LATTICES EMBEDDABLE IN THREE-GENERATED LATTICES

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Dedicated to the memory of E. Tamás Schmidt

ABSTRACT. We prove that every finite lattice L can be embedded in a three-generated *finite* lattice K. We also prove that every *algebraic* lattice with accessible cardinality is a *complete* sublattice of an appropriate *algebraic* lattice K such that K is completely generated by three elements. Note that ZFC has a model in which all cardinal numbers are accessible. Our results strengthen P. Crawley and R. A. Dean's 1959 results by adding finiteness, algebraicity, and completeness.

DEDICATION TO E. TAMÁS SCHMIDT'S TWO BEST PAPERS IN ACTA SCIENTIARUM MATHEMATICARUM (SZEGED)

This section is a tribute to *E. Tamás Schmidt* (1936–2016). Note that there will be a memorial issue dedicated to him in Algebra Universalis with more details. Note also that his carefully edited website at `http://www.math.bme.hu/~schmidt/` is still available and will be available for a long time; it is the best source to keep his memory alive.

On February 24, 2016, the Institute of Mathematics at the Budapest University of Technology and Economics, jointly with the Alfréd Rényi Institute of Mathematics, celebrated Professor Emeritus E. Tamás Schmidt's eightieth birthday. Besides the honoree, many Hungarian algebraists and even some foreign colleagues took part in this event. Professor Schmidt was an outstanding lattice theorist with many brilliant mathematical ideas and deep results, and he was also a kind, friendly person, respected and liked by all of us. Soon after this celebration, we were shocked by the sad news: on March 14, 2016, E. Tamás Schmidt passed away.

Professor Schmidt was my scientific advisor in 1984, when I obtained my CSc (in today's terminology, PhD) degree. From 1986 to 2000, he supported most Hungarian algebraists, including me, from his famous "mammoth OTKA" (Hungarian Scientific Research Fund) projects. Not much later, Tamás became my most frequent coauthor. Our collaboration was particularly fruitful in 2008–2013, when twelve of our thirteen joint papers appeared.

Although Tamás lived in Budapest, he had good connections to the Bolyai Institute and its mathematical journal, Acta Scientiarum Mathematicarum. In 2008, he was awarded the Béla Szőkefalvi-Nagy Medal of the Bolyai Institute; see `http://www.acta.hu/acta/`. He has published eighteen papers in Acta Sci. Math..

Date: December 12, 2015; last revision October 18, 2016.

2000 *Mathematics Subject Classification.* Primary 06B99, secondary 06B15.

Key words and phrases. Three-generated lattice, equivalence lattice, partition lattice, complete lattice embedding, inaccessible cardinal.

This research was supported by NFSR of Hungary (OTKA), grant number K 115518.

Below, I mention only two of these papers: in my view, the two most important ones.

In 1963, Grätzer and Schmidt [11] proved that every algebraic lattice can be represented as the congruence lattice $\mathrm{Con}(A)$ of an abstract algebra A. This very important theorem, the *Grätzer-Schmidt theorem*, became well-known soon for all universal algebraists and lattice theorists; now it belongs to the foundations of these branches of mathematics. Note that even the present paper, whose topic seems to be far from congruence lattices, uses the Grätzer-Schmidt theorem.

In 1981, Schmidt [21] proved that if D is a distributive algebraic lattice in which the intersection of any two compact elements is compact, then D can be represented as the congruence lattice $\mathrm{Con}(L)$ of a lattice L. For every lattice L, we know from a 1942 result of Funayama and Nakayama [8] that $\mathrm{Con}(L)$ is a distributive algebraic lattice. The question whether every distributive algebraic lattice can be represented in this way was (*Dilworth's*) *Congruence Lattice Problem*, CLP in short. CLP was one of the most significant problems on lattices for more than half a century; see Grätzer [10] for an overview. Wehrung [24] provided a *negative answer*. However, there are many *positive results* stating that certain distributive algebraic lattices D can be represented; Schmidt [21] is the deepest of them. Note that Huhn [12] published another positive result on CLP in Acta Sci. Math., which cannot be sharpened by Růžička [20].

We will cherish E. Tamás Schmidt's memory.

1. Introduction and result

A lattice K is *three-generated* if there are $a, b, c \in K$ such that for every sublattice S of K, $\{a, b, c\} \subseteq S$ implies that $S = K$. We know from Crawley and Dean [2, Theorem 7] that every at most countably infinite lattice is a sublattice of a three-generated lattice. A complete lattice K is *completely* generated by a subset X if the only complete sublattice of K including X is K itself. By [2, Theorem 7] again, every lattice is a sublattice of a complete lattice completely generated by three elements. Our aim is to strengthen these statements by proving that every finite lattice is a sublattice of a *finite* three-generated lattice and every algebraic lattice of accessible cardinality is a *complete* sublattice of a completely three-generated algebraic lattice; see Corollaries 1.3 and 1.4 stated later in this section. Our method is entirely different from the one used by Crawley and Dean [2]. Actually, we are going to prove a theorem on equivalence lattices; our theorem combined with deep results from the literature will easily imply the corollaries.

Next, we recall some well-known definitions. An element a in a complete lattice L is *compact* if whenever $a \leq \bigvee X$, then X has a finite subset Y with $a \leq \bigvee Y$. A complete lattice is *algebraic* if each of its elements is the join of (possibly, infinitely many) compact elements. Most lattices related to algebraic and other mathematical structures are algebraic lattices. A cardinal κ is *inaccessible* if

(i) $\kappa > \aleph_0$,
(ii) for every cardinal λ, $\lambda < \kappa$ implies that $2^\lambda < \kappa$, and
(iii) for every set I of cardinals, if $|I| < \kappa$ and each member of I is less than κ, then $\sum\{\lambda : \lambda \in I\} < \kappa$.

For convenience, a cardinal λ will be called *accessible* if there is no inaccessible cardinal κ such that $\kappa \leq \lambda$. Our terminology is slightly different from the one used in Keisler and Tarski [14], since finite cardinals are accessible here. Note that there can be cardinals that are neither accessible, nor inaccessible. Inaccessible cardinals are also called strongly inaccessible. Inaccessible cardinals, if exist, are extremely large; see, for example, Kanamori [13, page 18] or Levy [16, pages 138–141]. The "everyday's cardinals" like $0, 1, 2, 3, \ldots$, $\aleph_0$, $\aleph_1$, $\aleph_2$, $\aleph_3$, $2^{\aleph_0}$, $2^{2^{\aleph_0}}$, etc. are accessible. We know from Kuratowski [15], see also [13, page 18] or [16], that ZFC has a model without inaccessible cardinals. That is, in some model of ZFC, all cardinals are accessible and belong to the scope of our results. Given a set A, the lattice of all equivalence relations on A with respect to set inclusion is denoted by $\mathrm{Equ}(A) = \langle \mathrm{Equ}(A); \subseteq \rangle$; it is called the *equivalence lattice* over A. Now, we are in the position to formulate the main result of the paper.

Theorem 1.1. *For every set A of accessible cardinality $|A| \geq 3$, there exist a set B and a complete sublattice K of* $\mathrm{Equ}(B)$ *such that K is completely generated by three of its elements and* $\mathrm{Equ}(A)$ *is isomorphic to a complete sublattice of K. Furthermore, we can choose a finite B if A is finite, and we can let $B = A$ otherwise.*

The role of $|A| \geq 3$ in Theorem 1.1 is to ensure that $|\mathrm{Equ}(A)| \geq 3$. Note that a finite lattice is completely generated by three elements if and only if it is three-generated in the usual sense. Note also that K in the theorem is necessarily a proper complete sublattice of $\mathrm{Equ}(B)$ if $|B| > 3$, because $\mathrm{Equ}(B)$ cannot be completely generated by three elements; see Strietz [23] or Zádori [25] for the finite case, and see the paragraph following the theorem in [3] for the infinite case. For the sake of another terminology, we rephrase Theorem 1.1 as follows. A *complete lattice embedding* is an injective map preserving arbitrary (possibly infinite) meets and joins.

Proposition 1.2. *If A is a set and $|A| \geq 3$ is an accessible cardinal, then* $\mathrm{Equ}(A)$ *has a complete lattice embedding in a complete sublattice K of some* $\mathrm{Equ}(B)$ *such that K is completely generated by only three elements. We can let $B = A$ if A is infinite, and we can choose a finite B if A is finite.*

Corollary 1.3. *Every finite lattice is the sublattice of some three-generated* finite *lattice.*

To point out the difference between this corollary and the afore-mentioned Crawley and Dean [2, Theorem 7], note that Corollary 1.3 refers to a construction that preserves finiteness. On the other hand, as opposed to [2, Theorem 7], our corollary does not include the case when the original lattice is countably infinite.

Proof of Corollary 1.3. By Pudlák and Tůma [19], we can embed our lattice in some $\mathrm{Equ}(A)$ such that A is finite. Thus, Theorem 1.1 is applicable. $\square$

Corollary 1.4. *Every algebraic lattice of accessible cardinality is a* complete *sublattice of an* algebraic *lattice K that is completely generated by three elements.*

Theorem 7 for complete lattices in Crawley and Dean [2] considers the "degree" of completeness, so it is more involved than its simplified form given at the beginning of this section. However, Corollary 1.4 adds two new features even to the original [2, Theorem 7]: here the sublattice is a *complete* sublattice and K is an *algebraic* lattice. Note that we can easily derive the simplified form of [2, Theorem 7] for a lattice L with accessible cardinality from Corollary 1.4 as follows: embed L in its *ideal lattice* $\mathfrak{I}(L)$ and apply Corollary 1.4 to $\mathfrak{I}(L)$; in this way, we obtain that L is (isomorphic to) a sublattice of K, where K is from Corollary 1.4.

Conversely, we do not see any straightforward way to derive Corollary 1.4 directly from [2, Theorem 7]. In particular, if K is a complete lattice completely generated by $\{x, y, z\}$, then there seems to be no reason why the algebraic lattice $\mathfrak{I}(K)$ should be completely generated by the set $\{\downarrow x, \downarrow y, \downarrow z\}$ of principal ideals, because $a = \bigvee\{b_i : i \in I\}$ in K need not imply that $\downarrow a$ equals $\bigvee\{\downarrow b_i : i \in I\} = \bigcup\{\bigvee\{\downarrow b_j : j \in J\} : J \subseteq I$ and J is finite$\}$.

Proof of Corollary 1.4. By the Grätzer–Schmidt theorem, see [11], we can assume that our lattice L is the congruence lattice of an algebra A. So L is a complete sublattice of $\mathrm{Equ}(A)$. Since $|L|$ is accessible, the construction in [11] shows that $|A|$ is also accessible. (This is not surprising, since the class of sets with accessible cardinalities is closed under "reasonable" constructions.) Hence, Theorem 1.1 yields that L is a complete sublattice of a complete sublattice K of some $\mathrm{Equ}(B)$ such that K is completely generated by three elements. It is well known, see Grätzer and Schmidt [11, Theorem 8] or Nation [18, Exercise 3.6], that complete sublattices of algebraic lattices are algebraic. Therefore, since $\mathrm{Equ}(B)$ is an algebraic lattice, so is K, as required. $\square$

1.1. **Outline, prerequisite, and method.** The rest of the paper is devoted to the proof of Theorem 1.1.

