LAMPS IN SLIM RECTANGULAR PLANAR SEMIMODULAR LATTICES

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Dedicated to László Kérchy on his seventieth birthday

Abstract. A planar (upper) semimodular lattice L is *slim* if the five-element nondistributive modular lattice M_3 does not occur among its sublattices. (Planar lattices are finite by definition.) *Slim rectangular lattices* as particular slim planar semimodular lattices were defined by G. Grätzer and E. Knapp in 2007. In 2009, they also proved that the congruence lattices of slim planar semimodular lattices with at least three elements are the same as those of slim rectangular lattices. In order to provide an effective tool for studying these congruence lattices, we introduce the concept of *lamps* of slim rectangular lattices and prove several of their properties. Lamps and several tools based on them allow us to prove in a new and easy way that the congruence lattices of slim planar semimodular lattices satisfy the two previously known properties. Also, we use lamps to prove that these congruence lattices satisfy four new properties including the *Two-pendant Four-crown Property* and the *Forbidden Marriage Property.*

Note on the dedication. Professor *László Kérchy* is the previous editor-in-chief of Acta Sci. Math. (Szeged). I have known him since 1967, when I enrolled in a high school where he was a second-year student with widely acknowledged mathematical talent. His influence had played a role that I became a student at the Bolyai (Mathematical) Institute in Szeged. Furthermore, he helped me to become one of his roommates in Loránd Eötvös University Dormitory in 1972. With my gratitude, I dedicate this paper to his seventieth birthday.

1. Introduction

The theory of planar semimodular lattices has been an intensively studied part of lattice theory since Grätzer and Knapp's pioneering [22]. The key role in the theory of these lattices is played by *slim* planar semimodular lattices defined in Section 2. The importance of slim planar semimodular lattices is surveyed, for example, in Czédli and Kurusa [10], Czédli and Grätzer [7], Czédli and Schmidt [12], and Grätzer and Nation [24]. *Slim rectangular lattices*, to be defined later, were introduced in Grätzer and Knapp [23] as particular slim planar semimodular lattices and they will play a crucial role in our proofs. The study of *congruence lattices*

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Con L of slim planar semimodular lattices L goes back to Grätzer and Knapp [23]. These congruence lattices are finite distributive lattices and, in addition, we know from Czédli [4] and Grätzer [16], [17], and [21] that they have special properties.

Target. One of our targets is to develop effective tools to derive the above-mentioned special properties in a new and easy way and to present four new properties. To accomplish that, we are going to define the *lamps* of a slim rectangular lattice L so that the set of lamps becomes a poset (partially ordered set) isomorphic to the poset $\mathrm{J}(\mathrm{Con}\,L)$ of join-irreducible congruences of L. The problem whether the properties recalled or proved in the present paper and in Czédli and Grätzer [8] characterize the congruence lattices of slim planar semimodular lattices remains open.

Outline. For comparison with our lamps, the rest of the present section mentions the known ways to describe $\mathrm{J}(\mathrm{Con}\,L)$ for a finite lattice L. In Section 2, we recall the concept of slim planar semimodular lattices, that of slim rectangular lattices, and that of their $\mathcal{C}_1$-diagrams. Also in Section 2, we introduce the concept of lamps of these lattices and prove our (Main) Lemma 2.11. This lemma provides the main tool for "illuminating" the congruence lattices of slim planar semimodular lattices. In Section 3, further tools are given and several consequences of (the Main) Lemma 2.11 are proved. In particular, this section proves that the congruence lattices of slim planar semimodular lattices satisfy both previously known properties, see Corollaries 3.3 and 3.6, and two new properties, see Corollaries 3.4 and 3.5. Section 4 defines the *Two-pendant Four-crown Property* and the *Forbidden Marriage Property*, and proves that the congruence lattices of slim planar semimodular lattices satisfy these two properties; see Theorem 4.3.

Comparison with earlier approaches to $\mathrm{J}(\mathrm{Con}\,L)$. Let L be a finite lattice; its *congruence lattice* is denoted by Con L. Since Con L is distributive, it is determined by the poset $\mathrm{J}(\mathrm{Con}\,L) = \langle \mathrm{J}(\mathrm{Con}\,L); \leq\rangle$ of its nonzero *join-irreducible elements*. There are three known ways to describe this poset.

First, one can use the *join dependency relation* defined on $J(L)$; see Lemma 2.36 of the monograph Freese, Ježek, and Nation [14], where this relation is attributed to Day [13].

Second, Grätzer [17] takes (and well describes) the prime-perspectivity relation on the set of *prime intervals* of L. His description becomes more powerful if L happens to be a slim planar semimodular lattice: for such a lattice, Grätzer's Swing Lemma applies, see [18] and see also Czédli, Grätzer, and Lakser [9] and Czédli and Makay [11].

Third, but only for a slim rectangular lattice L, Czédli [2] defined a relation on the set of *trajectories* of L while Grätzer [18] defined an analogous relation on the set of prime intervals. Theorem 7.3(ii) of Czédli [2] indicates that the dual of the approach based on join dependency relation could also have been used to derive a more or less similar description of $\mathrm{J}(\mathrm{Con}\,L)$.

A relation $\rho \subseteq X^2$ on a set X is a *quasiorder* (also called *preorder*) if it is reflexive and transitive. Each of the three approaches mentioned above defines only a quasiorder on a set X in the first step; this set consists of join-irreducible elements, prime intervals, or trajectories, respectively. In the next step, we have to form the quotient set $X/(\rho\cap\rho^{-1})$ and equip it with the quotient relation $\rho/(\rho\cap\rho^{-1})$ to obtain $\langle \mathrm{J}(\mathrm{Con}\,L); \leq\rangle$ up to isomorphism. For a slim rectangular lattice L, our lamps provide a more efficient description of $\mathrm{J}(\mathrm{Con}\, L)$ since we do not have to form a quotient set.

2. From diagrams to lamps

We assume familiarity with the rudiments of lattice theory. For concepts and notation not defined in this paper, the reader is referred to the monograph Grätzer [15] and (the freely available) Part I of Grätzer [19]. By definition, planar lattices are finite. A *slim planar semimodular lattice* is a planar (upper) semimodular lattice L such that one of the following three conditions holds:

(i) M_3, the five-element nondistributive modular lattice, is not a sublattice of L,
(ii) M_3 is not a cover-preserving sublattice of L,
(iii) $\mathrm{J}(L)$, the *set of nonzero join-irreducible elements* of L, is the union of two chains;

see Grätzer and Knapp [22] and Czédli and Schmidt [12], or the book chapter Czédli and Grätzer [7] for the equivalence of these three conditions for planar semimodular lattices (but not for other lattices.)

Let L be a slim planar semimodular lattice; we *always* assume that a planar diagram of L is fixed. The *left boundary chain* and the *right boundary chain* of L are denoted by $\mathrm{C}_{\mathrm{left}}(L)$ and $\mathrm{C}_{\mathrm{right}}(L)$, respectively. Here and at several other concepts occurring later, we heavily rely on the fact that the diagram of L is fixed; indeed, $\mathrm{C}_{\mathrm{left}}(L)$ and $\mathrm{C}_{\mathrm{right}}(L)$ depend on the diagram, not only on L.

Following Grätzer and Knapp [23], a slim planar semimodular lattice is called a *slim rectangular lattice* if $|L| \geq 4$, $\mathrm{C}_{\mathrm{left}}(L)$ has exactly one doubly irreducible element, $\mathrm{lc}(L)$, $\mathrm{C}_{\mathrm{right}}(L)$ has exactly one doubly irreducible element, $\mathrm{rc}(L)$, and these two doubly irreducibly elements are complementary, that is, $\mathrm{lc}(L) \vee \mathrm{rc}(L) = 1$ and $\mathrm{lc}(L) \wedge \mathrm{rc}(L) = 0$. Here $\mathrm{lc}(L)$ and $\mathrm{rc}(L)$ are called the *left corner* (element) and the *right corner* (element) of the rectangular lattice L. Note that $|L| \geq 4$ above can be replaced by $|L| \geq 3$. Note also that the definition of rectangularity does not depend on how the diagram is fixed since $\mathrm{lc}(L)$ and $\mathrm{rc}(L)$ are the only doubly irreducible elements of L.

Let us emphasize that a slim rectangular lattice is planar and semimodular by definition, whereby the title of the paper is redundant. The purpose of this redundancy is to give more information about the content of the paper.

As in Grätzer and Knapp [23], the (principal) ideals $\downarrow\mathrm{lc}(L)$ and $\downarrow\mathrm{rc}(L)$ are chains and they are called the *bottom left boundary chain* and the *bottom right boundary chain*, respectively, while the filters $\uparrow\mathrm{lc}(L)$ and $\uparrow\mathrm{rc}(L)$ are also chains, the *top left boundary chain* and *top right boundary chain*, respectively. The *lower boundary* and the *upper boundary* of L are $\downarrow\mathrm{lc}(L) \cup \downarrow\mathrm{rc}(L)$ and $\uparrow\mathrm{lc}(L) \cup \uparrow\mathrm{rc}(L)$, respectively. Note also that $\mathrm{J}(L) \cup \{0\}$ equals the lower boundary $\downarrow\mathrm{lc}(L) \cup \downarrow\mathrm{rc}(L)$, but for the set $\mathrm{M}(L)$ of non-unit *meet-irreducible elements*, we only have that $\uparrow\mathrm{lc}(L) \cup \uparrow\mathrm{rc}(L) \subseteq \mathrm{M}(L) \cup \{1\}$. For example, the lattices $S_7^{(n)}$ for $n \in \mathbb{N}^+ := \{1, 2, 3, \dots\}$, defined in Czédli [2] and presented here in Figure 1 for $n \leq 4$, are slim rectangular lattices.

If p and q are elements of a lattice such that $p \prec q$, then the *prime interval*, that is, the two-element interval $[p, q]$ is an *edge* of the diagram. Following Czédli [5], we need the following concepts.

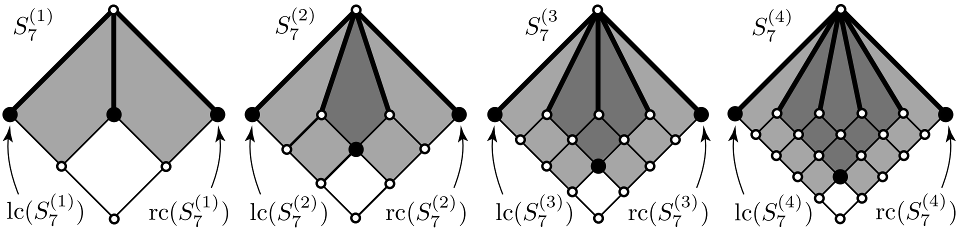

FIGURE 1. $S_7^{(1)}$, $S_7^{(2)}$, $S_7^{(3)}$, and $S_7^{(4)}$

Definition 2.1 (Types of diagrams)**.** The *slope* of the line $\{\langle x, x\rangle : x \in \mathbb{R}\}$ and that of the line $\{\langle x, -x\rangle : x \in \mathbb{R}\}$ are called *normal slopes*. This allows us to speak of lines, line segments, and edges of normal slopes. For example, an edge $[p, q]$ of a lattice diagram is of a normal slope iff the angle that this edge makes with a horizontal line is $\pi/4$ (45°) or $3\pi/4$ (135°). If this angle is strictly between $\pi/4$ and $3\pi/4$, then the edge is *precipitous*. For examples, vertical edges are precipitous. We say that a diagram of a slim rectangular lattice L belongs to $\mathcal{C}_1$ or, in other words, it is a $\mathcal{C}_1$*-diagram* if every edge $[p, q]$ such that $p \in \mathrm{M}(L) \setminus (\mathrm{C}_{\mathrm{left}}(L) \cup \mathrm{C}_{\mathrm{right}}(L))$ is precipitous and all the other edges are of normal slopes. That is, if every edge with a meet-irreducible bottom not on the boundary is precipitous and the rest of edges are of normal slopes. A $\mathcal{C}_1$-diagram of L belongs to $\mathcal{C}_2$ or, in other words, it is a $\mathcal{C}_2$*-diagram* if any two edges on the lower boundary are of the same geometric length.

The diagrams in Figures 1, 3, 7, and 11, $L_6 = L$ in Figure 6, and the diagrams denoted by L in Figures 2 and 4 are $\mathcal{C}_2$-diagrams. In addition to these $\mathcal{C}_2$-diagrams, the diagrams in Figures 5 and 6 are $\mathcal{C}_1$-diagrams while the diagram in Figure 10 can be a part of a $\mathcal{C}_1$-diagram. In fact, *all* diagrams of slim rectangular lattices in this paper are $\mathcal{C}_1$-diagrams. Note that we believe that only $\mathcal{C}_1$-diagrams can give satisfactory insight into the congruence lattices of slim rectangular lattices. For diagrams, drawn or not, let us agree in the following.

Convention 2.2. In the rest of the paper, all diagrams of slim rectangular lattices are assumed to be $\mathcal{C}_1$-diagrams. Furthermore, L will always denote a slim rectangular lattice with a fixed $\mathcal{C}_1$-diagram.

The reader may wonder how the new concepts in the following definition obtained their name; the explanation will be given in the paragraph preceding Definition 2.8.

Definition 2.3 (Lamps)**.** Let L be a slim rectangular lattice with a fixed $\mathcal{C}_1$-diagram.

(i) An edge $\mathfrak{n} = [p, q]$ of L is a *neon tube* if $p \in \mathrm{M}(L)$. The elements p and q are the *foot*, denoted by $\mathrm{Foot}(\mathfrak{n})$, and the *top* of this neon tube. Since p is meet-irreducible, a neon tube is determined by its foot.
(ii) If an edge $[p, q]$ is a neon tube such that p belongs to the boundary of L (equivalently, if $\downarrow p$ contains a doubly irreducible element), then $[p, q]$ is also called a *boundary lamp* (with a unique neon tube $[p, q]$). If $I = [p, q]$ is a boundary lamp, then p is called the *foot* of I and is denoted by $\mathrm{Foot}(I)$ while $\mathrm{Peak}(I) := q$ is the *peak* of I. Note the terminological difference: neon tubes have tops but lamps have peaks.

