LARGE SETS OF LATTICES WITHOUT ORDER EMBEDDINGS

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To the memory of Ervin Fried

Abstract. Let I and μ be an infinite index set and a cardinal, respectively, such that $|I| \leq \mu$ and, starting from $\aleph_0$, μ can be constructed in countably many steps by passing from a cardinal λ to 2^λ at successor ordinals and forming suprema at limit ordinals. We prove that there exists a system $X = \{L_i : i \in I\}$ of complemented lattices of cardinalities less than $|I|$ such that whenever $i, j \in I$ and $\varphi\colon L_i \to L_j$ is an order embedding, then $i = j$ and φ is the identity map of L_i. If $|I|$ is countable, then, in addition, X consists of finite lattices of length 10. Stating the main result in other words, we prove that the category of (complemented) lattices with order embeddings has a discrete full subcategory with $|I|$ many objects. Still in other words, the class of these lattices has large antichains (that is, antichains of size $|I|$) with respect to the quasiorder "embeddability". As corollaries, we trivially obtain analogous statements for partially ordered sets and semilattices.

1. Introduction and the main result

Although the main result we prove is a statement on lattices, the problem we deal with is meaningful for many other classes of algebras and structures, including partially ordered sets, *ordered sets* in short, and semilattices. A minimal knowledge of the rudiments of lattice theory is only assumed, as it is presented in any book on lattices or universal algebra; otherwise the paper is more or less self-contained for most algebraists.

As usual, $\aleph_0$ and $\omega = \omega_0$ denote the smallest infinite cardinal and ordinal, respectively. Otherwise, we follow the convention that cardinals are denoted by κ, λ, and μ, while α, β, and γ stand for ordinals. The cardinality of an ordinal α is denoted by $|\alpha|$. Define a transfinite sequence κ_α of cardinals as follows. Let $\kappa_0 = \aleph_0$. If κ_α is defined, then let $\kappa_{\alpha+1} = 2^{\kappa_\alpha}$. Finally, if α is a limit ordinal and κ_β is defined for all $\beta < \alpha$, then let $\kappa_\alpha = \sup\{\kappa_\beta : \beta < \alpha\}$ or, equivalently, let $\kappa_\alpha = \sum\{\kappa_\beta : \beta < \alpha\}$. Let us say that a cardinal λ is $\aleph_0$-*step accessible*, if there exists an ordinal α such that $|\alpha| \leq \aleph_0$ and $\lambda \leq \kappa_\alpha$. For example, $\kappa_{\omega^3+\omega+2}$ is an $\aleph_0$-step accessible cardinal.

For ordered sets A and B, a map $\varphi\colon A \to B$ is called an *order embedding* if φ is injective and $x \leq y \iff \varphi(x) \leq \varphi(y)$ holds for all $x, y \in A$. Note that an injective

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monotone map need not be an order embedding, since then the "$\Leftarrow$" implication above can fail. Let I be an index set, and let L_i be a lattice for $i \in I$. We say that $\{L_i : i \in I\}$ is a *system of lattices without order embeddings* if for all $i, j \in I$ and every order embedding $\varphi\colon L_i \to L_j$, we have that $i = j$ and that φ is the identity map id_{L_i}, defined by $x \mapsto x$ for all $x \in L_i$. (Note that $\{L_i : i \in I\}$ is actually a set but, in order to emphasize the implication $i \neq j \Rightarrow L_i \neq L_j$, we prefer to call it a *system* here and in analogous situations.) For a nonnegative integer n and a lattice L, $\mathrm{length}(L) = n$ means that L has an $(n+1)$-element chain (in other words, linearly ordered subset) but it has no $(n+2)$-element chain. Now we are in the position to formulate our main result.

Theorem 1.1. *Let I be an index set. If $|I|$ is an $\aleph_0$-step accessible cardinal, then there exists a system $\{L_i : i \in I\}$ of complemented lattices without order embeddings. Furthermore, this system can be chosen so that*

(i) $|L_i| < |I|$ *for all $i \in I$, provided $|I| \geq \aleph_0$, and*
(ii) $|L_i| < \aleph_0$ *and* $\mathrm{length}(L_i) = 10$ *for all $i \in I$, provided $|I| \leq \aleph_0$.*

Outline. In Section 2, we formulate some corollaries of Theorem 1.1 for lattices, semilattices, and ordered sets. Also, we give some historical comments on other structures. Section 3 is devoted to proofs and auxiliary statements. Lemma 3.2 of that section can be of separate interest in lattice theory. Lemmas 3.3 and 3.9 reveal some additional properties of the lattices L_i that can also be required.

2. Comments and corollaries

A lattice L will be called *order-rigid*, if the identity map $\mathrm{id}_L\colon L \to L$ is the only $L \to L$ order-embedding. For brevity, systems of lattices without order embeddings will also be called *order-rigid systems*. A lattice L is *order-rigid* if the singleton system $\{L\}$ is order-rigid. While rigidity in the sense of all lattice homomorphisms would only hold for the one-element system consisting of the one-element lattice, order-rigidity is an interesting property even for a single lattice. At the end of Section 3, we will derive the following statement from Theorem 1.1.

Corollary 2.1. *If λ is an $\aleph_0$-step accessible cardinal, then there exists an order-rigid complemented lattice L such that $\lambda < |L| \leq 2^{2^\lambda}$.*

Let LatE, SLatE, and POSetE denote the category of lattices, that of semilattices, and that of ordered sets, respectively, with lattice embeddings, semilattice embeddings, and order embeddings, respectively, as morphisms. (The suffix "E" comes from "embeddings".) Note that if A and B are lattices or meet-semilattices and $\varphi\colon A \to B$ is a meet-preserving injective map, then φ is necessarily an order embedding. Hence, the following statement obviously follows from Theorem 1.1. We express the absence of nontrivial embeddings in terms of category theory, and also in the language of quasiordered sets; this explains the terminology we use below.

