([9]) Assume that I is a distributive involution lattice and $\rho \in I$ such that $\rho \wedge \rho^* = 0$ and $\rho \vee \rho^* = 1$. Then*

$$u : I \to L^2, \quad \gamma \mapsto \langle (\gamma \wedge \rho) \vee (\gamma^* \wedge \rho^*), (\gamma \wedge \rho^*) \vee (\gamma^* \wedge \rho) \rangle$$

is an isomorphism. The inverse of u is the isomorphism

$$v : L^2 \to I, \quad \langle \alpha, \beta \rangle \mapsto (\alpha \wedge \rho) \vee (\beta \wedge \rho^*).$$

Now let A be a lattice or, more generally, assume that A has a lattice reduct such that the basic operations of A are monotone with respect to the lattice order. Denoting the lattice order by ρ, we have $\rho \wedge \rho^* = 0$ and $\rho \vee \rho^* = 1$ in Quord(A). Put $I = \text{Quord}(A)$, then $L = \text{Con}^2(A)$. Since Quord(A) is distributive by [8], Theorem F applies and gives a satisfactory description of (members of) Quord(A):

Corollary G. *([9], Szabó [16]) The quasiorders of a lattice A are exactly the relations of the form $(\alpha \wedge \rho) \vee (\beta \wedge \rho^*)$ where $\alpha, \beta \in \text{Con}(A)$.*

From this result it is quite straightforward to derive

Corollary H. *([9], Szabó [16], for finite lattices [7]) Every compatible (partial) order γ of a lattice A is induced by a subdirect representation of A as a subdirect product of A_1 and A_2 such that $\langle x, y \rangle \in \gamma$ iff $x_1 \leq y_1$ in A_1 and $x_2 \geq y_2$ in A_2. Conversely, any relation derived from a subdirect decomposition this way is a compatible order of A.*

Note that describing the compatible orders is an interesting task also for semilattices; this was done by Kolibiar [13]. A very deep result of Tischendorf and Tůma [17] combined with Theorem F and the distributivity of Quord(A) easily yield

Corollary I. *([16]) An involution lattice I is isomorphic to* Quord(A) *for some lattice A iff I is algebraic, distributive and $x \wedge x^* = 0$, $x \vee x^* = 1$ hold for some $x \in I$.*

For lattices A the fact $\mathrm{Quord}(A) \cong \mathrm{Con}^2(A)$ can be stated in a stronger form. Let us fix a prevariety $\mathcal{L}$ of lattices. I.e., $\mathcal{L}$ is a class closed under forming sublattices, homomorphic images and finite direct products. $\mathcal{L}$ will be considered a category in which the morphisms are the *surjective* lattice homomorphisms. The category of all involution lattices with all homomorphisms will be denoted by $\mathcal{V}$. For $A, B \in \mathcal{L}$ and a morphism $f : A \to B$, let

$$\mathrm{Quord}(f) : \mathrm{Quord}(B) \to \mathrm{Quord}(A), \quad \gamma \mapsto \{\langle x, y\rangle \in A^2 : \langle f(x), f(y)\rangle \in \gamma\}$$

and

$$\mathrm{Con}^2(f) : \mathrm{Con}^2(B) \to \mathrm{Con}^2(A) \quad \langle \alpha, \beta\rangle \mapsto \langle \hat{f}(\alpha), \hat{f}(\beta)\rangle,$$

where $\hat{f}(\delta) = \{\langle x, y\rangle \in A^2 : \langle f(x), f(y)\rangle \in \delta\}$. Then Quord and Con^2 are contravariant $\mathcal{L} \to \mathcal{V}$ functors. For $A \in \mathcal{L}$ let

$$\tau_A : \mathrm{Quord}(A) \to \mathrm{Con}^2(A), \quad \gamma \mapsto \langle (\gamma \wedge \rho) \vee (\gamma^* \wedge \rho^*), (\gamma \wedge \rho^*) \vee (\gamma^* \wedge \rho)\rangle$$

and

$$\nu_A : \mathrm{Con}^2(A) \to \mathrm{Quord}(A), \quad \langle \alpha, \beta\rangle \mapsto (\alpha \wedge \rho) \vee (\beta \wedge \rho^*),$$

where ρ is the lattice order of A.