(iii) Assume that $q \in L$ is the top of a neon tube whose foot is not on the boundary $\mathrm{C_{left}}(L) \cup \mathrm{C_{right}}(L)$ of L, and let

$$\beta_q := \bigwedge\{p_i : [p_i, q] \text{ is a neon tube and } p_i \notin \mathrm{C_{left}}(L) \cup \mathrm{C_{right}}(L)\}. \tag{2.1}$$

Then the interval $I := [\beta_q, q]$ is an *internal lamp* of L. The prime intervals $[p_i, q]$ such that $p_i \in \mathrm{M}(L)$ but $p_i \notin \mathrm{C_{left}}(L) \cup \mathrm{C_{right}}(L)$ are the *neon tubes* of this lamp. If I is an internal lamp, then either I is a neon tube and we say that I has a unique neon tube, or I has more than one neon tubes. The element q is the *peak* of the lamp I and it is denoted by $\mathrm{Peak}(I)$ while $\mathrm{Foot}(I) := \beta_q$ is the *foot* of I.

(iv) The *lamps* of L are its boundary lamps and its internal lamps. Clearly, for every lamp I of L,

$$\mathrm{Foot}(I) = \bigwedge\{\mathrm{Foot}(\mathfrak{n}) : \mathfrak{n} \text{ is a neon tube of } I\}. \tag{2.2}$$

Since a slim rectangular lattice L has only two doubly irreducible elements, $\mathrm{lc}(L)$ and $\mathrm{rc}(L)$, the non-containment in (2.1) is equivalent to the condition that "$\downarrow p_i$ does not contain a doubly irreducible element". Therefore, the concept of lamps does not depend on the diagram of L. Note that under reasonable assumptions, the $\mathcal{C}_1$-diagram of L is unique, and so is its $\mathcal{C}_2$-diagram; see Czédli [5]. To help the reader in finding the lamps in our diagrams, let us agree to the following.

Convention 2.4. In our diagrams of slim rectangular lattices, the feet of lamps are exactly the black-filled elements. (Except possibly for Figure 10, which can be but need not be the whole lattice in question.) The thick edges are always neon tubes (but there can be neon tubes that are not thick edges).

Note that in (the slim rectangular lattices of) Figures 1, 2, 4, and 11, the neon tubes are exactly the thick edges. In addition to the fact that neon tubes are easy to recognize as edges with bottom elements in $\mathrm{M}(L)$, there is another way to recognize them even more easily; the following remark follows from definitions.

Remark 2.5. Neon tubes in a $\mathcal{C}_1$-diagram of a slim rectangular lattice (and so in our figures) are exactly the precipitous edges and the edges on the upper boundary $\uparrow\mathrm{lc}(L) \cup \uparrow\mathrm{rc}(L)$.

Regions of a slim rectangular lattice L are defined as closed planar polygons surrounded by some edges of (the fixed diagram) of L; see Kelly and Rival [26] for an elaborate treatise on these regions, or see Czédli and Grätzer [7]. Note that every interval of a planar lattice determines a region; possibly of area 0 if the interval is a chain. The affine plane on which diagrams are drawn is often identified with $\mathbb{R}^2$ via the classical coordinatization. By the *full geometric rectangle* of a slim rectangular lattice L with a fixed diagram we mean the closed geometric rectangle whose boundary is the union of all edges belonging to $\mathrm{C_{left}}(L) \cup \mathrm{C_{right}}(L)$. It is a rectangle indeed since we allow $\mathcal{C}_1$-diagrams only. Smaller geometric rectangles are also relevant; this is why we have the second part of the following definition.

Definition 2.6 (geometric shapes associated with lamps)**.** Keeping Convention 2.2 in mind, let $I = [p, q] = [\mathrm{Foot}(I), \mathrm{Peak}(I)]$ be a lamp of L.

(i) The *body* of I, denoted by $\mathrm{Body}(I) = \mathrm{Body}([p, q])$ is the region determined by $[p, q]$. Note that $\mathrm{Body}(I)$ is a line segment if I has only one neon tube, and (by Remark 2.5) it is a quadrangle of positive area having two precipitous upper edges and two lower edges of normal slopes otherwise.

(ii) Assume that I is an internal lamp, and define r as the meet of all lower covers of q. Then the interval $[r, q]$ is a region; this region is denoted by $\mathrm{CircR}(I) = \mathrm{CircR}([p, q])$ and it is called the *circumscribed rectangle* of I.

For example, if I is the only internal lamp of $S_7^{(n)}$, then $\mathrm{CircR}(I)$ is the full geometric rectangle of $S_7^{(n)}$ for all $n \in \mathbb{N}^+$ while $\mathrm{Body}(I)$ is the dark-grey area for $n \in \{2, 3, 4\}$ in Figure 1. To see another example, if E is the lamp with two neon tubes labelled by e on the left of Figure 2, then $\mathrm{Body}(I)$ is the dark-grey area and the vertices of $\mathrm{CircR}(I)$ are x, y, z, and t. Note that by the dual of Czédli [3, Proposition 3.13],

$$r \text{ in Definition 2.6(ii) can also be defined as the meet of the leftmost lower cover and the rightmost lower cover of } q. \tag{2.3}$$

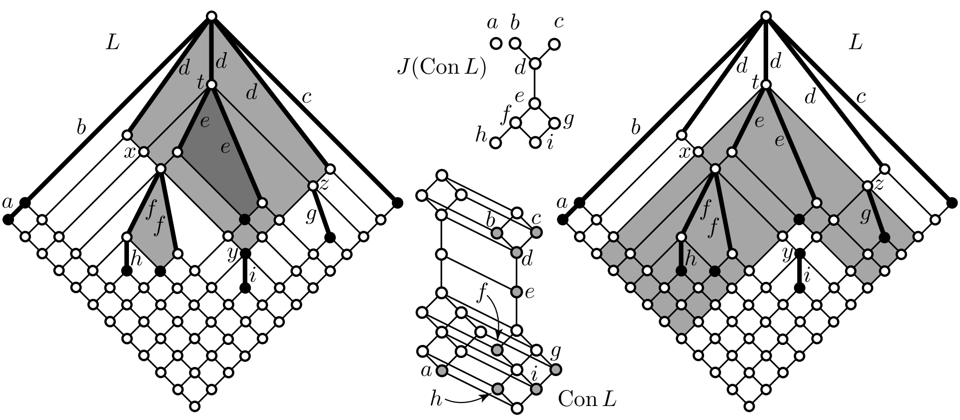

FIGURE 2. $\mathrm{Lamp}(L) \cong \mathrm{J}(\mathrm{Con}\, L)$, whence $\mathrm{Lamp}(L)$ determines $\mathrm{Con}\, L$

Definition 2.7 (line segments associated with lamps)**.** Let $I := [p, q]$ be a lamp of a slim rectangular lattice L with a fixed $\mathcal{C}_1$-diagram, and let F stand for the full geometric rectangle of L. Let $\langle p_x, p_y\rangle \in \mathbb{R}^2$ and $\langle q_x, q_y\rangle \in \mathbb{R}^2$ be the geometric points corresponding to $p = \mathrm{Foot}(I)$ and $q = \mathrm{Peak}(I)$. As usual, $\mathbb{R}^+$ will stand for the set of non-negative real numbers. We define the following four (geometric) line segments of normal slopes; see Figure 3 where these line segments are dashed edges of normal slopes.

$$\mathrm{LRoof}(I) := \{\langle \xi, \eta\rangle \in F : (\exists t \in \mathbb{R}^+)\,(\xi = q_x - t \text{ and } \eta = q_y - t)\},$$
$$\mathrm{RRoof}(I) := \{\langle \xi, \eta\rangle \in F : (\exists t \in \mathbb{R}^+)\,(\xi = q_x + t \text{ and } \eta = q_y - t)\},$$
$$\mathrm{LFloor}(I) := \{\langle \xi, \eta\rangle \in F : (\exists t \in \mathbb{R}^+)\,(\xi = p_x - t \text{ and } \eta = p_y - t)\},$$
$$\mathrm{RFloor}(I) := \{\langle \xi, \eta\rangle \in F : (\exists t \in \mathbb{R}^+)\,(\xi = p_x + t \text{ and } \eta = p_y - t)\}.$$

These line segments are called the *left roof*, the *right roof*, the *left floor*, and the *right floor* of I, respectively. Note that $\mathrm{LRoof}(I)$ and $\mathrm{LFloor}(I)$ lie on the same geometric line if and only if I is a boundary lamp on the left boundary, and analogously for $\mathrm{RRoof}(I)$ and $\mathrm{RFloor}(I)$. We defined the *roof of* I and the *floor of* I as follows:

$$\mathrm{Roof}(I) := \mathrm{LRoof}(I) \cup \mathrm{RRoof}(I), \text{ and}$$
$$\mathrm{Floor}(I) := \mathrm{LFloor}(I) \cup \mathrm{RFloor}(I);$$

they are $\boldsymbol{\wedge}$-shaped broken lines.

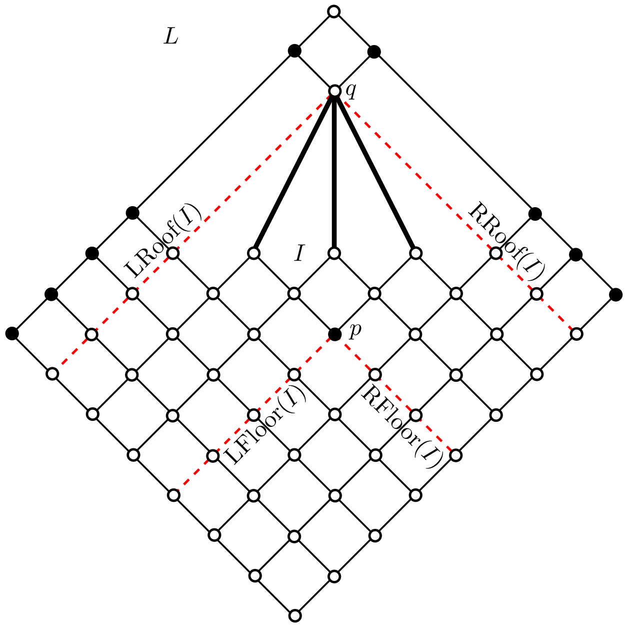

FIGURE 3. Four line segments associated with I

In real life, neon tubes and lamps are for *illuminating* in the sense of emitting light beams. Our lamps do this only downwards with normal slopes; the photons they emit can only go to the directions $\langle 1, -1\rangle$ and $\langle -1, -1\rangle$. Definition 2.8 below describes this more precisely. Note at this point that in addition to brightening with light, "illuminating" also means intellectual enlightening, that is, making things clear for human mind. This explains that lamps and neon tubes occur in our terminology just introduced: by *illuminating* a part of a $\mathcal{C}_1$-diagram in a visual geometrical way like in physics, lamps also *illuminate* the congruence structure by enlightening it in intellectual sense. Convention 2.2 is still in effect.

Definition 2.8 (Illuminated sets)**.** Let $I := [\mathrm{Foot}(I), q] = [\mathrm{Foot}(I), \mathrm{Peak}(I)]$ be a lamp of L. A geometric point $\langle x, y\rangle$ of the full geometric rectangle of L is *illuminated by* I *from the left* if the lamp has a neon tube $[p_i, q]$ such that the edge $[p_i, q]$ as a geometric line segment has a nonempty intersection with the half-line $\{\langle x - t, y + t\rangle : 0 \le t \in \mathbb{R}\}$. Similarly, a point $\langle x, y\rangle$ of the full geometric rectangle of L is *illuminated by* I *from the right* if the half-line $\{\langle x + t, y + t\rangle : 0 \le t \in \mathbb{R}\}$ has a nonempty intersection with at least one of the neon tubes of I. If $\langle x, y\rangle$ is illuminated from the left or from the right, then we simply say that this point is *illuminated by the lamp* I. The set of points illuminated by the lamp I, that of points illuminated by I from the right, and that of points illuminated by I from the left are denoted by

$$\left.\begin{aligned} \mathrm{Lit}(I) &= \mathrm{Lit}([\mathrm{Foot}(I), \mathrm{Peak}(I)]), \\ \mathrm{LeftLit}(I) &= \mathrm{LeftLit}([\mathrm{Foot}(I), \mathrm{Peak}(I)]), \text{ and} \\ \mathrm{RightLit}(I) &= \mathrm{RightLit}([\mathrm{Foot}(I), \mathrm{Peak}(I)]) \end{aligned}\right\} \tag{2.4}$$

respectively. The acronym "Lit" and its variants come from "light" rather than from "illuminate". (The shape of "Ill" coming from "illuminate" would be less satisfactory and would heavily depend on the font used.) Let us emphasize that, say, LeftLit(I) consist of points illuminated from the *right*; the notation is explained by the fact that the geometric points of LeftLit(I) are on the left of (and down from) I. Note that LeftLit(I) is of positive geometric area if and only if I is not a boundary lamp on the left boundary, and analogously for RightLit(I). Finally, we also define

$$\mathrm{Lit}^+(I) := \mathrm{Lit}(I) \setminus \mathrm{Floor}(I). \tag{2.5}$$

Note that Lit(I) is LeftLit$(I) \cup$ RightLit(I). By Definition 2.7 and 2.8,

Lit(I) is (topologically) bordered by Roof(I), Floor(I), and appropriate line segments of $\mathrm{C}_{\mathrm{left}}(L)$ and $\mathrm{C}_{\mathrm{right}}(L)$, and so it is bordered by line segments of normal slopes. (2.6)

Note also that the intersection LeftLit$(I) \cap$ RightLit(I) can be of positive (geometric) area in the plane and that both LeftLit(I) and RightLit(I) are of positive area if and only if I is an internal lamp.

