A PROPERTY OF MEETS IN SLIM SEMIMODULAR LATTICES AND ITS APPLICATION TO RETRACTS

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Dedicated to Professor László Stachó on the occasion that he received the Béla Szőkefalvi-Nagy Medal of Acta Scientiarum Mathematicarum for the year 2021

ABSTRACT. Slim semimodular lattices were introduced by G. Grätzer and E. Knapp in 2007, and they have intensively been studied since then. These lattices can be given by $\mathcal{C}_1$-diagrams, defined by the author in 2017. We prove that if x and y are incomparable elements in such a lattice L, then their meet has the property that the interval $[x \wedge y, x]$ is a chain, this chain is of a normal slope in every $\mathcal{C}_1$-diagram of L, and except possibly for x, the elements of this chain are meet-reducible.

In the direct square K_1 of the three-element chain, let X_1 and A_1 be the set of atoms and the sublattice generated by 0 and the coatoms, respectively. Denote by K_2 the unique eight-element lattice embeddable in K_1. Let A_2 be the sublattice of K_2 consisting of 0, 1, the meet-reducible atom, and the join-reducible coatom. Let X_2 stand for the singleton consisting of the doubly reducible element of K_2. For $i = 1, 2$, we apply the above-mentioned property of meets to prove that whenever K_i is a sublattice and S_i is a retract of a slim semimodular lattice, then $A_i \subseteq S_i$ implies that $X_i \subseteq S_i$.

1. INTRODUCTION

A lattice L is *slim* if it is finite and the set $\mathrm{J}(L)$ of its (nonzero) join-irreducible elements is the union of two chains. We know from [11, Lemma 2.2] that slim lattices are planar. Slim semimodular lattices were introduced by G. Grätzer and E. Knapp in 2007 in a different but equivalent way. At the time of writing, four dozen publications have been devoted to slim semimodular lattices; see the (extended) arXiv version of Czédli [6] for their list[1]. Many of these publications are freely available and explain what motivates the study of these lattices and what connections these lattices have with other parts of mathematics; we only refer to Czédli and Kurusa [9] and the book chapter Czédli and Grätzer [8] for surveys.

For an algebra A, the idempotent endomorphisms of A are called the *retractions* of A. That is, a retraction of A is a homomorphism $f : A \rightarrow A$ such that $f(f(x)) = f(x)$ for all $x \in A$. If $f : A \rightarrow A$ is a retraction, then the subalgebra $f(A) = \{f(x) : x \in A\}$ is called a *retract* of A. Retracts are similarly defined and frequently occur in some categories of other structures. For example, they

[1]See `http://www.math.u-szeged.hu/~czedli/m/listak/publ-psml.pdf` for updates.

are important for posets (partially ordered sets); we only mention Rival [20] and Zádori [21]. Apart from some obvious cases like vector spaces over a field (where every subspace is a retract) and monounary algebras, whose retracts have nice properties by Jakubíková–Studenovská and Pócs [17], we do not know much about retracts in general.

For lattices, retractions and retracts have already been investigated in some papers including Boyu [1], Czédli [5], [7], and Czédli and Molkhasi [10], but we still know little about them. In particular, there are only a few lattices the retracts of which are well understood.

Goal and outline. In Section 2, we formulate and prove Theorem 2.4 on meets in slim semimodular lattices and slim lattices. The first part of the theorem is easy and purely algebraic; the second part has some visual ingredients based on $\mathcal{C}_1$-diagrams introduced in Czédli [3]. Section 2 recalls the necessary details about these diagrams. Even though Section 2 begins with mentioning some monographs, the paper is intended to be self-contained for most lattice theorists, who need not look into these monographs. In Section 3, we apply Theorem 2.4 to prove that the retracts of slim semimodular lattices have two particular absorption properties.

2. A property of meets in slim semimodular lattices

Without seeking for completeness, we mention that the book chapter Czédli and Grätzer [8], the monographs Grätzer [13] and the freely available Nation [19], and the freely available Part I (`https://doi.org/10.1007/978-3-319-38798-7`) of the monograph Grätzer [14] provide reasonable introduction to lattice theory and in particular, to planar semimodular lattices.

Before formulating the main result of the paper, we recall the concept of $\mathcal{C}_1$-diagrams. These diagrams together with even more specific diagrams were introduced in Czédli [3], and they have already proved to be efficient tools to study slim semimodular lattices; see, for example, Czédli [4].

Definition 2.1. (A) We always assume that a coordinate system is fixed on the plane. For a ray (half-line), line, or vector r on the plane, the *slope angle* $\mathrm{sl}^{\angle}(r)$ is the angle measured from the positive half of the x-axis to r. For example, if $r = \{(x, -x) : 0 \le x \in \mathbb{R}\}$, then $\mathrm{sl}^{\angle}(r) = 3\pi/4$ or, in another standard scale, $135°$.

(B) Lines, rays, and edges (of a lattice diagram) with slope angle $\pi/4$ ($45°$) and those with slope angle $3\pi/4$ ($135°$) are said to be of *normal slopes*. If r is an edge, line, or ray, then r is said to be *precipitous* if $\pi/4 < \mathrm{sl}^{\angle}(r) < 3\pi/4$.

(C) Let L be a slim semimodular lattice; we always assume that a planar diagram of L is fixed. The left boundary chain and the right boundary chain of L are denoted by $B_{\mathrm{left}}(L)$ and $B_{\mathrm{right}}(L)$, respectively, while $\mathrm{Bnd}(L) := B_{\mathrm{left}}(L) \cup B_{\mathrm{right}}(L)$ is the *boundary* of L. (Their dependence on the diagram will not cause any trouble since the diagram is fixed.) The set of non-unit meet-irreducible elements of L is denoted by $\mathrm{M}(L)$.

(D) The planar diagram of L is a $\mathcal{C}_1$-*diagram* if every edge $[a, b]$ such that $a \in \mathrm{M}(L) \setminus \mathrm{Bnd}(L)$ is precipitous and all other edges are of normal slopes.