Only basic knowledge of lattice theory is assumed. For example, a small part of each of the books Burris and Sankappanavar [1], Davey and Priestley [7], Grätzer [9], McKenzie, McNulty, and Taylor [17], and Nation [18] is sufficient.

Our approach towards Theorem 1.1 has three main ingredients but two of them are not explicitly stated. First, we need the fact that $\mathrm{Equ}(A)$ is completely generated by four elements if $|A|$ is accessible; see [3] and [4] for the infinite case and Strietz [23] and Zádori [25] for the finite case. Second, [5] and [6] give the vague idea that we need some auxiliary graphs. Third, the appropriate graphs given in Figure 1 are taken from Skublics [22]. His graphs are symmetric; this explains why they are more appropriate for our plan than those in [5]. (Actually, we have not checked whether the graphs from [5] could be used here.) Note that we cannot use the statements of [5], [6], and [22] in the present setting, because [5] and [22] are related only to the particular case $|A_0| = 2$ of Lemma 2.2. Hence, we borrow only the ideas and some methods from these sources without explicit further reference.

2. Auxiliary statements and the proof

2.1. **An easy lemma.** The least equivalence and the largest equivalence on a set X are denoted by Δ_X and ∇_X, respectively. For $Y \subseteq X$ and an equivalence $\Theta \in \mathrm{Equ}(X)$, the restriction $\Theta \cap Y^2$ of Θ to Y will be denoted by $\Theta\rceil_Y$.

Lemma 2.1. *For a non-empty subset A of a set B, let $\Theta \in \mathrm{Equ}(B)$ be such that $\Theta\rceil_A = \Delta_A$. Then the map $g\colon \mathrm{Equ}(A) \to \mathrm{Equ}(B)$, defined by $\mu \mapsto (\mu \cup \Delta_B) \vee \Theta$, where the join is taken in $\mathrm{Equ}(B)$, is a complete lattice embedding.*

Proof. Let $\boldsymbol{\varepsilon}_i \in \mathrm{Equ}(B)$ for $i \in I$. For $u, v \in B$, by a $(\bigcup_{i\in I} \boldsymbol{\varepsilon}_i)$-*sequence* of length n from u to v we mean a sequence $x_0 = u$, $x_1, \dots, x_{n-1}$, $x_n = v$ of elements of B such that $\langle x_{j-1}, x_j\rangle \in \bigcup_{i\in I} \boldsymbol{\varepsilon}_i$ for all $j \in \{1, \dots, n\}$. For $\mu \in \mathrm{Equ}(A)$, since $(\mu \cup \Delta_B) \setminus \mu \subseteq \Theta$, $\langle u, v\rangle \in g(\mu)$ iff there is a $(\Theta \cup \mu)$-sequence from u to v. In order to show that g preserves joins, assume that $\mu_i \in \mathrm{Equ}(A)$ for $i \in I$. We need to show that $g(\bigvee_{i\in I} \mu_i) = \bigvee_{i\in I} g(\mu_i)$. Actually, since g is clearly order-preserving, it suffices to show that $g(\bigvee_{i\in I}\mu_i)\le\bigvee_{i\in I}g(\mu_i)$. Let $\langle u,v\rangle\in g(\bigvee_{i\in I}\mu_i)$. This is witnessed by a $(\Theta\cup\bigvee_{i\in I}\mu_i)$-sequence from u to v. By the description of joins in $\mathrm{Equ}(A)$, there is a (usually longer) $(\Theta\cup\bigcup_{i\in I}\mu_i)$-sequence from u to v, which is also a $(\bigcup_{i\in I}(\Theta\cup\mu_i))$-sequence and, thus, a $(\bigcup_{i\in I}g(\mu_i))$-sequence. Hence, $\langle u,v\rangle\in\bigvee_{i\in I}g(\mu_i)$, and we conclude that g preserves joins.

For $\mu\in\mathrm{Equ}(A)$, instead of using long $(\mu\cup\Theta)$-sequences, we can describe $g(\mu)$ as follows. Namely, we claim that for distinct $u,v\in B$, we have that $\langle u,v\rangle\in g(\mu)$ if and only if one of the following five possibilities holds:

$$\begin{cases} u\in A,\ v\in A,\ \text{and } \langle u,v\rangle\in\mu;\\ u\notin A,\ v\in A,\ \text{and } (\exists x\in A)\,(\langle u,x\rangle\in\Theta \text{ and } \langle x,v\rangle\in\mu);\\ u\in A,\ v\notin A,\ \text{and } (\exists y\in A)\,(\langle u,y\rangle\in\mu \text{ and } \langle y,v\rangle\in\Theta);\\ u\notin A,\ v\notin A,\ \text{and } (\exists\langle x,y\rangle\in\mu)\,(\langle u,x\rangle\in\Theta \text{ and } \langle y,v\rangle\in\Theta);\\ u\notin A,\ v\notin A,\ \text{and } \langle u,v\rangle\in\Theta. \end{cases} \tag{2.1}$$

The "if part" is trivial. In order to prove the "only if" part, assume that $u\neq v$ are elements of B and $\langle u,v\rangle\in g(\mu)$. Take a *shortest* $(\mu\cup\Theta)$-sequence P: $x_0=u$, $x_1,\dots,x_{n-1}$, $x_n=v$ from u to v. Since $u\neq v$, this sequence is repetition-free, that is, $|\{x_0,\dots,x_n\}|=n+1$. Hence, by $\mu\cap\Theta=\mu\cap\Theta\rceil_A=\mu\cap\Delta_A=\Delta_A$, each pair $\langle x_{i-1},x_i\rangle$ of two consecutive elements belongs either to μ, or to Θ, but not to both. Let $\pi_i=\mu$ if $\langle x_{i-1},x_i\rangle\in\mu$, and let $\pi_i=\Theta$ otherwise. The sequence $\vec\pi:=\langle\pi_1,\pi_2,\dots,\pi_n\rangle$ is the *pattern* of P. Since P is a shortest sequence from u to v, neither $\langle\mu,\mu\rangle$, nor $\langle\Theta,\Theta\rangle$ is a subsequence of $\vec\pi$. If a pair belongs to $\Theta\setminus\Delta_B$, then at least one of its components is outside A. This gives that $\langle\mu,\Theta,\mu\rangle$ is not a subsequence of $\vec\pi$. Hence, $\vec\pi$ is one of the patterns $\langle\mu\rangle$, $\langle\Theta,\mu\rangle$, $\langle\mu,\Theta\rangle$, $\langle\Theta,\mu,\Theta\rangle$, and $\langle\Theta\rangle$. These patterns correspond to the possibilities (that is, lines) in (2.1). This proves (2.1). Observe that, by the transitivity of Θ and the assumption $\Theta\rceil_A=\Delta_A$,

$$\text{for each } b\in B\setminus A,\text{ there is at most one } a\in A \text{ such that } \langle b,a\rangle\in\Theta. \tag{2.2}$$

Next, in order to show that g preserves meets, let $\mu_i\in\mathrm{Equ}(A)$ for $i\in I$. It suffices to show that $\bigwedge_{i\in I}g(\mu_i)\le g(\bigwedge_{i\in I}\mu_i)$, because the converse inequality follows from the fact that g is order-preserving. So let $\langle u,v\rangle\in\bigwedge_{i\in I}g(\mu_i)$. We can assume that $\langle u,v\rangle\notin\Theta$, since otherwise $\langle u,v\rangle\in g(\bigwedge_{i\in I}\mu_i)$ is trivial. According to (2.1), there are four possibilities; we deal only with the case $u\notin A$ and $v\notin A$ since the rest of the cases are even more simple. For each $i\in I$, (2.1) gives a pair $\langle x_i,y_i\rangle\in\mu_i$ such that $\langle u,x_i\rangle\in\Theta$ and $\langle y_i,v\rangle\in\Theta$. (2.2) yields that neither x_i, nor y_i depends on i. Hence, there is a common member $\langle x,y\rangle$ of all μ_i such that $\langle u,x\rangle$ and $\langle y,v\rangle$ are in Θ. Thus, $\langle x,y\rangle\in\bigwedge_{i\in I}\mu_i$, implying that $\langle u,v\rangle\in g(\bigwedge_{i\in I}\mu_i)$. Therefore, g preserves the meets.

Finally, for $\mu\in\mathrm{Equ}(A)$, (2.1) yields that $g(\mu)\rceil_A=\mu$. Hence, g is injective, completing the proof of Lemma 2.1. $\square$

Note that an earlier version of this paper proved essentially the same lemma but with a different approach. Namely, an isomorphism (which turns out to be $g^{-1}\colon g(\mathrm{Equ}(A))\to\mathrm{Equ}(A)$ in the present setting) was constructed as the composite of the usual isomorphism given by the folkloric Correspondence Theorem from, say, Burris and Sankappanavar [1, Theorem 6.20] and another isomorphism. Although the present elementary proof is not shorter, it seems to be easier to follow.

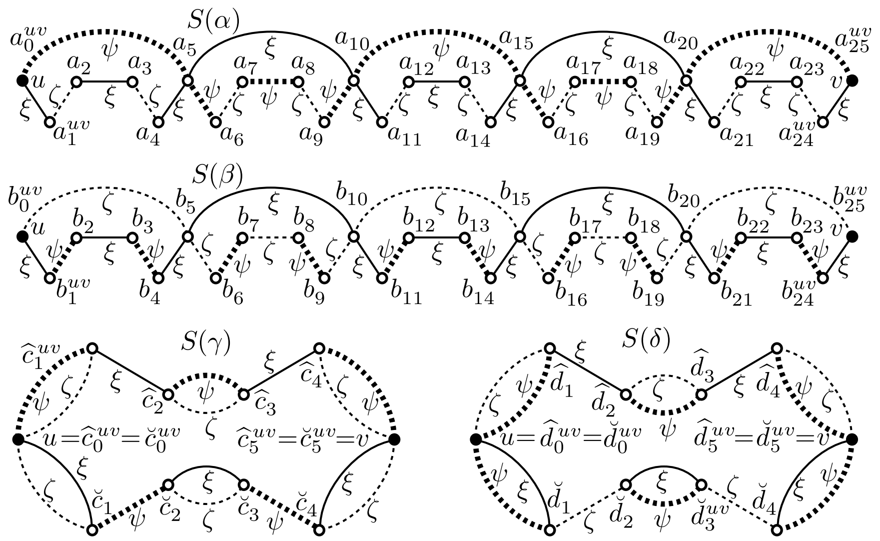

FIGURE 1. The auxiliary graphs

2.2. **Blowing equivalence lattices up with auxiliary graphs.** The graphs $S(\alpha)$, …, $S(\delta)$ given in Figure 1 (but here we consider them without the superscripts uv) will be called *auxiliary graphs*; they are the key gadgets in our construction. Vertices distinct from u and v in these graphs are called *internal vertices* while u and v are said to be *side vertices*. Before formulating a lemma on these graphs, we need some easy definitions. Let A_0 be a set and let $\alpha_0, \beta_0, \gamma_0, \delta_0 \in \mathrm{Equ}(A_0)$. We define a larger set A_1 and equivalences $\xi_1, \psi_1, \zeta_1 \in \mathrm{Equ}(A_1)$ as follows. First, we fix a well-ordering on A_0 for convenience. A pair $\langle u, v\rangle \in A_0^2$ is a *nontrivial pair* if $u \neq v$. If $\langle u, v\rangle \in A_0^2$ is a nontrivial pair such that u precedes v with respect to the fixed well-ordering, then $\langle u, v\rangle \in A_0^2$ is said to be an *eligible pair*. For each eligible pair $\langle u, v\rangle \in \alpha_0$, we insert a copy of $S(\alpha)$ such that its black-filled left vertex is identified with u while its black-filled right vertex with v. The vertices of $S(\alpha)$ are $a_0, a_1, \dots, a_{25}$.

