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Lattice Generation of Small Equivalences of a Countable Set

Dedicated to George Grätzer on his 60th birthday

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Abstract. Given a countable set A, let $\mathrm{Equ}(A)$ denote the lattice of equivalences of A. We prove the existence of a four-generated sublattice Q of $\mathrm{Equ}(A)$ such that Q contains all atoms of $\mathrm{Equ}(A)$. Moreover, Q can be generated by four equivalences such that two of them are comparable. Our result is a reasonable generalization of Strietz [5, 6] from the finite case to the countable one; and in spite of its essentially simpler proof it asserts more for the countable case than [2, 3].

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Given a set A, let $\mathrm{Equ}(A)$ denote the lattice of equivalences of A. For a finite A, $\mathrm{Equ}(A)$ is generated by its atoms. Strietz [5, 6] has shown that $\mathrm{Equ}(A)$ is four-generated, provided A is finite and has at least three elements. Moreover, Zádori [7] has shown that $\mathrm{Equ}(A)$ is $(1+1+2)$-generated if $|A| \geqslant 7$ (cf. also Strietz [6] for $|A| \geqslant 10$); i.e., we can assume that exactly two of the four generators are comparable. These results have recently been generalized for many infinite sets A (for all A such that no inaccessible cardinal is $\leqslant |A|$) in [2, 3], but in spite of the complicated construction and long proof, complete lattices with infinitary lattice operations (even to produce the atoms when A is countable) were necessary in this generalization. Analogous investigations for (involution) lattices of quasiorders can be found in [1]. For an overview on equivalence lattices the reader is referred to Rival and Stanford [4].

The purpose of this short note is, firstly, to extend Strietz and Zádori's above-mentioned result for the countable case such that only binary lattice operations are allowed, and, secondly, to present a proof for the countable case that is essentially simpler than [2, 3].

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THEOREM 1. *Let A be a countable set. Then there is a four-generated sublattice Q of* $\mathrm{Equ}(A)$ *such that Q contains all the atoms of* $\mathrm{Equ}(A)$. *Moreover, Q can be generated by a four-element subset of type $1+1+2$, and also by a four-element antichain.*

Notice that three equivalences would be insufficient; this follows from Zádori [7, Lemma 2 and p. 580]. Q in Theorem 1 cannot be the sublattice $\mathrm{Equ}_{\mathrm{fin}}(A)$ generated by the atoms of $\mathrm{Equ}(A)$, for $\mathrm{Equ}_{\mathrm{fin}}(A)$ is not finitely generated. Since $|A| = |\mathrm{Equ}_{\mathrm{fin}}(A)|$ for any infinite A, Theorem 1 cannot hold for sets with more than countably many elements.

Proof. Let $\mathbf{N}_0 = \{0,1,2,3,\dots\}$, $A = \{a_i^k \colon k \in \mathbf{N}_0,\ 0 \leqslant i \leqslant k+12\} \cup \{b_i^k \colon k \in \mathbf{N}_0,\ 0 \leqslant i \leqslant k+11\}$. The subsets $L^k = \{a_i^k \colon i \leqslant k+12\} \cup \{b_i^k \colon i \leqslant k+11\}$, $k \in \mathbf{N}_0$, constitute a partition of A. For p and q in A the smallest equivalence collapsing p and q will be denoted by $\langle p, q\rangle$. This is an atom in $\mathrm{Equ}(A)$ if $p \neq q$. The letters x and y (and only these two) will always be used as index variables that take all permitted values from $\mathbf{N}_0$. This convention allows us to use brief notations like $\langle a_y^x, b_{18-y}^{x-1} \colon y < 4\rangle$, which denotes the sum of all those atoms $\langle a_j^i, b_{18-j}^{i-1}\rangle$ that satisfy the explicit condition(s) (now $j < 4$) and the implicit conditions (expressing that both elements belong to A, currently $j \leqslant i+12$ and $0 \leqslant 18-j \leqslant i-1+11$). (The explicit condition can be missing.) The lattice operations will be denoted by $+$ (join) and $\cdot$ (meet). We define four equivalences:

$$\alpha = \langle a_y^x, a_{y+1}^x\rangle + \langle b_y^x, b_{y+1}^x\rangle, \qquad \beta = \langle a_y^x, b_y^x\rangle + \langle b_6^x, a_6^{x+1}\rangle,$$

$$\gamma = \langle a_{y+1}^x, b_y^x\rangle + \langle b_4^x, a_2^{x+1}\rangle + \langle b_5^x, a_3^{x+1}\rangle + \langle b_{x+7}^x, a_{x+11}^{x+1}\rangle + \langle b_{x+8}^x, a_{x+12}^{x+1}\rangle$$

and

$$\delta = \langle a_0^x, a_{x+12}^x\rangle + \langle b_0^x, b_{x+11}^x\rangle.$$