(E) If an interval $[u, v]$ is a chain and all edges of $[u, v]$ are of the same slope, then this common slope is the slope of the interval and the slope angle of $[u, v]$ is denoted by $\mathrm{sl}^{\angle}([u, v])$. Otherwise, the slope of $[u, v]$ and the notation $\mathrm{sl}^{\angle}([u, v])$

are undefined. If $[u,v]$ has a slope and $\mathrm{sl}^{\angle}([u,v]) \in \{\pi/4, 3\pi/4\}$, then $[u,v]$ is *of a normal slope*.

In connection with (D) above, note that the reader can (but need not) find an alternative definition of $\mathcal{C}_1$-diagrams in Grätzer [15]. By Czédli [3, Theorem 5.5(ii)], each slim semimodular lattice has a $\mathcal{C}_1$-diagram. This allows us to agree upon the following convention.

Convention 2.2. We assume that every slim semimodular lattice occurring in the paper has a *fixed* $\mathcal{C}_1$-diagram.

To ease the language of statements and proofs, we introduce the following concepts and notations.

Definition 2.3. For elements a and b in a lattice L, we use $a \parallel b$ to denote that a and b are incomparable. We say that $c = a \wedge b$ is a *nontrivial meet* if $a \parallel b$. We call $c = a \wedge b$ a $\mathcal{C}_1$-*regular meet* if

    (1) L is a slim semimodular lattice and $c = a \wedge b$ is a nontrivial meet,
    (2) the intervals $[c,a]$ and $[c,b]$ are chains,
    (3) $[c,a]$ and $[b,a]$ are of normal slopes and $\mathrm{sl}^{\angle}([c,a]) \neq \mathrm{sl}^{\angle}([c,b])$, and
    (4) every element of $([c,a] \setminus \{a\}) \cup (c,b] \setminus \{b\})$ is meet-reducible.

The conditions above will be referenced by 2.3(1), ..., 2.3(4). Now Convention 2.2 and Definition 2.3 allow us to formulate the main result of the paper.

Theorem 2.4.
    (A) *In a slim lattice, 2.3(2) holds for every nontrivial meet $c = a \wedge b$.*
    (B) *In a slim semimodular lattice, every nontrivial meet is a $\mathcal{C}_1$-regular meet.*

Proof of Theorem 2.4. Let L be a slim lattice. There are no nontrivial meets in chains, whereby we can assume that L is not a chain. By slimness, $\mathrm{J}(L) = U \cup V$ where U and V are *disjoint* chains. Indeed, if they are not disjoint, then we can replace V by $V \setminus U$, which is nonempty since L is not a chain. We can write that $U = \{u(1), u(2), \ldots, u(m)\}$ and $V = \{v(1), \ldots, v(n)\}$ where $u(1) < u(2) < \cdots < u(m)$ and $v(1) < v(2) < \cdots < v(n)$. For $x \in L$, let i_x denote the largest (meaningful) subscript i such that $u(i) \leq x$; if there is no such i then $i_x := 0$. Similarly, j_x stands for the largest subscript j such that $v(j) \leq x$; again, $j_x := 0$ if there is no such j. Since each element is a join of join-irreducible elements,

$$x = u(i_x) \vee v(j_x), \quad \text{and} \quad x \leq y \iff (i_x \leq i_y \text{ and } j_x \leq j_y). \qquad (2.1)$$

Observe that $u(i) \leq x \wedge y$ if and only if $u(i) \leq x$ and $u(i) \leq y$. Analogous observation holds for $v(j)$. Hence, it follows from (2.1) that, for any $x, y \in L$,

$$i_{x \wedge y} = \min\{i_x, i_y\} \quad \text{and} \quad j_{x \wedge y} = \min\{j_x, j_y\}. \qquad (2.2)$$

Now assume that $a, b \in L$ such that a and b are incomparable, in notation, $a \parallel b$. Using (2.1), we obtain that either $i_a < i_b$ and $j_a > j_b$, or $i_a > i_b$ and $j_a < j_b$. By symmetry, we can assume that $i_a < i_b$ and $j_a > j_b$. Let $c := a \wedge b$. It follows from (2.2) that $i_c = i_a$ and $j_c = j_b$. Now it is clear by (2.1) that for any $x \in [c,a]$, $i_x = i_a$. Therefore, $[c,a]$ is a chain by the first half of (2.1). So is $[c,b]$ by symmetry. This proves part (A).

To prove part (B), we need to recall the structure theorem of slim semimodular lattices; it is Czédli [2, Theorem 3.7] (see also Czédli [3, Lemma 5.7] for diagrams) combined with Czédli and Schmidt [12]. But first we need some concepts and
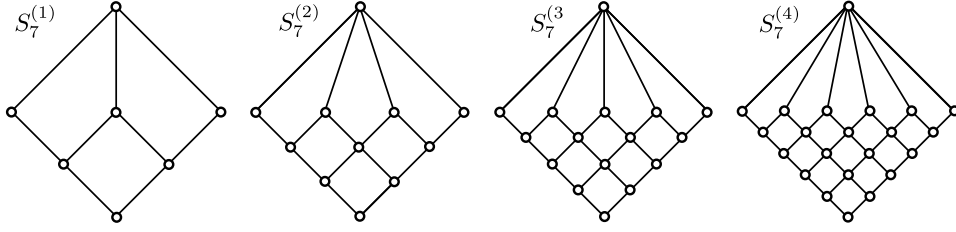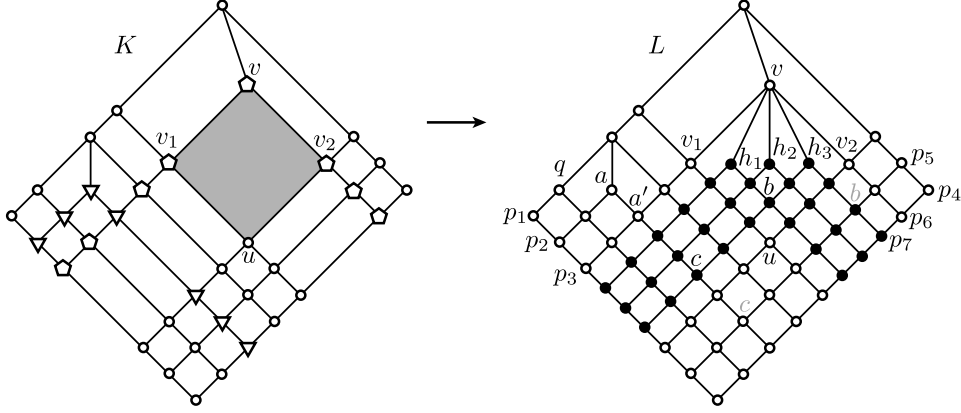
FIGURE 1. $S_7^{(n)}$ for $n \in \{1, 2, 3, 4\}$

