INFINITELY MANY NEW PROPERTIES OF THE CONGRUENCE LATTICES OF SLIM SEMIMODULAR LATTICES

GÁBOR CZÉDLI

Dedicated to Professor Ágnes Szendrei, who has recently become a member of the Hungarian Academy of Sciences

Abstract. Slim planar semimodular lattices (SPS lattices or slim semimodular lattices for short) were introduced by G. Grätzer and E. Knapp in 2007. More than four dozen papers have been devoted to these (necessarily finite) lattices and their congruence lattices since then. In addition to distributivity, the congruence lattices of SPS lattices satisfy seven known properties. Out of these seven properties, the first two were published by G. Grätzer in 2016 and 2020, the next four by the present author in 2021, and the seventh jointly by G. Grätzer and the present author in 2022.

Here we give two infinite families of new properties of the congruence lattices of SPS lattices. These properties are independent. We also present stronger versions of these properties but not all of them are independent, and improve three out of the seven previously known properties. The approach is based on lamps, which we introduced in a 2021 paper. In addition to using the 2021 results, we need to prove some easy new lemmas on lamps.

1. Introduction

By a *slim semimodular lattice* or, in another terminology, an *SPS lattice* we mean a finite planar semimodular lattice that does not contain an M_3-sublattice (equivalently, does not contain a cover-preserving M_3 sublattice); M_3 stands for the five element modular lattice of length 2. This is the original 2007 definition from Grätzer and Knapp [10]. In 2011, Czédli and Schmidt [5] gave a related other definition: a lattice L is *slim* if it is finite and the poset (= partially ordered set) $\mathrm{J}(L)$ of the join-irreducible elements of L is the union of two chains. (Note that $0 \notin \mathrm{J}(L)$ by definition.) Slim lattices are automatically planar; this justifies our terminology: slim semimodular (that is: slim and semimodular) lattices are the same as Grätzer and Knapp's SPS lattices.

Since 2007, more than four dozen papers have been devoted to these lattices. The list of these papers was first given in the appendix of my arXiv paper

`http://arxiv.org/abs/2107.10202` ;

this is a permanent list but out of date; the updated list is here:

`http://www.math.u-szeged.hu/~czedli/m/listak/publ-psml.pdf` .

2010 *Mathematics Subject Classification.* 06C10 **February 18, 2023**.

Key words and phrases. Slim rectangular lattice, slim semimodular lattice, planar semimodular lattice, congruence lattice, lattice congruence, Crown with Two Fences Property CTF(n), Crown with Diamonds and Emeralds Property CDE(n), Three-pendant Three-crown Property, lamp.

This research was supported by the National Research, Development and Innovation Fund of Hungary, under funding scheme K 134851.

This list shows that slim semimodular lattices have useful connections with geometry, group theory, combinatorics, and finite model theory. Most of the motivations to study these lattices are explained in the survey section of Czédli and Kurusa [4], open access at the time of writing.

In 2016, Grätzer [7] and [8] asked *what the congruence lattices of slim semimodular lattices are*. These congruence lattices are finite distributive lattices, and seven of their additional properties have been known from Grätzer [8] and [9], Czédli [2], and Czédli and Grätzer [3]. The new properties we are going to prove here and the seven old ones strengthen our feeling that Grätzer's above-mentioned problem is difficult.

Outline. In Section 2, we define two infinite sets of properties of the congruence lattices of slim semimodular lattices. In fact, we define both sets in two versions, and then we formulate the main result, Theorem 2.9. Section 2 also contains a reformulation of the main result in Corollary 2.11 and a stronger form of an earlier property in Proposition 2.12. Section 3 presents some easy lemmas on lamps and proves the results presented in the previous section. Section 4 contains an observation about the independence of an infinite set of properties. Moreover, it excludes some variants as further properties, and points out which old properties have been strengthened.

2. Results

Notation 2.1. The ring of residue classes modulo n will be denoted by $\mathbb{Z}_n := \{0, 1, 2, \dots, n-1\}$. So for $i, j \in \mathbb{Z}_n$, $i+j$ and $i-j$ are understood modulo n; for example, $2-3=4$ in $\mathbb{Z}_5$. The (ring) ideal $\{0, 2, 4, \dots, 2n-2\}$ of $\mathbb{Z}_{2n}$ will be denoted by $2\mathbb{Z}_{2n}$. The *subscripts* we use later will belong to $\mathbb{Z}_n$ or $\mathbb{Z}_{2n}$. (A part of the proof of Observation 4.4, where SCDE(k) rather than SCDE(n) is considered, is a self-explanatory exception.) Additions and subtractions in subscripts are understood in $\mathbb{Z}_n$ or $\mathbb{Z}_{2n}$.

Notation 2.2. For an element u in a poset P, let $\downarrow u = \downarrow_P u := \{x \in P : x \le u\}$, $\Downarrow u = \Downarrow_P u := \{x \in P : x < u\}$, and $\downarrow^{\prec} u = \downarrow^{\prec}_P u := \{x \in P : x \prec u\}$. Note that $\downarrow^{\prec} u$ is the (possibly empty) set of lower covers of u in P.

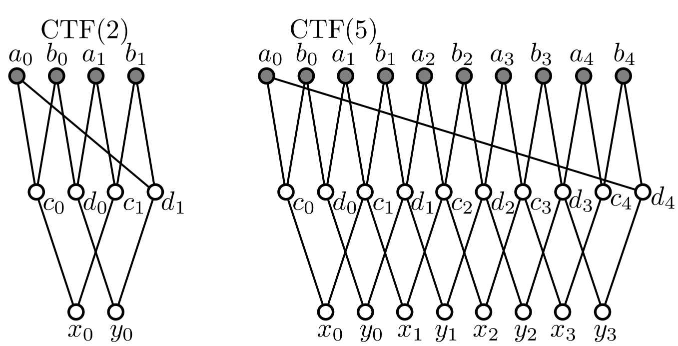

Figure 1. CTF(2) and CTF(5)

