Lattices with many congruences are planar

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Dedicated to the memory of Ivan Rival

Abstract. Let L be an n-element finite lattice. We prove that if L has more than 2^{n-5} congruences, then L is planar. This result is sharp, since for each natural number $n \geq 8$, there exists a non-planar lattice with exactly 2^{n-5} congruences.

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1. Aim and introduction

Our goal is to prove the following statement.

Theorem 1.1. *Let L be an n-element finite lattice. If L has more than 2^{n-5} congruences, then it is a planar lattice.*

In order to point out that this result is sharp, we will also prove the following easy remark. An n-element finite lattice L is *dismantlable* if there is a sequence $L_1 \subset L_2 \subset \cdots \subset L_n = L$ of its sublattices such that $|L_i| = i$ for every $i \in \{1, \ldots, n\}$; see Rival [13]. We know from Kelly and Rival [9] that every finite planar lattice is dismantlable.

Remark 1.2. For each natural number $n \geq 8$, there exists an n-element non-dismantlable lattice $L(n)$ with exactly 2^{n-5} congruences; this $L(n)$ is non-planar.

We know from Freese [5] that an n-element lattice L has at most $2^{n-1} = 16 \cdot 2^{n-5}$ congruences. In other words, denoting the lattice of congruences of L by $\mathrm{Con}(L)$, we have that $|\mathrm{Con}(L)| \leq 2^{n-1}$. For $n \geq 5$, the second largest number of the set

$$\mathrm{ConSizes}(n) := \{|\mathrm{Con}(L)| : L \text{ is a lattice with } |L| = n\}$$

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is $8 \cdot 2^{n-5}$ by Czédli [3], while Kulin and Mureşan [10] proved that the third, fourth, and fifth largest numbers of $\mathrm{ConSizes}(n)$ are $5 \cdot 2^{n-5}$, $4 \cdot 2^{n-5}$, and $\frac{7}{2} \cdot 2^{n-5}$, respectively. Since both [3] and Kulin and Mureşan [10] described the lattices witnessing these numbers, it follows from these two papers that $|\mathrm{Con}(L)| \geq \frac{7}{2} \cdot 2^{n-5}$ implies the planarity of L. So, [3], [10], and even their precursor, Mureşan [11] have naturally lead to the conjecture that if an n-element lattice L has *many* congruences with respect to n, then L is necessarily planar. However, the present paper needs a technique different from Kulin and Mureşan [10], because a [10]-like description of the lattices witnessing the sixth, seventh, eighth, ..., k-th largest numbers in $\mathrm{ConSizes}(n)$ seems to be hard to find and *prove*; we do not even know how large is k. Fortunately, we can rely on the powerful characterization of planar lattices given by Kelly and Rival [9].

Note that although an n-element finite lattice with "*many*" (that is, more than 2^{n-5}) congruences is necessarily planar by Theorem 1.1, an n-element planar lattice may have only very *few* congruences even for large n. For example, the n-element modular lattice of length 2, denoted usually by M_{n-2}, has only two congruences if $n \geq 5$. On the other hand, we know, say, from Kulin and Mureşan [10] that there are a lot of lattices L with many congruences, whereby a lot of lattices belong to the scope of Theorem 1.1.

Outline and prerequisites

Section 2 recalls some known facts from the literature and, based on these facts, proves Remark 1.2 in three lines. The rest of the paper is devoted to the proof of Theorem 1.1.

Due to Section 2, the reader is assumed to have only little familiarity with lattices. Apart from some figures from Kelly and Rival [9], which should be at hand while reading, the present paper is more or less self-contained modulo the above-mentioned familiarity. Note that [9] is an open access paper at the time of this writing; see `http://dx.doi.org/10.4153/CJM-1975-074-0`.

2. Some known facts about lattices and their congruences

In the whole paper, *all lattices are assumed to be finite* even if this is not repeated all the time. For a finite lattice L, the set of nonzero join-irreducible elements, that of nonunit meet-irreducible elements, and that of doubly irreducible (neither 0, not 1) elements will be denoted by $\mathrm{J}(L)$, $\mathrm{M}(L)$, and $\mathrm{Irr}(L) = \mathrm{J}(L) \cap \mathrm{M}(L)$, respectively. For $a \in \mathrm{J}(L)$ and $b \in \mathrm{M}(L)$, the unique lower cover of a and the unique (upper) cover of b will be denoted by a^- and b^+, respectively. For $a, b \in L$, let $\mathrm{con}(a, b)$ stand for the smallest congruence of L such that $\langle a, b\rangle \in \mathrm{con}(a, b)$. For $x, y \in \mathrm{J}(L)$, let $x \equiv_{\mathrm{con}} y$ mean that $\mathrm{con}(x^-, x) = \mathrm{con}(y^-, y)$. Then $\equiv_{\mathrm{con}}$ is an equivalence relation on $\mathrm{J}(L)$, and the corresponding quotient set will be denoted by

