Some varieties and convexities generated by fractal lattices

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Dedicated to the memory of András P. Huhn (1947–1985)

Abstract. Let L be a lattice. If for each $a < b \in L$ there is a lattice embedding of $\varphi : L \to [a, b]$ then L is called a *semifractal*. If, in addition, $0, 1 \in L$ and φ can always be chosen such that $\varphi(0) = a$ and $\varphi(1) = b$ then L is said to be a *0–1-semifractal*. Now let L be a bounded lattice. If for each $a_1 < b_1 \in L$ and $a_2 < b_2 \in L$ there is a lattice embedding $\psi : [a_1, b_1] \to [a_2, b_2]$ with $\psi(a_1) = a_2$ and $\psi(b_1) = b_2$ then we say that L is a *quasifractal*. If ψ can always be chosen an isomorphism or, equivalently, if L is isomorphic to each of its nontrivial intervals then L will be called a *fractal lattice* or, shortly, a *fractal*. Although there is an obvious hierarchy of these notions and we construct 0–1-semifractals which are not quasifractals it remains only a conjecture that the above notions are distinct.

A variety generated by a fractal lattice is called *fractal generated*, and analogous terminology applies for the rest of our new notions. We show that semifractal generated nondistributive lattice varieties cannot be of residually finite length. This will easily imply that there are exactly continuously many lattice varieties which are not semifractal generated. On the other hand, for each prime field F, the variety generated by all subspace lattices of vector spaces over F is shown to be fractal generated. These countably many varieties and the class $\mathcal{D}$ of all distributive lattices are the only known fractal generated lattice varieties at present. Four distinct countable distributive fractal lattices will be given such that each of them generates $\mathcal{D}$. After showing that each lattice can be embedded in a quasifractal, continuously many quasifractals will be given such that each of them has the cardinality $\aleph_0$ and generates the variety of all lattices.

The last section of the paper is devoted to an application. A class of lattices is called a *convexity* if it is closed under taking homomorphic images, convex sublattices and direct products. This notion is due to Ervin Fried. Each nontrivial lattice variety includes the variety generated by the two element lattice, which is a minimal variety. The question if the same is true for convexities goes back to Jakubík [17]. Using appropriate semifractals we give many convexities which include no minimal convexity.

1. Introduction and the main theorem

In colloquial usage, a *fractal* is a geometric shape that is self-similar (at least approximately) to its arbitrarily small parts, cf. Wikipedia [24]. Nature has many objects that approximate fractals. These objects include river networks, systems of

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blood vessels and pulmonary vessels, cauliflower or broccoli, snow flakes, mountains and lightning bolts. Fractals are frequently used tools to make mathematics popular for wider audience. Therefore it is quite natural to investigate the related notion in algebra and, first of all, in lattice theory. (For general reference on this theory the reader can resort to, e.g., Grätzer [10] or [11] or Crawley and Dilworth [3].)

In lattice theory, self-similarity will, of course, mean isomorphisms. But what should we understand by small parts and should we consider all of these small parts or only some of them? The one element lattice, also called *trivial lattice*, should of course be excluded. We have decided to consider all parts except the trivial ones. An obvious resp. straightforward argument shows that a lattice with more than two elements cannot be isomorphic to all of its sublattices resp. convex sublattices. Hence we take the only possibility to give the following

Definition 1.1. By a *fractal lattice*, or shortly *fractal*, we mean a lattice which is isomorphic to each of its nontrivial (i.e., at least two element) intervals.

To provide some examples and to fix some notations we remark that the trivial lattice $\mathbf{1}$, the two element lattice $\mathbf{2}$ and the poset $\mathbf{Q}_{[0,1]} = (\mathbf{Q}_{[0,1]}, \leq)$ of rational numbers between zero and one are fractal lattices. Except for $\mathbf{2}$, all nontrivial fractal lattices are infinite and they have 0 and 1. Although infinitely many fractals will be given in the present paper and they are appropriate to derive a theorem on convexities of lattices in the last section, we do not know sufficiently many of them. In order to derive a better theorem, and also to make some of our statements stronger by weakening the conditions, we introduce some weaker notions as well.

Definition 1.2. A lattice L is called a *semifractal* if for each $a < b \in L$ there is a lattice embedding $L \to [a, b]$. If L is a bounded lattice such that for each $a < b \in L$ there is a lattice embedding $L \to [a, b]$ which maps $\{0, 1\}$ to $\{a, b\}$ then L is called a *0–1-semifractal.* If L is a bounded lattice and for each $a_1 < b_1 \in L$ and $a_2 < b_2 \in L$ there is a lattice embedding $[a_1, b_1] \to [a_2, b_2]$ which maps $\{a_1, b_1\}$ to $\{a_2, b_2\}$ then we say that L is a *quasifractal.* For brevity, the term *β-fractal* will mean an arbitrary element of the set {fractal, quasifractal, semifractal, 0–1-semifractal}; the meaning of a β-fractal is of course fixed within a context.

There are obvious inclusions among these kinds of lattices: {semifractals} $\supseteq$ {0–1-semifractals} $\supseteq$ {quasifractals} $\supseteq$ {fractals}. We conjecture that all these inclusions are proper but we can prove {0–1-semifractals} $\neq$ {quasifractals} only.

Any fractal $L = (L, \vee, \wedge)$ can be considered as an algebra $(L, \vee, \wedge, t)$ where t is a ternary operation such that $t(x, y, -) : L \to L,\ z \mapsto t(x, y, z)$ is an $L \to [x, y]$ isomorphism when $x < y$ and it is the identical $L \to L$ map otherwise. (Notice that the choice of t is not unique in general but this does not create any problem.) The treatment for other β-fractals is similar. Since $(L, \vee, \wedge, t)$ is first-order axiomatizable, we obtain the following statement easily from well-known theorems of Löwenheim and Skolem, and Łoś, cf., e.g., Bell and Slomson [1].

Proposition 1.3. *The ultraproduct of any set of β-fractals is a β-fractal again. If a lattice variety $\mathcal{V}$ can be generated by an infinite β-fractal then for any infinite cardinal α, $\mathcal{V}$ is generated by a β-fractal of power α.*

A variety $\mathcal{V}$ is said to be *nontrivial* if it contains a nontrivial lattice. Let us call a nontrivial variety *β-fractal generated* if it can be generated by a β-fractal. Proposition 1.3 yields that a variety is β-fractal generated if and only if it is generated by a β-fractal of power $\aleph_0$ or less. We say that a lattice variety $\mathcal{V}$ is *of residually finite length* if there exists an $n \in \mathbf{N}$ such that every chain in every subdirectly irreducible lattice of $\mathcal{V}$ has at most n elements.

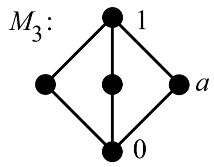

Figure 1

Now we introduce a notion which may look too technical here but it will be quite relevant in the last section where the main theorem is applied.

Definition 1.4. A lattice L will be called an *M_3-simple lattice* if $|L| \geq 3$ and for each chain $x < y < z$ of L there is an embedding $M_3 \to L$ such that $0 \mapsto x$, $a \mapsto y$ and $1 \mapsto z$, cf. Figure 1.

Clearly, M_3-simple lattices are simple. We say that a bounded lattice L has a *spanning* M_3 if M_3 is a 0–1-sublattice of L. Let P be the set of prime numbers and for $p \in P \cup \{0\}$ let F_p be the prime field of characteristic p. The lattice variety generated by the subspace lattices of all vector spaces over F_p will be denoted by $\mathcal{V}_p$. Now we formulate the main result of the paper.