Definition 2.3. For posets P_1 and P_2, a function $f\colon P_1 \to P_2$ is an *order embedding* if for all $x, y \in P_1$, $x \le y$ in P_1 if and only if $f(x) \le f(y)$ in P_2. Note that an order embedding is necessarily an injective function. If f is an order embedding and, for all $x, y \in P_1$, $x \prec y$ in P_1 implies that $f(x) \prec f(y)$ in P_2, then f is a *cover-preserving embedding.*

Definition 2.4. For an integer $n \ge 2$, the *Crown with Two Fences of order* n is the $(6n-2)$-element poset

$$\begin{aligned}\mathrm{CTF}(n) = \{a_i : i \in \mathbb{Z}_n\} \cup \{b_i : i \in \mathbb{Z}_n\} \cup \{c_i : i \in \mathbb{Z}_n\} \cup \{d_i : i \in \mathbb{Z}_n\} \\ \cup \{x_i : i \in \mathbb{Z}_n \setminus \{n-1\}\} \cup \{y_i : i \in \mathbb{Z}_n \setminus \{n-1\}\}\end{aligned}$$

such that the edges are exactly the following:

$$\begin{aligned}&c_i \prec a_i,\ c_i \prec b_i,\ d_i \prec b_i, \text{ and } d_i \prec a_{i+1} \text{ for } i \in \mathbb{Z}_n, \text{ and}\\ &x_j \prec c_j,\ x_j \prec c_{j+1},\ y_j \prec d_j, \text{ and } y_j \prec d_{j+1} \text{ for } j \in \mathbb{Z}_n \setminus \{n-1\}\end{aligned}$$

where the additions in the subscripts are understood in $\mathbb{Z}_n$. For CTF(2) and CTF(5), see Figure 1.

As usual, $\mathrm{J}(D)$ stands for the poset of nonzero join-irreducible elements of D, and we use Notations 2.1 and 2.2.

Definition 2.5. Let $n \ge 2$ be an integer, and let D be a finite distributive lattice.

(A) We say that D satisfies the *CTF(n)-property* if there is no cover-preserving embedding $f\colon \mathrm{CTF}(n) \to \mathrm{J}(D)$ such that the f-image of every maximal element of CTF(n) is a maximal element of $\mathrm{J}(D)$. (The maximal elements of CTF(n) in Figure 1 are grey-filled.)

(B) We say that D satisfies the *SCTF(n)-property* (in other words, the *strong* CTF(n)-property) if the poset $(\mathrm{J}(D);\le)$ does not have (not necessarily distinct) maximal elements $a_0, b_0, a_1, b_1, \dots, a_{n-1}, b_{n-1}$ and (not necessarily distinct) additional elements $c_0, d_0, c_1, d_1, \dots, c_{n-1}, d_{n-1}$ such that, for all $i \in \mathbb{Z}_n$ and $j \in \mathbb{Z}_n \setminus \{n-1\}$,

$$a_i \ne a_{i+1}, \quad b_i \ne b_{i+1}, \quad c_i \ne c_{i+1}, \quad d_i \ne d_{i+1}, \tag{2.1}$$

$$c_i \prec a_i, \quad c_i \prec b_i, \quad d_i \prec b_i, \quad d_i \prec a_{i+1}, \tag{2.2}$$

$$\downarrow^{\prec} c_j \cap \Downarrow c_{j+1} \ne \emptyset, \quad \Downarrow c_j \cap \downarrow^{\prec} c_{j+1} \ne \emptyset, \tag{2.3}$$

$$\downarrow^{\prec} d_j \cap \Downarrow d_{j+1} \ne \emptyset, \quad \text{and} \quad \Downarrow d_j \cap \downarrow^{\prec} d_{j+1} \ne \emptyset; \tag{2.4}$$

the additions in the subscripts are understood in $\mathbb{Z}_n$.

The maximal elements of CTF(n) and their lower covers together form a so-called *$2n$-crown* while both $\{c_i : i \in \mathbb{Z}_n\} \cup \{x_i : i \in \mathbb{Z}_n \setminus \{n-1\}\}$ and $\{d_i : i \in \mathbb{Z}_n\} \cup \{y_i : i \in \mathbb{Z}_n \setminus \{n-1\}\}$ are called *fences.*

The poset occurring in the following definition is illustrated by Figures 2 and 3; Figure 3 includes three pictures copied from `www.clker.com`.

Definition 2.6. For an integer $n \ge 3$, the *Crown with Diamonds and Emeralds of order* n is the $4n$-element poset

$$\begin{aligned}\mathrm{CDE}(n) = \{a_i : i \in 2\mathbb{Z}_{2n}\} \cup \{b_{i+1} : i \in 2\mathbb{Z}_{2n}\} \\ \cup \{d_i : i \in 2\mathbb{Z}_{2n}\} \cup \{e_i : i \in 2\mathbb{Z}_{2n}\}\end{aligned}$$

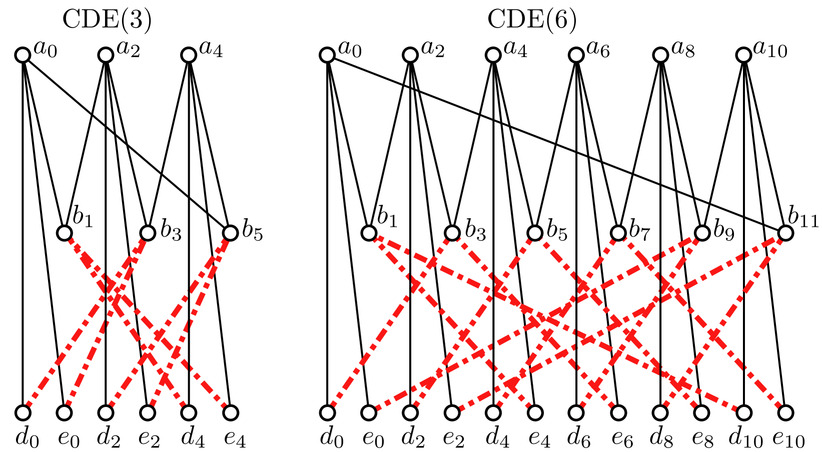

FIGURE 2. CDE(3) and CDE(6)