$$\mathrm{Q}(L) := \mathrm{J}(L)/{\equiv_{\mathrm{con}}}. \tag{2.1}$$

As an obvious consequence of Freese, Ježek and Nation [6, Theorem 2.35] or Nation [12, Corollary to Theorem 10.5], for every finite lattice L,

$$|\mathrm{Con}(L)| \leq 2^{|\mathrm{Q}(L)|} \leq 2^{|\mathrm{J}(L)|}; \tag{2.2}$$

more explanation of this fact and (2.3) below will be given later. The situation simplifies for distributive lattices; it is well known that

$$\text{if } L \text{ is a finite } \textit{distributive} \text{ lattice, then } |\mathrm{Con}(L)| = 2^{|\mathrm{J}(L)|}. \tag{2.3}$$

Next, having no explicit reference at hand, we give a possible way how to extract (2.2) and (2.3) from the literature; the reader may skip over these details. A *quasiordered set* is a structure $\langle A; \leq\rangle$ where $\leq$ is a quasiordering, that is, a reflexive and symmetric relation on A. For example, if we let $a \leq_{\mathrm{con}} b$ mean $\mathrm{con}(a^-, a) \leq \mathrm{con}(b^-, b)$, then $\langle \mathrm{J}(L); \leq_{\mathrm{con}}\rangle$ is a quasiordered set. A subset X of $\langle A; \leq\rangle$ is *hereditary*, if $(\forall x \in X)(\forall y \in A)(y \leq x \Rightarrow y \in X)$. The set of all hereditary subsets of $\langle A; \leq\rangle$ with respect to set inclusion forms a lattice $\mathrm{Hered}(\langle A; \leq\rangle)$. Freese, Ježek and Nation [6, Theorem 2.35] can be reworded as $\langle \mathrm{Con}(L); \subseteq\rangle \cong \mathrm{Hered}(\langle \mathrm{J}(L); \leq_{\mathrm{con}}\rangle)$. To recall this theorem in a form closer to [6, Theorem 2.35], for the $\equiv_{\mathrm{con}}$-blocks of $a, b \in \mathrm{J}(L)$, we define the meaning of $a/{\equiv_{\mathrm{con}}} \leq_{\mathrm{con}} b/{\equiv_{\mathrm{con}}}$ as $a \leq_{\mathrm{con}} b$. In this way, we obtain a poset (partially ordered set) $\langle \mathrm{Q}(L); \leq_{\mathrm{con}} /{\equiv_{\mathrm{con}}}\rangle$. With this notation, the original form of [6, Theorem 2.35] states that $\mathrm{Con}(L) \cong \mathrm{Hered}(\mathrm{Q}(L); \leq_{\mathrm{con}} /{\equiv_{\mathrm{con}}})$, which implies (2.2).

Since $\equiv_{\mathrm{con}}$ will play an important role later, recall that for intervals $[a, b]$ and $[c, d]$ in a lattice L, $[a, b]$ *transposes up* to $[c, d]$ if $b \wedge c = a$ and $b \vee c = d$. This relation between the two intervals will be denoted by $[a, b] \nearrow [c, d]$. We say that $[a, b]$ *transposes down* to $[c, d]$, in notation $[a, b] \searrow [c, d]$ if $[c, d] \nearrow [a, b]$. We call $[a, b]$ and $[c, d]$ *transposed intervals* if $[a, b] \searrow [c, d]$ or $[a, b] \nearrow [c, d]$. It is well known and easy to see that

$$\text{if } [a, b] \text{ and } [c, d] \text{ are transposed intervals, then } \mathrm{con}(a, b) = \mathrm{con}(c, d). \tag{2.4}$$