Theorem 1.5. (1) *For each $p \in P \cup \{0\}$, $\mathcal{V}_p$ is generated by a simple countable fractal lattice with a spanning M_3. Further, $\mathcal{V}_p$ contains continuously many simple countable 0–1-semifractals with a spanning M_3 such that none of them is a quasifractal and each of them generates $\mathcal{V}_p$.*

(2) *There are at least four nonisomorphic fractal lattices with power $\leq \aleph_0$ such that each of them generates the variety $\mathcal{D}$ of distributive lattices.*

(3) *There is a countable distributive 0–1-semifractal which is not a quasifractal.*

(4) *Let $\mathcal{V}$ be a variety of lattices. If $\mathcal{V}$ is of residually finite length, then all semifractals in $\mathcal{V}$ are distributive.*

(5) *There are continuously many lattice varieties which are not semifractal generated.*

(6) *Each lattice with at least three elements has a 0–1-embedding into an appropriate M_3-simple quasifractal. Moreover, there are continuously many pairwise nonisomorphic M_3-simple quasifractals such that each of them has only $\aleph_0$ elements and generates the variety of all lattices.*

Notice that the $\mathcal{V}_p$, $p \in P \cup \{0\}$, are distinct and they are just the minimal nondistributive modular congruence varieties, cf. Freese [5] or Corollary 14 in Freese, Herrmann and Huhn [6]. Since "length is at most n" is a first-order property, it follows by the famous $\mathbf{P}_s\mathbf{HSP}_u$ lemma of Jónsson [18] that every finite lattice generates a variety of residually finite length. Theorem 1.5 leaves many natural questions open, we mention only a few.

Problems 1.6. (1) Is the variety of all lattices fractal generated?

(2) Is the variety of modular lattices β-fractal generated?

(3) Given a β-fractal generated variety $\mathcal{V}$, what is the number of pairwise nonisomorphic countable β-fractals such that each of them generates $\mathcal{V}$? (Some particular cases are answered by Theorem 1.5.)

(4) What is the cardinality of the set of β-fractal generated varieties?

2. Lemmas and proofs

Let $\mathbf{Q}_{[0,1)} = \mathbf{Q}_{[0,1]} \setminus \{1\}$, a sublattice of the fractal $\mathbf{Q}_{[0,1]}$. For any lattice L, a function $g : \mathbf{Q}_{[0,1)} \to L$ is called a *sectionally constant function* if there is an $n \in \mathbf{N}$ and there are rational numbers $0 = a_0 < a_1 < a_2 < \cdots < a_n = 1$ such that for each $i \in \{1, 2, \ldots, n\}$ the restriction of g to the left closed right open interval $[a_{i-1}, a_i)$ is a constant function. Clearly, the set $C(\mathbf{Q}_{[0,1)}, L)$ of sectionally constant $\mathbf{Q}_{[0,1)} \to L$ functions is a lattice, in fact, a sublattice of the direct power $L^{\mathbf{Q}_{[0,1)}}$ of L. Notice that the particular case $L = \mathbf{2}$ of this construction is well-known, cf. Lemma 10 of Section 10 in Grätzer [10].

Lemma 2.1. *If L is a fractal lattice then so is $C(\mathbf{Q}_{[0,1)}, L)$.*

Proof. Suppose $g < h$ in $C(\mathbf{Q}_{[0,1)}, L)$. Then there is a $k \in \mathbf{N}$ and there are

$$0 \le a_1 < b_1 \le a_2 < b_2 \le a_3 < b_3 \le \cdots \le a_k < b_k \le 1$$

in $\mathbf{Q}_{[0,1)}$ such that both g and h are distinct constant functions on each $[a_i, b_i)$, $i \in \{1, \ldots, k\}$, and, with the notation $T = \bigcup_{i=1}^{k} [a_i, b_i)$, g and h are the same functions on $\mathbf{Q}_{[0,1)} \setminus T$. (Notice that $\mathbf{Q}_{[0,1)} \setminus T$ can be empty.) Clearly, T is (order) isomorphic to $\mathbf{Q}_{[0,1)}$. Let $\alpha : \mathbf{Q}_{[0,1)} \to T$ be an isomorphism. For $i \in \{1, \ldots, k\}$, let β_i be a $[g(a_i), h(a_i)] \to L$ isomorphism. We claim that $\gamma : [g, h] \to C(\mathbf{Q}_{[0,1)}, L)$, $t \mapsto t^*$ where $t^*(x) = \beta_i(t(\alpha(x)))$ for $x \in \alpha^{-1}([a_i, b_i))$ is an isomorphism. Indeed, let $\delta : C(\mathbf{Q}_{[0,1)}, L) \to [g, h]$, $t \mapsto t^\sharp$ where $t^\sharp(x) = g(x) = h(x)$ for $x \in \mathbf{Q}_{[0,1)} \setminus T$ and $t^\sharp(x) = \beta_i^{-1}(t(\alpha^{-1}(x)))$ for $x \in [a_i, b_i)$. It is straightforward to check that γ and δ are monotone and they are inverses of each other. Therefore γ is a lattice isomorphism. $\square$

Proof. Now we prove parts (2) and (3) of Theorem 1.5. Since any nontrivial distributive lattice generates $\mathcal{D}$, it suffices to present four distinct nontrivial distributive fractals of power at most $\aleph_0$. The first two of them are $\mathbf{2}$, the two element lattice, and $\mathbf{Q}_{[0,1]}$, the set or rational numbers between 0 and 1.

The third one is $C(\mathbf{Q}_{[0,1)}, \mathbf{2})$. First of all, it is a fractal by Lemma 2.1. However, this follows from well-known results on Boolean algebras, too. Indeed, Grätzer [10] shows that $C(\mathbf{Q}_{[0,1)}, \mathbf{2})$ is, up to isomorphism, the unique countable atomless boolean lattice, cf. Lemma 10, Thm. 20 and Cor. 23 in Section 10 of [10]. In other words, the theory of countable atomless boolean lattices is $\aleph_0$-categorical. This easily implies that $C(\mathbf{Q}_{[0,1)}, \mathbf{2})$ is a fractal lattice.

The fourth fractal is $C(\mathbf{Q}_{[0,1)}, \mathbf{Q}_{[0,1]})$. It is distinct from the previous ones since it is neither a chain nor a complemented lattice.

Now let K be the subset of all those $g \in C(\mathbf{Q}_{[0,1)}, \mathbf{Q}_{[0,1]})$ that are monotone $\mathbf{Q}_{[0,1)} \to \mathbf{Q}_{[0,1]}$ function. Clearly, K is a 0–1-sublattice of $C(\mathbf{Q}_{[0,1)}, \mathbf{Q}_{[0,1]})$. Instead of an arbitrary $\beta_i : [g(a_i), h(a_i)] \to \mathbf{Q}_{[0,1]}$ from the proof of Lemma 2.1 we define

$$\beta_i(x) = \frac{x - g(a_i)}{h(a_i) - g(a_i)}, \quad \text{then} \quad \beta_i^{-1}(x) = g(a_i) + (h(a_i) - g(a_i))x.$$

For $i < j$, we have $g(a_i) \le g(a_j)$ and $h(a_i) \le h(a_j)$, whence

$$\beta_j^{-1}(x) - \beta_i^{-1}(x) = (1 - x)(g(a_j) - g(a_i)) + (h(a_j) - h(a_i))x \ge 0.$$

This easily implies that whenever $t \in C(\mathbf{Q}_{[0,1)}, \mathbf{Q}_{[0,1]})$ is monotone then so is $t^\sharp$. Thus the restriction of δ from the proof of Lemma 2.1 is a lattice embedding. Since $\delta(0) = g$ and $\delta(1) = h$, K is a 0–1-semifractal. Now define $g, c, d \in K$ by

$$g(x) = \begin{cases} 0 & \text{if } x < \frac{1}{2} \\ \frac{2}{3} & \text{if } x \ge \frac{1}{2} \end{cases}, \quad c(x) = \begin{cases} \frac{1}{3} & \text{if } x < \frac{1}{2} \\ \frac{2}{3} & \text{if } x \ge \frac{1}{2} \end{cases}, \quad d(x) = \begin{cases} 0 & \text{if } x < \frac{1}{2} \\ 1 & \text{if } x \ge \frac{1}{2} \end{cases},$$

and let $h(x) = g(x) + \frac{1}{3}$. Then c is a complement of d in the interval $[g, h]$. Since only 0 and 1 have complements is K, we conclude that there is no 0–1-embedding of $[g, h]$ into $[0, 1]$. Therefore K is not a quasifractal. $\square$

Proof. Now we prove part (4) of Theorem 1.5. Let us say that v/u is a *low critical quotient* of a bounded subdirectly irreducible lattice K if $u < v < 1 = 1_K$ and (u, v) generates the least nontrivial congruence of K.