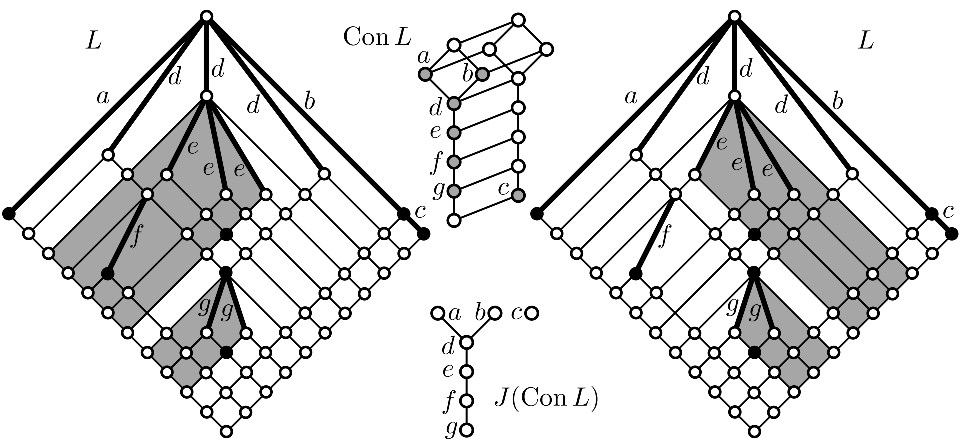

FIGURE 4. Illustrating (2.4) and Lamp$(L) \cong$ J(Con L)

For example, each of $S_7^{(1)}$, $S_7^{(2)}$, $S_7^{(3)}$, and $S_7^{(4)}$ of Figure 1 has a unique internal lamp, namely, the interval spanned by the black-filled enlarged element in the middle and 1. The illuminated set of this lamp is the "$\mathbf{\Lambda}$-shaped" grey-filled hexagon (containing light-grey and dark-grey points). Also, $S_7^{(n)}$ has exactly two boundary lamps and the illuminated set of each of these two lamps is the whole geometric rectangle of $S_7^{(n)}$, for every $n \in \mathbb{N}^+$. If E denotes the lamp with two e-labelled neon tubes on the right of Figure 2, then Lit(I) is the "$\mathbf{\Lambda}$-shaped" grey-filled hexagon. In Figure 4, let E and G denote the lamps consisting of the e-labelled edges and the g-labelled edges, respectively. On the left of this figure, $\mathrm{LeftLit}(E)$ and $\mathrm{LeftLit}(G)$ are the grey-filled trapezoids while the grey-filled trapezoids on the right are $\mathrm{RightLit}(E)$ and $\mathrm{RightLit}(G)$. Let us note that, for any slim rectangular lattice L,

$$\text{two distinct internal lamps can never have the same peak,} \tag{2.7}$$

but all the three lamps, two boundary and one internal, have the same peak in $S_7^{(n)}$.

Next, we introduce some relations on the set of lamps. Even though not all of these relations are applied in the subsequent sections, these relations and Lemma 2.11 will hopefully be useful in future investigation of the congruence lattices of slim planar semimodular lattices; for example, in Czédli and Grätzer [8].

Definition 2.9 (Relations defined for lamps)**.** Let L be a slim rectangular lattice with a fixed $\mathcal{C}_1$-diagram. The set of lamps of L will be denoted by $\mathrm{Lamp}(L)$. On this set, we define eight irreflexive binary relations; seven of them geometrically, based on Definitions 2.6–2.8, and one in a purely algebraic way; these relations will soon be shown to be equal. For $I, J \in \mathrm{Lamp}(L)$,

(i) let $\langle I, J\rangle \in \boldsymbol{\rho}_{\mathrm{Body}}$ mean that $I \neq J$, $\mathrm{Body}(I) \subseteq \mathrm{Lit}(J)$, and I is an internal lamp;
(ii) let $\langle I, J\rangle \in \boldsymbol{\rho}_{\mathrm{CircR}}$ mean that I is an internal lamp, $\mathrm{CircR}(I) \subseteq \mathrm{Lit}(J)$, and $I \neq J$;
(iii) let $\langle I, J\rangle \in \boldsymbol{\rho}_{\mathrm{alg}}$ mean that $\mathrm{Peak}(I) \leq \mathrm{Peak}(J)$ but $\mathrm{Foot}(I) \nleq \mathrm{Foot}(J)$;
(iv) let $\langle I, J\rangle \in \boldsymbol{\rho}_{\mathrm{LRBody}}$ mean that I is an internal lamp, $I \neq J$, and

$$\mathrm{Body}(I) \subseteq \mathrm{LeftLit}(J) \ \text{ or } \ \mathrm{Body}(I) \subseteq \mathrm{RightLit}(J);$$

(v) let $\langle I, J\rangle \in \boldsymbol{\rho}_{\mathrm{LRCircR}}$ mean that I is an internal lamp, $I \neq J$, and

$$\mathrm{CircR}(I) \subseteq \mathrm{LeftLit}(J) \ \text{ or } \ \mathrm{CircR}(I) \subseteq \mathrm{RightLit}(J);$$

(vi) let $\langle I, J\rangle \in \boldsymbol{\rho}_{\mathrm{foot}}$ mean that $I \neq J$, $\mathrm{Foot}(I) \in \mathrm{Lit}(J)$, and I is an internal lamp;
(vii) let $\langle I, J\rangle \in \boldsymbol{\rho}_{\mathrm{infoot}}$ mean that $I \neq J$ and $\mathrm{Foot}(I)$ is in the geometric (or, in other words, topological) interior of $\mathrm{Lit}(J)$; and, finally,
(viii) let $\langle I, J\rangle \in \boldsymbol{\rho}_{\mathrm{in+foot}}$ mean that $\mathrm{Foot}(I) \in \mathrm{Lit}^+(J)$.

Remark 2.10. In each of (i), (ii),…,(viii) of Definition 2.9, I is an internal lamp and $I \neq J$; this follows easily from other stipulations even where this is not explicitly stipulated.

Now we are in the position to formulate the main result of this section. The congruence generated by a pair $\langle x, y\rangle$ of elements will be denoted by $\mathrm{con}(x, y)$. If $\mathfrak{p} = [x, y]$ is an interval, then we can write $\mathrm{con}(\mathfrak{p})$ instead of $\mathrm{con}(x, y)$.

Lemma 2.11 (Main Lemma)**.** *If L is a slim rectangular lattice with a fixed $\mathcal{C}_1$-diagram, then the following four statements hold.*

(i) *The relations $\boldsymbol{\rho}_{\mathrm{Body}}$, $\boldsymbol{\rho}_{\mathrm{CircR}}$, $\boldsymbol{\rho}_{\mathrm{alg}}$, $\boldsymbol{\rho}_{\mathrm{LRBody}}$, $\boldsymbol{\rho}_{\mathrm{LRCircR}}$, $\boldsymbol{\rho}_{\mathrm{foot}}$, $\boldsymbol{\rho}_{\mathrm{infoot}}$, and $\boldsymbol{\rho}_{\mathrm{in+foot}}$ are equal, that is, $\boldsymbol{\rho}_{\mathrm{Body}} = \boldsymbol{\rho}_{\mathrm{CircR}} = \cdots = \boldsymbol{\rho}_{\mathrm{in+foot}}$.*
(ii) *The reflexive transitive closure of $\boldsymbol{\rho}_{\mathrm{alg}}$ is a partial ordering of the set* $\mathrm{Lamp}(L)$ *of all lamps of L; we denote this reflexive transitive closure by $\leq$.*

(iii) *The poset* $\langle \text{Lamp}(L); \leq\rangle$ *is isomorphic to the poset* $\langle \text{J}(\text{Con}\, L); \leq\rangle$ *of nonzero join-irreducible congruences of* L *with respect to the ordering inherited from* $\text{Con}\, L$*, and the map* $\varphi\colon \text{Lamp}(L) \to \text{J}(\text{Con}\, L)$*, defined by* $[p, q] \mapsto \text{con}(p, q)$*, is an order isomorphism.*

(iv) *If* $I \prec J$ *(that is,* I *is covered by* J*) in* $\text{Lamp}(L)$*, then* $\langle I, J\rangle \in \boldsymbol{\rho}_{\text{alg}}$.

The proof of this lemma heavily relies upon Czédli [2] and [5]. Before presenting this proof, we need some preparations. First, we need to recall the *multifork structure theory* from Czédli [2]. Minimal regions of a planar lattice are called *cells*. Every slim planar semimodular lattice, in particular, every slim rectangular lattice L is a 4-*cell lattice*, that is, its cells are formed by four edges; see Grätzer and Knapp [22]. In a slim planar semimodular lattice, the cells are of the from $[a \wedge b, a \vee b]$ such that $a \parallel b$ and $[a \wedge b, a \vee b] = \{a \wedge b, a, b, a \vee b\}$. (Without slimness, 4-cells need not be intervals; this is exemplified by the 5-element modular lattice of length 2.) This cell is said to be a *distributive cell* if the principal ideal $\downarrow(a \vee b)$ is a distributive lattice. Let $n \in \mathbb{N}^+$. To obtain the *multifork extension* (*of rank* n) of L *at a distributive* 4-*cell* J means that we replace J by a copy of $S_7^{(n)}$ and keep adding new elements while going to the southeast and the southwest to preserve semimodularity. This is visualized by Figure 5, where L is drawn on the left, J is the grey-filled 4-cell, $n = 3$, and the slim rectangular lattice L' we obtain by the multifork extension of L at J of rank 3 is L', drawn on the right. The new elements, that is, the elements of $L' \setminus L$, are the pentagon-shaped ones. (Czédli [2] gives a more detailed definition of a multifork extension, which we do not need here since Figures 5 and 6 are sufficient for our purposes. Note that [2] uses the term "n-fold" rather than "of rank n".)

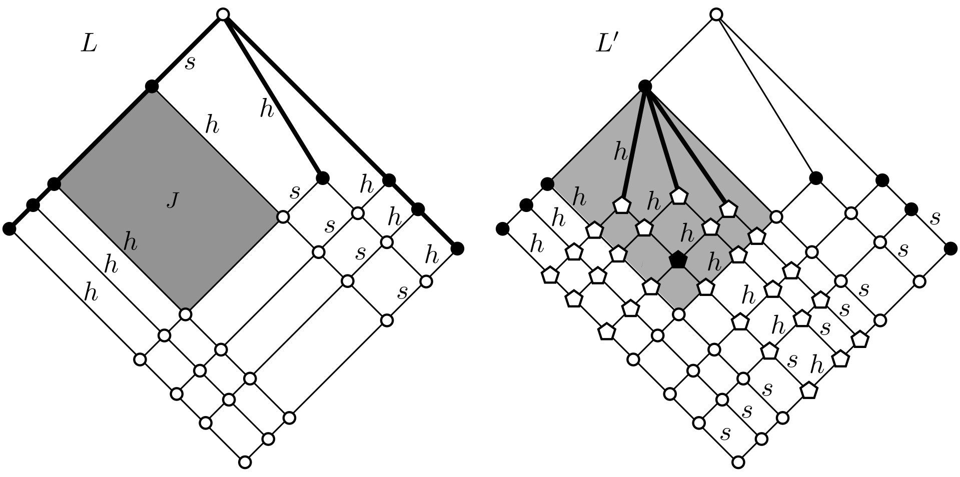

FIGURE 5. L' is the multifork extension of rank 3 at the 4-cell J

By a *grid* we mean the direct product of two finite nonsingleton chains or a $\mathcal{C}_1$-diagram of such a direct product. Note that a slim rectangular lattice is distributive if and only if it is a grid. Note also that, in a slim rectangular lattice with a fixed $\mathcal{C}_1$-diagram,

$$\text{a 4-cell } I = [p, q] \text{ is distributive if and only if every edge in the ideal } \downarrow q \text{ is of a normal slope;} \tag{2.8}$$

the "only if" part of (2.8) is Corollary 6.5 of Czédli [5] while the "if" part follows easily by using that if all edges of $\downarrow q$ are of normal slopes then $\downarrow q$ is a (sublattice of a) grid.

Lemma 2.12 (Theorem 3.7 of Czédli [2]). *Each slim rectangular lattice is obtained from a grid by a sequence of multifork extensions at distributive 4-cells, and every lattice obtained in this way is a slim rectangular lattice.*

Figure 6 illustrates how we can obtain in six steps the lattice L defined by Figure 2 from the initial grid L_0. For $i = 1, 2, \ldots 6$, L_i is obtained from L_{i-1} by a multifork extension at the grey-filled 4-cell of L_{i-1}. We still need one important concept, which we recall from Czédli [2]; it was originally introduced in Czédli and Schmidt [12].