Theorem J. *τ is a natural equivalence from the functor* Quord *to the functor* Con^2*. The inverse of τ is $\nu : \mathrm{Con}^2 \to \mathrm{Quord}$.*

2. Results and proofs

As mentioned before, we intend to describe the natural equivalences Quord $\to$ Con^2. One natural equivalence, τ, is given in Theorem J. Evidently, the map $\psi \to \psi \circ \tau$ from the class of $\mathrm{Con}^2 \to \mathrm{Con}^2$ natural equivalences to the class of Quord $\to \mathrm{Con}^2$ natural equivalences is a bijection. Therefore it suffices to describe the class $T(\mathcal{L})$ of natural equivalences from the contravariant functor $\mathrm{Con}^2 : \mathcal{L} \to \mathcal{V}$ to the same functor. We are able to describe $T(\mathcal{L})$ for some very small prevarieties $\mathcal{L}$ only. The fact that $|T(\mathcal{L})|$ heavily depends on $\mathcal{L}$ for these small $\mathcal{L}$ indicates that we are far from describing $T(\mathcal{L})$ for all $\mathcal{L}$.

From now on let $\mathcal{L}$ be a prevariety *consisting of finite lattices only.* Let $\mathcal{S} = \mathcal{S}(\mathcal{L})$ be the class of subdirectly irreducible lattices belonging to $\mathcal{L}$. Note that the one-element lattice is not considered subdirectly irreducible. A pair $\mathcal{D} = \langle \mathcal{D}_1, \mathcal{D}_2\rangle$ of subclasses of $\mathcal{S}$ is said to be an *H-partition of $\mathcal{S}$* if $\mathcal{D}_1 \cup \mathcal{D}_2 = \mathcal{S}$, $\mathcal{D}_1 \cap \mathcal{D}_2 = \emptyset$, and for any $i = 1, 2$, $A \in \mathcal{D}_i$ and $B \in \mathcal{S}$ if B is a homomorphic image of A then $B \in \mathcal{D}_i$. An H-partition $\mathcal{D}$ is called trivial if $\mathcal{D}_1 = \emptyset$ or $\mathcal{D}_2 = \emptyset$. Since the $\mathcal{D}_i$ are closed under isomorphism and we consider finite lattices only, the H-partitions of $\mathcal{S}$ form a set.

We always have at least two natural equivalences from Con^2 to Con^2. The identical $\mathrm{Con}^2 \to \mathrm{Con}^2$ natural equivalence will be denoted by id; id_A is the identical $\mathrm{Con}^2(A) \to \mathrm{Con}^2(A)$ map for each $A \in \mathcal{L}$. Defining $\mathrm{inv}_A : \mathrm{Con}^2(A) \to \mathrm{Con}^2(A)$, $x \to x^*$, it is easy to see that $\mathrm{inv} : \mathrm{Con}^2 \to \mathrm{Con}^2$ is also a natural equivalence.

With an H-partition $\mathcal{D} = \langle \mathcal{D}_1, \mathcal{D}_2 \rangle$ we associate a transformation (in fact a natural equivalence) $\psi = \psi(\mathcal{D}) : \mathrm{Con}^2 \to \mathrm{Con}^2$ as follows. Let $A \in \mathcal{L}$ and choose $\alpha_1, \alpha_2 \in \mathrm{Con}(A)$ such that $\alpha_1 \wedge \alpha_2 = 0$, A/α_1 is isomorphic to a (finite) subdirect product of some lattices from $\mathcal{D}_1$ and A/α_2 is isomorphic to a (finite) subdirect product of some lattices from $\mathcal{D}_2$. (The case $\alpha_i = 1$ is allowed since the empty subdirect product is defined to be the one-element lattice. We will show soon that α_1 and α_2 exist and they are uniquely determined.) Let

$$\psi_A : \mathrm{Con}^2(A) \to \mathrm{Con}^2(A) \quad \langle \gamma, \delta \rangle \mapsto \langle (\gamma \vee \alpha_1) \wedge (\delta \vee \alpha_2), (\delta \vee \alpha_1) \wedge (\gamma \vee \alpha_2) \rangle.$$

Conversely, given a natural equivalence $\psi : \mathrm{Con}^2 \to \mathrm{Con}^2$, we define $\mathcal{D} = \mathcal{D}(\psi) = \langle \mathcal{D}_1, \mathcal{D}_2 \rangle$ by $\mathcal{D}_1 = \{A \in \mathcal{S} : \psi_A = \mathrm{id}_A\}$ and $\mathcal{D}_2 = \{A \in \mathcal{S} : \psi_A = \mathrm{inv}_A\}$.