Let $\mathcal{V}$ be a lattice variety which contains a nondistributive semifractal L. We have to show that for each n an appropriate subdirectly irreducible lattice of $\mathcal{V}$ has a chain consisting of more than n elements.

First we assume that L is modular. We define a sequence of sublattices K_n, $n \in \mathbf{N}_0$. The largest element of K_n will be denoted by w_n. Since L is modular but not distributive, we can choose a sublattice K_0 of L such that K_0 is isomorphic to M_3, cf. Figure 1. Let us fix a low critical quotient v_0/u_0 in K_0. Since L is a semifractal, we can choose a sublattice K_1 in the interval $[u_0, v_0]$ such that K_1 is isomorphic to M_3. Let us fix a low critical quotient v_1/u_1 in K_1 and continue: if K_n has already been chosen with a low critical quotient v_n/u_n then let $K_{n+1} \cong M_3$ be a sublattice of the interval $[u_n, v_n]$ and let v_{n+1}/u_{n+1} be a low critical quotient of K_{n+1}.

Let T_n be the sublattice generated by $K_0 \cup \cdots \cup K_n$. By Zorn Lemma, there is a maximal congruence Θ_n of T_n such that Θ_n does not collapse u_n and v_n. Since $[u_n, v_n] \subseteq [u_i, v_i]$ for $i \le n$, $(u_i, v_i) \notin \Theta_n$. Clearly, $B_n = T_n/\Theta_n$ is a subdirectly irreducible lattice in $\mathcal{V}$. We claim that

$$v_0/\Theta_n > v_1/\Theta_n > v_2/\Theta_n > \cdots > v_n/\Theta_n$$

holds in B_n, which does the job. Since $v_0 > v_1 > \cdots > v_n$, we have $v_0/\Theta_n \ge v_1/\Theta_n \ge \cdots \ge v_n/\Theta_n$. Now, by way of contradiction, suppose that $v_i/\Theta_n = v_{i+1}/\Theta_n$ for some $i < n$. Since $v_i \ge w_{i+1} > v_{i+1}$ and the Θ_n-classes are convex sublattices, (w_{i+1}, v_{i+1}) belongs to Θ_n and also to the restriction $\Theta_n|_{K_{i+1}}$ of Θ_n to

K_{i+1}. But v_{i+1}/u_{i+1} is a critical quotient of K_{i+1}, so we conclude

$$(u_{i+1}, v_{i+1}) \in \Theta_n|_{K_{i+1}} \subseteq \Theta_n ,$$

a contradiction.

The same argument with N_5, the five element nonmodular lattice, instead of M_3 works when L is not modular. $\square$

Proof. Now we prove part (5) of Theorem 1.5. Let P denote the set of prime numbers. For $q \in P$ let $\mathrm{Sub}(F_q^3)$ stand for the lattice of all subspaces of the three-dimensional vector space over F_q. For a subset T of P, let $\mathcal{V}_T$ be the variety generated by $\{\mathrm{Sub}(F_q^3) : q \in T\}$. Since "length ≤ 3" is a first-order property, the classical $\mathbf{P}_s\mathbf{H}\mathbf{S}\mathbf{P}_u$ lemma of Jónsson [18] yields that $\mathcal{V}_T$ is of residually finite length. Part (4) of Theorem 1.5 gives that $\mathcal{V}_T$ is not semifractal generated.

The identity ε_p in Herrmann and Huhn [13] (or $\Delta(p, 1)$ of [16]) holds in $\mathrm{Sub}(F_q^3)$ iff $1 + 1 + \cdots + 1$ (the sum of p copies of the unit element of F_q) is an invertible element of F_q iff $p \neq q$. Therefore ε_p holds in $\mathcal{V}_T$ iff $p \notin T$. This shows that the map $T \mapsto \mathcal{V}_T$ from the set of all subsets of P to the set of lattice varieties that are not semifractal generated is injective. $\square$

Let B be a subalgebra of an algebra A. With a temporary terminology, B will be called a *homogeneously unique subalgebra of* A if for any subalgebra C of A such that $C \cong B$ each isomorphism $\varphi : B \to C$ can be extended to an $A \to A$ automorphism. When speaking of homogeneously unique 0–1-sublattices then lattices are considered as algebras of type $\{\vee, \wedge, 0, 1\}$. For a field F let

$$F^n = \{(a_1, \dots, a_n) : a_1, \dots, a_n \in F\}$$

denote the n-dimensional vector space over F, and let $\mathrm{Sub}(F^n)$ stand for its subspace lattice. Part (C) of the following lemma is due to von Neumann [21]; we will shortly give the embedding for the reader's convenience and for later reference.

Lemma 2.2. (A) *Suppose that* A_n *is a subalgebra of* A_{n+1} *for all* $n \in \mathbf{N}$. *If* A_n *is a homogeneously unique subalgebra of* A_{n+1} *for all but finitely many* $n \in \mathbf{N}$ *then, up to isomorphism, the directed union* $\bigcup_{n\in\mathbf{N}} A_n$ *does not depend on the choice of the* $A_n \to A_{n+1}$ *embeddings.*

(B) *If, for all* $n \in \mathbf{N}$, A_n *is a homogeneously unique subalgebra of* A_{n+1}, B_n *is a homogeneously unique subalgebra of* B_{n+1} *and* B_n *is a homogeneously unique subalgebra of* A_n, *then* $\bigcup_{n\in\mathbf{N}} B_n$ *can be embedded in* $\bigcup_{n\in\mathbf{N}} A_n$.

(C) *If* F *is a field and* $k, n \in \mathbf{N}$ *then* $\mathrm{Sub}(F^n)$ *is a 0–1-sublattice of* $\mathrm{Sub}(F^{kn})$.

(D) *If* F *is a prime field and* $k, n \in \mathbf{N}$ *with* $3 \leq n$ *then* $\mathrm{Sub}(F^n)$ *is a homogeneously unique 0–1-sublattice of* $\mathrm{Sub}(F^{kn})$.

Proof. (A) Since the directed union does not depend on finitely many members, we can assume that A_n is a homogeneously unique subalgebra of A_{n+1} for all $n \in \mathbf{N}$. We will write homomorphisms on the left, so for $\alpha : X \to Y$ and $\beta : Y \to Z$ the composite map is denoted by $\beta \circ \alpha : X \to Z$ and, for $x \in X$, we have $(\beta \circ \alpha)(x) = \beta(\alpha(x))$.