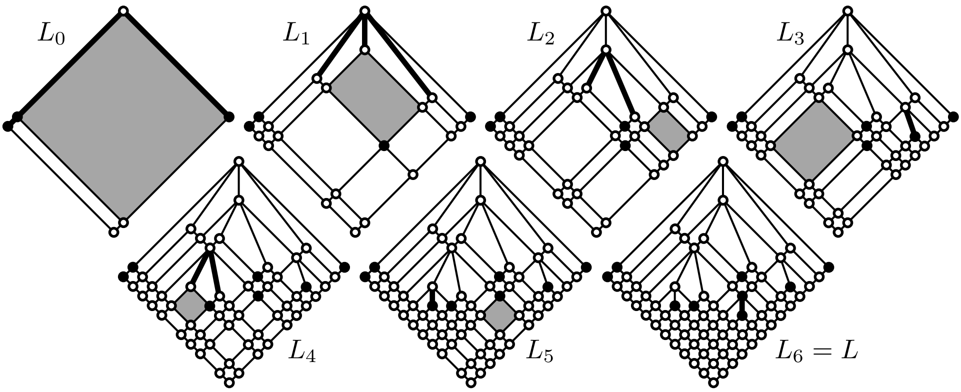

FIGURE 6. Illustrating Lemma 2.12: a sequence of multifork extensions

Definition 2.13 (Trajectories). Let L be a slim rectangular lattice with a fixed $\mathcal{C}_1$-diagram. The set of its *edges*, that is, the set of its prime intervals is denoted by Edge(L). We say that $\mathfrak{p}, \mathfrak{q} \in$ Edge(L) are *consecutive edges* if they are opposite sides of a 4-cell. Maximal sequences of consecutive edges are called *trajectories*. In other words, the blocks of the least equivalence relation on Edge(L) including the consecutiveness relation are called trajectories. If a trajectory has an edge on the upper boundary (equivalently, if it has an edge that is the unique neon tube of a boundary lamp), then this trajectory a *straight trajectory*. Otherwise, it is a *hat trajectory*. Each trajectory has a unique edge that is a neon tube; this edge is called the *top edge* of the trajectory. The top edge of a trajectory u will be denoted by TopE(u) while Traj(L) will stand for the set of trajectories of L.

For example, if L is the lattice on the left of Figure 5, then it has eight trajectories. One of the eight trajectories is a hat trajectory and consists of the h-labelled edges. There are seven straight trajectories and one of these seven consists of the s-labelled edges. Note that all neon tubes of L are drawn by thick lines. On the right of the same figure, only the neon tubes of the new lamp are thick. Also, L' has exactly four hat trajectories and seven straight trajectories. One of the hat trajectories consists of the h-labelled edges while the s-labelled edges form a straight trajectory.

Proof of Lemma 2.11. Let L be a slim rectangular lattice with a fixed $\mathcal{C}_1$-diagram. To prove part (i), we need some preparations. We know from Lemma 2.12 that L is obtained by a

sequence $L_0, L_1, \dots, L_k = L$ such that L_0 is a grid and L_i is a multifork extension of L_{i-1} at a distributive 4-cell H_i of L_{i-1} for $i \in \{1, \dots, k\}$. (2.9)

This is illustrated by Figure 6. In $\mathrm{Lamp}(L_0)$, there are only boundary lamps. It is clear by definitions that each internal lamp arises from the replacement of the distributive 4-cell H_i of L_{i-1}, grey-filled in Figures 5 and 6, by a copy of $S_7^{(n_i)}$, for some $i \in \{1, \dots, k\}$ and $n_i \in \mathbb{N}^+$. For later reference, we mention that

each internal lamp originates from a multifork extension. Furthermore, if a lamp K comes by a multifork extension at a 4-cell H_i, then $\mathrm{CircR}(I)$ is the geometric region determined by H_i; (2.10)

the second half of (2.10) follows from (2.8) and Convention 2.2. The meaning of (2.10) is exemplified by the lamp with pentagon-shaped black-filled foot on the right of Figure 5. This lamp with three thick neon tubes is the lamp that the multifork extension of L to L' at J has just brought. Similarly, the new lamp is the one with thick neon tube(s) in each of L_1, …, $L_6 = L$ in Figure 6. Keeping Convention 2.2 in mind, it is clear that if $[p, q]$ is the new lamp that the multifork extension of L_{i-1} brings into L_i, then $[\mathrm{lc}(L_i)] \wedge p, p]$ and $[\mathrm{rc}(L_i)] \wedge p, p]$ are chains with all of their edges of normal slopes. Observe that

no geometric line segment that consists of some edges of L_i disappears at further multifork extension steps, (2.11)

but there can appear more vertices on it. This fact, Convention 2.2, (2.8), (2.10), and the fact that L_{i-1} is a sublattice of its multifork extension L_i for all $i \in \{1, \dots, k\}$ yield that

if $I = [p, q] \in \mathrm{Lamp}(L)$, then the lowest point of $\mathrm{LRoof}(I)$ and that of $\mathrm{LFloor}(I)$ are $\mathrm{lc}(L) \wedge q$ and $\mathrm{lc}(L) \wedge p$, respectively, and the intervals $[\mathrm{lc}(L) \wedge q, q]$ and $[\mathrm{lc}(L) \wedge p, p]$ are chains. That is, $\mathrm{LRoof}(I)$ and $\mathrm{LFloor}(I)$ correspond to intervals that are chains. The edges of these chains are of the same normal slope. Similar statements hold for "right" instead of "left". (2.12)

For later reference, note another easy consequence of (2.9) and (2.11), or (2.12):

With reference to (2.9), assume that $i < j \le k$ and a lamp I is present in L_i. Then $\mathrm{Lit}(I)$ is the same in L_i as in L_j and L. (2.13)

For an internal lamp I, the leftmost neon tube and the (not necessarily distinct) rightmost neon tube of I are the right upper edge and the left upper edge of a 4-cell, respectively. Combining this fact, (2.10), and Observation 6.8(iii) of Czédli [5], we conclude that

the two upper (geometric) sides of the circumscribed rectangle $\mathrm{CircR}(I)$ of an internal lamp I are edges (that is, prime intervals) of L. (2.14)

Since $\mathrm{LeftLit}(J) \subseteq \mathrm{Lit}(J)$ and $\mathrm{RightLit}(J) \subseteq \mathrm{Lit}(J)$ for every $J \in \mathrm{Lamp}(L)$,

$$\boldsymbol{\rho}_{\mathrm{LRCircR}} \subseteq \boldsymbol{\rho}_{\mathrm{CircR}} \quad \text{and} \quad \boldsymbol{\rho}_{\mathrm{LRBody}} \subseteq \boldsymbol{\rho}_{\mathrm{Body}}. \tag{2.15}$$

Next, we are going to prove that

$$\boldsymbol{\rho}_{\mathrm{CircR}} \subseteq \boldsymbol{\rho}_{\mathrm{Body}} \quad \text{and} \quad \boldsymbol{\rho}_{\mathrm{LRCircR}} \subseteq \boldsymbol{\rho}_{\mathrm{LRBody}}. \tag{2.16}$$

To do so, assume that $\langle I, J\rangle \in \boldsymbol{\rho}_{\mathrm{CircR}}$ and $\langle I', J'\rangle \in \boldsymbol{\rho}_{\mathrm{LRCircR}}$. This means that $\mathrm{CircR}(I) \subseteq \mathrm{Lit}(J)$ and $\mathrm{CircR}(I') \subseteq \mathrm{LeftLit}(J')$ or $\mathrm{CircR}(I') \subseteq \mathrm{RightLit}(J')$, respectively. The definition of $S_7^{(n)}$ and multifork extensions, (2.10), and (2.11) yield that $\mathrm{Body}(I) \subseteq \mathrm{CircR}(I)$ and $\mathrm{Body}(I') \subseteq \mathrm{CircR}(I')$. Combining these inclusions with the earlier ones, we have that $\mathrm{Body}(I) \subseteq \mathrm{Lit}(J)$ and $\mathrm{Body}(I') \subseteq \mathrm{LeftLit}(J')$ or $\mathrm{Body}(I') \subseteq \mathrm{RightLit}(J')$. Hence, $\langle I, J\rangle \in \boldsymbol{\rho}_{\mathrm{Body}}$ and $\langle I', J'\rangle \in \boldsymbol{\rho}_{\mathrm{LRBody}}$, proving the validity of (2.16).

We claim that

$$\boldsymbol{\rho}_{\mathrm{Body}} \subseteq \boldsymbol{\rho}_{\mathrm{alg}}. \tag{2.17}$$

To show this, assume that $\langle I, J\rangle \in \boldsymbol{\rho}_{\mathrm{Body}}$, that is, $\mathrm{Body}(I) \subseteq \mathrm{Lit}(J)$, $I \neq J$, and I is an internal lamp. We know from (2.6) that $\mathrm{Lit}(J)$ is bordered by geometric line segments of normal slopes. Hence, Corollary 6.1 of Czédli [5] and $\mathrm{Body}(I) \subseteq \mathrm{Lit}(J)$ yield that $\mathrm{Peak}(I) \leq \mathrm{Peak}(J)$. Figure 1 and Convention 2.2 imply that $\mathrm{Foot}(I)$ is not on $\mathrm{Floor}(J)$, the lower geometric boundary of $\mathrm{Lit}(J)$. It is trivial by $\mathrm{Body}(I) \subseteq \mathrm{Lit}(J)$ that $\mathrm{Foot}(I)$ is not (geometrically and strictly) below $\mathrm{Floor}(J)$. Hence, the just mentioned Corollary 6.1 of [5] shows that $\mathrm{Foot}(I) \not\leq \mathrm{Foot}(J)$. We have obtained that $\langle I, J\rangle \in \boldsymbol{\rho}_{\mathrm{alg}}$. Thus, $\boldsymbol{\rho}_{\mathrm{Body}} \subseteq \boldsymbol{\rho}_{\mathrm{alg}}$, proving (2.17).

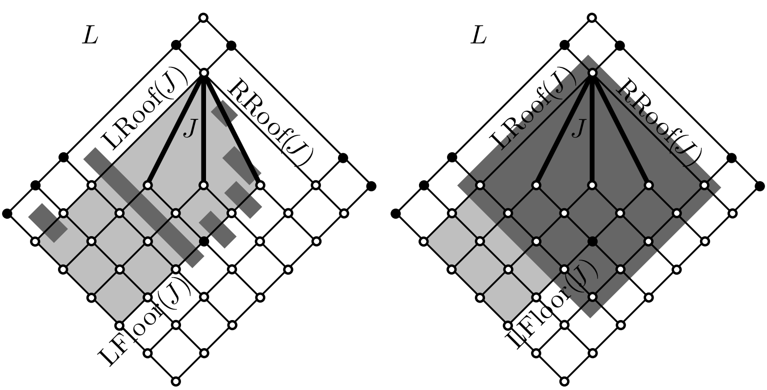

FIGURE 7. Proving that $\boldsymbol{\rho}_{\mathrm{foot}} \subseteq \boldsymbol{\rho}_{\mathrm{LRCircR}}$

By construction, see Figure 1, $\mathrm{CircR}(I)$ contains $\mathrm{Foot}(I)$ as a geometric point in its topological interior for every internal lamp $I \in \mathrm{Lamp}(L)$. This clearly yields the first inclusion in (2.18) below:

$$\boldsymbol{\rho}_{\mathrm{CircR}} \subseteq \boldsymbol{\rho}_{\mathrm{infoot}} \subseteq \boldsymbol{\rho}_{\mathrm{in+foot}} \subseteq \boldsymbol{\rho}_{\mathrm{foot}} \subseteq \boldsymbol{\rho}_{\mathrm{LRCircR}}. \tag{2.18}$$

The second and the third inclusions above are trivial. In order to show the fourth inclusion, $\boldsymbol{\rho}_{\mathrm{foot}} \subseteq \boldsymbol{\rho}_{\mathrm{LRCircR}}$, assume that $\langle I, J\rangle \in \boldsymbol{\rho}_{\mathrm{foot}}$. We know that $I \neq J$, I is an internal lamp, and $\mathrm{Foot}(I) \in \mathrm{Lit}(J)$. Since $\mathrm{Lit}(J) = \mathrm{LeftLit}(J) \cup \mathrm{RightLit}(J)$, left-right symmetry allows us to assume that the geometric point $\mathrm{Foot}(I)$ belongs to $\mathrm{LeftLit}(J)$. But, as it is clear from Figure 1 and (2.10), $\mathrm{Foot}(I)$ is in the (topological) interior of $\mathrm{CircR}(I)$. Hence,

$$\text{there is an open set } U \subseteq \mathbb{R}^2 \text{ such that } U \subseteq \mathrm{CircR}(I) \cap \mathrm{LeftLit}(J). \tag{2.19}$$

We claim that $\mathrm{CircR}(I) \subseteq \mathrm{LeftLit}(J)$. Suppose, for a contradiction, that this inclusion fails. We know from (2.6), (2.10), (2.11), (2.13), and Remark 2.5 that $\mathrm{LeftLit}(J)$ is surrounded by edges of normal slopes except its top right precipitous side. Also, we obtain from (2.8), (2.10), and (2.14) that $\mathrm{CircR}(I)$ is a rectangle whose sides are of normal slopes, and the upper two sides are edges (that is, prime intervals). These facts, (2.19), and $\mathrm{CircR}(I) \not\subseteq \mathrm{LeftLit}(J)$ yield that a side of $\mathrm{CircR}(I)$ crosses a side of $\mathrm{LeftLit}(J)$. But edges do not cross in a planar diagram, whence no side of $\mathrm{LeftLit}(J)$ crosses an upper edge of $\mathrm{CircR}(I)$. This fact excludes several possibilities how $\mathrm{CircR}(I)$ can be positioned relative to $\mathrm{LeftLit}(J)$. We use the diagram on the left of Figure 7 to explain what possibilities are excluded. In the figure, J is formed by the three thick neon tubes and $\mathrm{LeftLit}(J)$ is the light-grey area. Each of the six little dark-grey rectangles is a possibility forbidden for $\mathrm{CircR}(I)$ because of prohibited crossings. In other words, none of the six little dark-grey rectangles can be $\mathrm{CircR}(I)$. Since the collection of the six dark-grey rectangles represents generality, the only possibility is that the top vertex $\mathrm{Peak}(I)$ of $\mathrm{CircR}(I)$ is positioned like the top of the dark-grey rectangle on the right of Figure 7. (There can be lattice elements not indicated in the diagram, and we took into consideration that the top sides of $\mathrm{CircR}(J)$ are also edges and cannot be crossed and that (2.7) holds.) But then Corollary 6.1 of Czédli [5] yields that $\mathrm{Peak}(J) \leq \mathrm{Peak}(I)$. With reference to (2.9) and (2.10), let i and j denote the subscripts such that I and J appear first in L_i and L_j, respectively. Since none of the lattices $S_7^{(n)}$, $n \in \mathbb{N}^+$, is distributive, we obtain that $j > i$. This means that the elements of L_i are old when L_j is constructed as a multifork extension of L_{j-1}. Here by an old element of L_j we mean an element of L_{j-1}. It is clear by the concept of multifork extensions and that of $\mathrm{LeftLit}(J)$, (2.8), and (2.10) that no old element belongs to the topological interior of $\mathrm{Lit}(J)$. But $\mathrm{Foot}(I) \in L_i \subseteq L_{j-1}$ does belong to this interior, which is a contradiction showing that $\mathrm{CircR}(I) \subseteq \mathrm{LeftLit}(J)$. Thus, $\langle I, J\rangle \in \boldsymbol{\rho}_{\mathrm{LRCircR}}$, and we obtain that $\boldsymbol{\rho}_{\mathrm{foot}} \subseteq \boldsymbol{\rho}_{\mathrm{LRCircR}}$. This completes the proof of (2.18).