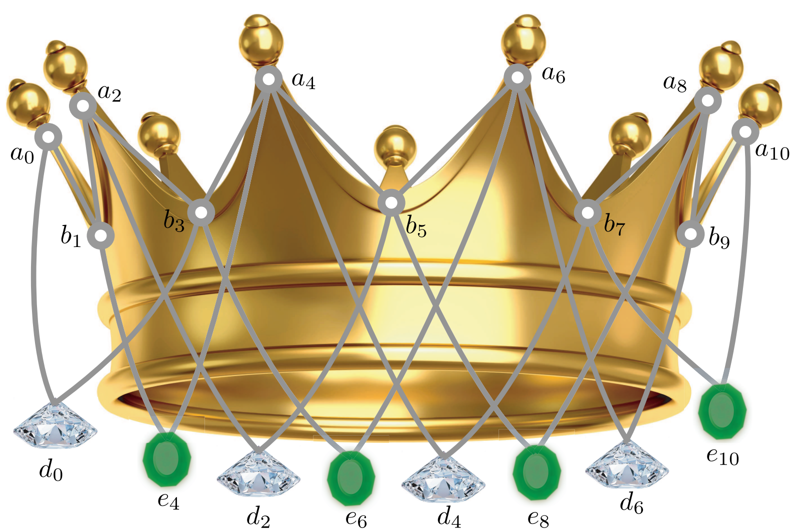

FIGURE 3. Motivating the terminology by a subposet of CDE(9)

such that the edges are

$$b_{i+1} \prec a_i,\ b_{i-1} \prec a_i,\ d_i \prec a_i,\ d_i \prec b_{i+3},\ e_i \prec a_i,\ \text{and } e_i \prec b_{i-3}$$

for $i \in 2\mathbb{Z}_{2n}$; the operations in the subscripts are understood in $\mathbb{Z}_{2n}$.

Definition 2.7. Let D be a finite distributive lattice; Notations 2.1 and 2.2 still apply and the operations in the subscripts are understood in $\mathbb{Z}_{2n}$.

(A) We say that D satisfies the CDE(n)-*property* if there is no order embedding $f\colon \mathrm{CDE}(n) \to \mathrm{J}(D)$ such that for all $i \in 2\mathbb{Z}_{2n}$ and $x \in \mathrm{CDE}(n)$, if $x \prec a_i$ then $f(x) \prec f(a_i)$.

(B) We say that D satisfies the SCDE(n)-*property* if the poset $(\mathrm{J}(D); \le)$ does not contain a system $(a_i, b_{i+1} : i \in 2\mathbb{Z}_{2n})$ of not necessarily distinct elements such that for all $i \in 2\mathbb{Z}_{2n}$,

$$a_i \neq a_{i+2}, \tag{2.5}$$

$$b_{i-1} \prec a_i, \quad b_{i+1} \prec a_i, \tag{2.6}$$

$$\downarrow^{\prec} a_i \cap \Downarrow b_{i+3} \neq \emptyset, \quad \text{and} \quad \downarrow^{\prec} a_i \cap \Downarrow b_{i-3} \neq \emptyset. \tag{2.7}$$

While Definitions 2.5(A) and 2.7(A) use order homomorphisms (in fact, order embeddings), Definitions 2.5(B) and 2.7(B) do not. With the help of order homomorphisms, Remark 2.8 below may shed more light on Definitions 2.5(B) and 2.7(B); however, note that *the reader can safely skip the following remark*.

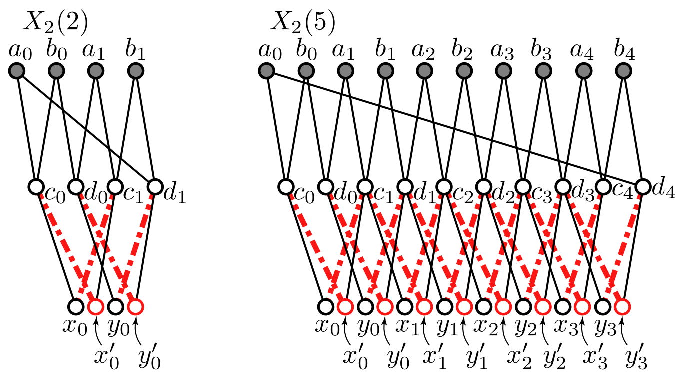

FIGURE 4. $X_2(2)$ and $X_2(5)$

Remark 2.8. It is possible to formulate Definitions 2.5(B) and 2.7(B) in terms of forbidden order homomorphisms that have special properties. In fact, we could do so in two ways; we only explain the two possibilities for Definition 2.5(B) since the case of Definition 2.7(B) is analogous.

First, let $X_1(n)$ be the subposet of $\mathrm{CTF}(n)$ that consists of the non-minimal elements of $\mathrm{CTF}(n)$. Then the $\mathrm{SCTF}(n)$ property means that there is no cover-preserving order homomorphism $f_1 \colon X_1(n) \to \mathrm{J}(D)$ such that f_1 has the properties that we obtain from (2.1), (2.3), and (2.4) by changing the elements $a_i, \dots, d_i$ to their f_1-images $f_1(a_i), \dots, f_1(d_i)$ for all meaningful subscripts i and, in addition, the f_1-images of the maximal elements of $X_1(n)$ are maximal in $\mathrm{J}(D)$.

Second, we can extend $\mathrm{CTF}(n)$ to a larger poset $X_2(n)$ by "doubling" x_i to an antichain $\{x_i, x'_i\}$ and y_j to an antichain $\{y_i, y'_i\}$ for all $i, j \in \mathbb{Z}_n \setminus \{n-1\}$ as it is shown, at least for $n \in \{2, 5\}$, in Figure 4. (Compare Figures 1 and 4.) As it is easy to see, the $\mathrm{SCTF}(n)$ property means that there is no order homomorphism $f_2 \colon X_2(n) \to \mathrm{J}(D)$ such that f_2 preserves the solid thin edges as covers and the "dot-dash-dot-dash" thick edges as "strictly less", it satisfies the properties that we obtain from (2.1) by changing the elements to their f_2-images, and the f_2-images of the maximal elements of $X_2(n)$ are maximal in $\mathrm{J}(D)$.