Let $\varphi_n : A_n \to A_{n+1}$ $(n \in \mathbf{N})$ and $\psi_n : A_n \to A_{n+1}$ $(n \in \mathbf{N})$ be two systems of embeddings. Let φ_n^{-1} denote the inverse of φ_n when φ_n is considered as an $A_n \to \varphi_n(A_n)$ map. Subsequent inverses in the proof of the lemma will be understood analogously. We define automorphisms $\gamma_n : A_n \to A_n$ via induction. Let γ_1 be the identical map. If γ_n is already defined then $\varphi_n(A_n)$ and $\psi_n(\gamma_n(A_n))$ are subalgebras of A_{n+1} and both are isomorphic to A_n. Now $\psi_n \circ \gamma_n \circ \varphi_n^{-1}$ is a $\varphi_n(A_n) \to \psi_n(\gamma_n(A_n))$ isomorphism, therefore we can choose an $A_{n+1} \to A_{n+1}$ automorphism γ_{n+1} such that γ_{n+1} extends $\psi_n \circ \gamma_n \circ \varphi_n^{-1}$. Clearly, $\gamma_{n+1} \circ \varphi_n = \psi_n \circ \gamma_n$.

If we think of A_n as a subset of A_{n+1} via identifying $x \in A_n$ with $\varphi_n(x)$ resp. $\psi_n(x)$ of A_{n+1} then it is straightforward to check that $\bigcup_{n\in\mathbf{N}} \gamma_n$ is an isomorphism between the two directed unions. (Indeed, we can easily show that $\bigcup_{n\in\mathbf{N}} \gamma_n^{-1}$ is the inverse mapping.) Another way to derive the same conclusion is to use the fact that direct limits are unique up to isomorphism in category theory, cf. e.g. pp. 76–77 in Freud [7].

(B) Let $\varphi_n : A_n \to A_{n+1}$ resp. $\psi_n : B_n \to B_{n+1}$ be the $x \mapsto x$ embedding and let $\delta_n : B_n \to A_n$ an arbitrary embedding $(n \in \mathbf{N})$. It suffices to define an embedding $\gamma_n : B_n \to A_n$ for each $n \in \mathbf{N}$ such that $\varphi_n \circ \gamma_n = \gamma_{n+1} \circ \psi_n$; indeed, then $\bigcup_{n\in\mathbf{N}} \gamma_n$ is a desired $\bigcup_{n\in\mathbf{N}} B_n \to \bigcup_{n\in\mathbf{N}} A_n$ embedding. Let $\gamma_1 = \delta_1$. If γ_n is already defined then let η_{n+1} be an automorphism of A_{n+1} extending the isomorphism

$$\varphi_n \circ \gamma_n \circ \psi_n^{-1} \circ \delta_{n+1}^{-1} : \delta_{n+1}(\psi_n(B_n)) \to \varphi_n(\gamma_n(B_n)),$$

and let $\gamma_{n+1} = \eta_{n+1} \circ \delta_{n+1}$. Then, for any $x \in B_n$, we have

$$\gamma_{n+1} \circ \psi_n(x) = \eta_{n+1} \circ \delta_{n+1} \circ \psi_n(x) = \varphi_n \circ \gamma_n \circ \psi_n^{-1} \circ \delta_{n+1}^{-1}(\delta_{n+1} \circ \psi_n(x)) = \varphi_n \circ \gamma_n(x),$$

so $\varphi_n \circ \gamma_n = \gamma_{n+1} \circ \psi_n$.

(C) Now, to show that $\mathrm{Sub}(F^n)$ is a 0–1-sublattice of $\mathrm{Sub}(F^{kn})$ for any $k, n \in \mathbf{N}$, consider the vector space embeddings

$$\pi_i : F^n \to F^{kn}, \quad \sum_{j=1}^{n} \alpha_j e_j \mapsto \sum_{j=1}^{n} \alpha_j e_{j+n(i-1)}$$

for $i \in \{1, \dots, k\}$. Then it is straightforward to check that

$$\kappa : \mathrm{Sub}(F^n) \to \mathrm{Sub}(F^{kn}), \quad M \mapsto \sum_{i=1}^{k} \pi_i(M) \tag{1}$$

is a lattice embedding preserving 0 and 1.

(D) Let $3 \le n \in \mathbf{N}$. A *spanning n-diamond* in a bounded modular lattice L is defined to be an $(n+1)$-tuple $\vec{a} = (a_0, a_1, \dots, a_n)$ satisfying $\bigvee_{i\ne j} a_i = 1$ and $a_j \wedge \bigvee_{i\notin\{j,k\}} a_i = 0$ for all $j \ne k \in \{0, 1, \dots, n\}$. This important concept is due to András P. Huhn [13] and [15] but occurs under several names and in equivalent versions in the literature, cf. e.g. Day and Kiss [4]. Let F be a prime field. It follows from Theorem 4.1 in Herrmann and Huhn [14] that $\mathrm{Sub}(F^n)$ is generated by a spanning n-diamond. (This is where we need the assumption that $n \geq 3$ and F is a prime field.) Now suppose that S and T are 0–1-sublattices of $\mathrm{Sub}(F^{kn})$, both being isomorphic to $\mathrm{Sub}(F^n)$, and let $\varphi : S \to T$ be an isomorphism.

Focusing our attention first on S, we have that S is generated by a spanning n-diamond $\vec{A} = (A_0, \ldots, A_n)$. Then $\vec{A}$ is also a spanning n-diamond of $\mathrm{Sub}(F^{kn})$. Hence, by a particular case of Lemma 1 in Herrmann and Huhn [13], F^{kn} is, up to isomorphism, of the form V^n such that $A_0 = \{(u, \ldots, u) : u \in V\}$ and $A_i = \{0\}^{i-1} \times V \times \{0\}^{n-i}$ for $1 \leq i \leq n$. It follows that V is k-dimensional and F^{kn} has a basis $\vec{\mathbf{e}} = (e_1, e_2, \ldots, e_{kn})$ such that

$$\begin{gathered} A_0 = [e_1 + e_{k+1} + \cdots + e_{(n-1)k+1},\, e_2 + e_{k+2} + \cdots + e_{(n-1)k+2},\, \ldots, \\ e_k + e_{2k} + \cdots + e_{nk}], \text{ and} \\ A_i = [e_{(i-1)k+1},\, e_{(i-1)k+2},\, \ldots,\, e_{ik}] \text{ for } 1 \leq i \leq n. \end{gathered}$$

Let $\vec{B} = \varphi(\vec{A})$, i.e. $B_i = \varphi(A_i)$ for $i = 0, \ldots, n$. Then $\vec{B}$ a spanning n-diamond which generates $T = \varphi(S)$. Applying Hermann and Huhn's result again we conclude that F^{kn} has a basis $\vec{\mathbf{f}} = (f_1, f_2, \ldots, f_{kn})$ such that the previous displayed formulas with A and e replaced by B and f are valid. Now the bijection $e_1 \mapsto f_1, \ldots, e_{kn} \mapsto f_{kn}$ extends to an automorphism of F^{kn} which induces an automorphism ψ of $\mathrm{Sub}(F^{kn})$. Since this ψ sends the genarating n-diamond $\vec{A}$ of S to the n-diamond $\vec{B}$ such that $\psi(A_i) = B_i = \varphi(A_i)$, we conclude that ψ extends φ. $\square$

Proof. Now we prove part (1) of Theorem 1.5. Let $F = F_p$ be a prime field. Suppose that $\vec{b} = (b_1, b_2, \ldots)$ and $\vec{c} = (c_1, c_2, \ldots)$ are two strictly increasing sequences in the lattice $(\mathbf{N}, |)$ where the lattice order is the divisibility relation. Then the directed unions $L_F(\vec{b}) = \bigcup_{n \in \mathbf{N}} \mathrm{Sub}(F^{b_n})$ and $L_F(\vec{c}) = \bigcup_{n \in \mathbf{N}} \mathrm{Sub}(F^{c_n})$ are uniquely defined (up to isomorphism) by Lemma 2.2. We say that $\vec{b}$ and $\vec{c}$ are cofinal with each other in $(\mathbf{N}, |)$ if for all i there exist k and ℓ such that $b_i \mid c_k$ and $c_i \mid b_\ell$.