Notice that α, β and γ are represented by horizontal, vertical and oblique (straight) edges, respectively, in Figure 1, while the dotted curves stand for δ. For example, $(p,q) \in \gamma$ iff p and q can be connected by a path consisting of oblique edges of the graph depicted in Figure 1. We will show that the sublattice Q generated α, β, γ and δ contains all atoms. Now $\delta < \alpha$. If we want a generating antichain, we can replace δ, by, say, $\delta' = \delta + \langle a_0^0, b_0^0\rangle$; then $\{\alpha,\beta,\gamma,\delta'\}$ is an antichain, and $[\alpha,\beta,\gamma,\delta'] \supseteq Q$ by $\delta = \delta' \cdot \alpha$.

All the equivalences we define in the sequel will belong, either we emphasize this or not, to Q. Given an equivalence Θ, we say that a subset $X \subseteq A$ is preserved by Θ if for all $p \in X$ the whole Θ-class of p is included in X. For later use we formulate an easy fact, to be referred to as *circle principle*: let $p = u_0, u_1, \dots, u_i = q, u_{i+1}, \dots, u_{i+j-1}, u_{i+j} = p$ be a circle in the graph

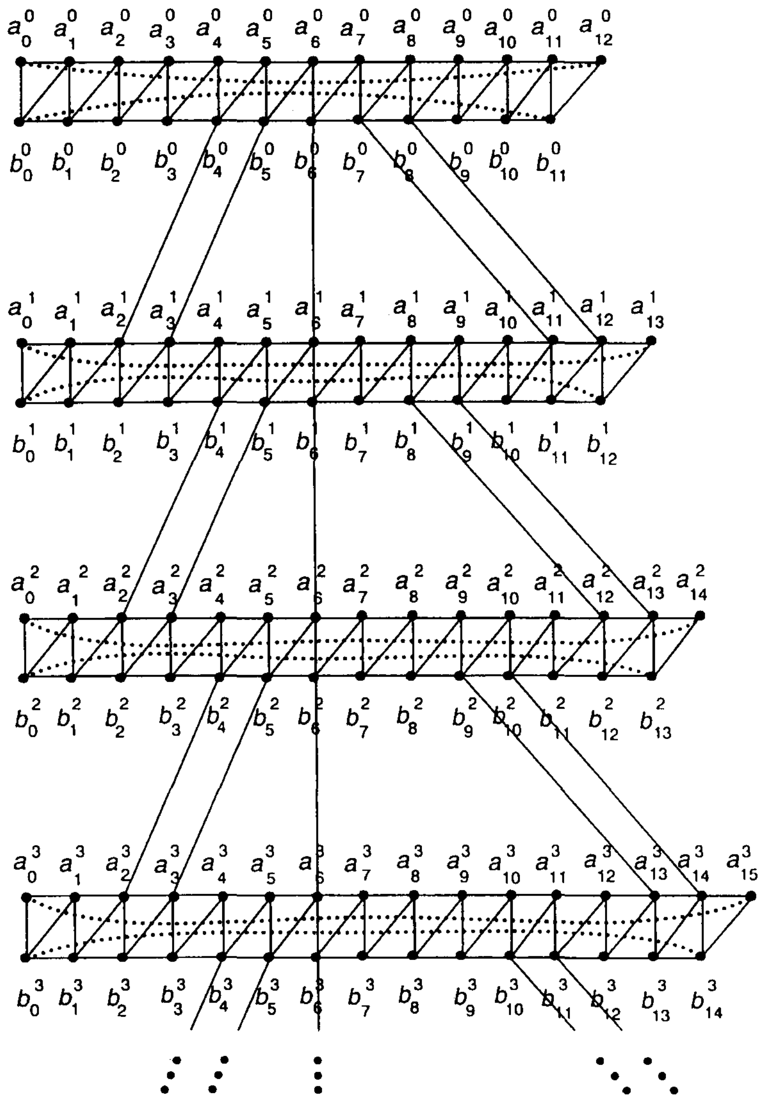

Figure 1.

depicted in Figure 1 such that $|\{u_0, u_1, \ldots, u_{i+j-1}\}| = i+j$; if all the $\langle u_{\ell-1}, u_\ell\rangle$ belong to Q, then so does $\langle p, q\rangle$.

Let I(k) denote the condition "$\langle p, q\rangle \in Q$ for all $p, q \in L^k$". We prove I(k) for all k via induction.