Note, however, that the use of f_1 would only be a trivial transcript making the notation more complicated while the use of f_2 would make the paper even more complicated. Therefore, we are not going to use f_1 and f_2 in the rest of the paper.

For $n \in \{3, 6\}$, the condition on the order embedding f in Definition 2.7(A) means that f preserves the thin solid edges of Figure 2 but need not preserve the bold "dot-dash-dot-dash" edges as covers.

As usual, the congruence lattice of a lattice L is denoted by $\mathrm{Con}\, L = (\mathrm{Con}\, L; \subseteq)$. Slim semimodular lattices are the same as SPS lattices. Now we are in the position to formulate the main result of the paper.

Theorem 2.9. *For every slim semimodular lattice L, the following two assertions hold.*

(A) *For every integer $n \geq 2$, $\mathrm{Con}\, L$ satisfies the $\mathrm{SCTF}(n)$-property.*
(B) *For every integer $n \geq 3$, $\mathrm{Con}\, L$ satisfies the $\mathrm{SCDE}(n)$-property.*

The following statement follows trivially from Theorem 2.9.

Corollary 2.10. *For every integer $n \geq 2$, the congruence lattice of a slim semimodular lattice satisfies the $\mathrm{CTF}(n)$-property as well as the $\mathrm{CDE}(n+1)$-property.*

Even though the congruence lattice of a lattice K (or another algebra) is known to influence the structure of K, the following corollary is perhaps unexpected.

Corollary 2.11. *Let K be a planar semimodular lattice. If there is an $n \geq 3$ and a system $(a_i, b_{i+1} : i \in 2\mathbb{Z}_{2n})$ of join-irreducible congruences of K such that (2.5), (2.6), and (2.7) hold in the poset $(\mathrm{J}(\mathrm{Con}\, K); \subseteq)$, then K contains a cover-preserving M_3-sublattice.*

Based on Grätzer and Knapp's 2007 definition of SPS lattices, it is a trivial task to derive Corollary 2.11 from Corollary 2.10, so no details will be given. In fact, Corollary 2.11 is only a reformulation of Part (B) of Theorem 2.9. One could easily formulate the analogous reformulations of Corollary 2.10 and Part (A) of Theorem 2.9.

The name of the following property will be explained in Remark 4.3(c). Note that a useful technical part of the statement below is postponed to Remark 3.7.

Proposition 2.12 (Strong Bipartite Maximal Elements Property, SBMEP in short)**.** *For a slim semimodular lattice K with at least three elements, $\mathrm{Max}(\mathrm{J}(\mathrm{Con}\, K))$ is the union of two disjoint nonempty sets M_0 and M_1 such that for each $i \in \{0, 1\}$ and for any two distinct elements $x, y \in M_i$, we have that at least one of $\Downarrow x \cap \downharpoonleft^{\prec} y$ and $\downharpoonleft^{\prec} x \cap \Downarrow y$ (equivalently, at least one of $\downarrow x \cap \downharpoonleft^{\prec} y$ and $\downharpoonleft^{\prec} x \cap \downarrow y$) is the empty set.*

3. Proving the results

To read this section, Czédli [2] and Czédli and Grätzer [3] should be kept near. At the time of writing, each of these two papers has an open access view at `http://www.acta.hu/` (or `https://doi.org/10.14232/actasm-021-865-y`) and `https://cgasa.sbu.ac.ir/article_101508.html`, respectively.

By Grätzer and Knapp [11], a slim semimodular lattice L is called a *slim rectangular lattice* if $|L| \geq 4$, L has exactly two doubly irreducible elements, and these two elements are complementary; see also [2, page 384]. For a slim rectangular lattice L, the reader is referred to [2, Definition 2.1] for a fixed $\mathcal{C}_1$-*diagram* of L, to [2, Definition 2.3] for *lamps* and their parts, to [2, Definitions 2.6 and 2.7] for line segments and geometric shapes associated with lamps, to [2, Definitions 2.8] for *illuminated sets*, and to [2, Definitions 2.9] for the relation $\rho_{\mathrm{Body}} = \rho_{\mathrm{infoot}}$ defined on $\mathrm{Lamp}(L)$; these definitions also contain the corresponding notations. Similarly, see

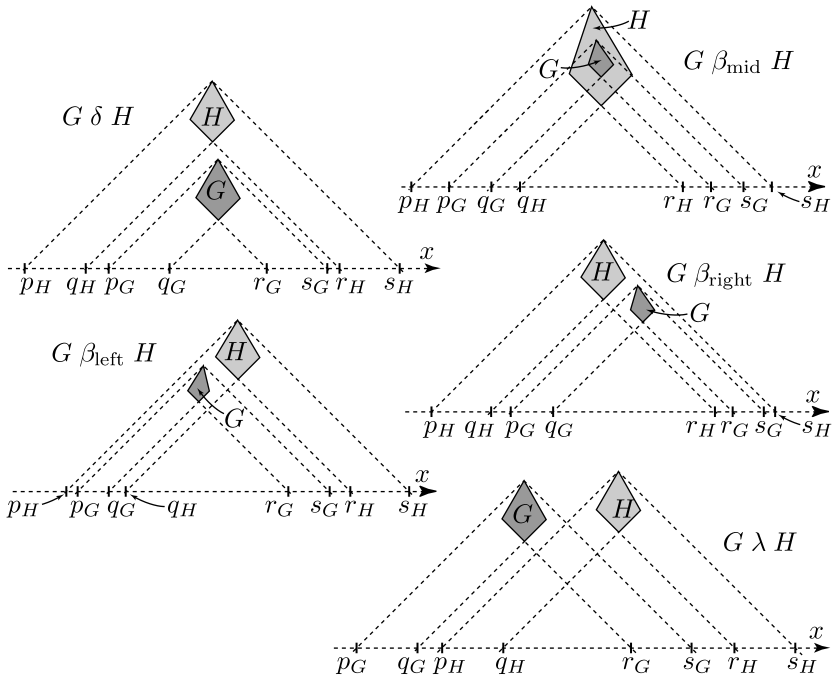

FIGURE 5. Possible positions of two lamps in the plane

[3, Definition 4.1(ii)] for the *coordinate quadruple* (p_I, q_I, r_I, s_I) of $I \in \mathrm{Lamp}(L)$. The following definition is illustrated by Figure 5, where the bodies of lamps are grey-filled. (Note that, as opposed to the figure, the body of a lamp can also be a precipitous line segment or a boundary line segment, but this causes no trouble.)