We claim that $(*)$ whenever $\vec{b}$ and $\vec{c}$ are cofinal with each other in $(\mathbf{N}, |)$ then $L_F(\vec{b})$ is (isomorphic to) $L_F(\vec{c})$.

Indeed, then we can choose a strictly increasing sequence $\vec{d} = (d_1, d_2, \ldots)$ in $(\mathbf{N}, |)$ as follows. Let $d_1 = b_1$. Let d_2 be the smallest c_i with $d_1 \mid c_i$ and $d_1 \neq c_i$. Let d_3 be the smallest b_j with $d_2 \mid b_j$ and $d_2 \neq b_j$. Let d_4 be the smallest c_k with $d_3 \mid c_k$ and $d_3 \neq c_k$. And so on we choose elements from $\vec{b}$ and $\vec{c}$ alternately. Now $\vec{d}^{\,(\mathrm{odd})} = (d_1, d_3, d_5, \ldots)$ is a subsequence of both $\vec{d}$ and $\vec{b}$, and in both cases $\vec{d}^{\,(\mathrm{odd})}$ is cofinal with the original sequence. This yields that $L_F(\vec{b}) \cong L_F(\vec{d}^{\,(\mathrm{odd})}) \cong L_F(\vec{d})$. Similarly, working with $\vec{d}^{\,(\mathrm{even})} = (d_2, d_4, d_6, \ldots)$ we obtain $L_F(\vec{c}) \cong L_F(\vec{d})$. Therefore $L_F(\vec{b}) \cong L_F(\vec{c})$, proving $(*)$.

We also claim that $(**)$ whenever $b_n \mid c_n$ for all n then $L_F(\vec{b})$ has a 0–1–embedding into $L_F(\vec{c})$.

This is a straightforward consequence of Lemma 2.2 and the fact that, after disregarding from finitely many members, we can assume that all the b_i and c_j are greater than 2.

Now we define continuously many 0–1-semifractals in $\mathcal{V}_p$. Let us consider a subset $H = \{p_1, p_2, p_3, \ldots\}$ of the set P of prime numbers such that $2 \in H$. Let $h_n = p_1^n p_2^{n-1} p_3^{n-2} \ldots p_n$. It is worth noting that $h_0 = 1$ and $h_n = h_{n-1} p_1 p_2 \ldots p_n$ for $n > 0$. Then $\vec{h} = (h_0, h_1, h_2 \ldots)$ is a strictly increasing sequence in $(\mathbf{N}, |)$, so we can consider the lattice $L_H = L_F(\vec{h})$. To make this definition unambiguous we assume that $p_1 < p_2 < \cdots$ when H is infinite while $p_1 < \cdots < p_n$ and $p_{kn+i} = p_i$ for $k \in \mathbf{N}$ and $i \in \{1, \ldots, n\}$ when $|H| = n \in \mathbf{N}$.

Consider a nontrivial interval $[u, v]$ in L_H. Then $u, v \in \mathrm{Sub}(F^{h_t})$ for some t. Keeping t fixed, let $A_i = [u, v] \cap \mathrm{Sub}(F^{h_{t+i}})$. Then $[u, v] = \bigcup_{i \in \mathbf{N}} A_i$. Let g_0 be the length of A_0 and define $g_i = g_{i-1} p_1 p_2 \ldots p_{t+i}$ for $i > 0$. This way we have a sequence $\vec{g} = (g_0, g_1, \ldots)$. Since any interval of finite length ℓ in a subspace lattice over F is isomorphic to $\mathrm{Sub}(F^\ell)$, we have $A_0 \cong \mathrm{Sub}(F^{g_0})$. We claim that for all $i \in \mathbf{N}_0$ we have

$$A_i \cong \mathrm{Sub}(F^{g_i}).$$

To show this, assume that $i > 0$ and $A_{i-1} \cong \mathrm{Sub}(F^{g_{i-1}})$. Using the fact that the particular embedding κ given in formula (1) has the property $\dim(\kappa(M)) = k \dim(M)$ for any $M \in \mathrm{Sub}(F^n)$, and the same is true for any other 0–1-embedding by Lemma 2.2[1], we conclude that $\dim(u)$ in $\mathrm{Sub}(F^{h_{t+i}})$ is $p_1 p_2 \ldots p_{t+i}$ times $\dim(u)$ in $\mathrm{Sub}(F^{h_{t+i-1}})$. The same holds for $\dim(v)$ and, consequently, for $\dim(v) - \dim(u)$. This implies that $A_i \cong \mathrm{Sub}(F^{g_{i-1} p_1 p_2 \cdots p_{t+i}}) = \mathrm{Sub}(F^{g_i})$, indeed.

The above arguments show that

$$[u, v] = \bigcup_{i \in \mathbf{N}} A_i \cong L_F(\vec{g}).$$

When $H = P$, the set of all primes, then $\vec{h}$ and $\vec{g}$ are cofinal with each other and $(*)$ yields $[u, v] \cong L_F(\vec{g}) \cong L_F(\vec{h}) = L_P$. This shows that L_P is a fractal. When $\{2\} \subseteq H \subset P$ then $h_n \mid g_n$ for all n and $(**)$ gives a 0–1-embedding of $L_H = L_F(\vec{h})$ into $L_F(\vec{g}) \cong [u, v]$. This shows that L_H is a 0–1-semifractal.

To prove both that $H_1 \neq H_2$ implies $L_{H_1} \neq L_{H_2}$ and that $H \neq P$ implies L_H is not a quasifractal, it suffices to show that, for any prime q, $\mathrm{Sub}(F^q)$ is always a sublattice of L_H but it is a 0–1-sublattice of L_H iff $q \in H$.

Take an n with $q \leq h_n$. Then $L(F^{h_n})$ has an iterval of length q. Hence $\mathrm{Sub}(F^q)$ can be embedded into $L(F^{h_n})$ and therefore also into L_H. If $q \in H$ then $q \mid h_n$ for some n. By Lemma 2.2 $\mathrm{Sub}(F^q)$ has a 0–1-embedding into $L(F^{h_n})$, and therefore also into L_H.

Now assume that $\mathrm{Sub}(F^q)$ is 0–1-embedded in L_H. According to Herrmann and Huhn [14], we can choose a spanning q-diamond $\vec{A} = (A_0, \ldots, A_q)$ which generates $\mathrm{Sub}(F^q)$. This q-diamond consists only of $q + 1$, i.e. finitely many, elements. So there is an h_n such that $L(F^{h_n})$ includes this spanning q-diamond. Like in the proof of part (D) of Lemma 2.2, from Lemma 1 in Herrmann and Huhn [13] we conclude that h_n, the dimension of F^{h_n} is a multiple of q. This implies $q \in H$.

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[1] Lemma 2.2 applies only for $g_0 \geq 3$. The case $g_0 = 1$ is trivial while the case $g_0 = 2$ follows easily from modularity.