Corollary 2.2. *If λ is an $\aleph_0$-step accessible cardinal, then each of the categories* LatE, SLatE, *and* POSetE *has a discrete full subcategory with λ many objects. In other terms, each of* LatE, SLatE, *and* POSetE *has an antichain of cardinality λ with respect to the quasiorder "embeddability".*

There are several earlier results of similar nature. For example, Duffus, Erdős, Nešetřil, and Soukup [2] and Nešetřil and Tardiff [7] deal with discrete full subcategories of graphs with the usual graph homomorphisms. Jakubíková-Studenovská [6], Pinus [8], and Primavesi and Thompson [9] consider antichains with respect to suitable embeddings. The titles of these papers but [8] indicate that the terminology is far from being unique. In order to find not only discrete full subcategories, one usually has to take a larger class of structures. For example, Hedrlín and Lambek [4] proved that each small category has a full embedding into the category of all semigroups with homomorphisms, while Fried and Sichler [3] proved an analogous result for some integral domains with 1-preserving homomorphisms. These results on other structures raise the problem if Theorem 1.1 and its corollaries hold for larger cardinalities or for more specific lattices.

3. Proofs and auxiliary statements

3.1. **Finite lattices.** A lattice is called a *ranked lattice* if all of its maximal chains are of the same finite length. This common length is the *length* of the lattice. For x in a ranked lattice, the order ideal $\downarrow x = \{y : y \le x\}$ is also a ranked lattice, and its length is the *height* of x, denoted by $h(x)$. The following lemma is almost trivial.

Lemma 3.1. *If A and B are ranked lattices of the same finite length and $\varphi\colon A \to B$ is an order embedding, then φ preserves height, that is, $h(\varphi(x)) = h(x)$ holds for all $x \in A$.*

Proof. Let $n = \text{length}(A)$. If $0 = a_0 \prec a_1 \cdots \prec a_n = 1$ is a maximal chain of A, then $\varphi(0) = \varphi(a_0) < \varphi(a_1) < \cdots < \varphi(a_n) = \varphi(1)$. Since $\text{length}(B) = n$, this is also a maximal chain, that is, $0 = \varphi(0) = \varphi(a_0) \prec \varphi(a_1) \prec \cdots \prec \varphi(a_n) = 1$. Hence, $h(a_i) = i = h(\varphi(a_i))$, for every $i \in \{0, 1, \ldots, n\}$. This implies the lemma, because each element of A belongs to some maximal chain. □

Figure 1 shows that modularity cannot be dropped from the following lemma.

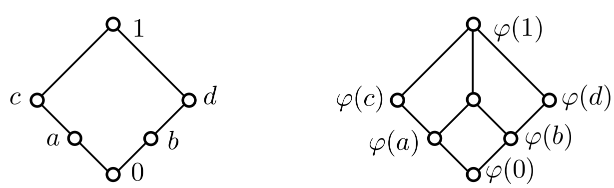

Figure 1. Lemma 3.2 would fail without modularity

Lemma 3.2. *If A and B are modular lattices of the same finite length, then every order embedding $\varphi\colon A \to B$ is a lattice embedding.*

Proof. Let $n = \text{length}(A) = \text{length}(B)$. We prove by induction on $h(b)$ that

$$\varphi(a \vee b) = \varphi(a) \vee \varphi(b) \tag{3.1}$$

holds for any $a, b \in A$. Since φ is monotone, we can assume that a and b are incomparable elements, in notation, $a \parallel b$. To handle the base of the induction, let $h(b) = 1$; that is, let b be an atom. By the dimension equation of modular lattices, see Birkhoff [1, Corollary to Theorem III.§3.15], $h(a \vee b) = h(a) + h(b) - h(a \wedge b) = h(a) + 1 - h(0) = h(a) + 1$. Since φ preserves incomparability and, by Lemma 3.1, height, the dimension equation also gives that $h(\varphi(a) \vee \varphi(b)) = h(\varphi(a)) + h(\varphi(b)) - h(\varphi(a) \wedge \varphi(b)) = h(a) + 1 - h(0) = h(a) + 1$. Hence, $\varphi(a) \vee \varphi(b)$ and $\varphi(a \vee b)$ are of the same height. Therefore, using that $\varphi(a) \vee \varphi(b) \leq \varphi(a \vee b)$ since φ is monotone, we obtain that (3.1) holds if b is an atom.

Next, assume that $h(b) > 1$ and that (3.1) holds for every element of smaller height. Pick a lower cover b_* of b, and let $c = a \vee b_*$. By the induction hypothesis, $\varphi(c) = \varphi(a) \vee \varphi(b_*)$. Since φ preserves height, it maps the order filter $\uparrow b_* = \{y \in A : y \geq b_*\}$ into the order filter $\uparrow\varphi(b_*)$, and these two filters are modular lattices of the same height $n - h(b_*)$. Applying (3.1) for the restriction $\varphi\rceil_{\uparrow b_*}$ of φ to $\uparrow b_*$ and using that b is an atom in $\uparrow b_*$, we obtain that $\varphi(c \vee b) = \varphi(c) \vee \varphi(b)$. Therefore, $\varphi(a \vee b) = \varphi(c \vee b) = \varphi(c) \vee \varphi(b) = \varphi(a) \vee \varphi(b_*) \vee \varphi(b) = \varphi(a) \vee \varphi(b)$, proving (3.1). Finally, (3.1) and its dual completes the proof. $\square$

Let Primes denote the *set of prime numbers*, and fix an integer $d \geq 3$. For $p \in$ Primes, the lattice of all subspaces of the d-dimensional vector space $\mathbb{Z}_p^d$ (over the prime field $\mathbb{Z}_p$) will be denoted by $M_p^{(d)}$. We know that $M_p^{(d)}$ is a complemented modular lattice of length d. (Note that $M_p^{(d)}$ is the lattice of submodules of $\mathbb{Z}_p^d$, and "module" is where the name "modular" came from.) The one-dimensional subspaces generated by $\langle 1, 1, \dots, 1\rangle$, $\langle 1, 0, \dots, 0\rangle$, $\langle 0, 1, 0, \dots, 0\rangle$, $\dots$, $\langle 0, \dots, 0, 1\rangle$ will be denoted by $e_0^{(d,p)}$, $e_1^{(d,p)}$, $e_2^{(d,p)}$, $\dots$, $e_d^{(d,p)}$, respectively. They are distinct atoms in $M_p^{(d)}$. It follows from the easier direction of the main result, Theorem 4.1, of Herrmann and Huhn [5] (and it is explicitely stated in [5, Lines 6–7 of page 113]) that

$$\{e_0^{(d,p)}, e_1^{(d,p)}, \dots, e_d^{(d,p)}\} \text{ generates the lattice } M_p^{(d)}. \tag{3.2}$$

Since $M_p^{(d)}$ has many lattice automorphisms, it cannot be a member of an order-rigid system. Therefore, we modify $M_p^{(d)}$ to obtain $\widehat{M_p^{(d)}}$ in the following way. First of all, take the direct product of the 2-element chain and the 3-element chain. To obtain an auxiliary lattice $T_j^{(d,p)}$ for $j \in \{0, \dots, d\}$, we glue a chain of length $(j+1)$ to the bottom and a chain of length $(d - j + 1)$ to the top of this direct product. For $d = 3$, these auxiliary lattices are also given in Figure 2. (They do not depend on p, which is here for technical reasons.)