Definition 3.1. Extending Czédli and Grätzer [3, Definition 4.1.(iii)] and Czédli [2, (4.1)], we define the following five relations for $G, H \in \mathrm{Lamp}(L)$; see Figure 5.

- $G\ \lambda\ H$, that is, G is *to the left of* H if $q_G \le p_H$ and $s_G \le r_H$;
- $G\ \delta\ H$, that is, G is *geometrically under* H if $q_H \le p_G$ and $s_G \le r_H$;
- $G\ \beta_{\rm mid}\ H$ if $p_H < p_G < q_G < q_H$ and $r_H < r_G < s_G < s_H$;
- $G\ \beta_{\rm left}\ H$ if $p_H \le p_G < q_G < q_H$, $s_G \le r_H$, and G is an internal lamp;
- $G\ \beta_{\rm right}\ H$ if $q_H \le p_G$, $r_H < r_G < s_G \le s_H$, and G is an internal lamp.

The notations λ and δ come from "Left" and "unDer", respectively. Clearly,

$$\lambda \text{ and } \delta \text{ are irreflexive and transitive relations.} \tag{3.1}$$

In the rest of the paper, unless otherwise stated explicitly,

$$L \text{ always denotes a slim rectangular lattice with a fixed } \mathcal{C}_1\text{-diagram.} \tag{3.2}$$

Lemma 3.2. *For L in as* (3.2)*, let J_0 and J_1 be two distinct members of* $\mathrm{Lamp}(L)$. *Then exactly one of the following five alternatives holds for some $i \in \{0, 1\}$; note that this i is uniquely determined.*

(i) $J_i\ \lambda\ J_{1-i}$,
(ii) $J_i\ \delta\ J_{1-i}$,
(iii) $J_i\ \beta_{\rm mid}\ J_{1-i}$,

(iv) $J_i\ \beta_{\text{left}}\ J_{1-i}$,
(v) $J_i\ \beta_{\text{right}}\ J_{1-i}$,

Furthermore, if J_0 and J_1 are incomparable in the poset $(\mathrm{Lamp}(L);\leq)$*, then the first two options are only possible.*

Proof. Those possible mutual geometric positions of J_0 and J_1 that are not listed in the lemma are ruled out by Lemma 3.8 of [2]. Alternatively, take a multifork extension sequence for L; see [2, (2.9) and Lemma 2.12]. Then each lamp originates from a multifork extension; see [2, (2.10)]. When the "younger" lamp comes to existence in this way, its body will be in the geometric area of a 4-cell. Since the alternatives listed in Lemma 3.2 take all possible positions of this 4-cell into account, it follows that there is a (unique) i such that one (and exactly one) of the five alternatives holds. Each of the last three alternatives leads to $(J_i, J_{1-i}) \in \rho_{\text{Body}}$, implying that $J_i < J_{i-1}$; see [2, Lemma 2.11]. Thus, if J_i and J_{i-1} are incomparable in the poset $\mathrm{Lamp}(L)$ (in notation, if $J_i \parallel J_{i-1}$), then only one of the first two alternatives can hold, completing the proof of Lemma 3.2. □

For L from (3.2) and $U \in \mathrm{Lamp}(L)$, $\mathrm{Roof}(U)$ is defined in [2, page 389]. Let $\downarrow_{\mathrm{g}}\mathrm{Roof}(U)$ denote the set of those geometric points of the full geometric rectangle (of the $\mathcal{C}_1$-diagram of L) that are on or below $\mathrm{Roof}(U)$. More precisely, a geometric point (x, y) (given in the usual coordinate system) of the full geometric rectangle belongs to $\downarrow_{\mathrm{g}}\mathrm{Roof}(U)$ if and only if $(x, y') \in \mathrm{Roof}(U)$ for some y' such that $y' \geq y$.

Lemma 3.3. *If L is as in* (3.2) *and $I \leq J$ holds in* $\mathrm{Lamp}(L)$*, then* $\downarrow_{\mathrm{g}}\mathrm{Roof}(I) \subseteq \downarrow_{\mathrm{g}}\mathrm{Roof}(J)$.

Proof. The case $I = J$ is trivial. If $I \prec J$, then $(I, J) \in \rho_{\text{Body}}$ by [2, Lemma 2.11], whence $\mathrm{Body}(I) \subseteq \mathrm{Lit}(J) \subseteq \downarrow_{\mathrm{g}}\mathrm{Roof}(J)$ gives the required inclusion $\downarrow_{\mathrm{g}}\mathrm{Roof}(I) \subseteq \downarrow_{\mathrm{g}}\mathrm{Roof}(J)$. Otherwise, the inclusion follows from its just-mentioned particular case by transitivity. □

Lemma 3.4. *For L as in* (3.2) *and $I, J, K \in$* $\mathrm{Lamp}(L)$*, if $I\ \delta\ J$ and $K \leq I$, then $K \not\leq J$. That is, if I is geometrically under J, then $\Downarrow I \cap \downarrow^{\prec} J \subseteq \downarrow I \cap \downarrow^{\prec} J = \emptyset$.*