Finally, the F^{h_n} are simple lattices whence so is their union, L_H. Since $\mathrm{Sub}(F^2)$ has a spanning M_3, so have all the L_H, $\{2\} \subseteq H \subseteq P$. Any variety is closed with respect to directed unions, so we have $L_H \in \mathcal{V}_p$. Since the F_p-vector spaces form a congruence permutable variety, the theory of Mal'cev conditions, cf. Wille [25] and Pixley [23], or cf. also [16], implies that $\mathcal{V}_p$ is generated by $\{\mathrm{Con}(F^{h_n}) : n \in \mathbf{N}\}$. But $\mathrm{Con}(F^{h_n}) \cong \mathrm{Sub}(F^{h_n})$ is a sublattice of L_H for any $n \in \mathbf{N}$, whence we conclude that L_H generates $\mathcal{V}_p$. $\square$

Notice that $L_{\{k\}}$, i.e. our 0–1-semifractal with $H = \{k\}$, is just von Neumann's example of a lattice that has a normalized dimension function without being of finite length, cf. [21]. Although the $L_{\{k\}}$ and any other L_H are far from being continuous or even from complete lattices, the metric completion $\overline{L_{\{k\}}}$ of $L_{\{k\}}$ is a continuous geometry and has the cardinality of continuum, cf. [21] (cf. also pages 161–162 in Crawley and Dilworth [3]). While, according to our proof, distinct values of k give distinct 0–1-semifractals $L_{\{k\}}$, von Neumann showed that $\overline{L_{\{k\}}}$ does not depend on k (cf. page vi in the foreword of [21]). Using his ideas it is trivial that each of our 0–1-semifractals L_H has a normalized dimension function.

While this paper was under refereeing, E. Tamás Schmidt and Luca Giudici remarked that many uncountable modular lattices obtained from regular or bisimple rings are fractals in our sense, cf. Hannah [12], Munn [20] and, for a very detailed overview, Giudici [9].

Now, to prepare the rest of the proof of Theorem 1.5, we give two lattice constructions. Let L be a lattice with a nontrivial interval $[a, b]$, and let $H = \{H_i : i \in I\}$ be a system of bounded lattices. (The H_i are not necessarily distinct.) Take an isomorphic copy K_i of H_i for each $i \in I$, disjoint from L and from K_j for $j \in L \setminus \{i\}$, and identify the least elements of these K_i with a and their greatest elements with b. This way we obtain a poset denoted by $L \mathbin{\bigcirc\!\!\!\!\!\lhd}_{[a,b]} H$.

For example, when $L = N_5$, b/a is the critical quotient of N_5, H_1 is the five element chain, $H_2 = H_1$, H_3 is obtained from the four element Boolean lattice by adding a new 0 and new 1 to it, $I = \{1, 2, 3\}$, and $H = \{H_i : i \in I\}$ then this poset, i.e. $N_5 \mathbin{\bigcirc\!\!\!\!\!\lhd}_{[a,b]} \{H_1, H_2, H_3\}$, is depicted on the left in Figure 2. Notice that a particular case of the $L \mathbin{\bigcirc\!\!\!\!\!\lhd}_{[a,b]} H$ construction occurs in Lihová [22].

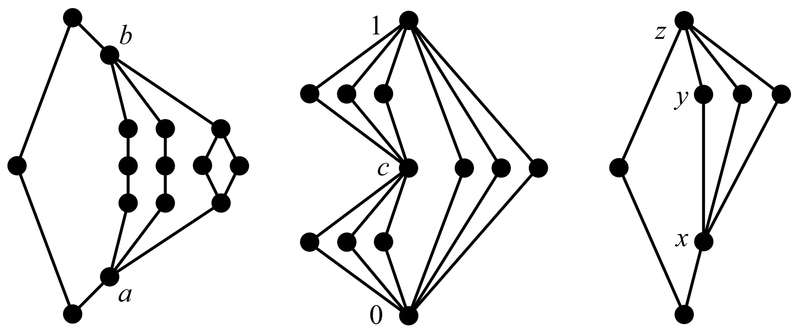

Figure 2

When we form $L \circledcirc_{[a,b]} H$ for all nontrivial intervals $[a,b]$ of L at the same time, adding disjoint elements to distinct intervals, then the poset we obtain is denoted by $L \circledcirc H$. For example, when $L = \mathbf{3} = \{0, c, 1\}$, the three element chain, $H_1 = M_3$, and $H = \{H_1\}$ then this poset, i.e. $\mathbf{3} \circledcirc \{M_3\}$, is depicted in the middle of Figure 2.

The following construction will be used to provide M_3-simplicity. For a lattice L and $x < y < z$ in L we obtain the poset $L \oplus_{x,y,z} M_3$ by identifying x, y, z of L with the respective elements $0, a, 1$ of M_3 in the disjoint union of L and M_3. For example, if $L = N_5$ and $0 < x < y < z = 1$ then $N_5 \oplus_{x,y,z} M_3$ is depicted on the right in Figure 2. When we form $L \oplus_{x,y,z} M_3$ for all choices of $x < y < z$ at the same time, using disjoint copies to distinct choices, then the poset obtained is denoted by $L \oplus M_3$.

Lemma 2.3. *All of the posets $L \circledcirc_{[a,b]} H$, $L \circledcirc H$, $L \oplus_{x,y,z} M_3$ and $L \oplus M_3$ are lattices.*

Proof. When L is $\mathbf{2}$, the two element lattice, then $A = \mathbf{2} \circledcirc_{[0,1]} H$ is obviously a lattice. Otherwise $L \circledcirc_{[a,b]} H = L \circledcirc_{[a,b]} \{A\}$, and a straightforward calculation shows that this is a lattice.

Let $\{b_\iota / a_\iota : \iota < \alpha\}$ be the set of nontrivial quotients of L. Here α is an ordinal and $a_\iota < b_\iota$ for all $\iota < \alpha$. Let $L_0 = L \circledcirc_{[a_0,b_0]} H$, for $\lambda < \alpha$ let $L_{\lambda+1} = L_\lambda \circledcirc_{[a_{\lambda+1},b_{\lambda+1}]} H$, and let $L_\lambda = \left(\bigcup_{\iota<\lambda} L_\iota\right) \circledcirc_{[a_\lambda,b_\lambda]} H$ when λ is a limit ordinal. Then $L \circledcirc H = \bigcup_{\lambda<\alpha} L_\lambda$. Since any directed union of lattices is a lattice, we conclude that $L \circledcirc H$ is a lattice.

It is straightforward again that $L \oplus_{x,y,z} M_3$ is a lattice, and we can resort to the previous direct limit argument to conclude that $L \oplus M_3$ is a lattice. $\square$

Proof. Now we prove part (6) of Theorem 1.5. For an arbitrary lattice X let $\mathcal{I}(X)$ denote the set of nontrivial intervals of X. So $\mathcal{I}(X)$ consists of lattices and distinct members of $\mathcal{I}(X)$ can be isomorphic. We say that X *extends* to another lattice Y if X is a 0–1-sublattice of Y. Let $L = L_0$ be an arbitrary lattice with $|L| \geq 3$. Define a sequence of lattices as follows: $L'_n = L_n \oplus M_3$ and let $L_{n+1} = L'_n \circledcirc \mathcal{I}(L'_n)$.

Notice that when passing from L'_n to L_{n+1} then each interval $[c,d]$ of L'_n is doubled in some but not in the exact sense. Namely, a new isomorphic copy of $[c,d]$ (together with other intervals) is glued to the "old" interval $[c,d]$ at c and d. The inner elements (i.e., those distinct from c and d) of this new copy form a convex subset of L_{n+1} but this is not the case for the "old" inner elements of the original interval $[c,d]$.