FIGURE 2. A 3-fold multifork extension

notations. In the rest of the proof, let L be a slim semimodular lattice. For $x \in \mathrm{J}(L)$ and $y \in \mathrm{M}(L)$, the unique lower cover of x and the unique upper cover of y are denoted by x^- and x^+, respectively. The elements of $\mathrm{J}(L) \cap \mathrm{M}(L)$ are called *doubly irreducible*. By a *corner* of a slim semimodular lattice L we mean a doubly irreducible element $u \in \mathrm{Bnd}(L)$ such that u^+ has exactly two lower covers. For example, L in Figure 2 has exactly two corners, p_1 and p_4, but q is neither a corner of L, nor a corner of the sublattice $L \setminus \{p_1, p_2, p_3\}$. By a *grid* we mean the direct product of two finite nonsingleton chains. The diagram of L is divided into so-called 4-*cells* by edges; these 4-cells are four-element lattices and they are also intervals of length 2. The top element of a 4-cell X is denoted by 1_X. A 4-cell X is a *distributive 4-cell* if the principal ideal $\downarrow 1_X := \{u \in L : u \leq 1_X\}$ is a distributive lattice. The lattice $S_7^{(n)}$ for $n = 1, 2, 3, 4$ is given in Figure 1; $S_7^{(n)}$ for $n \geq 5$ is analogously defined in Czédli [2]. Figure 2 shows what a *multifork extension*, introduced in [2], is. Namely, to construct a multifork extension L of a slim semimodular lattice K, we choose a positive integer n (we choose $n = 3$ in the figure) and a 4-cell of K (the grey one in the figure). Then we replace this 4-cell by a copy of $S_7^{(n)}$. Finally, proceeding to the lower left and the lower right directions, we add further elements to keep semimodularity; see on the right of Figure 2. The *new elements*, that is, the elements of $L \setminus K$ are the black-filled ones. If we want to specify n, then the multifork extension in question is called an *n-fold fork extension*; however, "1-fold" is usually dropped. Now the structure theorem

of slim semimodular lattices asserts the following.

> Each slim semimodular lattice and its $\mathcal{C}_1$-diagram can be obtained
> such that we take a grid, perform multifork extensions at distribu-
> tive 4-cells in a finite nonnegative number of steps, and then we
> omit some corners (possibly no corner) one by one. Furthermore,
> any lattice obtained in this way is a slim semimodular lattice.

$$(2.3)$$

Let us emphasize that multifork extensions in (2.3) are only allowed *before* omitting any corner. To illustrate (2.3), we refer to Figure 2 again. To obtain the lattice (and its $\mathcal{C}_1$-diagram) $M := L \setminus \{p_1, \ldots, p_7\}$, we start from the grid that is the direct product of a 4-element chain and a 5-element chain. The first step is a fork extension that adds the pentagon-shaped elements to the grid. The next step adds the ∇-shaped elements and results in K. From K, we obtain L by a 3-fold fork extension. Finally, we omit the corners $p_1, \ldots, p_7$, in this order, to obtain M. (The order of the corners can be different but not arbitrary since, say, p_2 only becomes a corner after omitting p_1, etc..)

Let L be a slim semimodular lattice, and keep Convention 2.2 in mind. For distinct elements b and c of L, let $\vec{\ell}_{bc}$ denote the ray (also called half-line) through c with initial point b; $\vec{\ell}_{bc}$ is a geometric object and it need not be an edge of the diagram. We write $b <_{\text{geom}} c$ to denote that $b \neq c$ and $\pi/4 \leq \text{sl}^{\angle}(\vec{\ell}_{bc}) \leq 3\pi/4$; see Definition 2.1. In other words, $b <_{\text{geom}} c$ means that $b \neq c$ and the line segment from b to c is of a normal slope or it is precipitous. Naturally, $b \leq_{\text{geom}} c$ means that $b <_{\text{geom}} c$ or $b = c$. For $b \in L$, we define the *upper cone* and the *lower cone* of b as the following subsets of the plane:

$$\text{Cone}^{\text{up}}(b) := \{c \in \mathbb{R}^2 : b \leq_{\text{geom}} c\} \text{ and } \text{Cone}_{\text{dn}}(b) := \{c \in \mathbb{R}^2 : c \leq_{\text{geom}} b\}.$$

We know from Czédli [3, Corollary 6.1] that, for any $x, y \in L$,

$$x \leq y \iff x \leq_{\text{geom}} y \iff x \in \text{Cone}_{\text{dn}}(y) \iff y \in \text{Cone}^{\text{up}}(x). \qquad (2.4)$$

Next, we recall a concept from Kelly and Rival [18]. For $u, v \in L$ such that $u \parallel v$, we say that u is *to the left of* v if u is on the left of some (equivalently, every) maximal chain containing v. We know from Kelly and Rival [18, Lemma 1.2 and Proposition 1.7] that, for all elements u, v in any planar lattice,

if $u \parallel v$, then u is to the left of v or v is to the left of u, and $\qquad (2.5)$

if C is a maximal chain of L, u is on the left while v is on the
right of C, and $u < v$, then $u \leq w \leq v$ holds for some $w \in C$. $\qquad (2.6)$

Now we are in the position to prove 2.4(B), that is part (B) of the theorem. In virtue of (2.3), it suffices to show that

(a) 2.4(B) holds for every grid,
(b) if 2.4(B) holds for a slim semimodular lattice K and L is obtained from K by a multifork extension at a distributive 4-cell, then 2.4(B) also holds for L, and
(c) if 2.4(B) holds for a slim semimodular lattice K and L is obtained from K by omitting a corner, then 2.4(B) holds for L, too.

The validity of (a) is trivial.