Theorem 1. *Given a prevariety $\mathcal{L}$ of finite lattices, the map $\mathcal{D} \mapsto \psi(\mathcal{D})$ from the set of H-partitions of S to the set of $\mathrm{Con}^2 \to \mathrm{Con}^2$ natural equivalences is a bijection. The map $\psi \mapsto \mathcal{D}(\psi)$ is the inverse of this bijection.*

Proof. First we make some observations for an arbitrary natural equivalence $\psi : \mathrm{Con}^2 \to \mathrm{Con}^2$. For $A, B \in \mathcal{L}$ and a surjective homomorphism $f : A \to B$ with kernel $\mu \in \mathrm{Con}(A)$ let $\hat{f}$ denote the canonical lattice embedding $\mathrm{Con}(B) \to \mathrm{Con}(A)$, $\alpha \mapsto \{\langle x, y \rangle : \langle f(x), f(y) \rangle \in \alpha\}$. Then $\mathrm{Con}^2(f) : \langle \alpha, \beta \rangle \mapsto \langle \hat{f}(\alpha), \hat{f}(\beta) \rangle$. Let us consider the following diagram

$$(1) \qquad \begin{array}{ccc} \mathrm{Con}^2(B) & \xrightarrow{\psi_B} & \mathrm{Con}^2(B) \\ {\scriptstyle \mathrm{Con}^2(f)}\big\downarrow & & \big\downarrow{\scriptstyle \mathrm{Con}^2(f)} \\ \mathrm{Con}^2(A) & \xrightarrow{\psi_A} & \mathrm{Con}^2(A) \end{array}$$

This diagram is commutative by the definition of a natural equivalence. Therefore, for any $\langle \gamma, \delta \rangle \in \mathrm{Con}^2(B)$ we have

$$(2) \qquad \mathrm{Con}^2(f)(\psi_B(\langle \gamma, \delta \rangle)) = \psi_A(\langle \hat{f}(\gamma), \hat{f}(\delta) \rangle).$$

Since $\psi_B(\langle 0, 0 \rangle) = \langle 0, 0 \rangle$, we obtain from (2) that $\psi_A(\langle \mu, \mu \rangle) = \langle \mu, \mu \rangle$. But any member of $\mathrm{Con}(A)$ is the kernel of an appropriate surjective homomorphism, so we obtain that

$$(3) \qquad \psi_A(\langle \beta, \beta \rangle) = \langle \beta, \beta \rangle$$

holds for every $\beta \in \mathrm{Con}(A)$. Now let $\psi_A^{(1)}(\langle \gamma, \delta \rangle)$ resp. $\psi_A^{(2)}(\langle \gamma, \delta \rangle)$ denote the first resp. second component of $\psi_A(\langle \gamma, \delta \rangle)$. Since ψ_A is monotone, $\psi_A(\langle \hat{f}(\gamma), \hat{f}(\delta) \rangle) \geq \psi(\langle \mu, \mu \rangle) = \langle \mu, \mu \rangle$. Therefore, factoring both sides of (2) by μ componentwise, we obtain

$$(4) \qquad \psi_B(\langle \gamma, \delta \rangle) = \big\langle \psi_A^{(1)}(\langle \hat{f}(\gamma, \hat{f}(\delta) \rangle)/\mu, \psi_A^{(2)}(\langle \hat{f}(\gamma, \hat{f}(\delta) \rangle)/\mu \big\rangle.$$

I.e., ψ_A determines ψ_B for any homomorphic image B of A. For $\langle \gamma, \beta \rangle \in \mathrm{Con}^2(A)$ such that $\langle \gamma, \beta \rangle \geq \langle \mu, \mu \rangle$, we can rewrite (4) with the help of (2) into the following form:

$$(5) \qquad \psi_A(\langle \gamma, \delta \rangle) = \big\langle \hat{f}(\psi_B^{(1)}(\langle \gamma/\mu, \delta/\mu \rangle)), \hat{f}(\psi_B^{(2)}(\langle \gamma/\mu, \delta/\mu \rangle)) \big\rangle.$$