After inserting a copy of this graph for $\langle u, v\rangle$, its vertices are denoted by $a_0^{uv} = u, a_1^{uv}, \dots, a_{24}^{uv}, a_{25}^{uv} = v$. The vertices $a_1^{uv}, \dots, a_{24}^{uv}$ are new elements; they are neither in A_0, nor in any other copy of an auxiliary graph that we add to A_0. Note that the superscript uv is indicated at some vertices like a_1^{uv} in Figure 1 but, in absence of space, not always. If we drop the superscripts uv, then the figure gives $S(\alpha)$; if we add these superscripts, then we obtain the actual copy $S^{uv}(\alpha)$ of $S(\alpha)$ that we have inserted for $\langle u, v\rangle$. Similarly, for $\boldsymbol{\varepsilon} \in \{\beta, \gamma, \delta\}$ and each eligible pair $\langle u, v\rangle \in \boldsymbol{\varepsilon}_0$, we insert a copy $S^{uv}(\boldsymbol{\varepsilon})$ of $S(\boldsymbol{\varepsilon})$ such that the left and right black-filled vertices are identified with u and v, respectively. Again, the superscripts ensure that, apart possibly from the black-filled vertices, we insert disjoint copies. After all these insertions, we obtain A_1, which is a superset of A_0. The edges of our auxiliary graphs are colored by ξ, ψ, and ζ. These edges and their colors are also added to A_0. So A_1 becomes a graph whose edges are colored by ξ, ψ, and ζ. This graph is denoted by A_1, in the same way as its vertex set. Note at this point that whenever the membership relation "$\in$" or the inclusion relation "$\subseteq$" is used in connection with a graph, we always mean the *vertex set* of the graph in question. Multiple edges between two vertices of A_1 may occur, and each edge has a unique color in $\{\xi, \psi, \zeta\}$. We define $\xi_1 \in \mathrm{Equ}(A_1)$ by the rule $\langle x, y\rangle \in \xi_1$ iff there is a path connecting x and y in the graph A_1 such that every edge of this path is ξ-colored. We define $\psi_1 \in \mathrm{Equ}(A_1)$ and $\zeta_1 \in \mathrm{Equ}(A_1)$ analogously by using ψ-colored paths and ζ-colored paths, respectively. Consider the ternary lattice terms

$$\begin{aligned} \widehat{\alpha} &= \big(\xi \wedge (\psi \vee \zeta)\big) \vee \big(\psi \wedge (\xi \vee \zeta)\big),\\ \widehat{\beta} &= \big(\xi \wedge (\zeta \vee \psi)\big) \vee \big(\zeta \wedge (\xi \vee \psi)\big),\\ \widehat{\gamma} &= \big(\xi \vee (\psi \wedge \zeta)\big) \wedge \big(\psi \vee (\xi \wedge \zeta)\big), \text{ and}\\ \widehat{\delta} &= \big(\xi \vee (\zeta \wedge \psi)\big) \wedge \big(\zeta \vee (\xi \wedge \psi)\big). \end{aligned} \tag{2.3}$$

Observe at this point that if we interchange ψ and ζ in our setting, then

$$\langle \widehat{\alpha}, \widehat{\beta}, \widehat{\gamma}, \widehat{\delta}, S(\alpha), S(\beta), S(\gamma), S(\delta)\rangle \ \text{ and } \ \langle \widehat{\beta}, \widehat{\alpha}, \widehat{\delta}, \widehat{\gamma}, S(\beta), S(\alpha), S(\delta), S(\gamma)\rangle$$

are also interchanged. We will frequently rely on this fact, called *ψ–ζ-symmetry*. Now, we are in the position to formulate the key lemma towards Theorem 1.1.

Lemma 2.2. *Let A_0 be a set with at least two elements and let $\alpha_0, \beta_0, \gamma_0, \delta_0 \in \mathrm{Equ}(A_0)$ be such that $\alpha_0 \leq \gamma_0 \vee \delta_0$ and $\beta_0 \leq \gamma_0 \vee \delta_0$. Let L_0 be the complete sublattice of $\mathrm{Equ}(A_0)$ completely generated by $\{\alpha_0, \beta_0, \gamma_0, \delta_0\}$. Consider the graph A_1 and the equivalences $\xi_1, \psi_1, \zeta_1 \in \mathrm{Equ}(A_1)$ defined above. For $\boldsymbol{\varepsilon} \in \{\alpha, \beta, \gamma, \delta\}$, let $\widehat{\boldsymbol{\varepsilon}}_1 := \widehat{\boldsymbol{\varepsilon}}(\xi_1, \psi_1, \zeta_1)$ and $\widehat{\boldsymbol{\varepsilon}}_2 := \widehat{\boldsymbol{\varepsilon}}_1 \wedge (\widehat{\gamma}_1 \vee \widehat{\delta}_1)$. Denote by L_2 the complete sublattice of $\mathrm{Equ}(A_1)$ completely generated by $\{\widehat{\alpha}_2, \widehat{\beta}_2, \widehat{\gamma}_2, \widehat{\delta}_2\}$. Then L_0 is isomorphic to L_2. Actually, there is a unique isomorphism $f\colon L_0 \to L_2$ such that, for $\boldsymbol{\varepsilon} \in \{\alpha, \beta, \gamma, \delta\}$, $f(\boldsymbol{\varepsilon}_0) = \widehat{\boldsymbol{\varepsilon}}_2$.*

Proof. Let $\{\rho, \tau\}$ be a two-element subset of $\{\xi, \psi, \zeta\}$. By a *$(\rho \cup \tau)$-path* we mean a path P in the graph A_1 such that each edge of P is ρ-colored or τ-colored. By the description of join in $\mathrm{Equ}(A_1)$ and the definition of ξ_1, ψ_1, and ζ_1, we have that, for $x, y \in A_1$,

(2.4) $\langle x, y\rangle \in \rho_1$ iff there is a ρ-colored path connecting the vertices x and y, and $\langle x, y\rangle \in \rho_1 \vee \tau_1$ iff there is a $(\rho \cup \tau)$-path between x and y.

As Figure 1 shows, none of the four auxiliary graphs has a "monochromatic" path from u to v. Thus, there is a monochromatic path from u to v neither within a single auxiliary graph $S^{uv}(\boldsymbol{\varepsilon})$, nor as a "detour path" through other auxiliary graphs. Therefore, (2.4) implies that

$$\xi_1\rceil_{A_0} = \psi_1\rceil_{A_0} = \zeta_1\rceil_{A_0} = \Delta_{A_0}. \tag{2.5}$$

For $\mu \in \mathrm{Equ}(A_1)$ and $X \subseteq A_1$, we say that X is a *μ-closed* subset of A_1 if the $\mu\rceil_X$-blocks are also μ-blocks. Equivalently, X is μ-closed if for all $x \in X$ and $y \in A_1$, $\langle x, y\rangle \in \mu$ implies that $y \in X$. For $\mu \in \mathrm{Equ}(A_1)$, $\boldsymbol{\varepsilon} \in \{\alpha, \beta, \gamma, \delta\}$, and an eligible pair $\langle u, v\rangle \in \boldsymbol{\varepsilon}_0$, we say that μ *perfectly restricts* to $S^{uv}(\boldsymbol{\varepsilon})$ if whenever B is a block of $\mu\rceil_{S^{uv}(\boldsymbol{\varepsilon})}$ such that $B \cap \{u, v\} = \varnothing$, then B is also a μ-block; equivalently, if every $\mu\rceil_{S^{uv}(\boldsymbol{\varepsilon})}$-block that is disjoint from $\{u, v\}$ is μ-closed. In this case, if μ and $\boldsymbol{\varepsilon}$ are clear from the context, the blocks of $\mu\rceil_{S^{uv}(\boldsymbol{\varepsilon})}$ are simply called the *restricted blocks.* A restricted block B is called a (restricted) *internal block* if $B \cap \{u, v\} = \varnothing$, and B is a (restricted) *side block* otherwise. Let us emphasize that whenever μ perfectly restricts to $S^{uv}(\boldsymbol{\varepsilon})$, then the restricted internal blocks are also μ-blocks but the restricted side blocks need not be μ-blocks. Note that, usually, we do not list the singleton $\mu\rceil_{S^{uv}(\boldsymbol{\varepsilon})}$-blocks in this case. We will often use the following trivial fact.

$$\text{If } B_1 \subseteq B_2 \subseteq S^{uv}(\boldsymbol{\varepsilon}),\ \mu \le \mu' \in \operatorname{Equ}(A_1),\ B_1 \text{ is a block of } \mu\rceil_{S^{uv}(\boldsymbol{\varepsilon})},\ \{u,v\} \cap B_2 = \varnothing, \text{ and } B_2 \text{ is } \mu'\text{-closed, then } B_1 \text{ is } \mu\text{-closed and so it is a } \mu\text{-block.} \tag{2.6}$$

Let $\langle u,v\rangle \in \alpha_0$ be an eligible pair. By (2.4), the $(\psi_1 \vee \zeta_1)\rceil_{S^{uv}(\alpha)}$-blocks are $\{a^{uv}_{1+5j}, a^{uv}_{2+5j}\}$ and $\{a^{uv}_{3+5j}, a^{uv}_{4+5j}\}$ for $j \in \{0,2,4\}$ and $\{u = a^{uv}_0,\ a^{uv}_5,\ a^{uv}_6,\ a^{uv}_7,\ a^{uv}_8,\ a^{uv}_9,\ a^{uv}_{10},\ a^{uv}_{15},\ a^{uv}_{16},\ a^{uv}_{17},\ a^{uv}_{18},\ a^{uv}_{19},\ a^{uv}_{20},\ a^{uv}_{25} = v\}$. (2.4) also yields that $\psi_1 \vee \zeta_1$ perfectly restricts to $S^{uv}(\alpha)$. Let

$$\sigma^{(1)} := \xi_1 \wedge (\psi_1 \vee \zeta_1).$$

The description of the $(\psi_1 \vee \zeta_1)\rceil_{S^{uv}(\alpha)}$-blocks, (2.4) applied to ξ_1, and (2.5) give that $\sigma^{(1)}\rceil_{S^{uv}(\alpha)}$ has only two non-singleton blocks, $\{a^{uv}_5, a^{uv}_{10}\}$ and $\{a^{uv}_{15}, a^{uv}_{20}\}$. We obtain from (2.4) that, for $j \in \{1,3\}$, the set $\{a^{uv}_{5j-1}, a^{uv}_{5j}, a^{uv}_{5j+5}, a^{uv}_{5j+6}\}$ is ξ_1-closed. Therefore, (2.6) yields that $\{a^{uv}_5, a^{uv}_{10}\}$ and $\{a^{uv}_{15}, a^{uv}_{20}\}$ are $\sigma^{(1)}$-closed. For $j \in \{0,2,4\}$, $\{a^{uv}_{1+5j}, a^{uv}_{2+5j}\}$ and $\{a^{uv}_{3+5j}, a^{uv}_{4+5j}\}$ are $(\psi_1 \vee \zeta_1)$-closed, because they are internal $(\psi_1 \vee \zeta_1)\rceil_{S^{uv}}$-blocks and $\psi_1 \vee \zeta_1$ perfectly restricts to $S^{uv}(\alpha)$. Hence, (2.6) yields that the singleton $\sigma^{(1)}\rceil_{S^{uv}(\alpha)}$-block $\{a^{uv}_{i+5j}\}$ is $\sigma^{(1)}$-closed for $i \in \{1,2,3,4\}$ and $j \in \{0,2,4\}$. Similarly, since $\{a^{uv}_{i+5j}\}$ is ξ_1-closed for $i \in \{1,2,3,4\}$ and $j \in \{1,3\}$ by (2.4), it is a $\sigma^{(1)}\rceil_{S^{uv}(\alpha)}$-block, whence it is $\sigma^{(1)}$-closed by (2.6). Consequently,

$$\sigma^{(1)} \text{ perfectly restricts to } S^{uv}(\alpha) \text{ and the restricted non-singleton blocks are } \{a^{uv}_5, a^{uv}_{10}\} \text{ and } \{a^{uv}_{15}, a^{uv}_{20}\}. \tag{2.7}$$