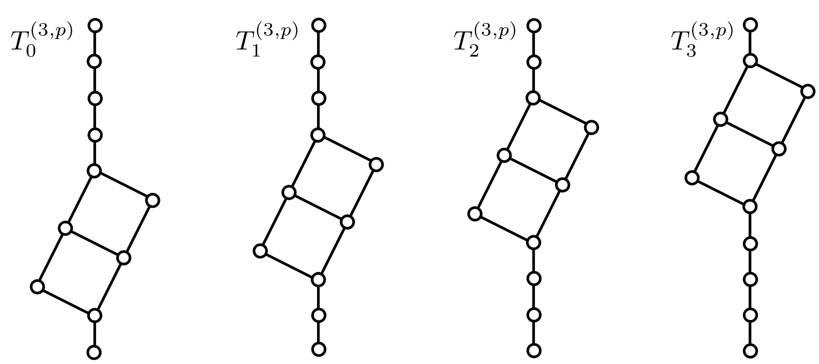

FIGURE 2. The auxiliary lattices $T_j^{(3,p)}$

Each of $T_0^{(d,p)}, \dots, T_3^{(d,p)}$ is a modular lattice of length $d + 5$. The chain of length $d + 5$ will be denoted by C_{d+5}; it consists of $d + 6$ elements. To obtain

$\widehat{M}_p^{(d)}$ from $M_p^{(d)}$, for each atom $a \in M_p^{(d)}$ we do the following. If $a = e_i^{(d,p)}$ for some $i \in \{0, 1, \dots, d\}$, then we replace the prime interval $[0, a]$ by $T_i^{(d,p)}$. That is, we identify the bottom and the top of $T_i^{(d,p)}$ with $0_{M_p^{(d)}}$ and a, respectively. If $a \notin \{e_0^{(d,p)}, \dots, e_d^{(d,p)}\}$, then we replace $[0, a]$ by C_{d+5} such that distinct copies of C_{d+5} are used for distinct atoms. That is, we identify the bottom and the top of C_{d+5} with $0_{M_p^{(d)}}$ and a, and the copies of C_{d+5} we use for distinct atoms a and b will be disjoint in the new lattice, except their bottom elements. This way, gluing a modular lattice of length $d+5$, $T_i^{(d,p)}$ or C_{d+5}, into each prime interval of the form $[0, a]$, as described above, we obtain the lattice $\widehat{M}_p^{(d)}$. Note that this lattice is a ranked lattice of length $2d+4$. Note also that $\widehat{M}_p^{(d)}$ a complemented lattice but it is not modular. Observe that $M_p^{(d)}$ is a sublattice of $\widehat{M}_p^{(d)}$, and

$$M_p^{(d)} = \{x \in \widehat{M}_p^{(d)} : h(x) = 0 \text{ or } h(x) \geq d+5\}. \tag{3.3}$$

Next, we state a bit more than Part (ii) of Theorem 1.1.

Lemma 3.3. *For each integer $d \geq 3$, $\{\widehat{M}_p^{(d)} : p \in \text{Primes}\}$ is an order-rigid system of ranked, finite, complemented lattices of length $2d+4$.*

Proof. Clearly, the lattices in question are ranked, finite, complemented lattices of length $2d+4$. Assume that $p, q \in$ Primes and that $\varphi\colon \widehat{M}_p^{(d)} \to \widehat{M}_q^{(d)}$ is an order embedding. By Lemma 3.1, φ is height-preserving. Hence, by (3.3), $\varphi(M_p^{(d)}) \subseteq M_q^{(d)}$. Therefore, the restriction $\psi = \varphi\rceil_{M_p^{(d)}}$ of φ to $M_p^{(d)}$ is an order embedding $\psi\colon M_p^{(d)} \to M_q^{(d)}$. It follows from Lemma 3.2 that $\psi\colon M_p^{(d)} \to M_q^{(d)}$ is a lattice embedding.

Next, we assert that

$$\varphi(e_i^{(d,p)}) = e_i^{(d,q)}, \text{ for } i \in \{0, 1, \dots, d\}. \tag{3.4}$$

We know that φ preserves height, whence $\varphi(e_i^{(d,p)})$ is an atom of $M_q^{(d)}$. Suppose, for a contradiction, that $\varphi(e_i^{(d,p)}) \notin \{e_0^{(d,q)}, \dots, e_d^{(d,q)}\}$. Since φ is monotone, $\varphi(\downarrow e_i^{(d,p)}) \subseteq \downarrow \varphi(e_i^{(d,p)})$. This gives

$$d+8 = |T_i^{(d,p)}| = |\downarrow e_i^{(d,p)}| = |\varphi(\downarrow e_i^{(d,p)})| \leq |\downarrow \varphi(e_i^{(d,p)})| = |C_{d+5}| = d+6.$$

This is a contradiction, proving that $\varphi(e_i^{(d,p)}) \in \{e_0^{(d,q)}, \dots, e_d^{(d,q)}\}$. Hence, there is a $j \in \{0, 1, \dots, d\}$ such that $\varphi(\downarrow e_i^{(d,p)}) \subseteq \downarrow e_j^{(d,q)}$. Applying Lemma 3.2 to the restriction $\varphi\rceil_{\downarrow e_j^{(d,p)}}$, we obtain that this restriction is a lattice embedding of $\downarrow e_j^{(d,p)}$ into $\downarrow e_j^{(d,q)}$. Therefore, $T_i^{(d,p)}$ has a lattice embedding into $T_j^{(d,q)}$. But each of these two lattices consists of $d+8$ elements, and we conclude that they are isomorphic. This clearly implies that $i = j$, proving (3.4).