Now we assert that

$$(\forall A \in \mathcal{S})(\psi_A = \mathrm{id}_A \text{ or } \psi_A = \mathrm{inv}_A). \tag{6}$$

Let $\mu \in \mathrm{Con}(A)$ be the monolith of A. To prove (6), first we observe that since ψ_A is monotone, bijective, and leaves $\langle \mu, \mu \rangle$ fixed, ψ_A permutes the subset

$$Y = \{\langle u, v \rangle : \langle u, v \rangle \not\geq \langle \mu, \mu \rangle\}$$

of $\mathrm{Con}^2(A)$. Since $\langle 1, 0 \rangle$ and $\langle 0, 1 \rangle$ are the only maximal elements of Y, ψ_A either interchanges these two elements or leaves both elements fixed. Suppose $\psi_A(\langle 0, 1 \rangle) = \langle 1, 0 \rangle$. (This assumption will soon imply $\psi_A = \mathrm{inv}_A$ while the case $\psi_A(\langle 0, 1 \rangle) = \langle 0, 1 \rangle$, not to be detailed, gives $\psi_A = \mathrm{id}_A$ analogously.) Let us compute, using (3) frequently: $\psi_A(\langle \mu, 1 \rangle) = \psi_A(\langle 0, 1 \rangle \vee \langle \mu, \mu \rangle) = \psi_A(\langle 0, 1 \rangle) \vee \psi_A(\langle \mu, \mu \rangle) = \langle 1, 0 \rangle \vee \langle \mu, \mu \rangle = \langle 1, \mu \rangle$; applying the involution operation to both sides we conclude $\psi_A(\langle 1, \mu \rangle) = \langle \mu, 1 \rangle$; for $\langle \alpha, \beta \rangle \geq \langle \mu, \mu \rangle$ we have $\psi_A(\langle \alpha, \beta \rangle) = \psi_A\big((\langle \mu, 1 \rangle \vee \langle \alpha, \alpha \rangle) \wedge (\langle 1, \mu \rangle \vee \langle \beta, \beta \rangle)\big) = (\psi_A(\langle \mu, 1 \rangle) \vee \psi_A(\langle \alpha, \alpha \rangle)) \wedge (\psi_A(\langle 1, \mu \rangle) \vee \psi_A\langle \beta, \beta \rangle)) = (\langle 1, \mu \rangle \vee \langle \alpha, \alpha \rangle) \wedge (\langle \mu, 1 \rangle \vee \langle \beta, \beta \rangle) = \langle \beta, \alpha \rangle$; for any $\gamma \in \mathrm{Con}(A)$ we obtain $\psi_A(\langle \gamma, 0 \rangle) = \psi_A(\langle 1, 0 \rangle \wedge \langle \gamma, \mu \rangle) = \psi_A(\langle 1, 0 \rangle) \wedge \psi_A(\langle \gamma, \mu \rangle) = \langle 0, 1 \rangle \wedge \langle \mu, \gamma \rangle = \langle 0, \gamma \rangle$; and $\psi_A(\langle 0, \gamma \rangle) = \langle \gamma, 0 \rangle$ follows similarly. Having taken all elements of $\mathrm{Con}^2(A)$ into consideration we have shown that $\psi_A = \mathrm{inv}_A$. This proves (6).

Armed with (4) and (6) we conclude that $\mathcal{D} = \mathcal{D}(\psi)$ is an H-partition, provided ψ is a natural equivalence.

Now let us assume that $\mathcal{D}$ is an H-partition, and let $\psi = \psi(\mathcal{D})$. We have to show that ψ is a natural equivalence. We claim that

(7) If $C \in \mathcal{S}$ is a homomorphic image of $A \in \mathcal{L}$ such that A is isomorphic to a subdirect product of finitely many $B_i \in \mathcal{D}_j$ then $C \in \mathcal{D}_j$.