Clearly, for an eligible pair $\langle u,v\rangle \in \beta_0$, ψ–ζ-symmetry turns (2.7) into

$$\sigma^{(1)} \text{ perfectly restricts to } S^{uv}(\beta) \text{ and the restricted non-singleton blocks are } \{b^{uv}_5, b^{uv}_{10}\} \text{ and } \{b^{uv}_{15}, b^{uv}_{20}\}. \tag{2.8}$$

Next, let $\langle u,v\rangle \in \gamma_0$ be an eligible pair. By (2.4), the $(\psi_1 \vee \zeta_1)\rceil_{S^{uv}(\gamma)}$-blocks are $\{\hat{c}^{uv}_2, \hat{c}^{uv}_3\}$ and $S^{uv}(\gamma) \setminus \{\hat{c}^{uv}_2, \hat{c}^{uv}_3\}$. Hence, applying (2.4) and (2.5) to ξ_1, we obtain that the non-singleton $\sigma^{(1)}\rceil_{S^{uv}(\gamma)}$-blocks are $\{u, \check{c}^{uv}_1\}$, $\{\check{c}^{uv}_2, \check{c}^{uv}_3\}$, and $\{\check{c}^{uv}_4, v\}$. Since $\{\check{c}^{uv}_2, \check{c}^{uv}_3\}$ is ξ_1-closed by (2.4), it is also $\sigma^{(1)}$-closed by (2.6). In what follows, in order to ease formulation, (2.4) will not always be mentioned when its first part is used. If $x \in S^{uv}(\gamma)$ and $\{x\}$ is a singleton $\sigma^{(1)}\rceil_{S^{uv}(\gamma)}$-block, then $x \in \{\hat{c}^{uv}_1, \hat{c}^{uv}_2, \hat{c}^{uv}_3, \hat{c}^{uv}_3\}$. Since this four-element set is ξ_1-closed, (2.6) gives that $\{x\}$ is $\sigma^{(1)}$-closed. Thus, we conclude that

$$\sigma^{(1)} \text{ perfectly restricts to } S^{uv}(\gamma) \text{ and the non-singleton restricted blocks are } \{u, \check{c}^{uv}_1\},\ \{\check{c}^{uv}_2, \check{c}^{uv}_3\}, \text{ and } \{\check{c}^{uv}_4, v\}. \tag{2.9}$$

By ψ–ζ-symmetry, (2.9) implies that for every eligible pair $\langle u,v\rangle \in \delta_0$,

$$\sigma^{(1)} \text{ perfectly restricts to } S^{uv}(\delta) \text{ and the non-singleton restricted blocks are } \{u, \check{d}^{uv}_1\},\ \{\check{d}^{uv}_2, \check{d}^{uv}_3\}, \text{ and } \{\check{d}^{uv}_4, v\}. \tag{2.10}$$

Next, let

$$\sigma^{(2)} := \psi_1 \wedge (\xi_1 \vee \zeta_1),$$

and let $\langle u,v\rangle \in \alpha_0$ be an eligible pair. By (2.4), the $(\xi_1 \vee \zeta_1)\rceil_{S^{uv}(\alpha)}$-blocks are $\{a^{uv}_{1+5j},\ a^{uv}_{2+5j}\}$ and $\{a^{uv}_{3+5j},\ a^{uv}_{4+5j}\}$ for $j \in \{1,3\}$, and $\{u,\ a^{uv}_1,\ a^{uv}_2,\ a^{uv}_3,\ a^{uv}_4,\ a^{uv}_5,$

a_{10}^{uv}, a_{11}^{uv}, a_{12}^{uv}, a_{13}^{uv}, a_{14}^{uv}, a_{15}^{uv}, a_{20}^{uv}, a_{21}^{uv}, a_{22}^{uv}, a_{23}^{uv}, a_{24}^{uv}, $v\}$. Thus, it follows from (2.4) that $\xi_1 \vee \zeta_1$ perfectly restricts to $S^{uv}(\alpha)$. We claim that

(2.11) $\sigma^{(2)}$ perfectly restricts to $S^{uv}(\alpha)$ and the non-singleton restricted blocks are $\{u, a_5^{uv}\}$, $\{a_{10}^{uv}, a_{15}^{uv}\}$, and $\{a_{20}^{uv}, v\}$.

The description of $(\xi_1 \vee \zeta_1)\rceil_{S^{uv}(\alpha)}$-blocks, (2.4), and (2.5) imply that (2.11) describes the $\sigma^{(2)}\rceil_{S^{uv}(\alpha)}$-blocks correctly. Since $\{a_9^{uv}, a_{10}^{uv}, a_{15}^{uv}, a_{16}^{uv}\}$ is ψ_1-closed, (2.6) yields that $\{a_{10}^{uv}, a_{15}^{uv}\}$ is $\sigma^{(2)}$-closed. Since $\xi_1 \vee \zeta_1$ perfectly restricts to $S^{uv}(\alpha)$, the two-element sets $\{a_{1+5j}^{uv}, a_{2+5j}^{uv}\}$ and $\{a_{3+5j}^{uv}, a_{4+5j}^{uv}\}$ are $(\xi_1 \vee \zeta_1)$-closed for $j \in \{1, 3\}$. Hence, by (2.6), the singleton $\sigma^{(2)}\rceil_{S^{uv}(\alpha)}$-block $\{x\}$ is $\sigma^{(2)}$-closed whenever x belongs to these two-element $(\xi_1 \vee \zeta_1)\rceil_{S^{uv}(\alpha)}$-blocks. If x belongs to the set $\{a_{1+5j}^{uv}, a_{2+5j}^{uv}, a_{3+5j}^{uv}, a_{4+5j}^{uv}\}$ for some $j \in \{0, 2, 4\}$, then $\{x\}$ is ψ_1-closed and so it is $\sigma^{(2)}$-closed by (2.6). This proves (2.11).

Next, for an eligible pair $\langle u, v\rangle \in \beta_0$, (2.4) yields that the $(\xi_1 \vee \zeta_1)\rceil_{S^{uv}(\beta)}$-blocks are $\{b_{2+5j}^{uv},\ b_{3+5j}^{uv}\}$ for $j \in \{0, 1, 2, 3, 4\}$ and $\{u,\ b_1^{uv},\ b_4^{uv},\ b_5^{uv},\ b_6^{uv},\ b_9^{uv},\ b_{10}^{uv},\ b_{11}^{uv},\ b_{14}^{uv},\ b_{15}^{uv},\ b_{16}^{uv},\ b_{19}^{uv},\ b_{20}^{uv},\ b_{21}^{uv},\ b_{24}^{uv},\ v\}$. Therefore, using (2.4) and (2.5), we obtain that all the $\sigma^{(2)}\rceil_{S^{uv}(\beta)}$-blocks are singletons. For $j \in \{0, 1, 2, 3, 4\}$, the set $\{b_{1+5j}^{uv}, b_{2+5j}^{uv}, b_{3+5j}^{uv}, b_{4+5j}^{uv}\}$ is ψ_1-closed. Hence, if x belongs to some of these four-element sets, then $\{x\}$ is $\sigma^{(2)}$-closed by (2.6). Similarly, the singletons $\{b_5^{uv}\}$, $\{b_{10}^{uv}\}$, $\{b_{15}^{uv}\}$, $\{b_{20}^{uv}\}$ are also $\sigma^{(2)}$-closed by (2.6), because they are ψ_1-closed. Thus,

(2.12) $\sigma^{(2)}$ perfectly restricts to $S^{uv}(\beta)$ and all restricted blocks are singletons.

Next, let $\langle u, v\rangle \in \gamma_0$ be an eligible pair. Using (2.4), we obtain that the $(\xi_1 \vee \zeta_1)\rceil_{S^{uv}(\gamma)}$-blocks are $\{\check{c}_2^{uv}, \check{c}_3^{uv}\}$ and $S^{uv}(\gamma) \setminus \{\check{c}_2^{uv}, \check{c}_3^{uv}\}$, and $\xi_1 \vee \zeta_1$ perfectly restricts to $S^{uv}(\gamma)$. Hence, (2.4) and (2.5) imply that the non-singleton $\sigma^{(2)}\rceil_{S^{uv}(\gamma)}$-blocks are $\{u, \widehat{c}_1^{uv}\}$, $\{\widehat{c}_2^{uv}, \widehat{c}_3^{uv}\}$, and $\{\widehat{c}_4^{uv},\ v\}$. Since $\{\widehat{c}_2^{uv}, \widehat{c}_3^{uv}\}$ is ψ_1-closed, it is $\sigma^{(2)}$-closed by (2.6). Similarly, since $\{\check{c}_1^{uv}, \check{c}_2^{uv}, \check{c}_3^{uv}, \check{c}_4^{uv}\}$ is ψ_1-closed, (2.6) yields that $\{x\}$ is $\sigma^{(2)}$-closed for $x \in \{\check{c}_1^{uv}, \check{c}_2^{uv}, \check{c}_3^{uv}, \check{c}_4^{uv}\}$. Therefore,

(2.13) $\sigma^{(2)}$ perfectly restricts to $S^{uv}(\gamma)$ and the non-singleton restricted blocks are $\{u, \widehat{c}_1^{uv}\}$, $\{\widehat{c}_2^{uv}, \widehat{c}_3^{uv}\}$, and $\{\widehat{c}_4^{uv},\ v\}$.