Note that $\mathrm{Con}(L)$ in (2.3) is a Boolean lattice. Even more is true: $\mathrm{Con}(L)$ is Boolean for every finite *modular* lattice; this follows from the characterization of lattice congruences given in Dilworth [4] and it is explicit in the monograph Crawley and Dilworth [2, 10.3 combined with 10.7]. Next, let L be a finite distributive lattice, and pick a maximal chain $0 \prec a_1 \prec \cdots \prec a_t = 1$ in L. Here t is the length of L, and it is well known that $t = |\mathrm{J}(L)|$; see, for example, Grätzer [7, Corollary 112 in page 114]. If $\mathrm{con}(a_{i-1}, a_i) = \mathrm{con}(a_{j-1}, a_j)$, then it follows from Crawley and Dilworth [2, 10.2 and 10.3] or from Grätzer [8] that there is a sequence of prime intervals (edges in the diagram) from $[a_{i-1}, a_i]$ to $[a_{j-1}, a_j]$ such that any two neighboring intervals in this sequence are transposed. (We know from Crawley and Dilworth [2, 10.4] that there exists such a sequence even of length two, but we do not need this fact.)

In the terminology of Adaricheva and Czédli [1], the existence of the above-mentioned sequence means that $[a_{i-1}, a_i]$ and $[a_{j-1}, a_j]$ belong to the same *trajectory*. Since no two distinct comparable prime intervals of L can belong to the same trajectory by [1, Proposition 6.1], it follows that $i = j$.

Hence, the congruences $\mathrm{con}(a_{i-1}, a_i)$, $i \in \{1, \dots, t\}$, are pairwise distinct. It is well known that a prime interval in a finite lattice generates a join-irreducible congruence; see, for example, Grätzer [7, page 213]. Hence, the $\mathrm{con}(a_{i-1}, a_i)$, $i \in \{1, \dots, t\}$, are atoms in $\mathrm{Con}(L)$ since $\mathrm{Con}(L)$ is Boolean. Clearly, $\bigvee_{i=1}^{t} \mathrm{con}(a_{i-1}, a_i)$ is $1_{\mathrm{Con}(L)}$, which implies that $|\mathrm{Con}(L)| = 2^t$. This proves (2.3) since $t = |\mathrm{J}(L)|$.

Next, a lattice is called *planar* if it is finite and has a Hasse-diagram that is a planar representation of a graph in the usual sense that any two edges can intersect only at a vertex. For lattices K and L, we say that L *contains K as a subposet* if there exists an injective map $\varphi\colon K \to L$ such that, for all $x, y \in K$, we have $x \leq y$ in K if and only if $\varphi(x) \leq \varphi(y)$ in L. If, in addition, $K \subseteq L$ and the inclusion map $\iota\colon K \to L$, defined by $x \mapsto x$, has the same property as φ above, then the lattice K *is a subposet of* L. If K is only a poset but need not be a lattice, then the same condition defines that K is (isomorphic to) a *subposet* of L. Let $\mathbb{N}_0$ and $\mathbb{N}^+$ denote the set $\{0, 1, 2, \dots\}$ of nonnegative integers and the set $\{1, 2, 3, \dots\}$ of positive integers, respectively. In their fundamental paper on planar lattices, Kelly and Rival [9] gave a set

$$\mathcal{L}_{\mathrm{KR}} = \{A_n, E_n, F_n, G_n, H_n : n \in \mathbb{N}_0\} \cup \{B, C, D\}$$

of finite lattices such that the following statement holds.

Proposition 2.1 (A part of Kelly and Rival [9, Theorem 1]**).** *A finite lattice L is planar if and only if neither L, nor its dual contains some lattice of $\mathcal{L}_{\mathrm{KR}}$ as a subposet.*

Note that the lattices A_n, F_n, G_n, and H_n are selfdual. Note also that Kelly and Rival [9] proved the minimality of $\mathcal{L}_{\mathrm{KR}}$, but we do not need this fact.

Next, we prove Remark 1.2. The *ordinal sum* of lattices L' and L'' is their disjoint union $L' \dot\cup L''$ such that for $x, y \in L' \dot\cup L''$, we have that $x \leq y$ if and only if $x \leq_{L'} y$, or $x \leq_{L''} y$, or $x \in L'$ and $y \in L''$.

Proof of Remark 1.2. Let $L(8)$ be the eight-element Boolean lattice. Also, for $n > 8$, let $L(n)$ be the ordinal sum of $L(8)$ and an $(n-8)$-element chain. Since $|\mathrm{J}(L_n)| = n - 5$, Remark 1.2 follows from (2.3). □

Note that $L(n)$ above occurs also in page 93 of Rival [13].