We consider L_n as a 0–1-sublattice of L'_n and L'_n as a 0–1-sublattice of L_{n+1} in the natural way. For $x \in L'_n$ when x is regarded as an element of L_{n+1} then we often denote this element by $(x,-)$. For $a < b \in L'_n$ and $c \leq x \leq d \in L'_n$ let $(a,b;c,x,d)$ denote the element x in the "flap" (i.e., interval) $[c,d]$ glued into the interval $[a,b]$. With this notation, L_{n+1} consists of the elements $(x,-)$ and the elements $(a,b;c,z,d)$. Notice that $(a,b;c,c,d) = (a,-)$ and $(a,b;c,d,d) = (b,-)$. We usually denote an element of L_{n+1} by a single letter like x when we do not want to specify if it is of the form $(x,-)$ or $(a,b;c,z,d)$.

Let L_ω be the (directed) union of these lattices. If $x < y < z$ in L_ω then $x < y < z$ in L_n for some n, whence there is an appropriate M_3 in $L'_n \subseteq L_\omega$. This shows that L_ω is M_3-simple.

To show that L_ω is a quasifractal, it suffices to show that for any $a < b$ in L_ω, there is a 0–1-embedding $[a, b] \to L_\omega$ and another 0–1-embedding $L_\omega \to [a, b]$. Let $U = [a, b]$. We can assume that $(a, b) \neq (0, 1)$. Choose an $m < \omega$ such that $a, b \in L_m$. For $n \geq m$ let $U_n = [a, b] \cap L_n$ and $U'_n = [a, b] \cap L'_n$. Clearly, $U = [a, b]$ is equal to $\bigcup_{n \geq m} U_n$.

In one direction, it suffices to define a 0–1-embedding $\varphi_n : U_n \to L_{n+1}$ for each $n \geq m$ such that φ_{n+1} extends φ_n. Indeed, in this case the union of these embeddings will be a desired $U \to L_\omega$ embedding.

When L_{m+1} is formed, a copy of U'_m is glued to the interval $[0, 1]$ of L'_m. But U'_m extends U_m, so there is a natural embedding φ_m of U_m into the new copy of U'_m just glued. This φ_m is an $U_m \to L_{m+1}$ 0–1-embedding. Now suppose $\varphi_n : U_n \to L_{n+1}$ is already defined. Clearly, $U'_n = U_n \oplus M_3$. First we extend φ_n to an embedding $\varphi'_n : U'_n \to L'_{n+1}$ that sends the copy of M_3 attached to $x < y < z \in U_n$ onto the copy of M_3 attached to $\varphi_n(x) < \varphi_n(y) < \varphi_n(z) \in L_{n+1}$. Now U_{n+1} consists of the elements $(x, -)$ for $x \in U'_n$ and the elements $(c, d; u, z, v)$ for $c < d \in U'_n$ and $u < z < v \in L'_n$. Let $\varphi_{n+1}(x, -) = (\varphi'_n(x), -)$ and $\varphi_{n+1}(c, d; u, z, v) = (\varphi'_n(c), \varphi'_n(d); u, z, v)$. (Here we used that $u, z, v \in L'_n$ are also elements of L'_{n+1}.) Then $\varphi_{n+1} : U_{n+1} \to L_{n+2}$ is a 0–1-embedding that extends φ_n, as desired.

Now, in the other direction, it suffices to define 0–1-embeddings $\psi_n : L_n \to U_{n+1}$ for each $n \geq m$ such that ψ_{n+1} extends ψ_n. Indeed, in this case the union of these embeddings will be a desired $L_\omega \to U$ embedding. When forming L_{m+1}, a copy of L'_m is glued to $[a, b]$, i.e. into U'_m. This copy becomes a 0–1-sublattice of U_{m+1}. But L'_m extends L_m, so we have a 0–1-embedding $\psi_m : L_m \to U_{m+1}$. Now suppose that $\psi_n : L_n \to U_{n+1}$ is already defined. We can extend it to a 0–1-embedding $\psi'_n : L'_n \to U'_{n+1}$ that sends the M_3 attached to $x < y < z \in L_n$ onto the M_3 attached to $\psi_n(x) < \psi_n(y) < \psi_n(z) \in U_{n+1}$, for any $x < y < z \in L_n$. Now we extend ψ'_n to an embedding $\psi_{n+1} : L_{n+1} \to U_{n+2}$ as follows. For $x \in L'_n$, $c < d \in L'_n$ and $u < z < v \in L'_n$, let $\psi_{n+1}(x, -) = (\psi'_n(x), -)$ and $\psi_{n+1}(c, d; u, z, v) = (\psi'_n(c), \psi'_n(d); u, z, v)$. (Here we used that $u, z, v \in L'_n$ are also elements of L'_{n+1}.) This embedding extends ψ_n, as desired.

We have seen that each lattice L with $|L| \geq 3$ extends to an M_3-simple quasifractal L_ω. Now, given a chain $I = (I, \leq)$ and a lattice K_i for each $i \in I$, by the ordinal sum $\sum_{i \in I} K_i$ we mean $(\{(a, i) : i \in I,\ a \in K_i\}, \leq)$ where $(a, i) \leq (b, j)$ means $i < j$ or $i = j$ with $a \leq b$.

Now let T be a subset of P, the set of prime numbers. Let K_1 be the free lattice on $\aleph_0$ free generators and let K_p be $\mathrm{Sub}(F_p^3)$ for $p \in T$. Then $\{1\} \cup T$ has its natural order, so we can form $L = \sum_{i \in \{1\} \cup T} K_i$. I.e., $L = L(T)$ is the ordinal sum of the free lattice and the $\mathrm{Sub}(F_p^3)$ for $p \in T$.

Clearly, $L(T)$ is a countable lattice generating the variety of all lattices, whence so is $L_\omega = L(T)_\omega$. To show that different subsets T give different quasifractals $L(T)_\omega$ it suffices to show that $\mathrm{Sub}(F_p^3)$ is embeddable in $L(T)_\omega$ iff $p \in T$. To prove the nontrivial direction, assume that $p \notin T$. For any prime q, $\mathrm{Sub}(F_q^3)$ is generated by any of its spanning 3-diamonds by Herrmann and Huhn [14]. Hence $\mathrm{Sub}(F_p^3)$ cannot be embedded in $\mathrm{Sub}(F_q^3)$ for $q \neq p$, for otherwise the sublattice generated by a 3-diamond would consist of exactly

$$|\mathrm{Sub}(F_q^3)| = 1 + \frac{q^3-1}{q-1} + \frac{q^3-1}{q-1} + 1 = 2q^2 + 2q + 4$$

elements and exactly $|\mathrm{Sub}(F_p^3)| = 2p^2 + 2p + 4$ elements, a contradiction. The free lattice is semidistributive by Jónsson and Kiefer [19], therefore its modular sublattices are distributive, whence $\mathrm{Sub}(F_p^3)$ cannot be embedded in the free lattice.

We conclude that $\mathrm{Sub}(F_p^3)$ cannot be embedded in $L(T) = L$. By way of contradiction suppose now that $\mathrm{Sub}(F_p^3)$ is embeddable in L_ω. Then there is a smallest m such that either $\mathrm{Sub}(F_p^3)$ is embeddable in L'_m or it is embeddable in L_{m+1}.

In the first case some element u of $L'_m \setminus L_m$ must belong to (a copy of) $\mathrm{Sub}(F_p^3)$. Then u is either $\vee$-reducible or $\wedge$-reducible in $\mathrm{Sub}(F_p^3)$, which contradicts the fact that every element of $L'_m \setminus L_m$ is both $\vee$-irreducible and $\wedge$-irreducible in L'_m.