Next, we are going show that

$$\boldsymbol{\rho}_{\mathrm{alg}} \subseteq \boldsymbol{\rho}_{\mathrm{foot}}. \tag{2.20}$$

To verify this, assume that $\langle I, J\rangle \in \boldsymbol{\rho}_{\mathrm{alg}}$. This means that $\mathrm{Peak}(I) \leq \mathrm{Peak}(J)$ but $\mathrm{Foot}(I) \not\leq \mathrm{Foot}(J)$. Observe that $\mathrm{Foot}(I) \leq \mathrm{Peak}(I) \leq \mathrm{Peak}(J)$. We know that $\mathrm{Peak}(J)$ and $\mathrm{Foot}(J)$ are the vertices of the $\boldsymbol{\wedge}$-shaped $\mathrm{Roof}(J)$ and $\mathrm{Floor}(J)$, respectively, and $\mathrm{Roof}(J)$ and $\mathrm{Floor}(J)$ consist of line segments of normal slopes. Hence, it follows from Corollary 6.1 of Czédli [5] and $\mathrm{Foot}(I) \leq \mathrm{Peak}(J)$ that $\mathrm{Foot}(I)$ is geometrically below or on $\mathrm{Roof}(J)$. On the other hand, Corollary 6.1 of [5] and $\mathrm{Foot}(I) \not\leq \mathrm{Foot}(J)$ imply that $\mathrm{Foot}(I)$ is neither geometrically below, nor on $\mathrm{Floor}(J)$. So $\mathrm{Foot}(I)$ is above $\mathrm{Floor}(J)$. Therefore, $\mathrm{Foot}(I)$ is geometrically between $\mathrm{Floor}(J)$ and $\mathrm{Roof}(J)$. Thus, (2.6) gives that $\mathrm{Foot}(I) \in \mathrm{Lit}(J)$. This shows that $\langle I, J\rangle \in \boldsymbol{\rho}_{\mathrm{foot}}$, and we have shown the validity of (2.20).

The directed graph in Figure 8 represents what we have already shown. Each (directed) edge $\rho_1 \to \rho_2$ of the graph means that $\rho_1 \subseteq \rho_2$ has been formulated in the displayed equation given by the label of the edge. The directed graph is strongly connected, implying part (i).

Next, we turn our attention to parts (ii) and (iii). For a trajectory u of L, $\mathrm{TopE}(u)$ is a neon tube, so it belongs to a unique lamp; we denote this lamp by $\mathrm{Lmp}(u)$. Lamps are special intervals. Hence, in agreement with how the notation

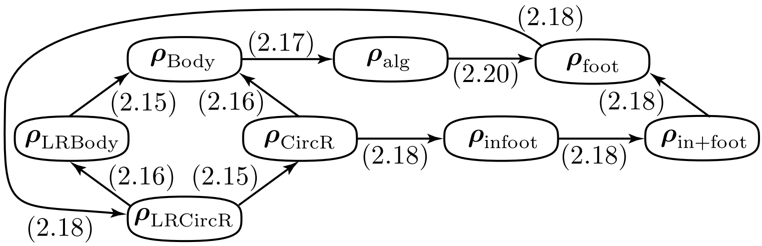

FIGURE 8. An overview of what has already been proved

con() was introduced right before Lemma 2.11, con(Lmp(u)) will denote the congruence con(Foot(Lmp(u)), Peak(Lmp(u))) generated by the interval Lmp(u). We claim that, for any trajectory u of L,

$$\mathrm{con}(\mathrm{TopE}(u)) = \mathrm{con}(\mathrm{Lmp}(u)). \tag{2.21}$$

To show (2.21), observe that this assertion is trivial if TopE(u) is the only neon tube of Lmp(u) since then the same prime interval generates the congruence on both sides of (2.21).

Hence, we can assume that Lmp(u) has more than one neon tubes; clearly, then Lmp(u) is an internal lamp. Let $p := \mathrm{Foot}(\mathrm{Lmp}(u))$, $q := \mathrm{Peak}(\mathrm{Lmp}(u))$, and let $[p_1, q] = \mathrm{TopE}(u)$, $[p_2, q]$, …, $[p_m, q]$ be a list of all neon tubes of Lmp(u). With this notation, (2.21) asserts that $\mathrm{con}(p_1, q) = \mathrm{con}(p, q)$. Since the congruence blocks of a finite lattice are intervals and $p \leq p_1 \leq q$, the inequality $\mathrm{con}(p_1, q) \leq \mathrm{con}(p, q)$ is clear. It follows in a straightforward way by inspecting the lattice $S_7^{(m)}$, see Figure 1, or it follows trivially by the Swing Lemma, see Grätzer [18] or Czédli, Grätzer and Lakser [9], that $\mathrm{con}(p_1, q) = \mathrm{con}(p_2, q) = \cdots = \mathrm{con}(p_m, q)$. Hence, $(p_i, q) \in \mathrm{con}(p_1, q)$ for $i \in \{1, \dots, m\}$, and we obtain that

$$(p, q) \overset{(2.2)}{=} (p_1 \wedge \cdots \wedge p_m, q \wedge \cdots \wedge q) \in \mathrm{con}(p_1, q). \tag{2.22}$$

This yields the converse inequality $\mathrm{con}(p_1, q) \geq \mathrm{con}(p, q)$ and proves (2.21).

As usual, the smallest element and largest element of an interval I will often be denoted by 0_I and 1_I, respectively. Following Czédli [2, Definition 4.3], we define two relations on Traj(L). For trajectories $u, v \in \mathrm{Traj}(L)$, we let

$$\langle u, v\rangle \in \boldsymbol{\sigma} \overset{\text{def}}{\iff} \begin{cases} 1_{\mathrm{TopE}(u)} \leq 1_{\mathrm{TopE}(v)},\ 0_{\mathrm{TopE}(u)} \not\leq 0_{\mathrm{TopE}(v)}, \\ \text{and } u \text{ is a hat trajectory.} \end{cases} \tag{2.23}$$

The second relation is $\boldsymbol{\tau}$, the reflexive transitive closure of $\boldsymbol{\sigma}$. We also need a third relation on Traj(L), namely, we need that $\boldsymbol{\Theta} := \boldsymbol{\tau} \cap \boldsymbol{\tau}^{-1}$. The $\boldsymbol{\Theta}$-block of a trajectory $u \in \mathrm{Traj}(L)$ will be denoted by $u/\boldsymbol{\Theta}$. Since $\boldsymbol{\tau}$ is a *quasiordering*, that is, a reflexive and transitive relation, the quotient set $\mathrm{Traj}(L)/\boldsymbol{\Theta}$ turns into a poset $\langle \mathrm{Traj}(L)/\boldsymbol{\Theta}, \boldsymbol{\tau}/\boldsymbol{\Theta}\rangle$ by defining

$$\langle u/\boldsymbol{\Theta}, v/\boldsymbol{\Theta}\rangle \in \boldsymbol{\tau}/\boldsymbol{\Theta} \overset{\text{def}}{\iff} \langle u, v\rangle \in \boldsymbol{\tau} \tag{2.24}$$

for $u, v \in \mathrm{Traj}(L)$; see, for example, (4.1) in Czédli [2]. We claim that, for any $u, v \in \mathrm{Traj}(L)$,

$$\langle u/\boldsymbol{\Theta}, v/\boldsymbol{\Theta}\rangle \in \boldsymbol{\tau}/\boldsymbol{\Theta} \iff \mathrm{Lmp}(u) \leq \mathrm{Lmp}(v). \tag{2.25}$$

As the first step towards (2.25), we show that, for any $u, v \in \mathrm{Lamp}(L)$,

$$u/\boldsymbol{\Theta} = v/\boldsymbol{\Theta} \iff \mathrm{Lmp}(u) = \mathrm{Lmp}(v). \tag{2.26}$$

The $\Leftarrow$ part of (2.26) is quite easy. Assume that $\mathrm{Lmp}(u) = \mathrm{Lmp}(v)$. Then it is clear by Figure 1 and (2.23) that $\langle u, v\rangle$ and $\langle v, u\rangle$ belong to $\boldsymbol{\sigma}$. Hence, both $\langle u, v\rangle$ and $\langle v, u\rangle$ are in $\boldsymbol{\tau}$, whence $u/\boldsymbol{\Theta} = v/\boldsymbol{\Theta}$.

To show the converse implication, assume that $u/\boldsymbol{\Theta} = v/\boldsymbol{\Theta}$. We also assume that $u \neq v$ since the case $u = v$ is trivial. The first assumption gives that $\langle u, v\rangle \in \boldsymbol{\tau}$, and thus there is a $k \in \mathbb{N}^+$ and there are elements $w_0 = u$, w_1, ..., $w_k = v$ such that $\langle w_{i-1}, w_i\rangle \in \boldsymbol{\sigma}$ for all $i \in \{1, \dots, k\}$. By (2.23), $u = u_0$ is a hat trajectory and $1_{\mathrm{TopE}(u)} = 1_{\mathrm{TopE}(w_0)} \leq 1_{\mathrm{TopE}(w_1)} \leq \cdots \leq 1_{\mathrm{TopE}(w_k)} = 1_{\mathrm{TopE}(v)}$. That is, $1_{\mathrm{TopE}(u)} \leq 1_{\mathrm{TopE}(v)}$. Since $\langle v, u\rangle$ is also in $\boldsymbol{\tau}$, we also have that v is a hat trajectory and $1_{\mathrm{TopE}(v)} \leq 1_{\mathrm{TopE}(u)}$. Hence, $1_{\mathrm{TopE}(v)} = 1_{\mathrm{TopE}(u)}$ and so $\mathrm{Peak}(\mathrm{Lmp}(u)) = \mathrm{Peak}(\mathrm{Lmp}(v))$. Thus, $\mathrm{Lmp}(u) = \mathrm{Lmp}(v)$ by (2.7), proving (2.26).

In the next step towards (2.25), (2.26) allows us to assume that $u/\boldsymbol{\Theta} \neq v/\boldsymbol{\Theta}$, which is equivalent to $\mathrm{Lmp}(u) \neq \mathrm{Lmp}(v)$. First, we show that

$$\text{if } \langle u, v\rangle \in \boldsymbol{\sigma}, \text{ then } \mathrm{Lmp}(u) \leq \mathrm{Lmp}(v). \tag{2.27}$$

Let $u = u_1, \dots, u_k$ and $v = v_1, \dots, v_t$ be the neon tubes of $\mathrm{Lmp}(u)$ and $\mathrm{Lmp}(v)$, respectively. With reference to (2.9) and (2.10), let i and j denote the subscripts such that $\mathrm{Lmp}(u)$ and $\mathrm{Lmp}(v)$ came to existence in L_i and L_j, respectively. We obtain from $\langle u, v\rangle \in \boldsymbol{\sigma}$ that $\mathrm{Peak}(\mathrm{Lmp}(u)) = 1_{\mathrm{TopE}(u)} \leq 1_{\mathrm{TopE}(v)} = \mathrm{Peak}(\mathrm{Lmp}(v))$. By (2.23), u is a hat trajectory, whence $\mathrm{Lmp}(u)$ is an internal lamp and so $i \geq 1$. Since none of the lattices $S_7^{(n)}$, $n \in \mathbb{N}^+$, is distributive, it follows that $i > j$. Since the sequence (2.9) is increasing, we obtain that $0_{\mathrm{TopE}(v)} \in L_{i-1}$. It is clear from (2.10) and the description of multifork extensions (see Figures 5 and 6, and see also Figure 1) that

$$\begin{aligned}&\text{if, according to (2.9), a lamp } K \text{ appears first in } L_\ell,\\ &x \in L_{\ell-1}, \text{ and } \mathrm{Foot}(K) \leq x, \text{ then } \mathrm{Peak}(K) \leq x.\end{aligned} \tag{2.28}$$

For the sake of contradiction, suppose that $\mathrm{Foot}(\mathrm{Lmp}(u)) \leq 0_{\mathrm{TopE}(v)}$. Applying (2.28) with $\ell = i$ and $K = I$, the containment $0_{\mathrm{TopE}(v)} \in L_{i-1}$ gives that $\mathrm{Peak}(\mathrm{Lmp}(u)) \leq 0_{\mathrm{TopE}(v)}$. But then $0_{\mathrm{TopE}(u)} < 1_{\mathrm{TopE}(u)} = \mathrm{Peak}(\mathrm{Lmp}(u)) \leq 0_{\mathrm{TopE}(v)}$, which is a contradiction since $0_{\mathrm{TopE}(u)} \nleq 0_{\mathrm{TopE}(v)}$ by $\langle u, v\rangle \in \boldsymbol{\sigma}$. Hence, $\mathrm{Foot}(\mathrm{Lmp}(u)) \nleq 0_{\mathrm{TopE}(v)}$. Combining this with $\mathrm{Foot}(\mathrm{Lmp}(v)) \leq 0_{\mathrm{TopE}(v)}$, see (2.2), we obtain that $\mathrm{Foot}(\mathrm{Lmp}(u)) \nleq \mathrm{Foot}(\mathrm{Lmp}(v))$. We have already seen that $\mathrm{Peak}(\mathrm{Lmp}(u)) \leq \mathrm{Peak}(\mathrm{Lmp}(v))$, whereby $\langle \mathrm{Lmp}(u), \mathrm{Lmp}(v)\rangle \in \boldsymbol{\rho}_{\mathrm{alg}}$. This yields that $\mathrm{Lmp}(u) \leq \mathrm{Lmp}(v)$ since "$\leq$" is the reflexive transitive closure of $\boldsymbol{\rho}_{\mathrm{alg}}$. Thus, we have shown the validity of (2.27).