As usual, we say that $\mu \in \mathrm{Equ}(A_1)$ *collapses* a subset X of A_1 if $X \times X \subseteq \mu$. Let $\langle u, v\rangle \in \delta_0$ be an eligible pair. Clearly, $\xi_1 \vee \zeta_1$ collapses the whole $S^{uv}(\delta)$. Thus, $\sigma^{(2)}\rceil_{S^{uv}(\delta)} = \psi_1\rceil_{S^{uv}(\delta)}$. Since $\{\check{d}_2^{uv}, \check{d}_3^{uv}\}$ and $\{\widehat{d}_2^{uv}, \widehat{d}_3^{uv}\}$ are ψ_1-closed, they are $\sigma^{(2)}$-closed by (2.6). This fact and (2.5) give that

(2.14) $\sigma^{(2)}$ perfectly restricts to $S^{uv}(\delta)$ and the restricted blocks are $\{u,\ \check{d}_1^{uv},\ \widehat{d}_1^{uv}\}$, $\{\check{d}_2^{uv}, \check{d}_3^{uv}\}$, $\{\widehat{d}_2^{uv}, \widehat{d}_3^{uv}\}$, and $\{\check{d}_4^{uv}, \widehat{d}_4^{uv}, v\}$.

Now, we are ready to prove several observations on the restrictions of $\widehat{\alpha}_1$, $\widehat{\beta}_1$, $\widehat{\gamma}_1$, and $\widehat{\delta}_1$ to the auxiliary graphs. For $\boldsymbol{\varepsilon}, \mu \in \{\alpha, \beta, \gamma, \delta\}$ and an eligible pair $\langle u, v\rangle \in \mu_0$, if μ_0 is the only member of $\{\alpha_0, \beta_0, \gamma_0, \delta_0\}$ that contains $\langle u, v\rangle$, then our forthcoming observations are conveniently visualized by Figure 2. In this case, the meaning of the edges in the figure is that for $x, y \in S^{uv}(\mu)$,

(2.15) $\langle x, y\rangle \in \widehat{\boldsymbol{\varepsilon}}_1\rceil_{S^{uv}(\mu)}$ iff there is an $\widehat{\boldsymbol{\varepsilon}}_1$-colored path in $S^{uv}(\mu)$ connecting x and y in the figure.

However, if more than one of α_0, β_0, γ_0, and δ_0 contains $\langle u, v\rangle$, then some additional edges connecting u and v should be added to Figure 2 in order to make (2.15) valid;

this situation will be clarified by (2.33) and (2.34). Note that, for $\mu \in \{\alpha, \beta, \gamma, \delta\}$, Figure 2 gives only the vertex set of $S^{uv}(\mu)$ but not the original edges.

We know that $\widehat{\alpha}_1 = \sigma^{(1)} \vee \sigma^{(2)}$. We claim that, for an eligible pair $\langle u, v\rangle \in \alpha_0$,

(2.16) $\widehat{\alpha}_1$ perfectly restricts to $S^{uv}(\alpha)$ and the only restricted non-singleton block is $\{u, a_5^{uv}, a_{10}^{uv}, a_{15}^{uv}, a_{20}^{uv}, v\}$.

We derive (2.16) from (2.7) and (2.11) as follows. Clearly, $\widehat{\alpha}_1$ collapses $\{u, a_5^{uv}, a_{10}^{uv}, a_{15}^{uv}, a_{20}^{uv}, v\}$. If x does not belong to this six-element subset, then $\{x\}$ is $\sigma^{(1)}$-closed by (2.7) and it is $\sigma^{(2)}$-closed by (2.11). Hence, by the well-known description of join in $\mathrm{Equ}(A_1)$, $\{x\}$ is $\widehat{\alpha}_1$-closed. This yields (2.16). Next, for an eligible $\langle u, v\rangle \in \beta_0$, we claim that

(2.17) $\widehat{\alpha}_1$ perfectly restricts to $S^{uv}(\beta)$. The subsets are $\{b_5^{uv}, b_{10}^{uv}\}$ and $\{b_{15}^{uv}, b_{20}^{uv}\}$ are restricted blocks, and $\{u, v\}$ is a restricted block if and only if $\langle u, v\rangle \in \alpha_0$. The rest of the restricted blocks are singletons.

It follows from (2.8), (2.12), and the description of join in $\mathrm{Equ}(A_1)$ that $\{b_5^{uv}, b_{10}^{uv}\}$, $\{b_{15}^{uv}, b_{20}^{uv}\}$, and $\{b_i^{uv}\}$ for $5 \nmid i$ (non-divisible) are $\widehat{\alpha}_1\rceil_{S^{uv}(\beta)}$-blocks and they are $\widehat{\alpha}_1$-closed. The "if" part is a trivial consequence of (2.16) and the construction of A_1. Thus, (2.17) without its "only if part" holds. In order to show the "only if part" of (2.17), it suffices to show that

$$\widehat{\alpha}_1\rceil_{A_0} = \alpha_0. \tag{2.18}$$

The elements of A_0 and those of $A_1 \setminus A_0$ will be called *old elements* and *new elements*, respectively. Note that the new elements are exactly the internal vertices of the auxiliary graphs used in the construction. We obtain from (2.16) and the construction of A_1 that $\widehat{\alpha}_1\rceil_{A_0} \supseteq \alpha_0$. In order to show the converse inclusion, it suffices to prove by induction on n that whenever u' and v' are old elements connected by a $(\sigma^{(1)} \cup \sigma^{(2)})$-sequence T: $x_0 = u'$, $x_1, \dots, x_{n-1}$, $x_n = v'$ of length n, then $\langle u', v'\rangle \in \alpha_0$. The base of the induction is the case of $n = 0$, which trivially holds by reflexivity. So, we assume that $n \ge 1$. We can assume also that

(2.19) T is a *repetition-free* sequence, that is, $x_i \neq x_j$ for $0 \le i < j \le n$,

because the repetition-free case trivially implies the general case. Transitivity combined with the induction hypothesis allows us to assume that

$$\text{if } \langle x_{i-1}, x_i\rangle \in \sigma^{(k)}, \text{ then } \langle x_i, x_{i+1}\rangle \in \sigma^{(3-k)} \setminus \sigma^{(k)}, \tag{2.20}$$

for $i \in \{1, \dots, n-1\}$ and $k \in \{1, 2\}$. By (2.7)–(2.14), x_1 is a new element. Hence, there is an $\boldsymbol{\varepsilon} \in \{\alpha,\ \beta,\ \gamma,\ \delta\}$ and there exists either an eligible pair $\langle u', w\rangle \in \boldsymbol{\varepsilon}_0$ such that x_1 is an internal vertex of the auxiliary graph $S^{u'w}(\boldsymbol{\varepsilon})$, or an eligible pair $\langle w, u'\rangle \in \boldsymbol{\varepsilon}_0$ such that x_1 is an internal vertex of $S^{wu'}(\boldsymbol{\varepsilon})$. By left-right symmetry, we can disregard the second alternative; that is, x_1 is an internal vertex of $S^{u'w}(\boldsymbol{\varepsilon})$. It follows from (2.8) and (2.12) that $\boldsymbol{\varepsilon} \neq \beta$. Since $x_n = v'$ is an old element, there exists a smallest i such that $i > 1$ and x_i is an old element. If $\boldsymbol{\varepsilon} = \gamma$, then (2.9) and (2.13) yield that $x_1 \in \{\widehat{c}_1^{u'w}, \check{c}_1^{u'w}\}$. Actually, either $\langle u', x_1\rangle = \langle u', \check{c}_1^{u'w}\rangle \in \sigma^{(1)}$, or $\langle u', x_1\rangle = \langle u', \widehat{c}_1^{u'w}\rangle \in \sigma^{(2)}$. Since x_i is an old element, $i > 1$. However, we know from (the perfectness part of) (2.9) and (2.13) that $\{\check{c}_1^{u'w}\}$ is $\sigma^{(2)}$-closed while $\{\widehat{c}_1^{u'w}\}$ is $\sigma^{(1)}$-closed. Hence, (2.19) and (2.20) give a contradiction, which excludes that $\boldsymbol{\varepsilon} = \gamma$. Next, assume that $\boldsymbol{\varepsilon} = \delta$. We obtain from (2.10) and (2.14) that the internal vertex x_1 belongs to $\{\widehat{d}_1^{u'w}, \check{d}_1^{u'w}\}$. If $x_1 = \widehat{d}_1^{u'w}$, then $\langle u', x_1\rangle \notin \sigma^{(1)}$ by

(2.10), whence (2.20) gives that $\langle x_1, x_2\rangle \in \sigma^{(1)}$. Since $\{x_1\} = \{\widehat{d}_1^{u'w}\}$ is $\sigma^{(1)}$-closed by (2.10), we obtain that $x_2 = x_1$, contradicting (2.19). Hence, $x_1 = \breve{d}_1^{u'w}$. There is a $k \in \{1, 2\}$ such that $\langle x_1, x_2\rangle \in \sigma^{(k)}$. Since $\langle u', x_1\rangle$ is also in $\sigma^{(k)}$ by (2.10) and (2.14), transitivity yields that $\langle u', x_2\rangle \in \sigma^{(k)}$. So u', x_2, $\dots$, x_{n-1}, $x_n = v'$ is a shorter sequence, and the required $\langle u', v'\rangle \in \alpha_0$ follows from the induction hypothesis. We are left with the case $\boldsymbol{\varepsilon} = \alpha$. By (2.7) and $x_1 \in S^{u'w}(\alpha) \setminus \{u'\}$, we have that $\langle u', x_1\rangle \notin \sigma^{(1)}$. Hence, $\langle u', x_1\rangle \in \sigma^{(2)}$, (2.11), and (2.19) give that $x_1 = a_5^{u'w}$. We know from (2.20) that $\langle x_1, x_2\rangle \in \sigma^{(1)} \setminus \sigma^{(2)}$. Combining this with (2.7) and (2.19), we obtain that $\langle x_1, x_2\rangle = \langle a_5^{u'w}, a_{10}^{u'w}\rangle \in \sigma^{(1)}$. Similarly, successive applications of (2.7), (2.11), (2.19), and (2.20) give that $\langle x_2, x_3\rangle = \langle a_{10}^{u'w}, a_{15}^{u'w}\rangle \in \sigma^{(2)} \setminus \sigma^{(1)}$, $\langle x_3, x_4\rangle = \langle a_{15}^{u'w}, a_{20}^{u'w}\rangle \in \sigma^{(1)} \setminus \sigma^{(2)}$, and $\langle x_4, x_5\rangle = \langle a_{20}^{u'w}, x_5\rangle \in \sigma^{(2)} \setminus \sigma^{(1)}$. However, $\langle w, a_{20}^{u'w}\rangle \in \sigma^{(2)}$ also holds by (2.11). By transitivity, $\langle w, x_5\rangle \in \sigma^{(2)}$. Thus, w, x_5, $\dots$, $x_n = v'$ is a $(\sigma^{(1)} \cup \sigma^{(2)})$-sequence of length $n - 4$ connecting the old elements w and v'. Hence, $\langle w, v'\rangle \in \alpha_0$ follows from the induction hypothesis. Since $\langle u', w\rangle \in \boldsymbol{\varepsilon}_0 = \alpha_0$, transitivity implies that $\langle u', v'\rangle \in \alpha_0$. This yields (2.18). Now, (2.18) completes the proof of (2.17).

Next, for an eligible $\langle u, v\rangle \in \gamma_0$, (2.9), (2.13), and (2.18) imply the following.