Proof. If K is a boundary lamp, then it is a maximal element of $(\mathrm{Lamp}(L);\leq)$ by [2, Lemma 3.2] and so the required $K \not\leq J$ follows trivially. Thus, we can assume that K is an internal lamp. Since $\Downarrow I \subseteq \downarrow I$, the "$\subseteq$" is trivial. For a subset X of the plane, let $\mathrm{GInt}(X)$ stand for the geometric (in other words, topological) *interior* of X. Assume that $I\ \delta\ J$ and $K \leq I$. Lemma 3.3 gives that $\downarrow_{\mathrm{g}}\mathrm{Roof}(K) \subseteq \downarrow_{\mathrm{g}}\mathrm{Roof}(I)$. Hence, $\mathrm{Foot}(K) \in \downarrow_{\mathrm{g}}\mathrm{Roof}(I)$ and $\mathrm{Body}(K) \subseteq \downarrow_{\mathrm{g}}\mathrm{Roof}(I)$. Actually, we have that $\mathrm{Foot}(K) \in \mathrm{GInt}(\downarrow_{\mathrm{g}}\mathrm{Roof}(I))$ since there is at least one precipitous edge with bottom at or above $\mathrm{Foot}(K)$ and this edge is included in $\mathrm{Body}(K) \subseteq \downarrow_{\mathrm{g}}\mathrm{Roof}(I)$. Since $I\ \delta\ J$, $\mathrm{GInt}(\downarrow_{\mathrm{g}}\mathrm{Roof}(I)) \cap \mathrm{GInt}(\mathrm{Lit}(J)) = \emptyset$. Hence, $\mathrm{Foot}(K) \notin \mathrm{GInt}(\mathrm{Lit}(J))$, that is, $(K, J) \notin \rho_{\text{infoot}}$. Therefore, the required $K \not\leq J$ follows by [2, Lemma 2.11]. □

Lemma 3.5. *If L is as in* (3.2)*, $I, J, K \in$* $\mathrm{Lamp}(L)$*, $J\ \delta\ K$, and $I \leq J$, then $I\ \delta\ K$.*

Proof. Apply Lemma 3.3. □

The following easy lemma is illustrated by Figure 8 in Czédli and Grätzer [3].

Lemma 3.6. *Assume that L is as in* (3.2)*, A_0, A_1, A_2 and B_1 are from* $\mathrm{Lamp}(L)$*, $A_0\ \lambda\ A_1\ \lambda\ A_2$, $B_1 \prec A_0$, and $B_1 \prec A_2$. Then $B_1\ \delta\ A_1$.*

Proof. By [2, Lemma 2.11], $(B_1, A_0), (B_1, A_2) \in \rho_{\mathrm{Body}}$. Hence $\mathrm{Body}(B_1)$ is a subset of $\mathrm{Lit}(A_0) \cap \mathrm{Lit}(A_2)$, see [3, Figure 8], and we obtain that $B_1\ \delta\ A_1$. □

Armed with the lemmas verified so far, we are ready to prove the results stated in Section 2.

Proof of Proposition 2.12. Since no two maximal elements are comparable, it makes no difference whether we deal with $\Downarrow x \cap \downarrow^{\prec} y$ and $\downarrow^{\prec} x \cap \Downarrow y$ or we consider $\downarrow x \cap \downarrow^{\prec} y$ and $\downarrow^{\prec} x \cap \downarrow y$. We know from the main result of Grätzer and Knapp [11], which is recalled in, say, Czédli and Grätzer [3, Lemma 3.7], that

$$\left.\begin{array}{l}\text{for each slim semimodular lattice } K \text{ with at least three elements}\\ \text{there exists a slim rectangular lattice } L \text{ such that } \mathrm{Con}\, K \cong \mathrm{Con}\, L.\end{array}\right\}\tag{3.3}$$

Hence, it suffices to deal with slim *rectangular* lattices rather than with slim semimodular lattices. By [2, Lemma 2.11(iii)], $(\mathrm{J}(\mathrm{Con}\, L); \leq) \cong (\mathrm{Lamp}(L); \leq)$. Hence, instead of $\mathrm{Max}(\mathrm{J}(\mathrm{Con}\, L))$, it suffices to deal with $\mathrm{Max}(\mathrm{Lamp}(L))$. By [2, Lemma 3.2], $\mathrm{Max}(\mathrm{Lamp}(L))$ is the set of boundary lamps. Let M_0 and M_1 be the set of left boundary lamps and that of right boundary lamps, respectively. By left–right symmetry, it is sufficient to show that for any two distinct left boundary lamps X and Y, at least one of $\downarrow X \cap \downarrow^{\prec} Y$ and $\downarrow^{\prec} X \cap \downarrow Y$ is empty. But this follows immediately from Lemma 3.4 since $X\ \delta\ Y$ or $Y\ \delta\ X$. □

Remark 3.7. The proof above implies that under the assumptions of Proposition 2.12, M_0 and M_1 can be chosen so that there is a slim rectangular lattice L and isomorphisms $f_1\colon \mathrm{Lamp}(L) \to \mathrm{Con}\, L$ and $f_2\colon \mathrm{Con}\, L \to \mathrm{Con}\, K$ such that $M_0 = f_2(f_1(\{\text{left boundary lamps}\}))$ and $M_1 = f_2(f_1(\{\text{right boundary lamps}\}))$.

Proof of Theorem 2.9. Clearly, we can assume that $|L| \geq 3$. Hence, by (3.3), we can assume that L is a slim *rectangular* lattice; see (3.2).

By way of contradiction, suppose that Part (A) of the theorem fails for some integer $n \geq 2$. By [2, Lemma 2.11(iii)], $(\mathrm{J}(\mathrm{Con}\, L); \leq) \cong (\mathrm{Lamp}(L); \leq)$. Hence, there are $A_i, B_i, C_i, D_i \in \mathrm{Lamp}(L)$ for $i \in \mathbb{Z}_n$ that satisfy (2.1)–(2.4) (after replacing the under-case letters with the corresponding capital letters, of course) and the A_i and B_i are maximal lamps. By [2, Lemma 3.2], A_i and B_i are boundary lamps for all $i \in \mathbb{Z}_n$. Using (2.2) and Remark 3.7, left-right symmetry allows us to assume that $A_0, \dots, A_{n-1}$ are left boundary lamps while $B_0, \dots, B_{n-1}$ are right boundary lamps. (There can be other boundary lamps but they cause no trouble.) Let $A_i\ \lambda_{\mathrm{bnd}}\ A_j$ mean that A_i is to the left of A_j on the left upper boundary of L. (Note that $A_i\ \lambda_{\mathrm{bnd}}\ A_j$ implies that $A_i \neq A_j$.) Similarly, $B_i\ \lambda_{\mathrm{bnd}}\ B_j$ means that B_i is to the left of B_j on the right upper boundary of L. Clearly, $A_i\ \lambda_{\mathrm{bnd}}\ A_j \iff A_i\ \delta\ A_j$ for left boundary lamps while $B_i\ \lambda_{\mathrm{bnd}}\ B_j \iff B_j\ \delta\ B_i$ for right boundary lamps. It follows from (2.1) that