Assume that K and L are as described in (b) and 2.4(B) holds for K; see Figure 2 where L is obtained from K by a 3-fold multifork extension. From left to right, the new meet-irreducible elements are labeled by $h_1, \ldots, h_n = h_3$. Since the multifork

extension is performed at a distributive 4-cell with top element denoted by v, the principal ideals $\downarrow_K v$, $\downarrow_L h_1$, $\ldots$, $\downarrow_L h_n$ are grids. Let $c = a \wedge b$ be a nontrivial meet in L; we have to show that it is a $\mathcal{C}_1$-regular meet in L. There are several cases.

Case 1. $a, b \in K$. Then $c \in K$, the chain $C := [c, a]_K$ of K is of a normal slope, and its elements except possibly a are meet-reducible. The edges (as line segments) of C can be divided into shorter edges by some new elements $x_1, \ldots, x_t$ in L, but C as a line segment and so its slope remain the same. It is clear by construction and Figure 2 that none of $h_1, \ldots, h_n$ can divide an old edge. Hence $\{x_1, \ldots, x_t\} \cap \{h_1, \ldots, h_n\} = \emptyset$. Since all the new elements but $h_1, \ldots, h_n$ are meet-reducible in L, so are $x_1, \ldots, x_n$. Since a meet-reducible element of K is also meet-reducible in L and a and b play a symmetrical role , we conclude that $c = a \wedge b$ is a $\mathcal{C}_1$-regular meet.

Case 2. $a, b \in L \setminus K$: since $\downarrow_K v$ is distributive, it is a grid. Hence, $\downarrow_L h_1$, $\ldots$, $\downarrow_L h_n$ are also grids; see Figure 2. Let $T := \downarrow_L h_1 \cup \cdots \cup \downarrow_L h_n$. Then T is a meet-subsemilattice and a subdiagram of a larger grid G. This G is not a subset of L but no problem: $c = a \wedge b$ is a $\mathcal{C}_1$-regular meet in G by (a), and T is a common meet-subsemilattice of L and G. Since h_1, $\ldots$, h_n are maximal elements in T, they are not in $([c, a] \setminus \{a\}) \cup ([c, b] \setminus \{b\})$, and it follows easily that $c = a \wedge b$ is a $\mathcal{C}_1$-regular meet in L.

Case 3. $a \in K$ and $b \in L \setminus K$. Two possible positions of b and $c := a \wedge b$ are given in Figure 2, one with black letters and another one with light-grey letters. Since b is a new element, $b < v$. Denote by v_1 and v_2 the lower covers of v in K; see Figure 2. Using (2.4) and reflecting the diagram across a vertical axis if necessary, we can assume that a is to the left of b, as in the figure. Since $b < v$ and $a \parallel b$, we have that $v \not\leq a$. Hence, there are two subcases.

In the first subcase, we assume that $a < v$. Observe that

$$\downarrow_L v \setminus \{v\} = \downarrow_L v_1 \cup \downarrow_L v_2 \cup \downarrow_L h_1 \cup \cdots \cup \downarrow_L h_n, \text{ and}$$
$$\text{every principal ideal on the right of the equality sign} \tag{2.7}$$
$$\text{is a chain with a normal slope or a grid.}$$

As in Case 2, we can extend $T := \downarrow_L v \setminus \{v\}$ to a grid G such that T is a common meet-subsemilattice of G and L. Then we can conclude the $\mathcal{C}_1$-regularity of $c = a \wedge b$ in L basically in the same way as in Case 2.

In the second subcase, we assume that $a \parallel v$. Since $b < v$ gives the existence of a maximal chain containing both b and v, we obtain that a is to the left of v. Let $a' := a \wedge v$, and observe that $a' \wedge b = (a \wedge v) \wedge b = a \wedge (v \wedge b) = a \wedge b = c$. This fact, $a' < v$, and the previous subcase yield that $c = a' \wedge b$ is a $\mathcal{C}_1$-regular meet. In particular, the interval $[c, b]$ has the required properties. So does $[c, a']$, but it is only a part of $[c, a]$. We know from the already proven part 2.4(A) that $[c, a]$ is a chain. Thus, a' is comparable with all elements of $[c, a]$, and it follows that $[c, a]$ is the union of $[c, a']$ and $[a', a]$. By its definition, a' is meet-reducible. Since part 2.4(B) holds for K and $a' = a \wedge v$, the intervals $[a', a]$ and $[a', v]$ are of normal slopes and $\mathrm{sl}^{\angle}(a', a]) \neq \mathrm{sl}^{\angle}(a', v])$. There are only two normal slopes, whence to show that $\mathrm{sl}^{\angle}(c, a']) = \mathrm{sl}^{\angle}(a', a])$, it suffices to exclude that $\mathrm{sl}^{\angle}(c, a']) = \mathrm{sl}^{\angle}(a', v])$. Suppose to the contrary that $\mathrm{sl}^{\angle}(c, a']) = \mathrm{sl}^{\angle}(a', v])$. Then c lies on the geometrical line of a normal slope through a' and v. (So, as opposed to what Figure 2 shows, c is on the line segment from p_3 to v.) Therefore, by the definition of multifork extensions,

the principal filter $\uparrow_L c$ cannot contain any new element. This contradicts $b \in \uparrow_L c$ and concludes Case 3.

Cases 1–3 yield the validity of (b).

To prove (c), assume that w is a corner of K and $L = K \setminus \{w\}$. Let $c = a \wedge b$ be a nontrivial meet in L. Note that $w \in \mathrm{Bnd}(K)$, $w \notin \{a, b, c\} \subseteq L$ and $c = a \wedge b$ is a $\mathcal{C}_1$-regular meet in K. In particular, $[c, a]_K$ and $[c, b]_K$ are chains of normal slopes. To see that $[c, a]_L = [c, a]_K$, it suffices to show that $w \notin [c, a]_K$. But this is obvious since otherwise $[c, a]_K$ would make an "orthogonal turn" at w and it could not be of a normal slope in K. Similarly, $[c, b]_L = [c, b]_K$. Thus, $[c, a]_L$ and $[c, b]_L$ are chains of normal slopes.