(2.21) $\widehat{\alpha}_1$ perfectly restricts to $S^{uv}(\gamma)$. There is no singleton restricted block, and $\{\widehat{c}_2^{uv}, \widehat{c}_3^{uv}\}$ and $\{\breve{c}_2^{uv}, \breve{c}_3^{uv}\}$ are the only restricted blocks disjoint from $\{u, v\}$. The subsets $\{u, \widehat{c}_1^{uv}, \breve{c}_1^{uv}\}$ and $\{v, \widehat{c}_4^{uv}, \breve{c}_4^{uv}\}$ are collapsed by $\widehat{\alpha}_1\rceil_{S^{uv}(\gamma)}$.

Similarly, for an eligible $\langle u, v\rangle \in \delta_0$, (2.10), (2.14), and (2.18) give that

(2.22) $\widehat{\alpha}_1$ perfectly restricts to $S^{uv}(\delta)$. There is no singleton restricted block, and $\{\widehat{d}_2^{uv}, \widehat{d}_3^{uv}\}$ and $\{\breve{d}_2^{uv}, \breve{d}_3^{uv}\}$ are the only restricted blocks disjoint from $\{u, v\}$. The subsets $\{u, \widehat{d}_1^{uv}, \breve{d}_1^{uv}\}$ and $\{v, \widehat{d}_4^{uv}, \breve{d}_4^{uv}\}$ are collapsed by $\widehat{\alpha}_1\rceil_{S^{uv}(\delta)}$.

Observe that (2.16), (2.17), (2.21), and (2.22) are satisfactorily visualized by Figure 2, provided (2.18) is taken into account. Hence, any reference to (the thick dotted $\widehat{\alpha}_1$-edges) of Figure 2 will be understood as a reference to (2.16), (2.17), (2.18), (2.21), and (2.22). By the ψ–ζ-symmetry mentioned before Lemma 2.2, $\widehat{\beta}_1$ is perfectly restricted to each auxiliary graph,

$$\widehat{\beta}_1\rceil_{A_0} = \beta_0, \tag{2.23}$$

and any reference to (the thick solid $\widehat{\beta}_1$-edges) of Figure 2 will be understood analogously as in case of $\widehat{\alpha}_1$-edges. Namely, the thick solid edges precisely describe the internal restricted blocks; if two side blocks are depicted, then they are collapsed iff (2.23) requires so.

The outer operation in $\widehat{\gamma}$ is a meet rather than a join. Hence, the analysis of $\widehat{\gamma}_1$ is easier than that of $\widehat{\alpha}_1$; this is why we are going to give less details below. It follows easily from (2.4) and the description of joins in $\mathrm{Equ}(A_1)$ that, for every eligible pair $\langle u, v\rangle$,

(2.24) If $\langle u, v\rangle$ belongs to γ_0 (resp., to δ_0), then $\xi_1 \vee (\psi_1 \wedge \zeta_1)$ perfectly restricts to $S^{uv}(\gamma)$ with restricted blocks $\{\breve{c}_2^{uv}, \breve{c}_3^{uv}\}$ and $S^{uv}(\gamma) \setminus \{\breve{c}_2^{uv}, \breve{c}_3^{uv}\}$ (resp., to $S^{uv}(\delta)$ with restricted blocks $\{\breve{d}_2^{uv}, \breve{d}_3^{uv}\}$ and $S^{uv}(\delta) \setminus \{\breve{d}_2^{uv}, \breve{d}_3^{uv}\}$).

Using (2.24), the assumptions $\alpha_0 \le \gamma_0 \vee \delta_0$ and $\beta_0 \le \gamma_0 \vee \delta_0$ of Lemma 2.2, and $\gamma_0 \cup \delta_0$ sequences in A_0, we conclude that $\langle u,v\rangle \in \alpha_0 \cup \beta_0$ implies that $\langle u,v\rangle \in \xi_1 \vee (\psi_1 \wedge \zeta_1)$. Furthermore, if $\boldsymbol{\varepsilon} \in \{\alpha,\beta\}$, then $\psi_1\rceil_{S^{uv}(\boldsymbol{\varepsilon})} \wedge \zeta_1\rceil_{S^{uv}(\boldsymbol{\varepsilon})} = \Delta_{S^{uv}(\boldsymbol{\varepsilon})}$, the least equivalence on $S^{uv}(\boldsymbol{\varepsilon})$. Hence, using (2.4), we obtain easily that, for every eligible $\langle u,v\rangle \in A_0^2$,

(2.25) If $\langle u,v\rangle$ belongs to α_0 (resp., to β_0), then $\xi_1 \vee (\psi_1 \wedge \zeta_1)$ perfectly restricts to $S^{uv}(\alpha)$ with restricted blocks $\{u, v, a_1^{uv}, a_{24}^{uv}\}$ and the internal blocks of $\xi_1\rceil_{S^{uv}(\alpha)}$ (resp., to $S^{uv}(\beta)$ with restricted blocks $\{u, v, b_1^{uv}, b_{24}^{uv}\}$ and the internal blocks of $\xi_1\rceil_{S^{uv}(\beta)}$).

Note that the internal ξ_1-blocks above are satisfactorily described by (2.4). It follows easily from (2.4) that each non-singleton $(\xi_1 \wedge \zeta_1)$-block is either of the form $\{\breve{c}_2^{uv}, \breve{c}_3^{uv}\}$, or it includes one of $\{u, \breve{c}_1^{uv}\}$ and $\{\breve{c}_4^{uv}, v\}$ for some eligible $\langle u,v\rangle \in \gamma_0$. Hence, similarly to our method yielding (2.18), a straightforward argument based on $(\psi_1 \cup (\xi_1 \wedge \zeta_1))$-sequences gives that

$$\text{for every } u,v \in A_0,\ \langle u,v\rangle \in \psi_1 \vee (\xi_1 \wedge \zeta_1) \text{ iff } \langle u,v\rangle \in \gamma_0. \tag{2.26}$$

Armed with (2.4) and (2.26) and still using $(\psi_1 \cup (\xi_1 \wedge \zeta_1))$-sequences, we obtain that, for any eligible pair $\langle u,v\rangle$,

(2.27) if $\langle u,v\rangle$ belongs to α_0 (resp., to β_0), then $\psi_1 \vee (\xi_1 \wedge \zeta_1)$ perfectly restricts to $S^{uv}(\alpha)$ and its restricted blocks are the $\psi_1\rceil_{S^{uv}(\alpha)}$-blocks except that the side blocks are collapsed iff $\langle u,v\rangle \in \gamma_0$ (resp., to $S^{uv}(\beta)$ and its restricted blocks are the $\psi_1\rceil_{S^{uv}(\beta)}$-blocks except that the side blocks are collapsed iff $\langle u,v\rangle \in \gamma_0$);

and

(2.28) if $\langle u,v\rangle$ belongs to γ_0 (resp., to δ_0), then $\psi_1 \vee (\xi_1 \wedge \zeta_1)$ perfectly restricts to $S^{uv}(\gamma)$ with restricted blocks $S^{uv}(\gamma) \setminus \{\widehat{c}_2^{uv}, \widehat{c}_3^{uv}\}$ and $\{\widehat{c}_2^{uv}, \widehat{c}_3^{uv}\}$ (resp., to $S^{uv}(\delta)$ with restricted blocks $\{\widehat{d}_2^{uv}, \widehat{d}_3^{uv}\}$, $\{\breve{d}_2^{uv}, \breve{d}_3^{uv}\}$, $\{u, \widehat{d}_1^{uv}, \breve{d}_1^{uv}\}$, and $\{v, \widehat{d}_4^{uv}, \breve{d}_4^{uv}\}$, but the last two collapse iff $\langle u,v\rangle \in \gamma_0$).

We know from (2.24) that $\gamma_0 \subseteq \xi_1 \vee (\psi_1 \wedge \zeta_1)$. Hence, (2.26) gives that

$$\text{for every } u,v \in A_0,\ \langle u,v\rangle \in \widehat{\gamma}_1 \text{ iff } \langle u,v\rangle \in \gamma_0. \tag{2.29}$$

We obtain from (2.6), (2.25) and (2.27) that, for every eligible pair $\langle u,v\rangle$,

(2.30) if $\boldsymbol{\varepsilon} \in \{\alpha,\beta\}$ and $\langle u,v\rangle \in \boldsymbol{\varepsilon}_0$, then $\widehat{\gamma}_1$ perfectly restricts to $S^{uv}(\boldsymbol{\varepsilon})$ with singleton restricted blocks except that $\langle u,v\rangle \in \widehat{\gamma}_1 \iff \langle u,v\rangle \in \gamma_0$ can yield a two-element restricted block.

Similarly, (2.6), (2.24), (2.28), and (2.29) yield that for every eligible pair $\langle u,v\rangle$,

(2.31) $\widehat{\gamma}_1$ perfectly restricts to $S^{uv}(\gamma)$ and the restricted blocks are $\{u, v, \widehat{c}_1^{uv}, \breve{c}_1^{uv}, \widehat{c}_4^{uv}, \breve{c}_4^{uv}\}$, $\{\widehat{c}_2^{uv}, \widehat{c}_3^{uv}\}$, and $\{\breve{c}_2^{uv}, \breve{c}_3^{uv}\}$; and

(2.32) $\widehat{\gamma}_1$ perfectly restricts to $S^{uv}(\delta)$ and the restricted blocks are $\{u, \widehat{d}_1^{uv}, \breve{d}_1^{uv}\}$, $\{v, \widehat{d}_4^{uv}, \breve{d}_4^{uv}\}$, $\{\widehat{d}_2^{uv}, \widehat{d}_3^{uv}\}$, and $\{\breve{d}_2^{uv}, \breve{d}_3^{uv}\}$, but the two side blocks collapse iff $\langle u,v\rangle \in \gamma_0$.

Now, (2.18), (2.23), (2.29), and ψ–ζ-symmetry give that

$$\text{for } u,v \in A_0 \text{ and } \boldsymbol{\varepsilon} \in \{\alpha,\beta,\gamma,\delta\}, \quad \langle u,v\rangle \in \widehat{\boldsymbol{\varepsilon}}_1 \iff \langle u,v\rangle \in \boldsymbol{\varepsilon}_0. \tag{2.33}$$

Note that (2.30)–(2.32) are visualized by Figure 2, provided (2.33) is taken into account. Therefore, by ψ–ζ-symmetry and the paragraph containing (2.23), for any $\boldsymbol{\varepsilon}, \mu \in \{\alpha, \beta, \gamma, \delta\}$ and every eligible pair $\langle u, v\rangle \in \mu_0$,

(2.34) $\widehat{\boldsymbol{\varepsilon}}_1$ perfectly restricts to $S^{uv}(\mu)$ and the restricted blocks are given by Figure 2 in the spirit of (2.15) so that an $\widehat{\boldsymbol{\varepsilon}}_1$-colored $\langle u, v\rangle$ edge has to be added iff $\langle u, v\rangle \in \boldsymbol{\varepsilon}_0$.