Next, we assert that, for any $u, v \in \mathrm{Traj}(L)$,

$$\text{if } \langle \mathrm{Lmp}(u), \mathrm{Lmp}(v)\rangle \in \boldsymbol{\rho}_{\mathrm{alg}}, \text{ then } \langle u, v\rangle \in \boldsymbol{\tau}. \tag{2.29}$$

To show this, let $v_1 = v, v_2, \dots, v_k$ be the neon tubes of $\mathrm{Lmp}(v)$. Assume that the pair $\langle \mathrm{Lmp}(u), \mathrm{Lmp}(v)\rangle$ belongs to $\boldsymbol{\rho}_{\mathrm{alg}}$. Then

$$\begin{aligned}&1_{\mathrm{TopE}(u)} = \mathrm{Peak}(\mathrm{Lmp}(u)) \leq \mathrm{Peak}(\mathrm{Lmp}(v)) =\\ &1_{\mathrm{TopE}(v_j)} \text{ for all } j \in \{1, \dots, k\},\end{aligned} \tag{2.30}$$

and we know from Remark 2.10 that $\mathrm{Lmp}(u)$ is an internal lamp. Hence, u is a hat trajectory. On the other hand, $0_{\mathrm{TopE}(u)} \nleq \mathrm{Foot}(\mathrm{Lmp}(v))$ since otherwise $\mathrm{Foot}(\mathrm{Lmp}(u)) \leq 0_{\mathrm{TopE}(u)} \leq \mathrm{Foot}(\mathrm{Lmp}(v))$ would contradict the containment

$\langle \mathrm{Lmp}(u), \mathrm{Lmp}(v)\rangle \in \boldsymbol{\rho}_{\mathrm{alg}}$. If we had that $0_{\mathrm{TopE}(u)} \le 0_{\mathrm{TopE}(v_j)}$ for all $j \in \{1,\dots,k\}$, then we would obtain that

$$0_{\mathrm{TopE}(u)} \le \bigwedge_{j\in\{1,\dots,k\}} 0_{\mathrm{TopE}(v_j)} \overset{(2.2)}{=} \mathrm{Foot}(\mathrm{Lmp}(v)),$$

contradicting $0_{\mathrm{TopE}(u)} \not\le \mathrm{Foot}(\mathrm{Lmp}(v))$. Hence, there exists a $j \in \{1,\dots,k\}$ such that $0_{\mathrm{TopE}(u)} \not\le 0_{\mathrm{TopE}(v_j)}$. Thus, using (2.30), (2.23), and the fact that u is a hat trajectory, we obtain that $\langle u, v_j\rangle \in \boldsymbol{\sigma}$. Hence, $\langle u, v_j\rangle \in \boldsymbol{\tau}$. Also, $\mathrm{Lmp}(v_j) = \mathrm{Lmp}(v)$ and (2.26) yield that $\langle v_j, v\rangle \in \boldsymbol{\tau}$. By transitivity, $\langle u, v\rangle \in \boldsymbol{\tau}$, proving (2.29).

By (2.24), statement (2.25) is equivalent to the statement that $\langle u, v\rangle \in \boldsymbol{\tau} \iff \mathrm{Lmp}(u) \le \mathrm{Lmp}(v)$. Since $\boldsymbol{\tau}$ on $\mathrm{Traj}(L)$ and "$\le$" on $\mathrm{Lamp}(L)$ are the reflexive transitive closures of $\boldsymbol{\sigma}$ and $\boldsymbol{\rho}_{\mathrm{alg}}$, respectively, (2.25) follows from (2.27) and (2.29).

Next, we are going to call a certain map a *quasi-coloring*. (The exact meaning of a quasi-coloring was introduced in Czédli [1] and [2] but we do not need it here.) Following Definition 4.3(iv) of [2], we define a map ξ from the set $\mathrm{Edge}(L)$ of edges of L to $\mathrm{Traj}(L)$ by the rule $\mathfrak{p} \in \xi(\mathfrak{p})$ for every $\mathfrak{p} \in \mathrm{Edge}(L)$. That is, ξ maps an edge to the unique trajectory containing it. By Theorem 4.4 of [2], ξ is a quasi-coloring. It follows from Lemma 4.1 of [2] that

$$\text{the posets } \langle \mathrm{J}(L); \le\rangle \text{ and } \langle \mathrm{Traj}(L)/\boldsymbol{\Theta}; \boldsymbol{\tau}/\boldsymbol{\Theta}\rangle \text{ are isomorphic.} \tag{2.31}$$

By (2.25) and its particular case, (2.26), the structures

$$\langle \mathrm{Traj}(L)/\boldsymbol{\Theta}, \boldsymbol{\tau}/\boldsymbol{\Theta}\rangle \text{ and } \langle \mathrm{Lamp}(L); \le\rangle \text{ are also isomorphic.} \tag{2.32}$$

We mentioned before (2.24) and we also know from (2.31) that $\langle \mathrm{Traj}(L)/\boldsymbol{\Theta}, \boldsymbol{\tau}/\boldsymbol{\Theta}\rangle$ is a poset. Thus, $\langle \mathrm{Lamp}(L); \le\rangle$ is also a poset by (2.32), proving part (ii) of Lemma 2.11.

Combining (2.31) and (2.32), it follows that $\langle \mathrm{J}(L); \le\rangle \cong \langle \mathrm{Lamp}(L); \le\rangle$, which is the first half of part (iii) of Lemma 2.11. From (2.6)–(2.8) of Czédli [1], or from the proof of (2.31) or that of Theorem 7.3(i) of [2], it is straightforward to extract that

$$\text{the map } \psi_1\colon \langle \mathrm{Traj}(L)/\boldsymbol{\Theta}; \boldsymbol{\tau}/\boldsymbol{\Theta}\rangle \to \langle \mathrm{J}(\mathrm{Con}\,L); \le\rangle \text{ defined by } u/\boldsymbol{\Theta} \mapsto \mathrm{con}(\mathrm{TopE}(u)) \text{ is a poset isomorphism.} \tag{2.33}$$

Observe that every lamp I is of the form $\mathrm{Lmp}(u)$ for some $u \in \mathrm{Traj}(L)$; indeed, we can choose u as $\xi(\mathfrak{p})$ for some (in fact, any) neon tube $\mathfrak{p}$ of I. By (2.25) and (2.26),

$$\text{the map } \psi_2\colon \langle \mathrm{Lamp}(L); \le\rangle \to \langle \mathrm{Traj}(L)/\boldsymbol{\Theta}; \boldsymbol{\tau}/\boldsymbol{\Theta}\rangle, \text{ defined by } \mathrm{Lmp}(u) \mapsto u/\boldsymbol{\Theta}, \text{ is also a poset isomorphism.} \tag{2.34}$$

Combining (2.21), (2.33), and (2.34), we obtain that $\varphi = \psi_2 \cdot \psi_1$. Hence, φ is also a poset isomorphism, proving part (iii) of Lemma 2.11.

Finally, if a partial ordering $\le$ is the reflexive transitive closure of a relation ρ, then the covering relation $\prec$ with respect to $\le$ is obviously a subset of ρ. This yields part (iv) and completes the proof of Lemma 2.11. □

3. Some consequences of Lemma 2.11 and some properties of lamps

Some statements of this section are explicitly about the congruence lattices of slim planar semimodular lattices; they are called corollaries since they are derived from Lemma 2.11. The rest of the statements of the section deal with lamps.

Convention 2.4 raises the (easy) question whether lamps are determined by their feet and how. For an element $u \in L \setminus \{1\}$, let u^+ denote the join of all covers of u, that is,

$$u^+ := \bigvee\{y \in L : u \prec y\}, \quad \text{provided} \quad u \neq 1. \tag{3.1}$$

Note that 1^+ is undefined. Note also that $\bigvee$ in (3.1) applies actually to one or two joinands since each $u \in L \setminus \{1\}$ has either a single cover, or it has exactly two covers by Grätzer and Knapp [22, Lemma 8]. Let $x \in L$ and define the element lifted(x) by induction on the number of elements of the principal filter $\uparrow x$ as follows.

$$\text{lifted}(x) = \begin{cases} 1, & \text{if } x = 1; \\ x^+, & \text{if there exists a } y \in \mathrm{M}(L) \text{ such that } y \prec x^+; \\ \text{lifted}(x^+), & \text{otherwise.} \end{cases} \tag{3.2}$$

Lemma 3.1. *If L is a slim rectangular lattice, then* $\text{Peak}(I) = \text{lifted}(\text{Foot}(I))$ *holds for every* $I \in \text{Lamp}(L)$*.*

Proof. The proof is trivial by Figure 1 and (2.10). □

Lemma 3.2 (Maximal lamps are boundary lamps)**.** *If L is a slim rectangular lattice, then the maximal elements of* $\text{Lamp}(L)$ *are exactly the boundary lamps.*

Proof. We know from Grätzer and Knapp [23] that each slim planar semimodular lattice L with at least three elements has a so-called congruence-preserving rectangular extension L'; see also Czédli [5] and Grätzer and Schmidt [25] for stronger versions of this result. Among other properties of this L', we have that $\text{Con}\,L' \cong \text{Con}\,L$. Hence, to simplify the notation,

$$\text{we can assume in the proof that } L \text{ is a slim rectangular lattice with a fixed } \mathcal{C}_1\text{-diagram.} \tag{3.3}$$

Note that the same assumption will be made in many other proofs when we know that $|L| \geq 3$. Armed with (3.3), Lemma 3.2 follows trivially from (2.10) and Lemma 2.11. □

The *set of maximal elements* of a poset P will be denoted by $\text{Max}(P)$.

Corollary 3.3 (P2 Property from Grätzer [21])**.** *If L is a slim planar semimodular lattice with at least three elements, then* $\text{Con}\,L$ *has at least two coatoms or, equivalently,* $\mathrm{J}(\text{Con}\,L)$ *has at least two maximal elements.*

Proof. Assume (3.3). The well-known representation theorem of finite distributive lattices (see, for example, Grätzer [15, Theorem 107]) easily implies that $\text{Con}\,L$ has at least two coatoms if an only if $\mathrm{J}(\text{Con}\,L)$ has at least two maximal elements. Hence, Lemma 3.2 applies. □

The covering relation in a poset P is denoted by $\prec$ or by $\prec_P$. For sets A_1, A_2, and A_3, the notation $A_3 = A_1 \dot\cup A_2$ will stand for the conjunction of $A_1 \cap A_2 = \emptyset$ and $A_3 = A_1 \cup A_2$.

Corollary 3.4 (Bipartite Maximal Elements Property). *Let L be a slim planar semimodular lattice with at least three elements and let $D := \mathrm{Con}\, L$. Then there exist nonempty sets* $\mathrm{LeftMax}(\mathrm{J}(D))$ *and* $\mathrm{RightMax}(\mathrm{J}(D))$ *such that*

$$\mathrm{Max}(\mathrm{J}(D)) = \mathrm{LeftMax}(\mathrm{J}(D)) \,\dot\cup\, \mathrm{RightMax}(\mathrm{J}(D))$$

and for each $x \in \mathrm{J}(D)$ and $y, z \in \mathrm{Max}(\mathrm{J}(D))$, if $x \prec_{\mathrm{J}(D)} y$, $x \prec_{\mathrm{J}(D)} z$, and $y \neq z$, then one of y and z belongs to $\mathrm{LeftMax}(\mathrm{J}(D))$ *while the other one belongs to* $\mathrm{RightMax}(\mathrm{J}(D))$. *Furthermore, when $\mathrm{J}(D) = \mathrm{J}(\mathrm{Con}\, L)$ is represented in the form* $\mathrm{Lamp}(L)$ *according to Lemma 2.11*(iii), *then* $\mathrm{LeftMax}(\mathrm{J}(D))$ *can be chosen so that its members correspond to the boundary lamps on the top left boundary chain of L while the members of* $\mathrm{RightMax}(\mathrm{J}(D)) = \max \mathrm{J}(D) \setminus \mathrm{LeftMax}(\mathrm{J}(D))$ *correspond to the boundary lamps on the top right boundary chain.*

Proof. Using assumption (3.3) again, we know from Lemma 3.2 that the maximal lamps are on the upper boundary. Let $\mathrm{LeftMax}(\mathrm{Lamp}(L))$ and $\mathrm{RightMax}(\mathrm{Lamp}(L))$ denote the set of boundary lamps on the top left boundary chain $\uparrow\mathrm{lc}(L)$ and those on the top right boundary chain $\uparrow\mathrm{rc}(L)$, respectively. Since L is rectangular, none of these two sets is empty. So these two sets form a partition of $\max \mathrm{Lamp}(L)$. Let us say that, for $I', J' \in \mathrm{Lamp}(L)$,

$\mathrm{Lit}(I')$ and $\mathrm{Lit}(J')$ are *sufficiently disjoint* if for every line segment S of positive length in the plane, if $S \subseteq \mathrm{Lit}(I') \cap \mathrm{Lit}(J')$, then S is of a normal slope. (3.4)

Clearly, if $\mathrm{Lit}(I')$ and $\mathrm{Lit}(J')$ are sufficiently disjoint, then no nonempty open set of $\mathbb{R}^2$ is a subset of $\mathrm{Lit}(I') \cap \mathrm{Lit}(J')$.