Indeed, by the assumptions there are $\gamma, \beta_1, \ldots, \beta_n \in \mathrm{Con}(A)$ such that $A/\beta_i \in \mathcal{D}_j$, $A/\gamma \cong C$ and $\bigwedge_{i=1}^n \beta_i = 0$. By distributivity we have $\gamma = \gamma \vee 0 = \gamma \vee \bigwedge_{i=1}^n \beta_i = \bigwedge_{i=1}^n (\gamma \vee \beta_i)$. Since C is subdirectly irreducible, γ is meet-irreducible in $\mathrm{Con}(A)$ and we obtain $\gamma = \gamma \vee \beta_i$, i.e. $\gamma \geq \beta_i$ for some i. Therefore $C \cong A/\gamma$ is a homomorphic image of $A/\beta_i \in \mathcal{D}_j$. This yields $C \in \mathcal{D}_j$, proving (7).

Now let $A \in \mathcal{L}$ and let $\alpha_1, \alpha_2 \in \mathrm{Con}(A)$ be the congruences from Theorem 1. (I.e., A/α_j is a subdirect product of some members of $\mathcal{D}_j$, $j = 1, 2$, and $\alpha_1 \wedge \alpha_2 = 0$.) We assert that

$$\alpha_1 \vee \alpha_2 = 1. \tag{8}$$

Suppose this is not the case. Then $A/(\alpha_1 \vee \alpha_2)$ is not the one-element lattice, whence it has a homomorphic image C in $\mathcal{S}$. (Indeed, $A/(\alpha_1 \vee \alpha_2)$ is a subdirect product of some lattices in $\mathcal{S}$ and C can be any of the factors of this subdirect decomposition.) But then, by (7), C belongs to $\mathcal{D}_j$ for $j = 1$ and $j = 2$ since it is a homomorphic image of A/α_j. This contradicts $\mathcal{D}_1 \cap \mathcal{D}_2 = \emptyset$, proving (8).

Now we claim that

(9) α_1 and α_2 exist and they are uniquely determined.

If $0 \in \mathrm{Con}(A)$ is meet-irreducible, i.e. $A \in \mathcal{S}$, then let $\langle \alpha_1, \alpha_2\rangle$ be $\langle 0,1\rangle$ or $\langle 1,0\rangle$ depending on $A \in \mathcal{D}_1$ or $A \in \mathcal{D}_2$, respectively. Otherwise 0 is the meet $\beta_1 \wedge \ldots \wedge \beta_k$ of some meet-irreducible congruences β_i, and we may put

$$\alpha_j = \bigwedge_{\substack{i=1\\ A/\beta_i \in \mathcal{D}_j}}^{k} \beta_i, \quad j = 1,2.$$

Now, having seen the existence, suppose that besides α_1, α_2 the pair α_1', α_2' also satisfies the corresponding definition. Hence there are congruences $\gamma_i, \gamma_j, \delta_k, \delta_\ell \in \mathrm{Con}(A)$ such that

$$\bigwedge_{i\in J} \gamma_i = \alpha_1, \ \bigwedge_{j \in J'} \gamma_j = \alpha_1', \ \bigwedge_{k\in K} \delta_k = \alpha_2, \ \bigwedge_{\ell \in K'} \delta_\ell = \alpha_2'.$$

and $A/\gamma_i, A/\gamma_j \in \mathcal{D}_1$, $A/\delta_k, A/\delta_\ell \in \mathcal{D}_2$. Put $\alpha_1'' = \alpha_1 \wedge \alpha_1'$ and $\alpha_2'' = \alpha_2 \wedge \alpha_2'$. From $\alpha_1 \wedge \alpha_2 = 0$ we have $\alpha_1'' \wedge \alpha_2'' = 0$. Since

$$\bigwedge_{i \in J \cup J'} \gamma_i = \alpha_1'', \ \bigwedge_{k\in K\cup K'} \delta_k = \alpha_2'',$$

the pair α_1'', α_2'' also meets the requirements of the definition. We obtain from (8) that $\alpha_1 \vee \alpha_2 = 1$ and $\alpha_1'' \vee \alpha_2'' = 1$. By distributivity, $\alpha_1 = \alpha_1 \wedge 1 = \alpha_1 \wedge (\alpha_1'' \vee \alpha_2'') = (\alpha_1 \wedge \alpha_1'') \vee (\alpha_1 \wedge \alpha_2'')$. But $\alpha_1 \wedge \alpha_2'' \le \alpha_1 \wedge \alpha_2 = 0$, whence $\alpha_1 = \alpha_1 \wedge \alpha_1''$. Hence $\alpha_1 = \alpha_1''$, and $\alpha_2 = \alpha_2''$ follows similarly. Therefore $\alpha_1 \le \alpha_1'$ and $\alpha_2 \le \alpha_2'$, and the reverse inequalities follow similarly. This yields (9).