The only meet-reducible element of K that turns into meet-irreducible in L is $x_0 := w^-$. Hence, we need to show that none of $[c, a]_K \setminus \{a\}$ and $[c, b]_K \setminus \{b\}$ contains x_0. By symmetry, it suffices to deal with $[c, a]_K \setminus \{a\}$, and we can assume that $w \in B_{\mathrm{left}}(K)$. For the sake of contradiction, suppose that $c \leq x_0 < a$. If $p \prec q$ in K then, for convenience, $p \nearrow_{\mathrm{sl}} q$ and $q \nwarrow_{\mathrm{sl}} p$ will stand for $\mathrm{sl}^{\angle}([p, q]) = \pi/4$ and $\mathrm{sl}^{\angle}([p, q]) = 3\pi/4$, respectively. Let x_1 denote the cover of x_0 that is distinct of w. Since we are in a $\mathcal{C}_1$-diagram, both $[x_0, w]$ and $[x_0, x_1]$ are of normal slopes and these slopes are different. Using that $w \in B_{\mathrm{left}}(L)$, the only possibility is that $w \nwarrow_{\mathrm{sl}} x_0$ and $x_0 \nearrow_{\mathrm{sl}} x_1$. Every element has at most two covers; this was proved by Grätzer and Knapp [16, Lemma 8] (and also follows from the definition of $\mathcal{C}_1$-diagrams). Therefore, since $w \notin [c, a]_K$ and $c \leq x_0 < a$, we have that $x_1 \in [c, a]_K$. From the facts that $[x_0, x_1]$ is an edge of the chain $[c, a]_K$, this chain is of a normal slope, and $x_0 \nearrow_{\mathrm{sl}} x_1$, we obtain that

$$\text{for every edge } p \prec q \text{ of } [c, a]_K, \text{ we have that } p \nearrow_{\mathrm{sl}} q. \tag{2.8}$$

We claim that

$$\text{if } p \prec q \text{ is an edge of } [c, a]_K \text{ and } q \in B_{\mathrm{left}}(K), \text{ then } p \in B_{\mathrm{left}}(K) \cap \mathrm{M}(K). \tag{2.9}$$

Suppose to the contrary that $p \notin B_{\mathrm{left}}(K)$. But $B_{\mathrm{left}}(K)$ is a maximal chain of K, whereby there is a unique $r \in B_{\mathrm{left}}(K)$ such that $r \prec q$. Since r and p are different lower covers of q, one of them is to the left of the other by (2.5). Since r is on the left boundary chain $B_{\mathrm{left}}(K)$, p cannot be to left of r. Hence, r is to the left of p, whereby (2.8) gives that $\mathrm{sl}^{\angle}([r, q]) < \mathrm{sl}^{\angle}([p, q]) = \pi/4$, contradicting Convention 2.2. This proves that $p \in B_{\mathrm{left}}(K)$.

To obtain a contradiction again, suppose that q is not the only cover of p, and pick a cover s of p such that $s \neq q$. Since $q \in B_{\mathrm{left}}(K)$, (2.5) yields that q is to the left of s. This fact and (2.8) yield that $\mathrm{sl}^{\angle}([p, s]) < \mathrm{sl}^{\angle}([p, q]) = \pi/4$, which is a contradiction proving (2.9).

Next, let y denote the unique cover of c that belongs to the chain $[c, a]_K$. Since $x_0 \in \mathrm{M}(L)$ but $c \notin \mathrm{M}(L)$, we obtain that $x_0 \neq c$ and so $c < x_0 \in [c, a]_K$. Hence $y \leq x_0$. Descending along the chain $[c, a]_K$ from $x_0 \in B_{\mathrm{left}}(K) \cap \mathrm{M}(K)$ down to y, we obtain by (2.9) that $c \in \mathrm{M}(K)$. This is a contradiction since the meet-reducibility of c in L implies its meet-reducibility in K. Therefore, (c) holds, and the proof of Theorem 2.4 is complete. $\qquad\square$

3. Two properties of retracts of slim semimodular lattices

To give a preliminary description of what sort of properties this section deals with, we define one of these properties visually as follows. We say that the retracts

of a lattice L satisfies property $P(9, 2)$ if whenever the lattice on the right of Figure 3 is a sublattice of L and S is a retract of L such that the black-filled elements are in S, then the star-shaped elements are also in S. More generally, the key concept for this section is presented in the following definition.

Definition 3.1. Let $A^\bullet$ and $X^\star$ be nonempty subsets of a lattice K. We say that a sublattice S of a lattice L satisfies the *absorption property* $\mathrm{AP}(K, A^\bullet, X^\star)$ if for every lattice embedding $g \colon K \to L$, the inclusion $g(A^\bullet) \subseteq S$ implies that $g(X^\star) \subseteq S$. If every retract of L satisfies $\mathrm{AP}(K, A^\bullet, X^\star)$, then we say that the *retracts* of L satisfy the absorption property $\mathrm{AP}(K, A^\bullet, X^\star)$.

Remark 3.2 (Remarks on Definition 3.1)**.** If $[A^\bullet]_K$ denotes the sublattice generated by $A^\bullet$ in K, then the absorption property $\mathrm{AP}(K, A^\bullet, X^\star)$ is clearly equivalent to $\mathrm{AP}(K, [A^\bullet]_K, [A^\bullet \cup X^\star]_K \setminus [A^\bullet]_K)$. The case $X^\star \subseteq [A^\bullet]_K$ is not interesting since every sublattice of any lattice satisfies it. Therefore, without losing anything, we always assume in this paper that

$$A^\bullet \text{ and } A^\bullet \cup X^\star \text{ are sublattices of } K \text{ and } A^\bullet \cap X^\star = \emptyset. \tag{3.1}$$

Furthermore, $\mathrm{AP}(K, A^\bullet, X^\star)$ in this paper is always given by a single diagram: the diagram of K in which the elements of $A^\bullet$ are drawn by large black-filled circles while $X^\star$ is the set of the star-shaped elements. If P stands for $\mathrm{AP}(K, A^\bullet, X^\star)$, then we can use the notation $K(P)$ for K.

Although $\mathrm{AP}(K, A^\bullet, X^\star)$ is not determined by $(|K|, |X^\star|)$, we include $(|K|, |X^\star|)$ in the notations of the absorption properties occurring in Figures 3 and 5. For example, property $P(8, 1)$ given in Figure 3 is the condition $\mathrm{AP}(K, A^\bullet, X^\star)$ where $K = \{a, b, c, d, x, y, z, t\}$ is the lattice given in the figure, $A^\bullet = \{a, b, c, d\}$, and $X^\star = \{y\}$.