Combining (3.2) and (3.4), we obtain that $\psi = \varphi\rceil_{M_p^{(d)}}$ maps the set of generators of $M_p^{(d)}$ onto the set of generators of $M_q^{(d)}$. Hence, the embedding $\varphi\rceil_{M_p^{(d)}}$ is surjective, whence it is a lattice isomorphism. Thus, $M_p^{(d)} \cong M_q^{(d)}$. For a prime r, let $f(r)$ denote the number of atoms of $M_r^{(d)}$, that is, the number of 1-dimensional subspaces of $\mathbb{Z}_r^d$. It belongs to the folklore that $f(r) = (r^d - 1)/(r-1) = \sum_{i=0}^{d-1} r^i$, which is a strictly increasing function of r. (The reason is simple: on $\mathbb{Z}_r^d \setminus \{0\}$, these subspaces induce a partition such that every block consists of $r-1$ points.) Hence, since $M_p^{(d)} \cong M_q^{(d)}$ gives $f(p) = f(q)$, we conclude $p = q$. Thus, by (3.2) and (3.4), ψ is the identity map of $M_p^{(d)}$.

Finally, since $\widehat{M}_q^{(d)} = \widehat{M}_p^{(d)}$ and φ is an injective self-map, it is a permutation of $\widehat{M}_p^{(d)}$ that acts identically on $M_p^{(d)}$. For each atom u of $M_p^{(d)}$, we have that $\varphi(\downarrow u) \subseteq \downarrow(\varphi(u)) = \downarrow u$, because φ is monotone. We know that $\varphi(u)$ is isomorphic to one of C_{d+5}, $T_0^{(d,p)}, \dots, T_d^{(d,p)}$. Applying Lemma 3.2 to $\varphi\rceil_{\downarrow u}$, we obtain that $\varphi\rceil_{\downarrow u} \colon \downarrow u \to \downarrow u$ is a lattice embedding. But none of the lattices C_{d+5}, $T_0^{(d,p)}, \dots, T_d^{(d,p)}$ has a lattice embedding into itself that is different form the identity map. (In fact, these lattices are order-rigid.) Hence, we conclude that φ acts identically on $\downarrow u$. Therefore, φ is the identity map of $\widehat{M}_p^{(d)}$. $\square$

3.2. **Cardinal sum of lattices and the class $\mathcal{G}(H)$.** A bounded lattice L is *atomic* if $\downarrow x$ contains an atom for each $x \in L \setminus \{0\}$. *Coatomic* lattices are defined dually. In what follows, H will always denote a subset of the set of all prime numbers. To capture all properties we need to prove Theorem 1.1, we define the class of "appropriate" lattices in the following way.

Definition 3.4. For $H \subseteq$ Primes, let $\mathcal{G}(H)$ be the class of all lattices L that satisfy

(i) L is an order-rigid, complemented, atomic, coatomic lattice and $|L| \geq 6$, and
(ii) for every $p \in H$, $\{\widehat{M}_p^{(3)}, L\}$ is a two element order-rigid system.

An order-rigid system of lattices belonging to $\mathcal{G}(H)$ will be called an *order-rigid system in* $\mathcal{G}(H)$. Let J be an index set with size at least 2, and let $L_j \in \mathcal{G}(H)$ for $j \in J$. For a moment, we can assume that these lattices are pairwise disjoint, since otherwise we can take their appropriate isomorphic copies. On the set of $L := \{0, 1, c\} \cup \bigcup\{L_j : j \in J\}$, where $\{0, 1, c\} \cap \bigcup\{L_j : j \in J\} = \varnothing$, we define an ordering by the following rule, where $\leq_j$ stands for the ordering of L_j:

$$x \leq y \overset{\text{def}}{\iff} \begin{cases} x \leq_j y, & \text{if } x \text{ and } y \text{ belong to the same } L_j, \\ x = 0 \text{ or } y = 1, & \text{otherwise.} \end{cases}$$

This way we obtain a new lattice $\langle L; \leq \rangle$. We call this lattice the *complemented cardinal sum* of the lattices L_j. It is denoted by $\sum_{j\in J}^{(\mathrm{c})} L_j$; see Figure 3 for an illustration. We will only form cardinal sums of (not necessarily disjoint but pairwise distinct) lattices that belong to $\mathcal{G}$; let us emphasize that we do not allow that $|J| \leq 1$, that is, at least two summands are always required

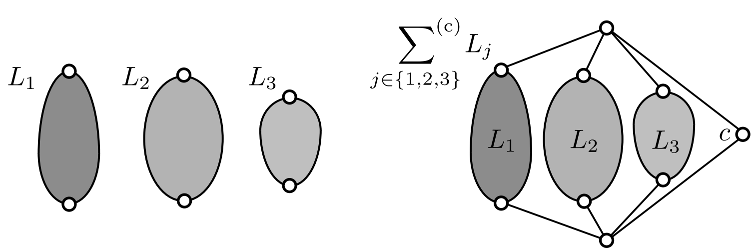

FIGURE 3. The complemented cardinal sum of three bounded lattices

Lemma 3.5.

(i) *$\mathcal{G}(H)$ is closed with respect to forming complemented cardinal sums of order-rigid systems.*

(ii) *Let $\{L_s : s \in S\}$ be an order-rigid system in $\mathcal{G}(H)$. Let $J \subseteq S$ and $K \subseteq S$ such that $|J| \geq 2$ and $|K| \geq 2$. If $\varphi\colon \sum_{j\in J}^{(c)} L_j \to \sum_{k\in K}^{(c)} L_k$ is an order embedding, then $J \subseteq K$ and, for every $x \in \big(\sum_{j\in J}^{(c)} L_j\big) \setminus \{c\}$, we have $\varphi(x) = x$. In particular, if $J = K$, then φ is necessarily the identity map.*

Proof. Clearly, complemented cardinal sums of lattices in $\mathcal{G}(H)$ are complemented, atomic, coatomic lattices with at least 6 elements. To prove Part (ii), let J and K be as in the lemma. Let $A = \sum_{j\in J}^{(c)} L_j$ and $B = \sum_{k\in K}^{(c)} L_k$. For $s \in S$, the top and the bottom of L_s are denoted by 0_s and 1_s, respectively. Suppose, for a contradiction, that $\varphi(1_A) \neq 1_B$. Since φ is monotone, $\varphi(A) \subseteq {\downarrow}\varphi(1_A)$, which excludes that $\varphi(1_A) \in \{c_B, 0_B\}$. Hence, there is a unique $k \in K$ with $\varphi(1_A) \in L_k$, and we have that $\varphi(A) \subseteq {\downarrow}1_k$ holds in B. There are three cases to consider.