$$\left.\begin{array}{l}\text{for every } i \in \mathbb{Z}_n,\ A_i\ \lambda_{\mathrm{bnd}}\ A_{i+1} \text{ or } A_{i+1}\ \lambda_{\mathrm{bnd}}\ A_i \text{ and,}\\ \text{furthermore, } B_i\ \lambda_{\mathrm{bnd}}\ B_{i+1} \text{ or } B_{i+1}\ \lambda_{\mathrm{bnd}}\ B_i.\end{array}\right\}\tag{3.4}$$

For $x, y \in \{a_i : i \in \mathbb{Z}_n\} \cup \{b_i : i \in \mathbb{Z}_n\} \cup \{c_i : i \in \mathbb{Z}_n\} \cup \{d_i : i \in \mathbb{Z}_n\}$, [2, Lemma 2.11] yields that

$$\text{if } x \prec y \text{ occurs in (2.2), then } (X, Y) \in \rho_{\mathrm{Body}} \text{ and so } \mathrm{Body}(X) \subseteq \mathrm{Lit}(Y).\tag{3.5}$$

We claim that

$$\text{for } j \in \mathbb{Z}_n \setminus \{n-1\},\ A_j\ \lambda_{\mathrm{bnd}}\ A_{j+1} \iff B_j\ \lambda_{\mathrm{bnd}}\ B_{j+1}.\tag{3.6}$$

We prove this by way of contradiction. Suppose that (3.6) fails. There are two cases. First, assume that $A_j \ \lambda_{\text{bnd}}\ A_{j+1}$ holds but $B_j \ \lambda_{\text{bnd}}\ B_{j+1}$ fails. Then (3.4) gives that $B_{j+1}\ \lambda_{\text{bnd}}\ B_j$. Since we know from (2.2) and (3.5) that $\text{Body}(C_j) \in \text{Lit}(A_j) \cap \text{Lit}(B_j)$ and $\text{Body}(C_{j+1}) \in \text{Lit}(A_{j+1}) \cap \text{Lit}(B_{j+1})$, we obtain that $C_j \ \delta\ C_{j+1}$. Hence, Lemma 3.4 yields that $\Downarrow C_j \cap \downharpoonleft^{\prec} C_{j+1} = \emptyset$, violating (2.3). Second, assume that $A_j\ \lambda_{\text{bnd}}\ A_{j+1}$ fails but $B_j\ \lambda_{\text{bnd}}\ B_{j+1}$ holds. Then, similarly to the first case, $A_{j+1}\ \lambda_{\text{bnd}}\ A_j$ by (3.4), it follows that $C_{j+1}\ \delta\ C_j$, and Lemma 3.4 yields that $\Downarrow C_{j+1} \cap \downharpoonleft^{\prec} C_j = \emptyset$, violating (2.3). We have proved (3.6).

Next, we claim that

$$\text{for } j \in \mathbb{Z}_n \setminus \{n-1\},\ B_j\ \lambda_{\text{bnd}}\ B_{j+1} \iff A_{j+1}\ \lambda_{\text{bnd}}\ A_{j+2}. \tag{3.7}$$

To prove this by way of contradiction, first we suppose that $B_j\ \lambda_{\text{bnd}}\ B_{j+1}$ holds but $A_{j+1}\ \lambda_{\text{bnd}}\ A_{j+2}$ fails. Then (3.4) gives that $A_{j+2}\ \lambda_{\text{bnd}}\ A_{j+1}$. Similarly to the argument given for (3.6), (2.2) and (3.5) yield that $D_{j+1}\ \delta\ D_j$. But then Lemma 3.4 yields that $\Downarrow D_{j+1} \cap \downharpoonleft^{\prec} D_j = \emptyset$, violating (2.4). Second, suppose that $B_j\ \lambda_{\text{bnd}}\ B_{j+1}$ fails while $A_{j+1}\ \lambda_{\text{bnd}}\ A_{j+2}$ holds. From (3.4), we have that $B_{j+1}\ \lambda_{\text{bnd}}\ B_j$. Then (2.2) and (3.5) yield that $D_j\ \delta\ D_{j+1}$. Hence, Lemma 3.4 leads to $\Downarrow D_j \cap \downharpoonleft^{\prec} D_{j+1} = \emptyset$, violating (2.4). This proves (3.7).

Let $A_i\ \rho_{\text{bnd}}\ A_j$ and $B_i\ \rho_{\text{bnd}}\ B_j$ mean that A_i is to the right of A_j and B_i is to the right of B_j on the upper boundary, respectively. That is, $A_i\ \rho_{\text{bnd}}\ A_j \iff A_j\ \lambda_{\text{bnd}}\ A_i$ and $B_i\ \rho_{\text{bnd}}\ B_j \iff B_j\ \lambda_{\text{bnd}}\ B_i$. It follows by (2.1) and (3.4) that that (3.6) and (3.7) remain valid if we replace λ_{bnd} by ρ_{bnd} in them.