3. A lemma on subposets that are lattices

While $\mathcal{L}_{\mathrm{KR}}$ consists of *lattices*, they appear in Proposition 2.1 as *subposets*. This fact causes some difficulties in proving our theorem; this section serves as a preparation to overcome them. The set of *join-reducible elements* of a lattice L will be denoted by $\mathrm{JRed}(L)$. Note that

$$\mathrm{JRed}(L) = L \setminus (\{0\} \cup \mathrm{J}(L)) = \{a \vee b : a, b \in L \text{ and } a \parallel b\}, \tag{3.1}$$

where $\parallel$ stands for incomparability, that is, $a \parallel b$ is the conjunction of $a \nleq b$ and $b \nleq a$. Similarly, $\mathrm{MRed}(L) = L \setminus (\{1\} \cup \mathrm{M}(L))$ denotes the set of *meet-reducible elements* of L.

Lemma 3.1. *Let L and K be finite lattices such that K is a subposet of L. Then the following four statements and their duals hold.*

(i) *If $a_1, \dots, a_t \in K$ and $t \in \mathbb{N}^+$, then $a_1 \vee_L \dots \vee_L a_t \leq a_1 \vee_K \dots \vee_K a_t$.*
(ii) *If $t, s \in \mathbb{N}^+$, $a_1, \dots, a_t, b_1, \dots, b_s \in K$, and $a_1 \vee_K \dots \vee_K a_t$ is distinct from $b_1 \vee_K \dots \vee_K b_s$, then $a_1 \vee_L \dots \vee_L a_t \neq b_1 \vee_L \dots \vee_L b_s$.*
(iii) *$|\mathrm{JRed}(L)| \geq |\mathrm{JRed}(K)|$ and, dually, $|\mathrm{MRed}(L)| \geq |\mathrm{MRed}(K)|$.*
(iv) *If $|\mathrm{JRed}(L)| = |\mathrm{JRed}(K)|$, $u_1, u_2, v_1, v_2 \in K$, $u_1 \parallel u_2$, $v_1 \parallel v_2$, and $u_1 \vee_K u_2 = v_1 \vee_K v_2$, then $u_1 \vee_L u_2 = v_1 \vee_L v_2$.*

Note that, according to (ii) and (iv), the distinctness of joins is generally preserved when passing from K to L, but equalities are preserved only under additional assumptions. The dual of a condition or statement (X) will often be denoted by $(\mathrm{X})^{\mathrm{d}}$; for example, the dual of Lemma 3.1(i) is denoted by Lemma $3.1(\mathrm{i})^{\mathrm{d}}$ or simply by $3.1(\mathrm{i})^{\mathrm{d}}$.

Proof of Lemma 3.1. Part (i) is a trivial consequence of the concept of joins as least upper bounds.

In order to prove (ii), assume that $a_1 \vee_L \dots \vee_L a_t = b_1 \vee_L \dots \vee_L b_s$. Part (i) gives that $a_i \leq_L a_1 \vee_L \dots \vee_L a_t = b_1 \vee_L \dots \vee_L b_s \leq_L b_1 \vee_K \dots \vee_K b_s$, for all $i \in \{1, \dots, t\}$. Since K is a subposet of L, $a_i \leq_K b_1 \vee_K \dots \vee_K b_s$. But $i \in \{1, \dots, t\}$ is arbitrary, whereby $a_1 \vee_K \dots \vee_K a_t \leq_K b_1 \vee_K \dots \vee_K b_s$. We have equality here, since the converse inequality follows in the same way. Thus, we conclude (ii) by contraposition.

Next, let $\{c_1, \dots, c_t\}$ be a repetition-free list of $\mathrm{JRed}(K)$. For each i in $\{1, \dots, t\}$, pick $a_i, b_i \in K$ such that $a_i \parallel b_i$ and $c_i = a_i \vee_K b_i$. That is,

$$\mathrm{JRed}(K) = \{c_1 = a_1 \vee_K b_1, \dots, c_t = a_t \vee_K b_t\}. \tag{3.2}$$

Since $a_i \parallel b_i$ holds also in L,

$$\{a_1 \vee_L b_1, \dots, a_t \vee_L b_t\} \subseteq \mathrm{JRed}(L). \tag{3.3}$$

The elements listed in (3.3) are pairwise distinct by part (ii). Therefore, $|\mathrm{JRed}(K)| = t \leq |\mathrm{JRed}(L)|$, proving part (iii).