The other case is when $\mathrm{Sub}(F_p^3)$ is embedded in L_{m+1} but not in L'_m. For $a < b$ and $c < d$ in L'_m let us call $E = E(a,b,c,d) = \{(a,b;c,z,d) : c \leq z \leq d\}$ a *flap*. This is just a recently inserted interval. If $c < z < d$ then $(a,b;c,z,d)$ is called an inner element of the flap. The elements of $\mathrm{Sub}(F_p^3)$ of height 1 resp. 2 will be called points resp. lines. Since $\mathrm{Sub}(F_p^3)$ is not a sublattice of L'_m and $\mathrm{Sub}(F_p^3)$ is generated by its points, there exists a point $u \in \mathrm{Sub}(F_p^3) \setminus L'_m$. This u is necessarily an inner element of some flap E.

Now if h, g are lines of $\mathrm{Sub}(F_p^3)$ with $h \wedge g = u$ then both h and g must belong to E. Indeed, otherwise $h \wedge g$ would be h or g or the bottom of E (which is $(a,-)$ in L_{m+1} and the zero of $\mathrm{Sub}(F_p^3)$). We have seen that if a point u belongs to E then any two lines whose intersection is u belongs to E. Therefore if u is in E then any lines above u is in E. The dual argument gives that if E contains a line then E contains all points below that line. It follows from these observations that for any point $v \neq u$, the line $u \vee v$ and then v belongs to E. But $\mathrm{Sub}(F_p^3)$ is generated by its points, so $\mathrm{Sub}(F_p^3)$ is a sublattice of E. This is a contradiction, for L'_m contains an interval isomorphic to E but we assumed that $\mathrm{Sub}(F_p^3)$ cannot be embedded in L'_m. □

3. Convexities of lattices

A class $\mathcal{V}$ of lattices is called a *convexity* if it is closed with respect to the operators $\mathbf{H}$ of forming homomorphic images, $\mathbf{C}$ of forming convex sublattices and $\mathbf{P}$ of forming direct products. This notion is due to Ervin Fried [8]. He showed that, given a class $\mathcal{V}$ of lattices, the least convexity containing this class is $\mathbf{HCP}\mathcal{V}$, and he asked how many convexities exist. The answer was given by Jakubík [17], who showed that convexities form a proper class and, apart from the fact that they do not constitute a set, they form a complete lattice with respect to inclusion. The least element of this lattice is the class of trivial lattices. Jakubík [17] also showed that $\mathbf{HCP}\{\mathbf{2}\}$ is an atom in this lattice. In other words, $\mathbf{HCP}\{\mathbf{2}\}$ is a so-called *minimal convexity*. He raised the question if there exists another minimal convexity or not. Although we still do not know the answer, this section of the paper will shed more light on the problem.

In some sense, it is not a surprise that $\mathbf{HCP}\{\mathbf{2}\}$ is an atom, for much more is true. Namely, if L is a lattice which has either a nontrivial distributive interval (in particular, if there are $a, b \in L$ with $a \prec b$) or a nontrivial distributive homomorphic image then $\mathbf{HCP}\{\mathbf{2}\} \subseteq \mathbf{HCP}\{L\}$. This follows easily from the prime ideal theorem. Notice that the above condition on L is not necessary. Indeed, Lihová [22] has recently given a lattice L with $\mathbf{HCP}\{\mathbf{2}\} \subset \mathbf{HCP}\{L\}$ such that, surprisingly, L is simple and it has no nontrivial distributive interval. She also generalized the classical Jónsson [18] lemma as follows: if $\mathcal{V}$ is any class of lattices then

$$\mathbf{HCP}\mathcal{V} = \mathbf{P}_s\mathbf{HCP}_u\mathcal{V}.$$

Knowing that the lattice of subvarieties of any variety of algebras is atomic, cf. e.g. Thm. 14.21 in Burris and Sankappanar [2], the following theorem is somewhat surprising.

Theorem 3.1. (A) *Let F be a simple 0–1-semifractal with a spanning M_3. (For example, let F be one of the fractals from part* (1) *of Theorem 1.5.) Then $\mathbf{HCP}\{\mathbf{2}\}$ is not a subclass of the convexity $\mathbf{HCP}\{F\}$.*

(B) *Let M be an M_3-simple semifractal. (For example, let M be one of the quasifractals from part* (6) *of Theorem 1.5.) Then no minimal convexity is a subclass of the convexity $\mathbf{HCP}\{M\}$.*

(C) *Let $\mathcal{C}$ be the collection of all lattice convexities $\mathcal{V}$ such that $\mathcal{V}$ has no minimal subconvexity. Then $\mathcal{C}$ is a proper class, not a set.*

Notice that if F and M are taken from Theorem 1.5 then the convexities in part (A) consist of modular lattices but this is not the case in part (B).

Proof. (A) By way of contradiction let us assume that $\mathbf{HCP}\{\mathbf{2}\} \subseteq \mathbf{HCP}\{F\}$. Lihová's result gives $\mathbf{2} \in \mathbf{HCP}_u\{F\}$. Hence there is an ultrapower K of F and a convex sublattice S of K such that $\mathbf{2}$ is a homomorphic image of S. Let Θ be the kernel of a surjective $S \to \mathbf{2}$ homomorphism. Observe that S is either a nontrivial interval of K or a directed union of nontrivial intervals. In both cases we obtain that the congruence Θ, restricted to an appropriate nontrivial interval J of K, has exactly two classes. By Proposition 1.3, K is a 0–1-semifractal. Further, K has a spanning M_3 since this is a first-order property. Therefore J has a spanning M_3 as well. Since Θ does not collapse the bottom and top elements of the spanning M_3 of J, the restriction of Θ to this M_3 has exactly two classes. But this is a contradiction, for M_3 is a simple lattice.

(B) First we claim that ($\ast\ast\ast$) if M is an M_3-simple semifractal and L is a subdirectly irreducible lattice in $\mathbf{HCP}\{M\}$ then L is also an M_3-simple semifractal and $|L| \geq |M|$.

The proof of ($\ast\ast\ast$) starts with applying Lihová's result again: there is an ultrapower K of M and a convex sublattice S of K such that L is a homomorphic image of S. Clearly, $|K| \geq |M|$, for the constant choice functions are distinct modulo any ultrafilter. Using Proposition 1.3 and the fact that M_3-simplicity is a first-order property we conclude that K is an M_3-simple semifractal. So is S, for these properties are inherited by convex sublattices. Since K is a semifractal, $|S| = |K|$. Finally, $L \cong S$, for S is simple. This proves $(***)$.

Now let $\mathcal{V}$ be an arbitrary nontrivial convexity with $\mathcal{V} \subseteq \mathbf{HCP}\{M\}$. Since $\mathcal{V}$ contains a nontrivial lattice, which must have a subdirectly irreducible homomorphic image, $\mathcal{V}$ contains a subdirectly irreducible lattice L. In view of $(***)$, L is an M_3-simple semifractal with $|L| \geq |M|$. Let M' be an ultrapower of L with $|M'| > |L|$. Then M' is again an M_3-simple semifractal by Proposition 1.3. Since $M' \in \mathbf{P}_u\mathcal{V} \subseteq \mathbf{HP}\mathcal{V} \subseteq \mathcal{V}$, we have that $\mathbf{HCP}\{M'\}$ is a subconvexity of $\mathcal{V}$. It follows from $|M| < |M'|$ by $(***)$ that $M \notin \mathbf{HCP}\{M'\}$. Hence $\mathbf{HCP}\{M'\} \subset \mathcal{V}$, showing that $\mathcal{V}$ is not minimal.

(C) It follows from Proposition 1.3 and part (6) of Theorem 1.5 that for each infinite cardinal α there is an M_3-simple semifractal M_α with $|M_\alpha| = \alpha$. Part (B) gives that the convexity $\mathbf{HCP}\{M_\alpha\}$ has no minimal subconvexity. These convexities are distinct, for $\alpha < \beta$ implies $M_\alpha \notin \mathbf{HCP}\{M_\beta\}$ in view of $(***)$. $\square$

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