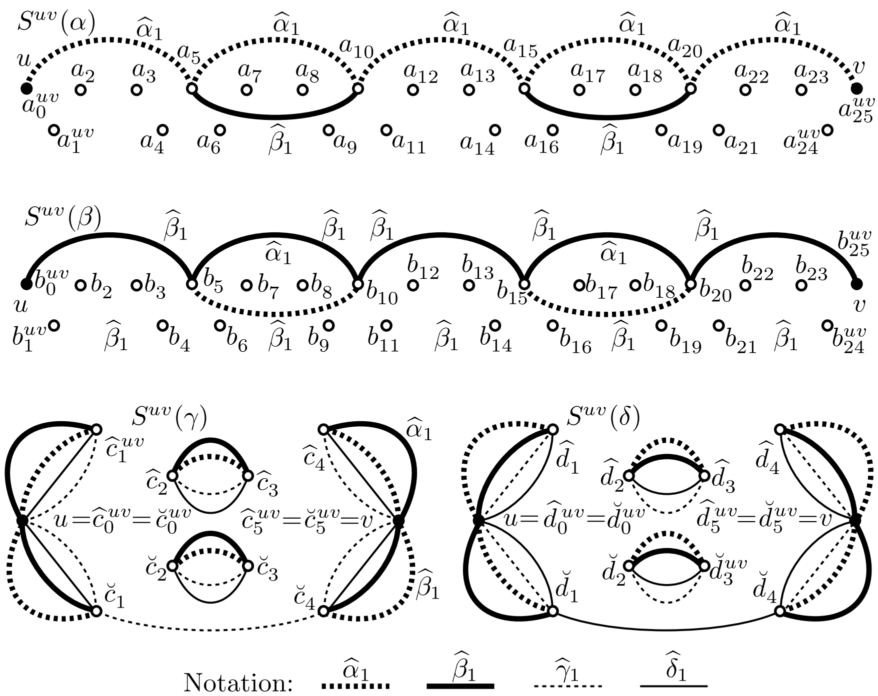

FIGURE 2. The restrictions of $\widehat{\alpha}_1$, $\widehat{\beta}_1$, $\widehat{\gamma}_1$, and $\widehat{\delta}_1$ to the auxiliary graphs; add an $\langle u, v\rangle$ edge if (2.34) requires so

Next, we turn our attention to $\widehat{\alpha}_2$, $\widehat{\beta}_2$, $\widehat{\gamma}_2$, and $\widehat{\delta}_2$. Clearly $\widehat{\gamma}_2 = \widehat{\gamma}_1$ and $\widehat{\delta}_2 = \widehat{\delta}_1$. Similarly to (2.34) and Figure 2, we need to explore how these equivalences are related to the auxiliary graphs. The situation is given by Figure 3. We claim that

$$\text{for } u, v \in A_0 \text{ and } \boldsymbol{\varepsilon} \in \{\alpha, \beta, \gamma, \delta\}, \quad \langle u, v\rangle \in \widehat{\boldsymbol{\varepsilon}}_2 \iff \langle u, v\rangle \in \boldsymbol{\varepsilon}_0, \tag{2.35}$$

For $\boldsymbol{\varepsilon} \in \{\alpha, \beta\}$, let $\langle u, v\rangle \in \boldsymbol{\varepsilon}_0$. Since $\boldsymbol{\varepsilon}_0 \leq \gamma_0 \vee \delta_0$ by the assumptions of Lemma 2.2, there exists a $(\gamma_0 \cup \delta_0)$-sequence from u to v. By (2.33), any two consecutive elements of this sequence is collapsed by $\widehat{\gamma}_1 \cup \widehat{\delta}_1$. By transitivity, $\langle u, v\rangle \in \widehat{\gamma}_1 \vee \widehat{\delta}_1$. Thus, using (2.33) again, we obtain that $\langle u, v\rangle \in \widehat{\boldsymbol{\varepsilon}}_1 \wedge (\widehat{\gamma}_1 \vee \widehat{\delta}_1) = \widehat{\boldsymbol{\varepsilon}}_2$. Conversely, if $\langle u, v\rangle \in \widehat{\boldsymbol{\varepsilon}}_2$, then $\widehat{\boldsymbol{\varepsilon}}_2 \leq \widehat{\boldsymbol{\varepsilon}}_1$ and (2.33) give that $\langle u, v\rangle \in \boldsymbol{\varepsilon}_0$. Hence, (2.35) holds for $\boldsymbol{\varepsilon} \in \{\alpha, \beta\}$. Therefore, (2.35) follows from (2.33), $\widehat{\gamma}_1 = \widehat{\gamma}_2$, and $\widehat{\delta}_1 = \widehat{\delta}_2$. Next, we claim that for any $\boldsymbol{\varepsilon}, \mu \in \{\alpha, \beta, \gamma, \delta\}$ and every eligible pair $\langle u, v\rangle \in \mu_0$,

(2.36) $\widehat{\boldsymbol{\varepsilon}}_2$ perfectly restricts to $S^{uv}(\mu)$ and the restricted blocks are given by Figure 3 in the spirit of (2.15) so that an $\boldsymbol{\varepsilon}$-colored $\langle u, v\rangle$ edge has to be added iff $\langle u, v\rangle \in \boldsymbol{\varepsilon}_0$.

In order to prove this, first we let $\mu \in \{\alpha, \beta\}$. If $x \in S^{uv}(\mu) \setminus \{u, v\}$, then $\{x\}$ is $\widehat{\gamma}_1$-closed and $\widehat{\delta}_1$-closed by (2.34); so $\{x\}$ is $(\widehat{\gamma}_1 \vee \widehat{\delta}_1)$-closed. By (2.6), $\{x\}$ is $\widehat{\varepsilon}_2$-closed. Thus, for $\mu \in \{\alpha, \beta\}$, (2.36) follows from (2.35). Second, we let $\mu \in \{\gamma, \delta\}$. Observe that the $\widehat{\alpha}_1$-edges and the $\widehat{\beta}_1$-edges in the $S^{uv}(\mu)$ part of Figure 2 are parallel to $\widehat{\gamma}_1$-edges (and also to $\widehat{\delta}_1$-edges). Therefore, for $\boldsymbol{\varepsilon} \in \{\alpha, \beta, \gamma, \delta\}$, the $\widehat{\varepsilon}_2\rceil_{S^{uv}(\mu)}$-blocks are exactly the $\widehat{\varepsilon}_1\rceil_{S^{uv}(\mu)}$-blocks. Hence, the internal $\widehat{\varepsilon}_2\rceil_{S^{uv}(\mu)}$-blocks are $\widehat{\varepsilon}_2$-closed, because they are $\widehat{\varepsilon}_1$-closed by (2.34) and $\widehat{\varepsilon}_2 \leq \widehat{\varepsilon}_1$; see also (2.6). This means that $\widehat{\varepsilon}_2$ perfectly restricts to $S^{uv}(\mu)$. Thus, (2.36) follows from (2.35).

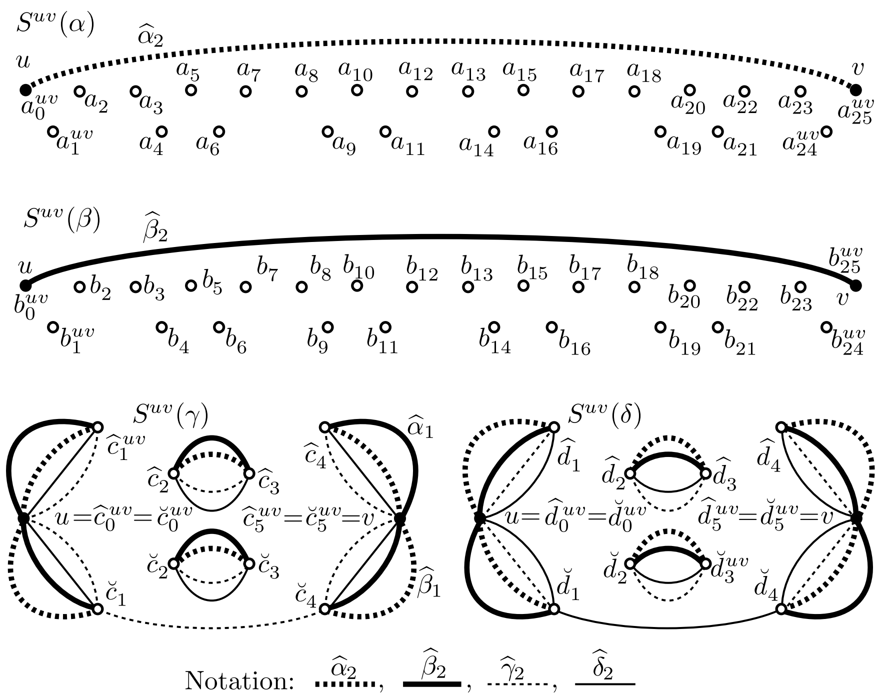

FIGURE 3. The restrictions of $\widehat{\alpha}_2$, $\widehat{\beta}_2$, $\widehat{\gamma}_2$, and $\widehat{\delta}_2$ to the auxiliary graphs; add an $\langle u, v\rangle$ edge if (2.35) requires so

Let Θ denote the smallest equivalence relation on A_1 collapsing each of the sets $\{u, \check{c}_1^{uv}, \widehat{c}_1^{uv}\}$, $\{v, \check{c}_4^{uv}, \widehat{c}_4^{uv}\}$, $\{\check{c}_2^{uv}, \check{c}_3^{uv}\}$, $\{\widehat{c}_2^{uv}, \widehat{c}_3^{uv}\}$ for eligible pairs $\langle u, v\rangle \in \gamma_0$ and $\{u, \check{d}_1^{uv}, \widehat{d}_1^{uv}\}$, $\{v, \check{d}_4^{uv}, \widehat{d}_4^{uv}\}$, $\{\check{d}_2^{uv}, \check{d}_3^{uv}\}$, and $\{\widehat{d}_2^{uv}, \widehat{d}_3^{uv}\}$, for eligible $\langle u, v\rangle \in \delta_0$. The two-element sets here are obviously Θ-blocks; they are the *two-element Θ-blocks.* Clearly, Θ is the join $\bigvee_{m\in M} \Theta_m$ of those equivalences Θ_m that correspond to the two-element and three-element sets listed above. For example, if $\langle u, v\rangle \in \gamma_0$ is an eligible pair, then the least equivalence collapsing $\{u, \check{c}_1^{uv}, \widehat{c}_1^{uv}\}$ is one of these Θ_m. Since $\widehat{\varepsilon}_2$ collapses the Θ-blocks by (2.36), it follows trivially that

$$\Theta \le \widehat{\varepsilon}_2 \text{ for all } \boldsymbol{\varepsilon} \in \{\alpha, \beta, \gamma, \delta\}, \text{ and } \Theta\rceil_{A_0} = \Delta_{A_0}. \tag{2.37}$$

Next, we claim that

$$\text{for } \boldsymbol{\varepsilon} \in \{\alpha, \beta, \gamma, \delta\}, \quad \widehat{\varepsilon}_2 = (\boldsymbol{\varepsilon}_0 \cup \Delta_{A_1}) \vee \Theta. \tag{2.38}$$

Since $\widehat{\varepsilon}_2 \supseteq \boldsymbol{\varepsilon}_0$ by (2.35), the inclusion $\widehat{\varepsilon}_2 \supseteq (\boldsymbol{\varepsilon}_0 \cup \Delta_{A_1}) \vee \Theta$ is clear by (2.37). In order to show the converse inclusion, assume that $\langle x, y\rangle \in \widehat{\varepsilon}_2$. We assume also that $\langle x, y\rangle$ belongs neither to $\boldsymbol{\varepsilon}_0$, nor to Θ, because otherwise the containment $\langle x, y\rangle \in (\boldsymbol{\varepsilon}_0 \cup \Delta_{A_1}) \vee \Theta$ would be evident. It follows from (2.36) that x neither belongs to a two-element Θ-block, which is $\widehat{\varepsilon}_2$-closed, nor it is an internal element of some $S^{uv}(\alpha)$ or $S^{uv}(\beta)$, which forms an $\widehat{\varepsilon}_2$-closed singleton. Hence, again by (2.36), there is an $x' \in A_0$ such that $\langle x', x\rangle \in \Theta$. Similarly, there is a $y' \in A_0$ such that $\langle y, y'\rangle \in \Theta$. By (2.37), $\langle x', x\rangle \in \widehat{\varepsilon}_2$ and $\langle y, y'\rangle \in \widehat{\varepsilon}_2$. By the transitivity of $\widehat{\varepsilon}_2$ and (2.35), we have that $\langle x', y'\rangle \in \boldsymbol{\varepsilon}_0$. This containment, $\langle x, x'\rangle \in \Theta$, and $\langle y', y\rangle \in \Theta$ give the required containment $\langle x, y\rangle \in (\boldsymbol{\varepsilon}_0 \cup \Delta_{A_1}) \vee \Theta$, proving (2.38).