Case 1. We suppose that $\varphi(0_A) = 0_B$ and 0_k is a φ-image. Let u denote its unique preimage, that is, $\varphi(u) = 0_k$. Since 0_k is comparable to all elements of ${\downarrow}1_k$ in B and φ is an order embedding, we obtain that u is comparable to all elements of A. Since A is a complemented lattice, $u \in \{0_A, 1_A\}$. The equality $\varphi(0_A) = 0_B$ excludes $u = 0_A$. Hence, $u = 1_A$, and $\varphi(A) = \varphi({\downarrow}u) \subseteq {\downarrow}\varphi(u) = {\downarrow}0_k = \{0_B, 0_k\}$. This contradicts $|A| \geq 6$. Hence, Case 1 cannot occur.

Case 2. We suppose that $\varphi(0_A) = 0_B$ but 0_k is not a φ-image. Define a new map $\varphi'\colon A \to B$ by

$$\varphi'(x) = \begin{cases} \varphi(x), & \text{if } x \neq 0_A,\\ 0_k, & \text{if } x = 0_A. \end{cases}$$

This is an order embedding, because 0_k is not a φ-image and $\varphi(A) \subseteq {\downarrow}1_k$. Note that φ' is actually an order-embedding of A into L_k. Pick a $j \in J$. Clearly, the restriction $\varphi'\rceil_{L_j}$ is an order embedding of L_j into L_k such that $0_k = \varphi'\rceil_{L_j}(0_A) < \varphi'\rceil_{L_j}(0_{L_j})$. Hence, $\varphi'\rceil_{L_j}$ is not the identity map, which contradicts the order-rigidity of the system $\{L_s : s \in S\}$. Consequently, we can exclude Case 2.

Case 3. We suppose that $\varphi(0_A) \neq 0_B$. Since $\varphi(0_A) < \varphi(1_A) \in L_k$, we conclude that $\varphi(A) \subseteq L_k$. Hence, for every $j \in J$, the restriction $\varphi\rceil_{L_j}$ is an order embedding of L_j into L_k, and we obtain the same contradiction as in the previous case.

Now, after excluding all possible cases, we see that $\varphi(1_A) \neq 1_B$ is impossible. Consequently, $\varphi(1_A) = 1_B$ and, by dualizing the argument above, $\varphi(0_A) = 0_B$.

Next, consider an arbitrary $j \in J$. The injectivity of φ excludes that $\varphi(0_j)$ or $\varphi(1_j)$ belongs to $\{0_B, 1_B\}$. Hence, there are $k_1, k_2 \in K$ such that $\varphi(0_j) \in L_{k_1}$ and $\varphi(1_j) \in L_{k_2}$. Since $\varphi(0_j) \leq \varphi(1_j)$, we have $k_1 = k_2$. That is, $\varphi(0_j), \varphi(1_j) \in L_k$ and $\varphi(L_j) \subseteq L_k$ for a unique $k \in K$. Thus, since $\varphi\rceil_{L_j}$ is an order embedding of L_j into L_k and $\{L_s : s \in S\}$ is an order-rigid system, we conclude that $j = k \in K$ and that $\varphi(x) = x$ for all $x \in L_j$. This, together with $\varphi(1_A) = 1_B$ and $\varphi(0_A) = 0_B$, implies Part (ii) of the lemma, because $j \in J$ was arbitrary.

To prove Part (i), consider an order-rigid system $\{L_j : j \in J\}$ (of pairwise distinct lattices) in $\mathcal{G}(H)$. Let $L = \sum_{j\in J}^{(c)} L_j$, and let $p \in H$. We have to show that $\{\widehat{M}_p^{(3)}, L\}$ is an order-rigid system. We know from Lemma 3.3 that $\widehat{M}_p^{(3)}$ is an order-rigid lattice. The order-rigidity of L follows from (the last sentence of) Part (ii). If we had an order embedding $L \to \widehat{M}_p^{(3)}$, then its restriction to any of the L_j, $j \in J$, would be an order embedding $L_j \to \widehat{M}_p^{(3)}$, which would contradict $p \in H$ and $L_j \in \mathcal{G}(H)$. Finally, we suppose, for a contradiction, that there exists an order-embedding $\varphi\colon \widehat{M}_p^{(3)} \to L$. Clearly,

$$\text{for every } x \in \widehat{M}_p^{(3)}, \quad |{\downarrow} x \cup {\uparrow} x| \geq 4. \tag{3.5}$$

This implies that $c = c_L$ is not in $\varphi(\widehat{M}_p^{(3)})$. Let a and b be distinct coatoms of $\widehat{M}_p^{(3)}$. Then $|{\uparrow} a| = 2$ and $|{\uparrow} b| = 2$ yield that $1_L \notin \{\varphi(a), \varphi(b)\}$. Since any two lines of a projective plane intersect in a point, $a \wedge b$ is not the least element of $\widehat{M}_p^{(3)}$. Hence, $\varphi(a)$ and $\varphi(b)$ has a nonzero lower bound in L. Thus, since φ is monotone, $\varphi(a)$ and $\varphi(b)$ belong to the same summand L_j. The bottom of L_j will be denoted by 0_j. Since b was arbitrary, we obtain that L_j contains the images of all coatoms. Using that each element of $\widehat{M}_p^{(3)} \setminus \{0,1\}$ is less than or equal to a coatom, we obtain that $\varphi\big(\widehat{M}_p^{(3)} \setminus \{0,1\}\big) \subseteq L_j$. Thus, since φ is monotone, $\varphi(\widehat{M}_p^{(3)}) \subseteq L_j \cup \{0_L, 1_L\}$.

Assume that $\varphi(1) = 1_L$. If we had an $u \in \widehat{M}_p^{(3)}$ such that $\varphi(u) = 1_j$, then ${\downarrow}\varphi(u) \cup {\uparrow}\varphi(u) = {\downarrow} 1_j \cup {\uparrow} 1_j \supseteq L_j \cup \{0_L, 1_L\} \supseteq \varphi(\widehat{M}_p^{(3)})$ would imply ${\downarrow} u \cup {\uparrow} u \supseteq \widehat{M}_p^{(3)}$, which would be a contradiction, because $\widehat{M}_p^{(3)}$ is a complemented lattice, but $u \neq 1$ and $|{\uparrow}\varphi(u)| = |{\uparrow} 1_j| = 2$ excludes $u = 0$. Hence, the map

$$\varphi'\colon \widehat{M}_p^{(3)} \to L, \text{ defined by } \varphi'(x) = \begin{cases} 1_j, & \text{if } x = 1, \\ \varphi(x), & \text{if } x \neq 1 \end{cases}$$

is still an order-embedding, but $\varphi'(1) \neq 1_L$. Observe that $\varphi'(\widehat{M}_p^{(3)}) \subseteq L_j \cup \{0_L\}$, since $\varphi(\widehat{M}_p^{(3)}) \subseteq L_j \cup \{0_L, 1_L\}$. If $\varphi(1) \neq 1_L$, then we simply let $\varphi' = \varphi$, and $\varphi'(\widehat{M}_p^{(3)}) \subseteq L_j \cup \{0_L\}$ holds again.