Now we are ready to prove that $\psi = \psi(\mathcal{D})$ is a natural equivalence. Suppose $f : A \to B$ is a surjective lattice homomorphism with kernel $\mu \in \mathrm{Con}(A)$; we have to show that the diagram (1) commutes. Consider the congruences $\alpha_1, \alpha_2 \in \mathrm{Con}(A)$ resp. $\alpha_1', \alpha_2' \in \mathrm{Con}(B)$ occurring in the definition of ψ_A resp. ψ_B. For $i = 1,2$, $(A/\mu)/((\alpha_i \vee \mu)/\mu) \cong A/(\alpha_i \vee \mu)$ can be decomposed into a subdirect product of finitely many members of $\mathcal{S}$. These subdirectly irreducible factors are homomorphic images of $A/(\alpha_i \vee \mu)$, so they are homomorphic images of A/α_i as well. By (7), they all belong to $\mathcal{D}_i$. Further, $(\alpha_1 \vee \mu) \wedge (\alpha_2 \vee \mu) = (\alpha_1 \wedge \alpha_2) \vee \mu = 0 \vee \mu = \mu$ yields $(\alpha_1 \vee \mu)/\mu \wedge (\alpha_2 \vee \mu)/\mu = 0$. Therefore we infer from (9) that $\alpha_1' = (\alpha_1 \vee \mu)/\mu$ and $\alpha_2' = (\alpha_2 \vee \mu)/\mu$.

Now let $\langle \gamma', \delta'\rangle \in \mathrm{Con}^2(B)$, and denote $\hat{f}(\gamma')$ and $\hat{f}(\delta')$ by γ and δ, respectively. Then $\mathrm{Con}^2(f)(\langle \gamma', \delta'\rangle) = \langle \gamma, \delta\rangle$. To check the commutativity of (1) we have to show that $\mathrm{Con}^2(f)$ sends $\psi_B(\langle \gamma', \delta'\rangle) = \big\langle (\gamma' \vee \alpha_1') \wedge (\delta' \vee \alpha_2'), (\delta' \vee \alpha_1') \wedge (\gamma' \vee \alpha_2')\big\rangle$ to $\psi_A(\langle \gamma, \delta\rangle) = \big\langle (\gamma \vee \alpha_1) \wedge (\delta \vee \alpha_2), (\delta \vee \alpha_1) \wedge (\gamma \vee \alpha_2)\big\rangle$. Since $\hat{f} : \mathrm{Con}(B) \to \mathrm{Con}(A)$ is a lattice homomorphism and sends $\alpha_i', \gamma', \delta'$ to $\alpha_i \vee \mu, \gamma, \delta$ respectively, $\mathrm{Con}^2(f)\big(\psi_B(\langle \gamma', \delta'\rangle)\big) = \big\langle (\gamma \vee \alpha_1 \vee \mu) \wedge (\delta \vee \alpha_2 \vee \mu), (\delta \vee \alpha_1 \vee \mu) \wedge (\gamma \vee \alpha_2 \vee \mu)\big\rangle$. But this equals $\psi_A(\langle \gamma, \delta\rangle)$ by $\gamma \ge \mu$ and $\delta \ge \mu$, indeed. We have seen that ψ is a natural transformation.