Finally, to prove part (iv), we assume its premise, and we let $t := |\mathrm{JRed}(K)| = |\mathrm{JRed}(L)|$. Choose $c_i, a_i, b_i \in K$ as in (3.2). Since $t = |\mathrm{JRed}(L)|$, part (ii) and (3.3) give that

$$\mathrm{JRed}(L) = \{a_1 \vee_L b_1, \dots, a_t \vee_L b_t\}. \tag{3.4}$$

As a part of the premise of (iv), $u_1 \parallel u_2$ has been assumed. Hence, $u_1 \vee_K u_2 \in \mathrm{JRed}(K)$, and so (3.2) yields a unique subscript $i \in \{1, \dots, t\}$ such that $c_i = u_1 \vee_K u_2 = v_1 \vee_K v_2$. Since $c_1, \dots, c_t$ is a repetition-free list of the elements of $\mathrm{JRed}(K)$, we have that $u_1 \vee_K u_2 \neq c_j = a_j \vee_K b_j$ for every $j \in \{1, \dots, t\} \setminus \{i\}$. So, for all $j \neq i$, part (ii) gives that $u_1 \vee_L u_2 \neq a_j \vee_L b_j$. But $u_1 \vee_L u_2 \in \mathrm{JRed}(L)$, whence (3.4) gives that $u_1 \vee_L u_2 = a_i \vee_L b_i$. Since the equality $v_1 \vee_L v_2 = a_i \vee_L b_i$ follows in the same way, we conclude that

$u_1 \vee_L u_2 = v_1 \vee_L v_2$, as required. This yields part (iv) and completes the proof of Lemma 3.1. ☐

Note that $a_i \vee_L b_i$ in the proof above can be distinct from c_i; this will be exemplified by Figures 1 and 2.

4. The rest of the proof

In this section, to ease our terminology, let us agree on the following convention. We say that a finite lattice L has *many congruences* if $|\mathrm{Con}(L)| > 2^{|L|-5}$. Otherwise, if $|\mathrm{Con}(L)| \le 2^{|L|-5}$, then we say that L has *few congruences.*

Lemma 4.1. *For every finite lattice L, the following two assertions holds.*

(i) *If $|\mathrm{JRed}(L)| \ge 4$ or $|\mathrm{MRed}(L)| \ge 4$, then L has few congruences.*
(ii) *If $|\mathrm{JRed}(L)| = 3$ and there are $p, q \in \mathrm{J}(L)$ such that $p \ne q$ and $\mathrm{con}(p^-, p) = \mathrm{con}(q^-, q)$, then L has few congruences.*

Proof. Let $n := |L|$. If $|\mathrm{JRed}(L)| \ge 4$, then (3.1) leads to $|\mathrm{J}(L)| \le n - 5$, and it follows by (2.2) that L has few congruences. By duality, this proves part (i). Under the assumptions of (ii), $p \equiv_{\mathrm{con}} q$, and we obtain from (2.1) that $|\mathrm{Q}(L)| \le |\mathrm{J}(L)| - 1 = n - 4 - 1 = n - 5$, and (2.2) implies again that L has few congruences. This proves the lemma. ☐

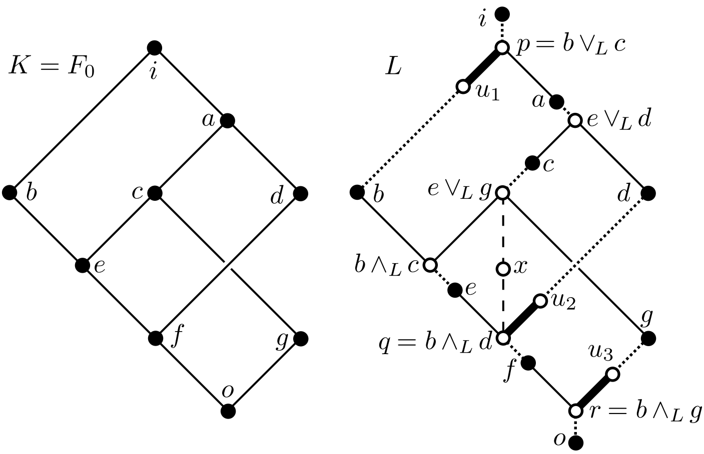

FIGURE 1. $K = F_0$ and an example for L containing K as a subposet

Lemma 4.2. *Let $K := F_0 \in \mathcal{L}_{\mathrm{KR}}$, see on the left in Figure 1. If K is a subposet of a finite lattice L, then L has few congruences.*