Applying the dual of the previous paragraph to φ', we obtain an order-embedding $\varphi''\colon \widehat{M}_p^{(3)} \to L$ such that $\varphi''(0) \neq 0_L$ and $\varphi''(\widehat{M}_p^{(3)}) \subseteq L_j$. This is a contradiction, because φ'' is an order embedding of $\widehat{M}_p^{(3)}$ to L_j, $p \in H$, and $L_j \in \mathcal{G}(H)$. $\square$

3.3. **Another auxiliary construction.** Let $A, B \in \mathcal{G}$. Temporarily, we assume that $A \cap B = \{1_A\} = \{0_B\}$; if this is not the case, then we replace B by an appropriate isomorphic copy. We define a new lattice $T(A,B) = \langle T(A,B); \leq \rangle$ by letting $T(A,B) = A \cup B \cup \{c\}$, where $c \notin A \cup B$, and defining the ordering by

$$x \leq y \overset{\text{def}}{\iff} \begin{cases} x, y \in A \text{ and } x \leq_A y, \text{ or } x, y \in B \text{ and } x \leq_B y, \text{ or} \\ x \in A \text{ and } y \in B, \text{ or } x = 0_A, \text{ or } y = 1_B, \text{ or } x = y. \end{cases}$$

This construction is illustrated in Figure 4. Note that $1_A = 0_B$ will usually be denoted by d.

Lemma 3.6. *Let A and B be bounded lattices and let p be a prime number. If $\varphi\colon \widehat{M}_p^{(3)} \to T(A,B)$ is an order embedding, then exactly one of the following possibilities holds.*

(i) *$\varphi\colon \widehat{M}_p^{(3)} \to A$ is an order embedding;*
(ii) *the map $\varphi_1\colon \widehat{M}_p^{(3)} \to A$, defined by $1 \mapsto 1_A = d$ and $x \mapsto \varphi(x)$ for $x \neq 1$, is an order embedding;*
(iii) *$\varphi\colon \widehat{M}_p^{(3)} \to B$ is an order embedding;*

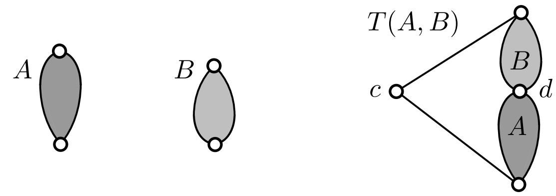

FIGURE 4. $T(A,B)$.

(iv) *the map* $\varphi_0\colon \widehat{M}_p^{(3)} \to B$*, defined by* $0 \mapsto 0_B = d$ *and* $x \mapsto \varphi(x)$ *for* $x \neq 0$*, is an order embedding.*

Proof. It follows from (3.5) that $\varphi(\widehat{M}_p^{(3)}) \subseteq A \cup B$. Since the φ-images of atoms form an antichain, either they all belong to A, or all belong to B. The same holds for the φ-images of coatoms. Since each atom of $\widehat{M}_p^{(3)}$ is incomparable with some coatom and the same holds for their φ-images, either φ maps all atoms and coatoms into A, or it maps them into B. We can assume the first possibility, because the second one can be settled by a dual argument. That is, φ maps all atoms and coatoms into A. If $\varphi(1) \leq d = 1_A$, then (i) holds, since φ is monotone. Thus, we can assume $\varphi(1) > 1_A$. Since φ is monotone and maps the coatoms of $\widehat{M}_p^{(3)}$ into A, and since $\widehat{M}_p^{(3)}$ is a coatomic lattice, we have $\varphi(\widehat{M}_p^{(3)} \setminus \{1\}) \subseteq A$. Therefore, to show that (ii) holds, it suffices to show that $d = 1_A \notin \varphi(\widehat{M}_p^{(3)})$. Suppose, for a contradiction, that $\varphi(x) = 1_A$ for some $x \in \widehat{M}_p^{(3)}$. We have $x \neq 1$ since $\varphi(1) > 1_A$. Since $\varphi(\widehat{M}_p^{(3)} \setminus \{1\}) \subseteq A = {\downarrow}1_A = {\downarrow}\varphi(x)$ and φ is an order embedding, we obtain that $\widehat{M}_p^{(3)} \setminus \{1\} \subseteq {\downarrow}x$. Hence, $\widehat{M}_p^{(3)} \setminus \{1\} \subseteq {\downarrow}y$ for some coatom $y \in {\uparrow}x$, which is clearly a contradiction. This proves the lemma. $\square$

Lemma 3.7. *Let* H *be set of prime numbers. If* $H_0 \subseteq H$ *and, for each* $p \in H_0$, $\{B_i^{(p)} : i \in J_p\}$ *is an order-rigid system in* $\mathcal{G}(H)$*, then* $\{T(\widehat{M}_p^{(3)}, B_i^{(p)}) : p \in H_0,\ i \in J_p\}$ *is an order-rigid system in* $\mathcal{G}(H \setminus H_0)$*.*