Using Theorem 2.4, we are going to prove two corollaries; $P(8, 1)$ and $P(9, 2)$ are given by Figure 3 and Definition 3.1.

FIGURE 3. Absorption properties $P(8, 1)$ and $P(9, 2)$, which are satisfied by the retracts of slim semimodular lattices

Corollary 3.3. *The retracts of every slim semimodular lattice satisfy $P(8, 1)$.*

Corollary 3.4. *The retracts of every slim semimodular lattice satisfy $P(9, 2)$.*

Proof of Corollary 3.3. With $\mathrm{AP}(K, A^\bullet, X^\star) := P(8, 1)$, let S be a retract of a slim semimodular lattice L, and let $g \colon K \to L$ be an embedding as in Definition 3.1. We can assume that g is the inclusion map. Then K is a sublattice of L and $A^\bullet \subseteq S$; we need to show that $y \in S$. Convention 2.2 applies. Pick a retraction map $f \colon L \to S$. For convenience, we will write $a', b', \ldots, z'$ instead of $f(a)$, $f(b)$, $\ldots$, $f(z)$. Of course, we can drop the apostrophe at black-filled elements, for example, $a' = a$.

Straightforward properties of f like $u \leq v \Rightarrow u' \leq v'$ will frequently be used without much explanation. For $u \in L$, the ray (as a geometric object) with initial point u and slope angle $\pi/4$, $3\pi/4$, $5\pi/4$, and $7\pi/4$ will be denoted by

$$\vec{\ell}(u \nearrow), \quad \vec{\ell}(\nwarrow u), \quad \vec{\ell}(\swarrow u), \quad \text{and} \quad \vec{\ell}(u \searrow), \tag{3.2}$$

respectively. For example, in L of Figure 2, v_1 is on $\vec{\ell}(\nwarrow u)$ (we will write $v_1 \in \vec{\ell}(\nwarrow u)$ to denote this adjacency), $u \in \vec{\ell}(v_1 \searrow)$, $b \in \vec{\ell}(c \nearrow)$, and $c \in \vec{\ell}(\swarrow b)$.

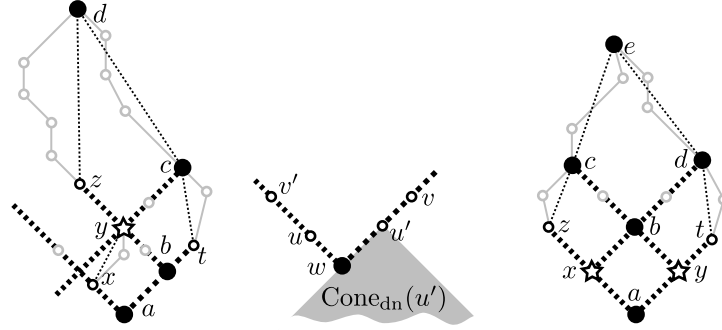FIGURE 4. Proving Corollary 3.3, (3.4), and Corollary 3.4

Let D_1 and D_2 be planar diagrams of the same planar lattice M. These diagrams are *similar* if there is an automorphism h of M that induces a $D_1 \to D_2$ map preserving the "left to" relation on pairs of incomparable elements. The diagram we obtain from D_2 by reflecting it across a vertical axis will be denoted by D_2'. If for every $u \in M \setminus \{0, 1\}$ there is a $v \in M$ with $v \parallel u$, then M is *glued sum indecomposable*. We know from Czédli [3, Proposition 5.1] that whenever D_1 and D_2 are planar diagrams of a glued sum indecomposable planar semimodular lattice, then D_1 is similar to D_2 or to D_2'. Therefore, we can assume that the "left to" relation is correctly indicated by Figure 4, because otherwise we could reflect the diagram across a vertical axis. In this figure, the relation "$<$" is indicated by dotted line segments. These dotted line segments need not be edges; there can be (but need not be) further elements and edges, drawn in light-grey, between the endpoints of a dotted line segment. For example, the segment of $\vec{\ell}(\nwarrow b)$ between b and z consists of edges by Theorem 2.4, but it is possible that, say, the line segment $[z, d]$ is neither an edge, nor it contains any element of L besides z and d. The thick dotted line segments are of normal slopes, and each of the thin line segments is precipitous or is of a normal slope. Applying Theorem 2.4 to $a = x \wedge t$, $b = z \wedge t$, and $y = z \wedge c$, we obtain that

$$x \in \vec{\ell}(\nwarrow a), \quad b, t \in \vec{\ell}(a \nearrow), \quad y, z \in \vec{\ell}(\nwarrow b), \quad c \in \vec{\ell}(y \nearrow); \tag{3.3}$$

see the left side of Figure 4. We claim the following, which is the key idea of the proof.

> If $u \parallel v$ in L, $w := u \wedge v \in S$, u is to the left of v,
> and $u' \parallel v'$, then $u' \in \vec{\ell}(\nwarrow w)$, $v' \in \vec{\ell}(w \nearrow)$, and both $\quad$ (3.4)
> $[w, u']$ and $[w, v']$ are chains of normal slopes.

To show this, observe that $w = w' = (u \wedge v)' = u' \wedge v'$. So both $w = u \wedge v$ and $w = u' \wedge v'$ are $\mathcal{C}_1$-regular meets by Theorem 2.4. In particular, $[w, u']$ and $[w, v']$

are chains of different normal slopes. Hence, it suffices to exclude that $u' \in \vec{\ell}(w\nearrow)$ and $v' \in \vec{\ell}(\searrow w)$. For the sake of contradiction, suppose that $u' \in \vec{\ell}(w\nearrow)$ and $v' \in \vec{\ell}(\searrow w)$, as in the middle of Figure 4. In the figure, the thick dotted rays are geometric half-lines and, say, the line segment from w to v' represents a chain of a normal slope. Since u is to the left of v and $w = u \wedge v$ is a $\mathcal{C}_1$-regular meet, $u \in \vec{\ell}(\searrow w)$ and $v \in \vec{\ell}(w\nearrow)$. In the figure, the light-grey area indicates $\mathrm{Cone_{dn}}(u')$. Since the dotted thick rays are of normal slopes, $\mathrm{Cone_{dn}}(u) \cap \mathrm{Cone_{dn}}(u') = \mathrm{Cone_{dn}}(w)$. Combining this equality with (2.4), we obtain that $u \wedge u' = w$. Hence, $u' = u' \wedge u' = u' \wedge u'' = (u \wedge u')' = w' = w$. On the other hand, $w \leq v$ yields that $w = w' \leq v'$. Thus, $u' = w \leq v'$ contradicts the assumption that $u' \parallel v'$, completing the proof of (3.4).