Clearly, ψ_A is monotone and preserves the operation *. So, in order to show that it is a lattice isomorphism, it suffices to show that $\psi_A \circ \psi_A = \mathrm{id}_A$. Let us compute for $\langle\gamma,\delta\rangle \in \mathrm{Con}^2(A)$, using first modularity, then (8) and distributivity:

$$\begin{aligned}\psi_A\circ\psi_A(\langle\gamma,\delta\rangle) = \psi_A\bigl(\bigl\langle(\gamma\vee\alpha_1)\wedge(\delta\vee\alpha_2),(\delta\vee\alpha_1)\wedge(\gamma\vee\alpha_2)\bigr\rangle\bigr) = \\ \bigl\langle(((\gamma\vee\alpha_1)\wedge(\delta\vee\alpha_2))\vee\alpha_1)\wedge(((\delta\vee\alpha_1)\wedge(\gamma\vee\alpha_2))\vee\alpha_2), \\ (((\delta\vee\alpha_1)\wedge(\gamma\vee\alpha_2))\vee\alpha_1)\wedge(((\gamma\vee\alpha_1)\wedge(\delta\vee\alpha_2))\vee\alpha_2)\bigr\rangle = \\ \bigl\langle((\gamma\vee\alpha_1)\wedge(\delta\vee\alpha_2\vee\alpha_1))\wedge((\delta\vee\alpha_1\vee\alpha_2)\wedge(\gamma\vee\alpha_2)), \\ ((\delta\vee\alpha_1)\wedge(\gamma\vee\alpha_2\vee\alpha_1))\wedge((\gamma\vee\alpha_1\vee\alpha_2)\wedge(\delta\vee\alpha_2))\bigr\rangle = \\ \bigl\langle((\gamma\vee\alpha_1)\wedge(\delta\vee 1))\wedge((\delta\vee 1)\wedge(\gamma\vee\alpha_2)), \\ ((\delta\vee\alpha_1)\wedge(\gamma\vee 1))\wedge((\gamma\vee 1)\wedge(\delta\vee\alpha_2))\bigr\rangle = \\ \bigl\langle(\gamma\vee\alpha_1)\wedge(\gamma\vee\alpha_2),(\delta\vee\alpha_1)\wedge(\delta\vee\alpha_2)\bigr\rangle = \\ \bigl\langle\gamma\vee(\alpha_1\wedge\alpha_2),\delta\vee(\alpha_1\wedge\alpha_2)\bigr\rangle = \bigl\langle\gamma\vee 0,\delta\vee 0\bigr\rangle = \langle\gamma,\delta\rangle,\end{aligned}$$

indeed. Thus, for every $A \in \mathcal{L}$, ψ_A is an isomorphism, whence $\psi = \psi(\mathcal{D})$ is a natural equivalence.

It is straightforward from the definitions that for any H-partition $\mathcal{D}$ we have $\mathcal{D}(\psi(\mathcal{D})) = \mathcal{D}$.

Now let us assume that ψ is a natural equivalence and let $\psi' = \psi(\mathcal{D}(\psi))$. We have to show that, for any $A \in \mathcal{L}$, $\psi_A = \psi'_A$. This is clear if $A \in \mathcal{S}$; assume this is not the case. Suppose A is a finite subdirect product of members of $\mathcal{D}_j$ for some $j = 1, 2$. We claim that

$$\psi_A = \mathrm{id}_A \text{ for } j = 1 \text{ and } \psi_A = \mathrm{inv}_A \text{ for } j = 2. \tag{10}$$

To show (10), observe that $0 = \bigwedge_{i=1}^n \beta_i$ holds in $\mathrm{Con}(A)$ for some β_i such that $A/\beta_i \in \mathcal{D}_j$ for all i. We will detail the case $j = 2$ only, for the case $j = 1$ is quite similar. For any $\langle\gamma,\delta\rangle \in \mathrm{Con}^2(A)$ we obtain $\langle\gamma,\delta\rangle \vee 0 = \langle\gamma,\delta\rangle \vee \bigwedge_{i=1}^n\langle\beta_i,\beta_i\rangle = \bigwedge_{i=1}^n(\langle\gamma,\delta\rangle \vee \langle\beta_i,\beta_i\rangle) = \bigwedge_{i=1}^n\langle\gamma\vee\beta_i,\delta\vee\beta_i\rangle$, i.e.,

$$\langle\gamma,\delta\rangle = \bigwedge_{i=1}^n \langle\gamma\vee\beta_i,\delta\vee\beta_i\rangle. \tag{11}$$

Since $\psi_{A/\beta_i} = \psi'_{A/\beta_i} = \mathrm{inv}_{A/\beta_i}$, (5) yields $\psi_A(\langle\gamma\vee\beta_i,\delta\vee\beta_i\rangle) = \langle\delta\vee\beta_i,\gamma\vee\beta_i\rangle$, whence (10) follows easily from (11).