Proof. Assume that $p_1, p_2 \in H_0$, $i_1 \in J_{p_1}$, $i_2 \in J_{p_2}$, and that $\psi\colon T(\widehat{M}_{p_1}^{(3)}, B_{i_1}^{(p_1)}) \to T(\widehat{M}_{p_2}^{(3)}, B_{i_2}^{(p_2)})$ is an order embedding. Let φ denote the restriction of ψ to $\widehat{M}_{p_1}^{(3)}$. Since $p_1 \in H_0 \subseteq H$ and $B_{i_2}^{(p_2)} \in \mathcal{G}(H)$, there is no order embedding $\widehat{M}_{p_1}^{(3)} \to B_{i_2}^{(p_2)}$. Hence we obtain from Lemma 3.6 that φ or φ_1 embeds $\widehat{M}_{p_1}^{(3)}$ into $\widehat{M}_{p_2}^{(3)}$, where φ_1 is defined in Lemma 3.6(ii). Applying Lemma 3.3 to φ or φ_1, we conclude that $p_1 = p_2$ and φ or φ_1 is the identity map. This implies, in both cases, that the restriction η of ψ to $B_{i_1}^{(p_1)}$ embeds $B_{i_1}^{(p_1)}$ into $B_{i_2}^{(p_2)}$. We let $p := p_1 = p_2$. Using that $\{B_i^{(p)} : i \in J_p\}$ is an order-rigid system, we obtain that $i_1 = i_2$ and η is the identity map. In particular, $\eta(d) = d$, which shows that φ rather than φ_1 embeds $\widehat{M}_{p_1}^{(3)}$ into $\widehat{M}_{p_2}^{(3)}$. We have obtained that, except possibly for d, ψ acts identically on $T(\widehat{M}_{p_1}^{(3)}, B_{i_1}^{(p_1)}) = T(\widehat{M}_{p_2}^{(3)}, B_{i_2}^{(p_2)})$. Using its injectivity, we conclude that ψ is the identity map. This proves that $K := \{T(\widehat{M}_p^{(3)}, B_i^{(p)}) : p \in H_0,\ i \in J_p\}$ is an order-rigid system.

Next, to show that $K \subseteq \mathcal{G}(H \setminus H_0)$, assume that $p \in H_0$, $i \in J_p$, and $q \in H \setminus H_0$; we have to show that $\{\widehat{M}_q^{(3)}, T(\widehat{M}_p^{(3)}, B_i^{(p)})\}$ is an order-rigid system. Since K is an order-rigid system, $T(\widehat{M}_p^{(3)}, B_i^{(p)})$ is an order-rigid lattice. By Lemma 3.3, so is $\widehat{M}_q^{(3)}$. If we had an order embedding $\varphi\colon T(\widehat{M}_p^{(3)}, B_i^{(p)}) \to \widehat{M}_q^{(3)}$, then Lemma 3.3, applied to the restriction of φ to $\widehat{M}_p^{(3)}$, would give $p = q$, which would contradict $p \in H_0$ and $q \in H \setminus H_0$. Hence, $T(\widehat{M}_p^{(3)}, B_i^{(p)})$ cannot be order-embedded into $\widehat{M}_q^{(3)}$. Finally, suppose, for a contradiction, that there is an order embedding $\varphi\colon \widehat{M}_q^{(3)} \to T(\widehat{M}_p^{(3)}, B_i^{(p)})$. Since there is no order embedding $\widehat{M}_q^{(3)} \to \widehat{M}_p^{(3)}$ by Lemma 3.3, Lemma 3.6 gives that $\widehat{M}_q^{(3)}$ is order-embeddable into $B_i^{(p)}$. This contradicts $q \in H$ and $B_i^{(p)} \in \mathcal{G}(H)$. Thus, $T(\widehat{M}_p^{(3)}, B_i^{(p)}) \in \mathcal{G}(H \setminus H_0)$ and $K \subseteq \mathcal{G}(H \setminus H_0)$. $\square$

3.4. **The rest of the proof.** The following lemma belongs to the folklore of set theory. However, for the reader's convenience, we give a short proof. The powerset lattice of a set J is denoted by $P(J) = \langle P(J); \subseteq\rangle = \langle \{X : X \subseteq J\}; \subseteq\rangle$, and we let $P_{\geq 2}(X) := \{X \in P(J) : |X| \geq 2\}$.

Lemma 3.8. *For an infinite set J, there is an antichain $I \subseteq P_{\geq 2}(J)$ with $|I| = 2^{|J|}$.*

Proof. Pick $J_1, J_2 \in P(J)$ such that $J = J_1 \cup J_2$, $J_1 \cap J_2 = \varnothing$, and $|J_1| = |J_2| = |J|$. Let $\eta\colon J_1 \to J_2$ be a bijective map. Clearly, $I = \{X \cup \big(J_2 \setminus \eta(X)\big) : X \in P(J_1)\}$ is an antichain of cardinality $|I| = |P(J_1)| = |P(J)| = 2^{|J|}$ and $I \subseteq P_{\geq 2}(J)$. $\square$

In some sense, the following lemma asserts more than Theorem 1.1.

Lemma 3.9. *If λ is an $\aleph_0$-step accessible cardinal and $H \subset$ Primes such that $|H| = |\text{Primes} \setminus H| = \aleph_0$, then there exists an order-rigid system $\{L_i : i \in I\}$ in $\mathcal{G}(H)$ such that $|I| = \lambda$ and, for all $i \in I$, $|L_i| < \max\{\lambda, \aleph_0\}$.*

Proof. Suppose, for a contradiction, that the lemma fails for some $\aleph_0$-step accessible cardinal. Let λ be the smallest $\aleph_0$-step accessible cardinal witnessing this failure. Pick an $H \subset$ Primes such that $|H| = |\text{Primes} \setminus H| = \aleph_0$ and the lemma fails for the pair $\langle \lambda, H\rangle$. Denote by α the smallest ordinal such that $\lambda \leq \kappa_\alpha$. Note that $|\alpha| \leq \aleph_0$, since λ is $\aleph_0$-step accessible. It follows from Lemma 3.3 that $0 < \alpha$. Thus, $\lambda > \aleph_0$.

First, we assume that α is a successor ordinal, that is, $\alpha = \beta + 1$ for some ordinal β. By the definition of α, we have $\kappa_\beta < \lambda$. Since the lemma holds for κ_β, there exists an order-rigid system $\{A_j : j \in J\}$ in $\mathcal{G}(H)$ such that $|J| = \kappa_\beta$ and, for all $j \in J$, $|A_j| < \kappa_\beta$. By Lemma 3.8 and $\lambda \leq \kappa_\alpha = 2^{\kappa_\beta}$, there is an antichain I in the powerset lattice $P(J)$ such that $|I| = \lambda$. Note that $\varnothing \notin I$. For $i \in I$, let $L_i = \sum^{(c)}_{j\in i} A_j$. By Lemma 3.5, $\{L_i : i \in I\}$ is an order-rigid system in $\mathcal{G}(H)$. For $i \in I$, we have $|L_i| = 3 + \sum_{j\in i} |A_j| \leq 3 + \sum_{j\in J} |A_j| \leq 3 + |J| \cdot \kappa_\beta = \kappa_\beta \cdot \kappa_\beta = \kappa_\beta < \lambda$. This is a contradiction, since the lemma fails for $\langle \lambda, H\rangle$.