Let $I, J \in \mathrm{LeftMax}(\mathrm{Lamp}(L))$ such that $I \neq J$. There are two easy ways to see that $\mathrm{Lit}(I)$ and $\mathrm{Lit}(J)$ are sufficiently disjoint: either we apply (2.12), or we use (2.13) with $\langle i, j\rangle = \langle 0, k\rangle$. For the sake of contradiction, suppose that $K \prec_{\mathrm{Lamp}(L)} I$ and $K \prec_{\mathrm{Lamp}(L)} J$. Then, by parts (i) and (iv) of Lemma 2.11, $\langle K, I\rangle \in \boldsymbol{\rho}_{\mathrm{Body}}$. Similarly, $\langle K, J\rangle \in \boldsymbol{\rho}_{\mathrm{Body}}$, and so we have that $\mathrm{Body}(K) \subseteq \mathrm{Lit}(I) \cap \mathrm{Lit}(J)$. Since K is an internal lamp, it contains a precipitous neon tube S, which contradicts the sufficient disjointness of $\mathrm{Lit}(I)$ and $\mathrm{Lit}(J)$. By Lemma 2.11 and left-right symmetry, we conclude Corollary 3.4. □

Corollary 3.5 (Dioecious Maximal Elements Property). *If L is a slim planar semimodular lattice, $D := \mathrm{Con}\, L$, $x \in \mathrm{J}(D)$, $y \in \mathrm{Max}(\mathrm{J}(D))$, and $x \prec_{\mathrm{J}(D)} y$, then there exists an element $z \in \mathrm{J}(D)$ such that $z \neq y$ and $x \prec_{\mathrm{J}(D)} z$.*

The adjective "dioecious" above is explained by the idea of interpreting $x \prec y$ as "x is a child of y".

Proof of Corollary 3.5. If $|L| < 3$, then $|\mathrm{J}(L)| \leq 1$ and the statement is trivial. Hence, we can assume that $|L| \geq 3$ and that L is rectangular; see (3.3). Assume that $I \in \mathrm{Lamp}(L)$, $J \in \mathrm{Max}(\mathrm{Lamp}(L))$, and $I \prec J$ in $\mathrm{Lamp}(L)$. We know from Lemma 3.2 that I is an internal lamp while J is a boundary lamp. For the sake of contradiction, suppose that J is the only cover of I in $\mathrm{Lamp}(L)$. By left-right symmetry and Lemma 3.2, we can assume that (the only neon tube of) J is on the top left boundary chain of L. With reference to (2.9), the illuminated sets of the lamps of L_0, which are boundary lamps, divide the full geometric rectangle of L_0 into pairwise sufficiently disjoint (topologically closed) rectangles $T_1, \dots, T_m$. That is, $T_1, \dots, T_m$ are the squares (that is, the 4-cells) of the initial grid L_0. By (2.13), the illuminated sets of the boundary lamps of L (rather than L_0) divide the full geometric rectangle of L into the same rectangles, and the same holds for all L_i, $i \in \{0, 1, \dots, k\}$. We know from (2.10) that, for some $i \in \{1, \dots, k\}$, each of the four sides of the rectangle $\mathrm{CircR}(I)$ is an edge in L_i. Since no two edges of L_i cross each other by planarity (see also Kelly and Rival [26, Lemma 1.2]), it follows from (2.11) that the sides (in fact, edges) of $\mathrm{CircR}(I)$ in L_i do not cross the sides of $T_1, \dots, T_m$. Hence, still in L_i, $\mathrm{CircR}(I)$ is fully included in one of the $T_1, \dots, T_m$. This also holds in L since $\mathrm{CircR}(I)$ is the same in L as in L_i by (2.10). Hence, there is lamp K on the top right boundary chain of L such that $\mathrm{CircR}(I) \subseteq \mathrm{Lit}(K)$. Hence, $\langle I, K\rangle \in \boldsymbol{\rho}_{\mathrm{CircR}}$, whence parts (i) and (ii) of Lemma 2.11 give that $I < K$ in $\mathrm{Lamp}(L)$. By finiteness, we can pick a lamp K' such that $I \prec K' \leq K$ in $\mathrm{Lamp}(L)$. We have assumed that J is the only cover of I, whereby $K' = J$ and so $J = K' \leq K$. The inequality here cannot be strict since both J and K belong to $\mathrm{Max}(\mathrm{Lamp}(L))$. Hence, $J = K$, but this is a contradiction since J is on the top left boundary chain of L while K is on the top right boundary chain. Since $\mathrm{J}(D) \cong \mathrm{Lamp}(L)$ by Lemma 2.11, we have proved Corollary 3.5. ☐

Corollary 3.6 (Two-cover Theorem from Grätzer [20])**.** *Let L be a slim planar semimodular lattice and let $D := \mathrm{Con}\, L$. Then for every element $x \in \mathrm{J}(D)$, the set $\{y \in \mathrm{J}(D) : x \prec_{\mathrm{J}(D)} y\}$ of covers of x in $\mathrm{J}(D)$ consists of at most two elements.*

Proof. Since the case $|L| < 3$ is trivial, we assume (3.3). It follows from Lemma 2.11 that we can work in $\mathrm{Lamp}(L)$ rather than in $\mathrm{J}(D)$. For lamps I and J of our slim rectangular lattice L, we define

$$\begin{aligned} I \prec_{\mathrm{left}} J &\overset{\mathrm{def}}{\iff} \mathrm{CircR}(I) \subseteq \mathrm{LeftLit}(L) \quad \text{and, similarly,} \\ I \prec_{\mathrm{right}} J &\overset{\mathrm{def}}{\iff} \mathrm{CircR}(I) \subseteq \mathrm{RightLit}(L). \end{aligned} \tag{3.5}$$

By parts (i) and (iv) of Lemma 2.11, for any $I, J \in \mathrm{Lamp}(L)$,

$$\text{if } I \prec J, \text{ then } I \prec_{\mathrm{left}} J \text{ or } I \prec_{\mathrm{right}} J; \tag{3.6}$$

note that $I \prec_{\mathrm{left}} J$ and $I \prec_{\mathrm{right}} J$ can simultaneously hold. Based on (2.10), a straightforward induction on i occurring in (2.9) yields that

for each $I \in \mathrm{Lamp}(L)$, there is at most one J in $\mathrm{Lamp}(L)$ such that $I \prec_{\mathrm{left}} J$. Similarly, $I \prec_{\mathrm{right}}$ K holds for at most one $K \in \mathrm{Lamp}(L)$. (3.7)

Finally, (3.6) and (3.7) imply Corollary 3.6. ☐

Definition 3.7. For non-horizontal parallel geometric *lines* T_1 and T_2, we say that T_1 is *to the left of* T_2 if T_i is of the form $\{\langle a_i, 0\rangle + t \cdot \langle v_x, v_y\rangle : t \in \mathbb{R}\}$ for $i \in \{1, 2\}$ such that $a_1 < a_2$. Here the vector $\langle v_x, v_y\rangle$ is the common direction of T_1 and T_2 while $(a_i, 0)$ is the intersection point of T_i and the x-axis. We denote by $T_1 \mathrel{\lambda} T_2$ that T_1 is left to T_2. For parallel *line segments* S_1 and S_2 of positive lengths, we say that S_1 is *to the left of* S_2, in notation, $S_1 \mathrel{\lambda} S_2$, if the line containing S_1 is to the left of the line containing S_2. Let us emphasize that if S_1 or S_2 is of zero length, then $S_1 \mathrel{\lambda} S_2$ fails! Next, let L be a slim rectangular lattice, and let J_0 and J_1 be distinct lamps of L. With reference to Definition 2.7, we say that J_0 and J_1 are *left separatory* lamps if there is a (unique) $i \in \{0, 1\}$ such that

$$\mathrm{LRoof}(J_i) \mathrel{\lambda} \mathrm{LRoof}(J_{1-i}) \mathrel{\lambda} \mathrm{LFloor}(J_i) \mathrel{\lambda} \mathrm{LFloor}(J_{1-i}). \tag{3.8}$$

Replacing the left roofs and left floors in (3.8) by right roofs and right floors, respectively, we obtain the concept of *right separatory* lamps. We say that J_0 and J_1 are *separatory lamps* if J_0 and J_1 are left separatory or right separatory. Finally, if the line segments $\mathrm{LFloor}(J_0)$ and $\mathrm{LFloor}(J_1)$ lie on the same line, then J_0 and J_1 are *left floor-aligned.* If $\mathrm{RFloor}(J_0)$ and $\mathrm{RFloor}(J_1)$ are segments of the same line, then J_0 and J_1 are *right floor-aligned.* They are *floor-aligned* if they are left floor-aligned or right floor-aligned.

Lemma 3.8. *If I and J are distinct lamps of a slim rectangular lattice, then these two lamps are neither separatory nor floor-aligned.*

Proof. To prove this by contradiction, suppose that the lemma fails. Let $J_0 := I$ and $J_1 := J$. Using (2.11), (2.14), and that $\mathrm{Foot}(J_j)$ is in the (topological) interior of $\mathrm{CircR}(J_j)$, for $j \in \{0, 1\}$, we can find an $i \in \{0, 1\}$ such that $\mathrm{LRoof}(J_i)$ or $\mathrm{LFloor}(J_i)$ crosses the upper right edge of $\mathrm{CircR}(J_{1-i})$ or left-right symmetrically. This contradicts planarity and completes the proof. Alternatively, we can use an induction on i as occurring in (2.9). $\square$

Since this section is intended to be a “toolkit”, we formulate the following lemma here; not only its proof but also its complete formulation are left to the next section.

Lemma 3.9. *If L is a slim rectangular lattice, then* $\mathrm{Lamp}(L) = \mathrm{Lamp}(L_k)$ *satisfies* (4.3).

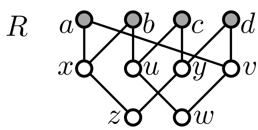

FIGURE 9. Two-pendant Four-crown

4. Two new properties of congruence lattices of slim planar semimodular lattices

For posets X and Y, we say that X is a *cover-preserving subposet* of Y if $X \subseteq Y$ and, for all $u, v \in X$, $u \leq_X v \iff u \leq_Y v$ and $u \prec_X v \iff u \prec_Y v$. The poset R given in Figure 9 will be called the *Two-pendant Four-crown*; it is a four-crown decorated with two “pendants”, z and w. This section is devoted to the following two properties.

Definition 4.1 (Two-pendant Four-crown Property)**.** Let D be a finite distributive lattice. We say that D satisfies the *Two-pendant Four-crown Property* if the poset R in Figure 9 is not a cover-preserving subposet of $\mathrm{J}(D)$ so that the maximal elements of R are maximal in $\mathrm{J}(D)$.

Definition 4.2 (Forbidden Marriage Property)**.** We say that a finite distributive lattice D satisfies the *Forbidden Marriage Property* if for every $x, y \in \mathrm{J}(D)$ and $z \in \mathrm{Max}(\mathrm{J}(D))$, if $x \neq y$, $x \prec_{\mathrm{J}(D)} z$, and $y \prec_{\mathrm{J}(D)} z$, then there is no $p \in \mathrm{J}(D)$ such that $p \prec_{\mathrm{J}(D)} x$ and $p \prec_{\mathrm{J}(D)} y$.

Now we are in the position to formulate the main theorem of the paper.

Theorem 4.3 (Main Theorem). *If L is a slim planar semimodular lattice, then*

(i) $\mathrm{Con}\,L$ *satisfies the the Forbidden Marriage Property, and*
(ii) $\mathrm{Con}\,L$ *satisfies the Two-pendant Four-crown Property.*

Remark 4.4. The smallest distributive lattice D that fails to satisfy the Forbidden Marriage Property is the eight-element lattice D_8 given in Czédli [4]. Now the result of [4], stating that D_8 cannot be represented as the congruence lattice of a slim planar semimodular lattice, becomes an immediate consequence of Theorem 4.3. In fact, part (i) of Theorem 4.3 is a generalization of the result in [4].

Remark 4.5. By the representation theorem of finite distributive lattices, there is a unique finite distributive lattice D_R such that $\mathrm{J}(D_R) \cong R$. Observe that D_R satisfies the properties in Corollaries 3.3–3.6, so all the previously known properties (and even the Forbidden Marriage Property), but D_R fails to satisfy the Two-pendant Four-crown Property. Therefore, Theorem 4.3(ii) is really a new result.

Remark 4.6. A straightforward calculation shows that the lattice D_R in Remark 4.5 consists of 56 elements. Furthermore, we are going to prove that every finite distributive lattice with less than 56 elements satisfies the Two-pendant Four-crown Property.