Now let $A \in \mathcal{L}$ be arbitrary and let $\langle\gamma,\delta\rangle \in \mathrm{Con}^2(A)$. Similarly to (11) we have

$$\langle\gamma,\delta\rangle = \langle\gamma\vee\alpha_1,\delta\vee\alpha_1\rangle \wedge \langle\gamma\vee\alpha_2,\delta\vee\alpha_2\rangle. \tag{12}$$

From (5) and (10) we obtain $\psi_A(\langle\gamma\vee\alpha_1,\delta\vee\alpha_1\rangle) = \langle\gamma\vee\alpha_1,\delta\vee\alpha_1\rangle$ and $\psi_A(\langle\gamma\vee\alpha_2,\delta\vee\alpha_2\rangle) = \langle\delta\vee\alpha_2,\gamma\vee\alpha_2\rangle$. Therefore $\psi_A(\langle\gamma,\delta\rangle) = \psi'_A(\langle\gamma,\delta\rangle)$ follows from 12, completing the proof. □

Since any finite lattice has a simple homomorphic image, we immediately obtain

Corollary 2. *Given a prevariety $\mathcal{L}$ of finite lattices, if two $\mathrm{Con}^2 \to \mathrm{Con}^2$ natural equivalences coincide on every simple lattice of $\mathcal{L}$ then they coincide on the whole $\mathcal{L}$.*

Now let $\mathcal{L}$ be a prevariety generated by a finite set K of finite lattices[1]. By a celebrated result of Jónsson [12], each subdirectly irreducible lattice in $\mathcal{L}$ is a

[1] $\mathcal{L}$ is just the class of finite lattices of the variety generated by K; this follows from Jónsson [12].

homomorphic image of a sublattice of some lattice in K. Therefore, apart from isomorphic copies, $\mathcal{S} = \mathcal{S}(\mathcal{L})$ is finite. This short argument proves

Corollary 3. *There is an algorithm which produces for any finite set K of finite lattices, as input, a description of the (necessarily finitely many) natural equivalences from the functor* $\mathrm{Quord} : \mathcal{L} \to \mathcal{V}$ *(or, equivalently, from the functor* $\mathrm{Con}^2 : \mathcal{L} \to \mathcal{V}$*) to the functor* $\mathrm{Con}^2 : \mathcal{L} \to \mathcal{V}$ *where $\mathcal{L}$ denotes the prevariety generated by K.*

In virtue of Corollary 3 it is quite easy to present some examples. Let $t(\mathcal{L}) = |T(\mathcal{L})|$, the number of natural equivalences from the functor $\mathrm{Quord} : \mathcal{L} \to \mathcal{V}$ to the functor $\mathrm{Con}^2 : \mathcal{L} \to \mathcal{V}$. By M_n and N_5 we denote the modular lattice of height two with exactly n atoms and the five-element nondistributive lattice, respectively. For $1 \le n < \infty$ let $\mathcal{L}_n$ resp. $\mathcal{L}'_n$ be the prevariety generated by M_{n+1} resp. $\{M_{n+1}, N_5\}$. Note that $\mathcal{L}_1$ is the class of finite distributive lattices. Clearly, $\mathcal{L}_{\aleph_0} = \bigcup_{n=1}^{\infty} \mathcal{L}_n$ and $\mathcal{L}'_{\aleph_0} = \bigcup_{n=1}^{\infty} \mathcal{L}'_n$ are prevarieties, too.

Example 4. *For $n = 1, 2, \dots, \aleph_0$, $t(\mathcal{L}_n) = t(\mathcal{L}'_n) = 2^n$.*

The straightforward proof, based on Corollary 3 and the aforementioned result of Jónsson, is left to the reader.

To conclude the paper with an open problem we mention that $t(\{\text{all finite lattices}\})$, $t(\{\text{all lattices}\})$ and $t(\{\text{all distributive lattices}\})$ are still unknown.

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