Let us mention at this pont that, with self-explanatory changes,

$$\begin{array}{l} \text{all what we have done in this proof so} \\ \text{far will be needed in the next proof.} \end{array} \tag{3.5}$$

We need to show that $y' = y$ since this would mean that $y \in S$. There are two cases discussed below; we are going to show that the first of these cases gives the required $y' = y$ while the second one cannot occur since it leads to a contradiction.

Case 1. We assume that $z' \parallel t'$. Applying (3.4) with $(u, v, w) := (z, t, b)$, we obtain that $[b, z']$ is a chain of a normal slope, $z' \in \vec{\ell}(\searrow b)$ and $t' \in \vec{\ell}(b\nearrow)$. Since $z' \in \vec{\ell}(\searrow b)$ and $t \in \vec{\ell}(b\nearrow) \setminus \{b\}$, we have that $t \notin \mathrm{Cone_{dn}}(z')$. This fact and (2.4) give that $z' \ngeq t$, implying that $z' \ngeq c$. Thus, either $z' \parallel c$ or $z' < c$. However, if $z' < c$, then $t' \leq c' = c$ leads to $d = d' = (z \vee t)' = z' \vee t' \leq c$, which is a contradiction. Therefore, $z' \parallel c$. Then $y' = (z \wedge c)' = z' \wedge c' = z' \wedge c$, so $y' = z' \wedge c$ is a $\mathcal{C}_1$-regular meet. By Theorem 2.4,

$$\text{either } y' \in \vec{\ell}(c\searrow), \text{ or } y' \in \vec{\ell}(\swarrow c). \tag{3.6}$$

Since $[b, z']$ is a chain of a normal slope and $b \leq y \leq z$ implies that $b = b' \leq y' \leq z'$, we obtain that $y' \in \vec{\ell}(\searrow b)$. The rays $\vec{\ell}(\searrow b)$ and $\vec{\ell}(c\searrow)$ are parallel. The line segment between y and c is orthogonal to these parallel rays, and $y \in \vec{\ell}(\searrow b)$. Hence the geometric distance of $\vec{\ell}(\searrow b)$ and $\vec{\ell}(c\searrow)$ is the distance of y and c, which is positive since $y \neq c$. Thus, $\vec{\ell}(\searrow b) \cap \vec{\ell}(c\searrow) = \emptyset$. This fact and $y' \in \vec{\ell}(\searrow b)$ excludes that $y' \in \vec{\ell}(c\searrow)$, whereby (3.6) implies that $y' \in \vec{\ell}(\swarrow c)$. So y' belongs to both $\vec{\ell}(\searrow b)$ and $\vec{\ell}(\swarrow c)$. But these two orthogonal rays only have one point in common, which is y. Therefore $y' = y$, as required. This completes the analysis of Case 1.

Case 2. We assume that z' and t' are comparable. Since their join and meet are $d = d'$ and $b = b'$, we have that $\{z', t'\} = \{d, b\}$. But $c = c' \geq t'$ since $c \geq t$, whereby $t' \neq d$ and we conclude that $z' = d$ and $t' = b$. There are two subcases.

First, assume that $x' \parallel b$. Since $b' = b$, (3.4) gives that $x' \in \vec{\ell}(\searrow a)$. Let C' be a maximal chain in the interval $[z, d]$, and take the chain $C := [a, b] \cup [b, z] \cup C'$; this is indeed a chain since $[a, b]$ and $[b, z]$ are chains (of different normal slopes) by Theorem 2.4. Now b and every element of $[a, b] \setminus \{a\}$ are strictly on the right of the geometric ray $\vec{\ell}(\searrow a)$. ("Strictly on the right of" means that "on the right of but not belonging to"; the meaning of "strictly on the left of" is analogous.) Since every edge of the subchain $C \cap [b, d]$ is precipitous or of a normal slope, every element of $C \cap [b, d]$ is strictly on the right of $\vec{\ell}(\searrow a)$, as the figure shows. Therefore,

$$\text{every element of } C \setminus \{a\} \text{ is strictly on the right of } \vec{\ell}(\searrow a). \tag{3.7}$$

Let $L' := [a, d]$, and observe that $a \leq x \leq d$ gives that $a = a' \leq x' \leq d' = d$, that is, $x' \in L'$. Now C is a maximal chain of the planar lattice L', b is on the left of C (since $b \in C$), and $x' \in \vec{\ell}(\searcher a)$ combined with (3.7) yield that x' is also on the left of C. It is visually clear and it has rigorously been proved in Kelly and Rival [18, Proposition 1.4] that the elements on the left of a maximal chain of a planar lattice form a (convex) sublattice. Consequently,

$$y' = (x \vee b)' = x' \vee b' = x' \vee b \text{ is on the left of } C. \qquad (3.8)$$

Since $z \in \vec{\ell}(\searcher y) \setminus \{y\}$, $c \in \vec{\ell}(y \nearrow) \setminus y$, and every edge of C' is precipitous or of a normal slope, it follows that, again in L',

$$c \text{ is strictly on the right of } C. \qquad (3.9)$$

Finally, using that $z' = d$, we obtain that $y' = (z \wedge c)' = z' \wedge c' = d \wedge c = c$. Hence, $y' = c$ and so (3.9) contradicts (3.8), proving that the subcase $x' \parallel b$ cannot occur.

As the second subcase, now we assume that x' and b are comparable. Since $a = a' = (x \wedge b)' = x' \wedge b' = x' \wedge b$, the just-mentioned comparability yields that $x' = a$. Using this equality and $t' = b$, we obtain that $c = c' = (x \vee t)' = x' \vee t' = a \vee b = b$, which is a contradiction excluding the second subcase. Therefore, Case 2 cannot occur.