Combining Lemma 2.1 and the second half of (2.37), we obtain that the map $g\colon \mathrm{Equ}(A_0) \to \mathrm{Equ}(A_1)$, defined by $\mu \mapsto (\mu \cup \Delta_{A_1}) \vee \Theta$, is a complete lattice embedding. Let $f = g\rceil_{L_0}$ be the restriction of g to L_0. Since L_0 is a complete sublattice of $\mathrm{Equ}(A_0)$, f is also a complete embedding. Clearly, with the temporary notation $L_2' := f(L_0)$, we have that $f\colon L_0 \to L_2'$ is a (complete) lattice isomorphism and L_2' is a complete sublattice of $\mathrm{Equ}(A_1)$. We know from (2.38) that, for $\boldsymbol{\varepsilon} \in \{\alpha, \beta, \gamma, \delta\}$, $f(\boldsymbol{\varepsilon}_0) = \widehat{\varepsilon}_2$. Hence, $\{\widehat{\alpha}_2, \widehat{\beta}_2, \widehat{\gamma}_2, \widehat{\delta}_2\} \subseteq L_2'$. This gives that $L_2 \subseteq L_2'$, because L_2 is the smallest complete sublattice of $\mathrm{Equ}(A_1)$ that includes $\{\widehat{\alpha}_2, \widehat{\beta}_2, \widehat{\gamma}_2, \widehat{\delta}_2\}$. In order to show the converse inclusion, observe that $f^{-1}(L_2)$ is a complete sublattice of $\mathrm{Equ}(A_0)$, because it is a complete sublattice of L_0 and L_0 is a complete sublattice of $\mathrm{Equ}(A_0)$. Since $f^{-1}(L_2)$ includes $\{f^{-1}(\widehat{\alpha}_2),\ f^{-1}(\widehat{\beta}_2),\ f^{-1}(\widehat{\gamma}_2),\ f^{-1}(\widehat{\delta}_2)\} = \{\alpha_0,\ \beta_0,\ \gamma_0,\ \delta_0\}$ and L_0 is the smallest complete sublattice of $\mathrm{Equ}(A_0)$ including $\{\alpha_0, \beta_0, \gamma_0, \delta_0\}$, $L_0 \subseteq f^{-1}(L_2)$. Hence, $L_2' = f(L_0) \subseteq f(f^{-1}(L_2)) = L_2$. Consequently, $L_2' = L_2$ and f is an $L_0 \to L_2$ isomorphism required by Lemma 2.2.

Finally, in order to see the uniqueness of f, let $h\colon L_0 \to L_2$ be an isomorphism such that, for all $\boldsymbol{\varepsilon} \in \{\alpha, \beta, \gamma, \delta\}$, $h(\boldsymbol{\varepsilon}_0) = \widehat{\varepsilon}_2$. The "equalizer set" $E := \{\mu \in L_0 : f(\mu) = h(\mu)\}$ of f and h is a complete sublattice of L_0, since both f and h, like any isomorphism, preserve arbitrary meets and joins. Since $\{\alpha_0, \beta_0, \gamma_0, \delta_0\} \subseteq E$, we obtain that $E = L_0$. Hence, $f = h$, completing the proof of Lemma 2.2. □

2.3. **The rest of the proof.** Now, armed with Lemma 2.2 and some earlier results, we are in the position to complete the proof of Theorem 1.1 in a short way.

Proof of Theorem 1.1. Assume that A is a set with accessible cardinality at least 3. For convenience, we can also assume that $|A|$ is not an even natural number, because otherwise we would construct a larger set A' from A by adding a new element and then we would use that $\mathrm{Equ}(A)$ is isomorphic to a sublattice of $\mathrm{Equ}(A')$.

For a finite A, it was first proved by Strietz [23] that $\mathrm{Equ}(A)$ is four-generated; however, we will use the four generators constructed by Zádori [25]. Note that these generators for $|A| = 33$ are reproduced in [3, Figure 1], and 33 is sufficiently large to indicate the general case for $|A|$ odd. Therefore, in order to be in harmony with the infinite case, we use ${}^{\mathrm{f}}\alpha_{[3]}$, ${}^{\mathrm{f}}\beta_{[3]}$, ${}^{\mathrm{f}}\gamma_{[3]}$, and ${}^{\mathrm{f}}\delta_{[3]}$ to denote Zádori's generators; see [3]. The superscript "f" indicates that we are dealing with the finite case. Alternatively, we can use a different system of generators from Zádori [25], which are also given for $|A| = 59$ in [4]; these generators will be denoted by ${}^{\mathrm{f}}\alpha_{[4]}$, ${}^{\mathrm{f}}\beta_{[4]}$, ${}^{\mathrm{f}}\gamma_{[4]}$, and ${}^{\mathrm{f}}\delta_{[4]}$. Whichever of the two systems is considered, the join of two appropriately chosen generators is clearly ∇_A. Namely,

$$\begin{gathered} {}^{\mathrm{f}}\alpha_{[3]} \vee {}^{\mathrm{f}}\beta_{[3]} = {}^{\mathrm{f}}\alpha_{[3]} \vee {}^{\mathrm{f}}\gamma_{[3]} = {}^{\mathrm{f}}\alpha_{[3]} \vee {}^{\mathrm{f}}\delta_{[3]} = {}^{\mathrm{f}}\beta_{[3]} \vee {}^{\mathrm{f}}\gamma_{[3]} = \nabla_A \\ \text{and } {}^{\mathrm{f}}\alpha_{[4]} \vee {}^{\mathrm{f}}\beta_{[4]} = {}^{\mathrm{f}}\alpha_{[4]} \vee {}^{\mathrm{f}}\gamma_{[4]} = {}^{\mathrm{f}}\beta_{[4]} \vee {}^{\mathrm{f}}\gamma_{[4]} = \nabla_A. \end{gathered}$$

Thus, we have many choices to fulfill the conditions $\alpha_0 \le \gamma_0 \vee \delta_0$ and $\beta_0 \le \gamma_0 \vee \delta_0$ of Lemma 2.2; let, say, $\langle \alpha_0, \beta_0, \gamma_0, \delta_0 \rangle = \langle {}^{\mathrm{f}}\gamma_{[3]}, {}^{\mathrm{f}}\delta_{[3]}, {}^{\mathrm{f}}\alpha_{[3]}, {}^{\mathrm{f}}\beta_{[3]} \rangle$.

If A is infinite but $|A|$ is accessible, then we know from [3] and [4] that $\mathrm{Equ}(A)$ is completely generated by four elements. Let $\{{}^{\mathrm{i}}\alpha_{[3]}, {}^{\mathrm{i}}\beta_{[3]}, {}^{\mathrm{i}}\gamma_{[3]}, {}^{\mathrm{i}}\delta_{[3]}\}$ and $\{{}^{\mathrm{i}}\alpha_{[4]}, {}^{\mathrm{i}}\beta_{[4]}, {}^{\mathrm{i}}\gamma_{[4]}, {}^{\mathrm{i}}\delta_{[4]}\}$ denote the complete generating sets constructed in [3] and [4], respectively. Here the superscript "i" comes from "infinite". Again, we can pick two generators whose join is ∇_A. Actually, we have the following equalities, but it suffices to see that one of the following six joins equals ∇_A:

$$\begin{gathered} {}^{\mathrm{i}}\alpha_{[3]} \vee {}^{\mathrm{i}}\beta_{[3]} = {}^{\mathrm{i}}\alpha_{[3]} \vee {}^{\mathrm{i}}\gamma_{[3]} = {}^{\mathrm{i}}\alpha_{[3]} \vee {}^{\mathrm{i}}\delta_{[3]} = \nabla_A \text{ and} \\ {}^{\mathrm{i}}\alpha_{[4]} \vee {}^{\mathrm{i}}\beta_{[4]} = {}^{\mathrm{i}}\alpha_{[4]} \vee {}^{\mathrm{i}}\gamma_{[4]} = {}^{\mathrm{i}}\beta_{[4]} \vee {}^{\mathrm{i}}\gamma_{[4]} = \nabla_A. \end{gathered}$$

Let, say, $\langle \alpha_0, \beta_0, \gamma_0, \delta_0 \rangle = \langle {}^{\mathrm{i}}\gamma_{[3]}, {}^{\mathrm{i}}\delta_{[3]}, {}^{\mathrm{i}}\alpha_{[3]}, {}^{\mathrm{i}}\beta_{[3]} \rangle$.

Next, if A is infinite, then let $A_0 = A$, and consider the set A_1 constructed before Lemma 2.2. Clearly, $|A_1| = |A|$. Let L_0 stand for the complete sublattice of $\mathrm{Equ}(A_0)$ completely generated by $\{\alpha_0, \beta_0, \gamma_0, \delta_0\}$. Denote by K' the complete sublattice of $\mathrm{Equ}(A_1)$ completely generated by its three-element subset $\{\xi_1, \psi_1, \zeta_1\}$. Since $L_0 = \mathrm{Equ}(A_0) = \mathrm{Equ}(A)$ and L_2 from Lemma 2.2 is clearly a complete sublattice of K', Lemma 2.2 gives that

$$\mathrm{Equ}(A) \text{ is isomorphic to a complete sublattice of } K'. \tag{2.39}$$

Now, let $B = A$. Since $|A_1| = |A| = |B|$, we have a (complete) lattice isomorphism $h\colon \mathrm{Equ}(A_1) \to \mathrm{Equ}(B)$. Clearly, $K := h(K')$ is isomorphic to K' and it is a complete sublattice of $\mathrm{Equ}(B)$. Also, K is completely generated by the three-element set $\{h(\xi_1), h(\psi_1), h(\zeta_1)\}$, and $\mathrm{Equ}(A)$ is isomorphic to a complete sublattice of K. This proves the theorem for A infinite.

If A is finite, then we can drop "complete" from the consideration above. Let $B = A_1$; it is finite by construction. By (2.39), $\mathrm{Equ}(A)$ is isomorphic to a sublattice of the three-generated K' and K' is a sublattice of $\mathrm{Equ}(B)$. Hence, we can let $K = K'$, which completes the proof of the theorem. ☐

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