Proof. Label the elements of $K = F_0$ as shown in Figure 1. A possible L is given on the right in Figure 1; the elements of K are black-filled. The diagram of L is understood as follows: for $y_1, y_2 \in L$, a *thick solid edge*, a *thin solid* *edge*, and a *thin dotted edge* ascending from y_1 to y_2 mean that, in the general case, we know that $y_1 \prec y_2$, $y_1 < y_2$, and $y_1 \leq y_2$, respectively. In a concrete situation, further relations can be fulfilled; for example, a thin dotted edge can happen to denote that $y_1 = y_2$. The two dashed edges and the element x as well as similar edges and elements can be present but they can also be missing. Note that $y_1 \leq y_2$ is understood as $y_1 \leq_L y_2$; for $y_1, y_2 \in K$, this is the same as $y_1 \leq_K y_2$ since K is a subposet of L. Since L in the figure carries a lot of information on the general case, the reader may choose to inspect Figure 1 instead of checking some of our computations that will come later. Note also that the convention above applies only for L; for K, every edge in the left of Figure 1 stands for covering.

Clearly, $|\text{JRed}(K)| = |\text{MRed}(K)| = 3$. Hence, Lemma 3.1(iii) gives that $|\text{JRed}(L)| \geq 3$ and $|\text{MRed}(L)| \geq 3$. We can assume that none of $|\text{JRed}(L)| \geq 4$ and $|\text{MRed}(L)| \geq 4$ holds, because otherwise Lemma 4.1(i) would immediately complete the proof. Hence,

$$|\text{JRed}(L)| = 3 \quad \text{and} \quad |\text{MRed}(L)| = 3. \tag{4.1}$$

Since $|\text{JRed}(K)| = |\text{MRed}(K)| = 3$ holds also for $K = E_0$, to be given later in Figure 2, note at this point that (4.1) will be valid in the proof of Lemma 4.3. Let $p := b \vee_L c \in L$, and let $u_1 \in L$ be a lower cover of p in the interval $[b, p]_L$. Also, let $q := b \wedge_L d$ and let $u_2 \in [q, d]_L$ be a cover of q. Finally, let $r := b \wedge_L g$, and let $u_3 \in [r, g]_L$ be a cover of r. Since we have formed the joins and the meets of incomparable elements in L such that the corresponding joins are pairwise distinct in K and the same holds for the meets, (4.1) and Lemma 3.1 imply that

$$\text{JRed}(L) = \{p, e \vee_L d, e \vee_L g, \} \quad \text{and} \quad \text{MRed}(L) = \{q, r, b \wedge_L c\}. \tag{4.2}$$

In order to justify some features of Figure 1, note that (4.2) implies easily that $a < p$, $c < e \vee_L d$, $q < e$, and $r < f$, but we will not use these inequalities. For example, we obtain $a < p$ as follows. Since $b \not\leq a$, we have that $p \not\leq a$. For the sake of contradiction, suppose that $a \parallel p$. Then $e \vee_L g \leq b \vee_L c = p < a \vee_L p$ and, by Lemma 3.1(i), $e \vee_L d \leq e \vee_K d = a < a \vee_L p$, whereby (4.2) gives that $a < a \vee_L p$ is strictly larger than every element of $\text{JRed}(L)$, which contradicts $a \vee_L p \in \text{JRed}(L)$. Hence, $a < p$.

Since $u_1 \prec_L p$, $u_1 \neq p$. If we had that $u_1 = e \vee_L d$, then

$$b \leq u_1 = e \vee_L d \overset{3.1(\text{i})}{\leq} e \vee_K d = a$$

would contradict $b \not\leq_K a$. Replacing $\langle d, a\rangle$ by $\langle g, c\rangle$, we obtain similarly that $u_1 \neq e \vee_L g$. Hence, (4.2) gives that $u_1 \in \text{J}(L)$. If we had that $u_2 = p$, then $b \leq p = u_2 \leq d$ would be a contradiction. Similarly, $u_2 = e \vee_L d$ would lead to $e \leq e \vee_L d = u_2 \leq d$ while $u_2 = e \vee_L g$ again to $e \leq e \vee_L g = u_2 \leq d$, which are contradictions. Hence, $u_2 \notin \text{JRed}(L)$ and so $0_L \leq q \prec_L u_2$ gives that $u_2 \in \text{J}(L)$. We have that $u_3 \neq p$, because otherwise $b \leq p = u_3 \leq g$ would be a contradiction. Similarly, $u_3 = e \vee_L d$ and $u_3 = e \vee_L g$ would lead to the contradictions $e \leq e \vee_L d = u_3 \leq g$ and $e \leq e \vee_L g = u_3 \leq g$, respectively. So,