Let $g_0^0 = \beta(\gamma+\delta)$ and $H_0^0 = \gamma(\beta+\delta)$. Using $\beta\gamma = \beta\delta = \gamma\delta = 0$ it is easy to see that $g_0^0 = \langle a_0^x, b_0^x\rangle$ and $H_0^0 = \langle a_{x+12}^x, b_{x+11}^x\rangle$. Let $\beta_0 := (\alpha + g_0^0)\beta$ and

$\gamma_0 = (\alpha + g_0^0)\gamma$. Notice that the restriction of β_0 (resp. γ_0) and that of β (resp. γ) to any L^k coincide. Define some further equivalences via induction:

$$h_{i+1}^0 = ((g_i^0 + \gamma_0)\alpha + g_i^0)\gamma_0, \qquad g_{i+1}^0 = ((h_{i+1}^0 + \beta_0)\alpha + h_{i+1}^0)\beta_0,$$

and

$$G_{i+1}^0 = ((H_i^0 + \beta_0)\alpha + H_i^0)\beta_0, \qquad H_{i+1}^0 = ((G_{i+1}^0 + \gamma_0)\alpha + G_{i+1}^0)\gamma_0.$$

Since α and g_0^0 preserve every L^k, so do β_0, γ_0, and the above-defined equivalences. Note that, for a single L^j and $(\beta_0, \gamma_0) = (\beta, \gamma)$, these expressions occur in Zádori [6]. We omit the straightforward induction that shows

$$\begin{aligned} g_i^0 &= \langle a_y^x, b_y^x \colon y \leqslant i \rangle, \quad i = 0, 1, 2, \ldots, \\ h_i^0 &= \langle a_y^x, b_{y-1}^x \colon 1 \leqslant y \leqslant i \rangle, \quad i = 1, 2, 3, \ldots, \\ H_i^0 &= \langle a_{x+12-y}^x, b_{x+11-y}^x \colon y \leqslant i \rangle, \quad i = 0, 1, 2, \ldots, \quad \text{and} \\ G_i^0 &= \langle a_{x+12-y}^x, b_{x+12-y}^x \colon 1 \leqslant y \leqslant i \rangle, \quad i = 1, 2, 3, \ldots . \end{aligned} \tag{1}$$

Hence

$$\begin{aligned} \langle a_i^0, b_i^0 \rangle &= g_i^0 \cdot G_{12-i}^0 \quad \text{for } i = 0, 1, \ldots, 11, \\ \langle a_i^0, b_{i-1}^0 \rangle &= h_i^0 \cdot H_{12-i}^0 \quad \text{for } i = 1, 2, \ldots, 12, \\ \langle a_i^0, a_{i+1}^0 \rangle &= \alpha(\langle a_i^0, b_i^0 \rangle + \langle a_{i+1}^0, b_i^0 \rangle) \quad \text{for } i = 0, 1, \ldots, 11, \end{aligned}$$

and

$$\langle b_i^0, b_{i+1}^0 \rangle = \alpha(\langle a_{i+1}^0, b_i^0 \rangle + \langle a_{i+1}^0, b_{i+1}^0 \rangle) \quad \text{for } i = 0, 1, \ldots, 10$$

all belong to Q. Therefore we obtain I(0) by the circle principle.

Now let us assume that $k > 0$ and I$(k-1)$ has already been proved. We define the following members of Q. Let

$$h_3^k = ((g_2^0 + \gamma_0)(\gamma + \langle b_4^{k-1}, b_5^{k-1} \rangle)\alpha + g_2^0)\gamma_0.$$

Using $\alpha\gamma = 0$ we can easily infer that

$$\begin{aligned} (\gamma + \langle b_4^{k-1}, b_5^{k-1} \rangle)\alpha &= \langle a_5^{k-1}, a_6^{k-1} \rangle + \langle b_4^{k-1}, b_5^{k-1} \rangle + \\ &\quad + \langle a_2^k, a_3^k \rangle + \langle b_1^k, b_2^k \rangle, \end{aligned}$$

and

$$\begin{aligned} (g_2^0 + \gamma_0)\alpha &= \langle a_0^x, a_1^x \rangle + \langle a_1^x, a_2^x \rangle + \langle a_2^x, a_3^x \rangle + \\ &\quad + \langle b_0^x, b_1^x \rangle + \langle b_1^x, b_2^x \rangle. \end{aligned}$$

This yields

$$\langle a_3^k, b_2^k \rangle \subseteq h_3^k \subseteq \langle a_y^k, b_{y-1}^k \colon 1 \leqslant y \leqslant 3 \rangle. \tag{2}$$