Consequently, α cannot be a successor ordinal. Hence, $\alpha = \sup\{\beta : \beta < \alpha\}$ and $\kappa_\alpha = \sup\{\kappa_\beta : \beta < \alpha\}$. By the choice of α, this implies $\lambda = \kappa_\alpha$ and, for all $\beta < \alpha$, $\kappa_\beta < \lambda$. Note that $|\alpha| = \aleph_0$, because $|\alpha| < \aleph_0$ would imply that α is a successor ordinal. Let H_1 be a subset of Primes$\setminus H$ such that $|H_1| = |(\text{Primes} \setminus H) \setminus H_1| = \aleph_0$. In other words, $\{H, H_1, \text{Primes} \setminus (H \cup H_1)\}$ is a partition of Primes with three infinite blocks. Since $|\{\beta : \beta < \alpha\}| = |\alpha| = \aleph_0 = |H_1|$, there exists a bijective map $\tau\colon H_1 \to \{\beta : \beta < \alpha\}$. We have $\{\kappa_\beta : \beta < \alpha\} = \{\kappa_{\tau(p)} : p \in H_1\}$. The lemma holds for these $\kappa_{\tau(p)}$. Hence, for each $p \in H_1$, we can pick an order-rigid set $\{B_i^{(p)} : i \in J_p\}$ in $\mathcal{G}(H \cup H_1)$ such that $|J_p| = \kappa_{\tau(p)}$ and, for all $i \in J_p$, $|B_i^{(p)}| < \kappa_{\tau(p)}$.

By Lemma 3.7, $K = \{T(\widehat{M}_p^{(3)}, B_i^{(p)}) : p \in H_1,\ \ i \in J_p\}$ is an order-rigid system in $\mathcal{G}(H)$. We have

$$|K| = \sum_{p\in H_1} |J_p| = \sum_{p\in H_1} \kappa_{\tau(p)} = \sum_{\beta<\alpha} \kappa_\beta = \sup\{\beta : \beta < \alpha\} = \kappa_\alpha = \lambda.$$

Furthermore, assume that $L = T(\widehat{M}_p^{(3)}, B_i^{(p)}) \in K$, that is, if $p \in H_1$ and $i \in J_p$. If L is infinite, then $|L| = |\widehat{M}_p^{(3)}| + |B_i^{(p)}| = |B_i^{(p)}| < \kappa_{\tau(p)} < \kappa_\alpha = \lambda$. This contradicts the assumption that the lemma fails for the pair $\langle \lambda, H\rangle$. □

Proof of Theorem 1.1. Lemmas 3.3 and 3.9. □

Proof of Corollary 2.1. For $\lambda \in \{0, 1\}$, we can take the 2-element lattice. Denoting the λ-element chain by $C_{\lambda-1}$, the lattice $L = \sum^{(c)}_{j\in\{1,2\}} C_{\lambda-1}$ proves the statement for $1 < \lambda < \aleph_0$. Thus, we assume that $\aleph_0 \le \lambda$. Since λ is $\aleph_0$-step accessible, there is a smallest ordinal α such that $\lambda \le \kappa_\alpha$ and $|\alpha| \le \aleph_0$. Let $\beta = \alpha + 1$. Clearly, κ_β is still an $\aleph_0$-step accessible cardinal and $\lambda < \kappa_\beta$. If α is a successor ordinal of the form $\alpha = \gamma + 1$, then $\kappa_\gamma < \lambda$ and $\kappa_\beta = 2^{2^{\kappa_\gamma}} \le 2^{2^\lambda}$. If α is a limit ordinal, then the minimality of α implies $\lambda = \kappa_\alpha$, and we again have $\kappa_\beta = 2^\lambda \le 2^{2^\lambda}$. By Theorem 1.1, there exists an order-rigid system $\{L_i : i \in I\}$ such that $|I| = \kappa_\beta$ and, for all $i \in I$, $|L_i| < \kappa_\beta$. It follows from Lemma 3.5 that $L := \sum^{(c)}_{i\in I} L_i$ is an order-rigid lattice. Finally,

$$\lambda < \kappa_\beta = \sum_{i\in I} 1 \le 3 + \sum_{i\in I} |L_i| = |L| \le \sum_{i\in I}(1 + |L_i|) \le |I| \cdot \kappa_\beta = \kappa_\beta \le 2^{2^\lambda}. \quad \square$$

References

[1] Birkhoff, G. (1979). *Lattice theory.* Corrected reprint of the 1967 third edition. American Mathematical Society Colloquium Publications, 25. American Mathematical Society, Providence, R.I., vi+418 pp. ISBN: 0-8218-1025-1 06-01 (46A40 47B55 54H12)

[2] Duffus, D., Erdős, P. L., Nešetřil, J. , Soukup, L. (2007). Antichains in the homomorphism order of graphs. *Comment. Math. Univ. Carolin.* **48**:571–583.

[3] Fried, E, Sichler, J. (1977). Homomorphisms of integral domains of characteristic zero. *Trans. Amer. Math. Soc.* **225**:163–182.

[4] Hedrlín, Z., J. Lambek, J. (1969). How comprehensive is the category of semigroups?. *J. Algebra* **11**:195–212.

[5] Herrmann, C., Huhn, A. P. (1976). Lattices of normal subgroups which are generated by frames. in: *Lattice theory (Proc. Colloq., Szeged, 1974)*, pp. 97–136. Colloq. Math. Soc. János Bolyai, Vol. 14, North-Holland, Amsterdam.

[6] Jakubíková-Studenovská, D. (1980). On weakly rigid monounary algebras. *Math. Slovaca* **30**:197–206.

[7] Nešetřil, J., Tardiff, C. (2003). On maximal finite antichains in the homomorphism order of directed graphs. *Discuss. Math. Graph Theory* **23**:325–332 .

[8] Pinus, A. G. (1993). *Boolean constructions in universal algebras.* Mathematics and its Applications, 242. Kluwer Academic Publishers Group, Dordrecht, viii+350 pp.

[9] Primavesi, A., Thompson, K. (2012). The embedding structure for linearly ordered topological spaces. *Topology Appl.* **159**:3103–3114.

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