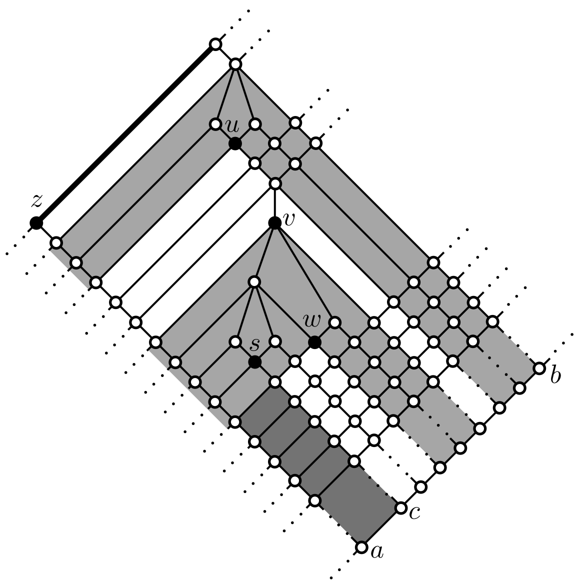

FIGURE 10. Illustrating the proof of (4.3)

Proof of Theorem 4.3. As usual, the case $|L| < 3$ is trivial. Hence, (3.3) is assumed. Also, we use the notation $D := \mathrm{Con}\,L$. By Lemma 2.11, we can also assume that $\mathrm{J}(D) = \mathrm{Lamp}(L)$. For $J_0, J_1 \in \mathrm{Lamp}(L)$, we say that

$$\text{the lamps } J_0 \text{ and } J_1 \text{ are } \textit{independent} \text{ if there is a (unique) } i \in \{0,1\} \text{ such that } \mathrm{Peak}(J_i) \le \mathrm{Foot}(J_{1-i}). \tag{4.1}$$

First, we deal with part (i), that is, with the Forbidden Marriage Property. In Figure 10, which is either L or only a part (in fact, an interval) of L, there are four internal lamps, S, U, V, and W; their feet are s, u, v, and w, respectively. In this figure, for example, U and V are independent but S and W are not. Actually, $\{S, W\}$ is the only two-element subset of $\{S, U, V, W\}$ whose two members are not independent. It follows from (2.6) and Corollary 6.1 of Czédli [5] that, using the terminology of (3.4),

$$\text{if } J_0 \text{ and } J_1 \text{ are independent lamps, then } \mathrm{Lit}(J_0) \text{ and } \mathrm{Lit}(J_1) \text{ are sufficiently disjoint.} \tag{4.2}$$

Now assume that Z is a boundary lamp. By left-right symmetry, we can assume that it is on the top left boundary chain; see Figure 10 where $z = \mathrm{Foot}(Z)$. The intersection of $\mathrm{Lit}(Z) = \mathrm{RightLit}(Z)$ with the right boundary chain $\mathrm{C}_{\mathrm{right}}(L)$ will be denoted by $E(Z)$; it is the (topologically closed) line segment with endpoints a and b in the Figure. Also, we define the following set of geometric points

$$F(Z) := \{q \in E(Z) : (\exists U \in \mathrm{Lamp}(L))\,(\langle U, Z\rangle \in \boldsymbol{\rho}_{\mathrm{CircR}} \text{ and } q \in \mathrm{Lit}(U))\}.$$

For example, $F(Z)$ in Figure 10 is the line segment with endpoints b and c. If $F(Z) = \emptyset$ or $F(Z)$ is a line segment with its upper endpoint being the same as that of $E(Z)$, then we say that *there is no gap in $F(Z)$*. For example, there is no gap in $F(Z)$ in Figure 10. With reference to (2.9) and (2.10), we claim that

$$\text{for } i = 0, 1, \dots, k, \text{ the lower covers of } Z \text{ in } \mathrm{Lamp}(L_i) \text{ are pairwise independent and there is no gap in } F(Z)\text{; still in } L_i. \tag{4.3}$$

Note that our boundary lamp Z is in L_0 and so (4.3) makes sense. Note also that (4.3) implies Lemma 3.9, because $L = L_k$.

We prove (4.3) by induction on i. The case $i = 0$ is trivial since Z has no lower cover in $\mathrm{Lamp}(L_0)$. In Figure 10, Z has three lower covers: U, V and W. (Since $S < W < Z$, S is not a lower cover.) Assume that this figure is the relevant part (that is, $\mathrm{Lit}(Z)$) of L_i for some $i \in \{0, 1, \dots, k-1\}$. Assume also that in L_{i+1}, Z obtains a new lower cover, T. We know from (i) and (iv) of Lemma 2.11 that $\langle T, Z\rangle \in \boldsymbol{\rho}_{\mathrm{CircR}}$. Due to (2.9) and (2.10), $\mathrm{CircR}(T)$ is a distributive 4-cell in the figure. For every lamp $G \in \mathrm{Lamp}(L_i)$ such that $G < Z$ (in particular, if $\langle G, Z\rangle \in \boldsymbol{\rho}_{\mathrm{CircR}}$), $\mathrm{CircR}(T)$ cannot be a 4-cell of $\mathrm{Lit}(G)$ since otherwise $\langle T, G\rangle \in \boldsymbol{\rho}_{\mathrm{CircR}}$ would lead to $T < G < Z$, contradicting $T \prec Z$. Also, there can be no $G \in \mathrm{Lamp}(L_i)$ such that $\langle G, Z\rangle \in \boldsymbol{\rho}_{\mathrm{CircR}}$ and G is to the "south-east" of T, because otherwise $\mathrm{CircR}(T)$ would not be distributive. It follows that T is one of the dark-grey cells in the figure, whereby even in L_{i+1}, the lower covers of Z remain pairwise independent and there is no gap in $F(Z)$. Since the figure clearly represents generality, we are done with the induction step from i to $i+1$. This proves (4.3).

Finally, Lemmas 2.11 and 3.2 translate part (i) of Theorem 4.3 to the following statement on $\mathrm{Lamp}(L)$:

$$\text{if } Z \text{ is a boundary lamp, } X \prec Z,\ Y \prec Z, \text{ and } X \neq Y, \text{ then there exists no } P \in \mathrm{Lamp}(L) \text{ such that } P \prec X \text{ and } P \prec Y. \tag{4.4}$$

To see this, assume the premise in (4.4). For the sake of contradiction, suppose that there does exist a P described in (4.4). By (i) and (iv) of Lemma 2.11(iv), $\langle P, X\rangle \in \boldsymbol{\rho}_{\mathrm{CircR}}$ and $\langle P, Y\rangle \in \boldsymbol{\rho}_{\mathrm{CircR}}$. Hence, $\mathrm{CircR}(P) \subseteq \mathrm{Lit}(X) \cap \mathrm{Lit}(Y)$, whereby $\mathrm{Lit}(X)$ and $\mathrm{Lit}(Y)$ are not sufficiently disjoint. On the other hand, we obtain from $L = L_k$ and (4.3) that $\mathrm{Lit}(X)$ and $\mathrm{Lit}(Y)$ are independent, whence they are sufficiently disjoint by (4.2). This is a contradiction proving (4.4) and part (i) of Theorem 4.3.

Next, we deal with part (ii). By way of contradiction, suppose that $D = \mathrm{Con}\, L$ fails to satisfy the Two-pendant Four-crown Property. This assumption and Lemma 2.11 yield that R is a cover-preserving subposet of $\mathrm{Lamp}(L) = \mathrm{J}(D)$ such that the four maximal elements of R are also maximal in $\mathrm{Lamp}(L)$; see Definition 4.1. For $a, b, \dots \in R$, the corresponding lamp will be denoted by A, B, ..., that is, by the capitalized version of the notation used in Figure 9. By Lemma 3.2, A, B, C, D are boundary lamps. Each of these four lamps is on the top left boundary chain or on the top right boundary chain.

By Corollary 3.4, any two consecutive members of the sequence A, B, C, D, A belong to different top boundary chains since they have a common lower cover. By left-right symmetry, we can assume that A and C are on the top left boundary chain while B and D on the top right one. We can assume that C is above A in the sense that $\mathrm{Foot}(A) < \mathrm{Foot}(C)$ since otherwise we can relabel R according to the "rotational" automorphism that restricts to $\{a, b, c, d\}$ as

$$\begin{pmatrix} a & b & c & d \\ c & d & a & b \end{pmatrix}.$$

Also, we can assume that D is above B since otherwise we can extend

$$\begin{pmatrix} a & b & c & d \\ a & d & c & b \end{pmatrix}$$

to a "reflection" automorphism of R and relabel R accordingly. Note that A and C are not necessarily neighboring boundary lamps, that is, we have $\mathrm{Peak}(A) \le \mathrm{Foot}(C)$ but $\mathrm{Peak}(A) < \mathrm{Foot}(C)$ need not hold. Similarly, we only have that $\mathrm{Peak}(B) \le \mathrm{Foot}(D)$. The situation is outlined in Figure 11. The feet of lamps in the figure are black-filled and any of the two distances marked by curly brackets can be zero. The illuminated sets $\mathrm{Lit}(X)$ and $\mathrm{Lit}(Y)$ are dark-grey while $\mathrm{Lit}(A)$ and $\mathrm{Lit}(C)$ are (dark and light) grey. Since $X \prec A$ and $X \prec B$, we know from (i) and (iv) of Lemma 2.11 that $\langle X, A\rangle \in \boldsymbol{\rho}_{\mathrm{Body}}$ and $\langle X, B\rangle \in \boldsymbol{\rho}_{\mathrm{Body}}$. Hence, $\mathrm{Body}(X) \subseteq \mathrm{Lit}(A) \cap \mathrm{Lit}(B)$, in accordance with the figure. Similarly, $Y \prec C$ and $Y \prec D$ lead to $\mathrm{Body}(Y) \subseteq \mathrm{Lit}(C) \cap \mathrm{Lit}(D)$, as it is indicated in Figure 11. Note that due to (2.6), the figure is satisfactorily correct in this aspect. Hence, the $\boldsymbol{\wedge}$-shaped $\mathrm{Lit}(Y)$ is above the $\boldsymbol{\wedge}$-shaped $\mathrm{Lit}(X)$. Thus, using (2.6), we obtain that

$$\mathrm{Lit}(X) \text{ and } \mathrm{Lit}(Y) \text{ are sufficiently disjoint;} \tag{4.5}$$

see (3.4) for this concept. On the other hand, $Z \prec X$ and $Z \prec Y$ together with (i) and (iv) of Lemma 2.11 gives that $\langle Z, X\rangle \in \boldsymbol{\rho}_{\mathrm{Body}}$ and $\langle Z, Y\rangle \in \boldsymbol{\rho}_{\mathrm{Body}}$. Hence, $\mathrm{Body}(Z) \subseteq \mathrm{Lit}(X) \cap \mathrm{Lit}(Y)$. Thus, since Z is an internal lamp by Lemma 3.2, $\mathrm{Lit}(X) \cap \mathrm{Lit}(Y)$ contains a precipitous neon tube. This contradicts (4.5). Note that there is another way to get a contradiction: since $\langle Z, X\rangle \in \boldsymbol{\rho}_{\mathrm{CircR}}$ and $\langle Z, Y\rangle \in \boldsymbol{\rho}_{\mathrm{CircR}}$, we have that $\mathrm{CircR}(Z) \subseteq \mathrm{Lit}(X) \cap \mathrm{Lit}(Y)$, which contradicts the fact that $\mathrm{CircR}(Z)$ is of positive area (two-dimensional measure) while $\mathrm{Lit}(X) \cap \mathrm{Lit}(Y)$ is of area 0. Both contradictions imply part (ii). The proof of Theorem 4.3 is complete. □

Proof of Remark 4.6. For the sake of contradiction, suppose that there exists a distributive lattice D such that D fails to satisfy the Two-pendant Four-crown

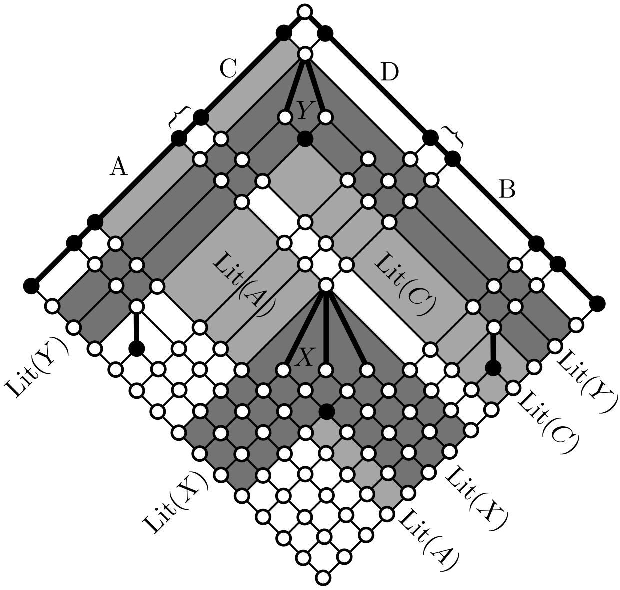

FIGURE 11. Illustrating the proof of Theorem 4.3(ii)

Property but $|D| < 56$. Let $Q := \mathrm{J}(D)$. By our assumption, R is a subposet of the poset Q. Denote by $\mathrm{DnSt}(R)$ and $\mathrm{DnSt}(Q)$ the lattice of down sets (that is, order ideals and the empty set) of R and Q, respectively. For $X \subseteq Q$, let $\downarrow_Q X := \{y \in Q : y \leq x \text{ for some } x \in X\}$. The map $\varphi\colon \mathrm{DnSt}(Q) \to \mathrm{DnSt}(R)$, defined by $X \mapsto X \cap R$ is surjective since, for each $Y \in \mathrm{DnSt}(R)$, we have that $\downarrow_Q Y \in \mathrm{DnSt}(Q)$ and $\varphi(\downarrow_Q Y) = Y$. Hence, using the structure theorem mentioned in Remark 4.5, $|D| = |\mathrm{DnSt}(Q)| \geq |\mathrm{DnSt}(R)| = |D_R| = 56$, which is a contradiction proving Remark 4.6. ☐

We conclude the paper with a last remark.

Remark 4.7. Lamps have several geometric properties. Many of these properties have already been mentioned, and there are some other properties of technical nature, too. These properties would allow us to represent the congruence lattices of slim planar semimodular lattices in a purely geometric (but quite technical) way. However, this does not seem to be more useful than our technique based on (2.9)–(2.10) and the tools presented in the paper.

Added on February 28, 2021. Since January 8, 2021, when the first version of the present paper was uploaded to arXiv, the tools developed here have been successfully used in Czédli [6] and Czédli and Grätzer [8].

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