We have seen that Case 1 implies the required $y' = y$ while Case 2 cannot occur. Thus, the proof of Corollary 3.3 is complete. $\qquad \square$

Proof of Corollary 3.4. The relevant illustration is on the right of Figure 4. After recalling (3.5), first we show that $\{x', y'\} \neq \{a, b\}$. Suppose the contrary. Then, since x and y play a symmetrical role, we can assume that $x' = a$ and $y' = b$. Since $t' \geq y' = b$, (2.4) gives that $t' \in \mathrm{Cone}^{\mathrm{up}}(b)$. But $\mathrm{Cone}^{\mathrm{up}}(b)$ is in the (topological) interior of $\mathrm{Cone}^{\mathrm{up}}(a)$, and both $\mathrm{Cone}^{\mathrm{up}}(a)$ and $\mathrm{Cone}^{\mathrm{up}}(b)$ are formed by rays of normal slopes. Hence $t' \in \mathrm{Cone}^{\mathrm{up}}(b)$ implies that $a \notin \vec{\ell}(\nearrow t')$ and $a \notin \vec{\ell}(t' \searcher)$. Thus, Theorem 2.4 excludes that $a = z' \wedge t'$ such that $z' \parallel t'$. However, $a = a' = (z \wedge t)' = z' \wedge t'$, whence $a \in \{z', t'\}$. Since $t' \geq b$ excludes that $a = t'$, we have that $z' = a$. But then $e = e' = (z \vee d)' = z' \vee d' = a \vee d = d$ is a contradiction. This proves that $\{x', y'\} \neq \{a, b\}$. This fact, $a = a' = (x \wedge y)' = x' \wedge y'$, and $b = b' = (x \vee y)' = x' \vee y'$ imply that $x' \parallel y'$. Applying (3.4) to $(u, v, w) := (x, y, a)$, we obtain that $x' \in \vec{\ell}(\searcher a)$ and $y' \in \vec{\ell}(a \nearrow)$.

If we had that $b = x'$, then $b = b' \geq y'$ would contradict that $x' \parallel y'$. (Alternatively, $x' \in \vec{\ell}(\searcher a)$ excludes that $b = x'$ by a trivial geometric reasoning.) If we had that $z' = x'$, then $c = c' = (z \vee b)' = z' \vee b' = x' \vee b' = (x \vee b)' = b' = b$ would be a contradiction. Hence, $x' \notin \{b, z'\}$. Thus, $x' = (z \wedge b)' = z' \wedge b' = z' \wedge b$ is a $\mathcal{C}_1$-regular meet by Theorem 2.4. In particular, $x' \in \vec{\ell}(\nearrow b)$ or $x' \in \vec{\ell}(b \searcher)$. The parallel (but oppositely oriented) rays $\vec{\ell}(\searcher a)$ and $\vec{\ell}(b \searcher)$ are disjoint, whereby $x' \in \vec{\ell}(\searcher a)$ excludes that $x' \in \vec{\ell}(b \searcher)$. Thus, $x' \in \vec{\ell}(\nearrow b)$.

Finally, since x and x' belong to both $\vec{\ell}(\searcher a)$ and $\vec{\ell}(\nearrow b)$ and these two orthogonal rays only have one geometric point in common, we obtain that $x' = x$, that is, $x \in S$, as required. The containment $y \in S$ follows by symmetry, completing the proof of Corollary 3.4. $\qquad \square$

An absorption property $\mathrm{AP}(K, A^\bullet, X^\star)$ becomes stronger if $X^\star$ is replaced by a larger subset of K. Therefore, the following example implies that none of Corollaries 3.3 and 3.4 can be strengthened by taking a larger $X^\star$.
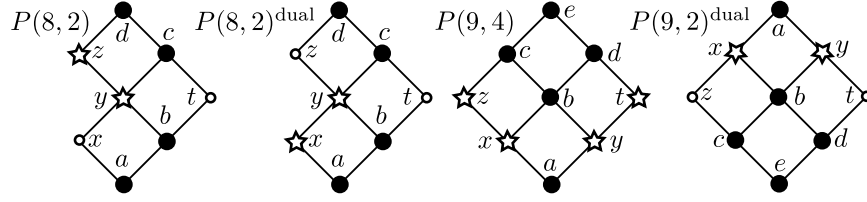
FIGURE 5. Some absorption properties that fail

Example 3.5. In general, the retracts of a slim semimodular lattice satisfy none of the absorption properties $P(8,2)$, $P(8,2)^{\text{dual}}$, $P(9,4)$, and $P(9,4)^{\text{dual}}$; see Figure 5.
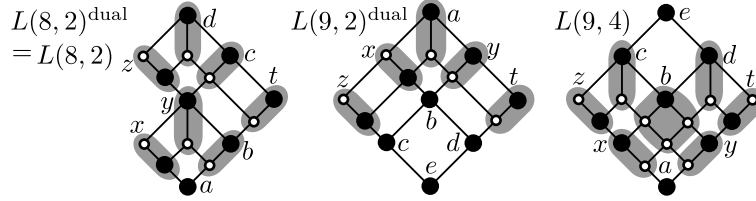

FIGURE 6. To the proof of Example 3.5

Proof. Consider the lattices $L(8,2) = L(8,2)^{\text{dual}}$, $L(9,4)$, and $L(9,4)^{\text{dual}}$ in Figure 6. They are slim semimodular lattices by (the last sentence of) (2.3). The embeddings of $K(P(8,2))$, ..., $K(P(9,4)^{\text{dual}})$ into the corresponding lattices are defined by the labeling, and the retract consists of the large black-filled elements in each case. For each of the lattices in Figure 6, define a map by the rule that

$$u \mapsto \begin{cases} u, & \text{if } u \text{ is a black-filled element,} \\ x, & \text{if } x \text{ is the black-filled element in the grey oval containing } u. \end{cases}$$

It is straightforward to check that this map is a retraction. Hence, we get an example showing that the corresponding absorption property fails.   □

Finally, we note that the sublattice $S = \{a,b,c,d,y\}$ of the slim semimodular lattice $K(P(8,1))$ in Figure 3 satisfies $P(8,1)$ and $P(9,2)$ but S is not a retract of $K(P(8,1))$. Hence, Corollaries 3.3 and 3.4 only give necessary conditions but not a characterization of retracts among the sublattices of slim semimodular lattices. In fact, we do not know such a characterization.

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