$u_3 \notin \mathrm{JRed}(L)$ by (4.2). Since $r \prec_L u_3$ excludes that $u_3 = 0$, we obtain that $u_3 \in \mathrm{J}(L)$. Since $u_3 = u_2$ would lead to

$$f = b \wedge_K d \overset{3.1(\mathrm{i})^{\mathrm{d}}}{\leq} b \wedge_L d = q \leq u_2 = u_3 \leq g, \tag{4.3}$$

which is a contradiction, we have that

$$u_1, u_2, u_3 \in \mathrm{J}(L) \quad \text{and} \quad u_2 \neq u_3. \tag{4.4}$$

Next, we claim that

$$[q, u_2] \nearrow [u_1, p] \quad \text{and} \quad [u_1, p] \searrow [r, u_3]. \tag{4.5}$$

Since $b \nleq a$, $b \nleq c$, and $b \nleq d$, none of $e \vee_L d$, $e \vee_L g$, and u_2 belongs to $[b, i]_L$. In particular, we obtain from $u_1 \prec p$ and (4.2) that

$$[b, u_1]_L \subseteq \mathrm{J}(L) \quad \text{and} \quad b \nleq u_2. \tag{4.6}$$

Suppose, for a contradiction, that $u_2 \leq u_1$, and pick a maximal chain in the interval $[u_2, u_1]$. So we pick a lower cover of u_1, then a lower cover of the previous lower cover, etc., and it follows from (4.6) that this chain contains b. Hence, $u_2 \leq b$, and we obtain that $q \prec_L u_2 \leq b \wedge_L d = q$, a contradiction. Hence, $u_2 \nleq u_1$. This means that $u_1 \wedge_L u_2 < u_2$. But $q \leq b \leq u_1$, so we have that $q \leq u_1 \wedge u_2 < u_2$. Since $q \prec_L u_2$, we obtain that $u_1 \wedge_L u_2 = q$. Similarly, $u_2 \leq d \leq p$ and $u_2 \nleq u_1$ give that $u_1 < u_1 \vee_L u_2 \leq p$, whereby $u_1 \prec_L p$ yields that $u_1 \vee_L u_2 = p$. The last two equalities imply the first half of (4.5). The second half follows basically in the same way, so we give less details. Based on (4.6), $u_3 \leq u_1$ would lead to $u_3 \leq b$ and $r \prec_L u_3 \leq b \wedge_L g = r$, whence $u_3 \nleq u_1$. Since $u_3 \leq g \leq c \leq b \vee_L c = p$ and $r = b \wedge_L g \leq b \leq u_1$, we obtain that $u_1 < u_1 \vee_L u_3 \leq p$ and $r \leq u_1 \wedge_L u_3 < u_3$. Hence the covering relations $u_1 \prec_L p$ and $r \prec_L u_3$ imply the second half of (4.5).

Finally, (2.4) and (4.5) give that $\mathrm{con}(q, u_2) = \mathrm{con}(u_1, p) = \mathrm{con}(r, u_3)$. Since (4.4) allows us to replace q and r by u_2^- and u_3^-, respectively, we obtain that $\mathrm{con}(u_2^-, u_2) = \mathrm{con}(u_3^-, u_3)$. But u_2 and u_3 are distinct elements of $\mathrm{J}(L)$ by (4.4), whereby (4.1) and Lemma 4.1(ii) imply that L has few congruences, as required. This completes the proof of Lemma 4.2. □

We still need another lemma.

Lemma 4.3. *Let $K := E_0 \in \mathcal{L}_{\mathrm{KR}}$, see on the left in Figure 2. If K is a subposet of a finite lattice L, then L has few congruences.*

Proof. This proof shows a lot of similarities with the earlier one. In particular, the same convention applies for the diagram of L in Figure 2 and, again, there can be several elements of L not indicated in the diagram. We have already noted that (4.1) holds in the present situation. Figure 2 shows how to pick $u_1, u_2 \in L$; they are covers of $a \wedge_L b$ in $[a \wedge_L b, a]$ and $b \wedge_L c$ in $[b \wedge_L c, c]$, respectively. As a counterpart of (4.2) and the paragraph following it, now we obtain in the same way from (4.1) and Lemma 3.1 that the comparabilities