Now define $g_{i+1}^k = ((h_{i+1}^k + \beta_0)\alpha + h_{i+1}^k)\beta_0$ $(i = 2, 3, \ldots)$, $h_{i+1}^k = ((g_i^k + \gamma_0)\alpha + g_i^k)\gamma_0$ for $i = 3, 4, \ldots$; and let $g_j^k = g_j^0$ for $j = 0, 1, 2$ and $h_j^k = h_j^0$ for $j = 1, 2$. Using (2), an easy induction (similar to what yielded (1)) gives

$$\begin{aligned} &\langle a_i^k, b_{i-1}^k \rangle \subseteq h_i^k \subseteq \langle a_y^k, b_{y-1}^k \colon 1 \leqslant y \leqslant i \rangle \quad \text{for } i = 3, 4, \ldots, \quad \text{and} \\ &\langle a_i^k, b_i^k \rangle \subseteq g_i^k \subseteq \langle a_y^k, b_y^k \colon y \leqslant i \rangle \quad \text{for } i = 3, 4, \ldots. \end{aligned} \tag{3}$$

We use a similar technique "from the right to the left". That is, let $H_0^k = H_0^0$, $G_1^k = G_1^0$, $H_1^k = H_1^0$,

$$G_2^k = ((H_1^0 + \beta_0)(\gamma + \langle b_{k+6}^{k-1}, b_{k+7}^{k-1} \rangle)\alpha + H_1^0)\beta_0,$$

$H_{i+1}^k = ((G_{i+1}^k + \gamma_0)\alpha + G_{i+1}^k)\gamma_0$ for $i = 1, 2, \ldots$ and $G_{i+1}^k = ((H_i^k + \beta_0)\alpha + H_i^k)\beta_0$ for $i = 2, 3, \ldots$. Then, similarly to (3), we conclude

$$\begin{aligned} &\langle a_{k+12-i}^k, b_{k+12-i}^k \rangle \\ &\quad \subseteq G_i^k \subseteq \langle a_{k+12-y}^k, b_{k+12-y}^k \colon 1 \leqslant y \leqslant i \rangle \quad (i = 2, 3, \ldots), \quad \text{and} \\ &\langle a_{k+12-i}^k, b_{k+11-i}^k \rangle \subseteq H_i^k \\ &\quad \subseteq \langle a_{k+12-y}^k, b_{k+11-y}^k \colon 0 \leqslant y \leqslant i \rangle \quad (i = 2, 3, \ldots). \end{aligned} \tag{4}$$

Using (1) for one of the factors on the righthand side and (3) or (4) for the other factor, we easily obtain

$$\langle a_i^k, b_i^k \rangle = g_i^k \cdot G_{k+12-i}^k \quad \text{for } i = 0, 1, \ldots, k + 11,$$

and

$$\langle a_i^k, b_{i-1}^k \rangle = h_i^k \cdot H_{k+12-i}^k \quad \text{for } i = 1, 2, \ldots, k + 12.$$

Armed with $\langle a_i^k, b_i^k \rangle \in Q$ and $\langle a_j^k, b_{j-1}^k \rangle \in Q$ for all meaningful i and j, we conclude $\mathrm{I}(k)$ the same way as we obtained $\mathrm{I}(0)$.

Now we know that $\mathrm{I}(k)$ holds for all $k \in \mathbf{N}_0$. It is easy to see that, for every $i \geqslant 1$,

$$\begin{aligned} &\langle b_6^{i-1}, a_6^i \rangle = \beta \cdot (\langle b_6^{i-1}, b_4^{i-1} \rangle + \gamma + \langle a_2^i, a_6^i \rangle) \quad \text{and} \\ &\langle b_4^{i-1}, a_2^i \rangle = \gamma \cdot (\langle b_4^{i-1}, b_6^{i-1} \rangle + \langle b_6^{i-1}, a_6^i \rangle + \langle a_6^i, a_2^i \rangle). \end{aligned} \tag{5}$$

In virtue of $\mathrm{I}(k)$ for all $k \in \mathbf{N}_0$ and (5), the circle principle applies for the whole A and we conclude $\langle p, q \rangle \in Q$ for all $p, q \in A$. □

References

1. Chajda, I. and Czédli, G. (1996) How to generate the involution lattice of quasiorders?, *Studia Sci. Math. (Budapest)*, to appear.

2. Czédli, G. (1996) Four-generated large equivalence lattices, *Acta Sci. Math. (Szeged)*, to appear.
3. Czédli, G. (1996) (1 + 1 + 2)-generated equivalence lattices, in preparation.
4. Rival, I. and Stanford, M. (1992) Algebraic aspects of partition lattices, in *Matroids and Applications*, Cambridge Univ. Press, Cambridge, pp. 106–122.
5. Strietz, H. (1975) Finite partition lattices are four-generated, in *Proc. Lattice Th. Conf. Ulm*, pp. 257–259.
6. Strietz, H. (1977) Über Erzeugendenmengen endlicher Partitionverbände, *Studia Sci. Math. Hungarica* **12**, 1–17.
7. Zádori, Z. (1986) Generation of finite partition lattices, in *Colloquia Math. Soc. J. Bolyai* **43**, Lectures in Universal Algebra (Proc. Conf. Szeged, 1983), North-Holland, Amsterdam, New York, pp. 573–586.