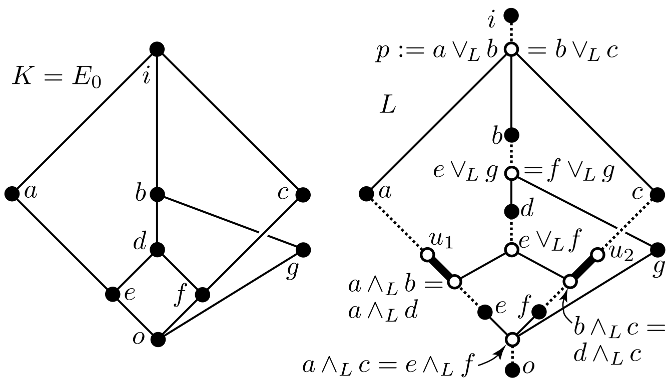

FIGURE 2. $K = E_0$ and an example for L containing K as a subposet

and incomparabilities on the right of Figure 2 are correctly depicted and

$$\left.\begin{aligned} \mathrm{JRed}(L) &= \{p := a \vee_L b = b \vee_L c,\ e \vee_L f,\ e \vee_L g = f \vee_L g\} \text{ and} \\ \mathrm{MRed}(L) &= \{a \wedge_L b = a \wedge_L d,\ b \wedge_L c = d \wedge_L c,\ a \wedge_L c = e \wedge_L f\}. \end{aligned}\right\} \tag{4.7}$$

Neither all the equalities above, nor all similar equalities like $d \wedge_L g = e \wedge_L f$, nor all features of the figure will be used, and there can be many more elements not indicated. Using (4.7) in the same way as we used (4.2) in the proof of Lemma 4.2 and the above-mentioned correctness of Figure 2, it follows that

$$\left.\begin{array}{l} [a \wedge_L b, a] \setminus \{a \wedge_L b\} \subseteq \mathrm{Irr}(L) \text{ and } [a, p] \setminus \{p\} \subseteq \mathrm{Irr}(L), \\ \text{whereby } u_1 \in \mathrm{J}(L) \text{ and } [u_1, p] \text{ is a chain.} \end{array}\right\} \tag{4.8}$$

Similarly to the argument verifying (4.5) (but p need not cover b and we need to use that $[u_1, p]$ is a chain), (4.8) implies that $[u_1^-, u_1] = [a \wedge_L b, u_1] \nearrow [b, p]$. Since $\langle a, u_1 \rangle$ and $\langle c, u_2 \rangle$ play symmetric roles, we obtain that $u_2 \in \mathrm{J}(L)$ and $[u_2^-, u_2] = [b \wedge_L c, u_2] \nearrow [b, p]$. Hence, (2.4) gives that $\mathrm{con}(u_1^-, u_1) = \mathrm{con}(b, p) = \mathrm{con}(u_2^-, u_2)$. Since u_1 and u_2 are distinct by Figure 2 and they belong to $\mathrm{J}(L)$ by (4.8) and the $\langle a, u_1 \rangle$–$\langle c, u_2 \rangle$-symmetry, (4.1) and Lemma 4.1(ii) imply that L has few congruences. This completes the proof of Lemma 4.3. □

Now, we are in the position to prove our theorem.

Proof of Theorem 1.1. Let L be an arbitrary non-planar finite lattice; it suffices to show that L has few congruences. By Proposition 2.1, there is a lattice K in Kelly and Rival's list $\mathcal{L}_{\mathrm{KR}}$ such that K is a subposet of L or the dual L^{dual} of L. Since $\mathrm{Con}(L^{\mathrm{dual}}) = \mathrm{Con}(L)$ and L^{dual} is non-planar either, we can assume that K is a subposet of L. A quick glance at the lattices of $\mathcal{L}_{\mathrm{KR}}$, see their diagrams in Kelly and Rival [9], shows that if $K \in \mathcal{L}_{\mathrm{KR}} \setminus \{E_0, F_0\}$, then $|\mathrm{JRed}(K)| \geq 4$ or $|\mathrm{MRed}(K)| \geq 4$. Hence, if $K \in \mathcal{L}_{\mathrm{KR}} \setminus \{E_0, F_0\}$, then Lemma 4.1(i) implies that L has few congruences, as required. If $K \in \{E_0, F_0\}$, then the same conclusion is obtained by Lemmas 4.2 and 4.3. This completes the proof of Theorem 1.1. □

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