Next, since either $A_0\ \lambda_{\text{bnd}}\ A_1$ or $A_0\ \rho_{\text{bnd}}\ A_1$ by (2.1) and (3.4), left-right symmetry allows us to assume that $A_0\ \lambda_{\text{bnd}}\ A_1$. Then we can argue as follows; when referencing (3.6) or (3.7) over implication signs, the value of j will be indicated.

$$\left.\begin{aligned} A_0\ \lambda_{\text{bnd}}\ A_1 &\overset{(3.6,j=0)}{\Longrightarrow} B_0\ \lambda_{\text{bnd}}\ B_1 \overset{(3.7,j=0)}{\Longrightarrow} A_1\ \lambda_{\text{bnd}}\ A_2 \\ &\overset{(3.6,j=1)}{\Longrightarrow} B_1\ \lambda_{\text{bnd}}\ B_2 \overset{(3.7,j=1)}{\Longrightarrow} A_2\ \lambda_{\text{bnd}}\ A_3 \\ &\overset{(3.6,j=2)}{\Longrightarrow} B_2\ \lambda_{\text{bnd}}\ B_3 \overset{(3.7,j=2)}{\Longrightarrow} A_3\ \lambda_{\text{bnd}}\ A_4\ \dots \\ &\overset{(3.6,j=3)}{\Longrightarrow} \quad \dots \quad \overset{(3.7,j=n-3)}{\Longrightarrow} A_{n-2}\ \lambda_{\text{bnd}}\ A_{n-1} \\ &\overset{(3.6,j=n-2)}{\Longrightarrow} B_{n-2}\ \lambda_{\text{bnd}}\ B_{n-1} \overset{(3.7,j=n-2)}{\Longrightarrow} A_{n-1}\ \lambda_{\text{bnd}}\ A_0. \end{aligned}\right\} \tag{3.8}$$

By the first four lines of (3.8) and the transitivity of λ_{bnd}, we have that $A_0\ \lambda_{\text{bnd}}\ A_{n-1}$. But this contradicts the last line of (3.8), where $A_{n-1}\ \lambda_{\text{bnd}}\ A_0$. We have proved Part (A) of Theorem 2.9.

To prove Part (B) of the theorem by way of contradiction, suppose that it fails. Note in advance that in many cases in the argument below, it suffices to verify some of our statements only for one value of the subscript i or j since the SCTF(n)-property is "rotationally symmetric". As in case of Part (A), there is an L satisfying (3.2) and there are A_i and B_{j+1} in Lamp(L) for $i, j \in 2\mathbb{Z}_{2n}$ such that (2.5), (2.6), and (2.7) hold. For $i \in 2\mathbb{Z}_{2n}$, we now from (2.5) and (2.6) that A_i and A_{i+2} are incomparable in Lamp(L), in notation, $A_i \parallel A_{i+2}$. Furthermore, by (2.6), $\downharpoonleft^{\prec} A_i \cap \downharpoonleft^{\prec} A_{i+2} \neq \emptyset$. Hence, the last sentence of Lemma 3.2 and Lemma 3.4 yield that

$$\text{for each } i \in 2\mathbb{Z}_{2n},\ A_i\ \lambda\ A_{i+2} \text{ or } A_{i+2}\ \lambda\ A_i. \tag{3.9}$$

We claim that

$$\text{for every } i \in 2\mathbb{Z}_{2n},\ A_i \text{ and } A_{i+4} \text{ are incomparable in } \mathrm{Lamp}(L). \tag{3.10}$$

To see this, we need to exclude both $A_i \leq A_{i+4}$ and $A_{i+4} \leq A_i$. First, suppose that $A_i \leq A_{i+4}$. Then $B_{i+1} \prec A_i \leq A_{i+4}$ by (2.6). So if $X \in {\Downarrow}B_{i+1}$, that is, $X < B_{i+1}$ for an $X \in \mathrm{Lamp}(L)$, then $X \not\leq A_{i+4}$, that is, $X \notin {\downharpoonleft}^{\prec} A_{i+4}$. Hence, ${\downharpoonleft}^{\prec} A_{i+4} \cap {\Downarrow}B_{i+4-3} = {\downharpoonleft}^{\prec} A_{i+4} \cap {\Downarrow}B_{i+1} = \emptyset$, violating (2.7). Second, suppose that $A_{i+4} \leq A_i$. Then $B_{i+3} \prec A_{i+4} \leq A_i$ by (2.6), whereby ${\downharpoonleft}^{\prec} A_i \cap {\Downarrow}B_{i+3} = \emptyset$, violating (2.7) again. We have shown the validity of (3.10). Observe that

$$\text{for every } i \in 2\mathbb{Z}_{2n},\ A_i \parallel B_{i+3} \text{ and } A_i \parallel B_{i-3}. \tag{3.11}$$

Indeed, if $A_i \leq B_{i+3}$ or $A_i \leq B_{i-3}$, then (2.6) gives that $A_i < A_{i+4}$ or $A_i \leq A_{i-4}$, contradicting (3.10). If $A_i > B_{i+3}$ or $A_i > B_{i-3}$, then ${\downharpoonleft}^{\prec} A_i \cap {\Downarrow}B_{i+3} = \emptyset$ or ${\downharpoonleft}^{\prec} A_i \cap {\Downarrow}B_{i-3} = \emptyset$ violates (2.7). We have shown (3.11). Next, we claim that

$$\text{for every } i \in 2\mathbb{Z}_{2n}, \text{ if } A_i \mathrel{\lambda} A_{i+2}, \text{ then } A_{i+2} \mathrel{\lambda} A_{i+4}. \tag{3.12}$$

To show this, suppose the contrary. Then, by (3.9), $A_i \mathrel{\lambda} A_{i+2}$ and $A_{i+4} \mathrel{\lambda} A_{i+2}$. For the geometric relation between A_i and A_{i+4}, (3.10) and the (last sentence of) Lemma 3.2 only allow four possibilities; we are going the exclude each of these four possibilities and then (3.12) will follow by way of contradiction.

First, let $A_{i+4} \mathrel{\delta} A_i$. Then $B_{i+3} < A_{i+4}$ by (2.6), and so Lemma 3.5 yields that $B_{i+3} \mathrel{\delta} A_i$. We claim that $B_{i+3} \mathrel{\delta} A_i$ leads to contradiction. Indeed, $B_{i+3} \mathrel{\delta} A_i$ together with Lemma 3.4 imply that ${\downharpoonleft}^{\prec} A_i \cap {\Downarrow}B_{i+3} = \emptyset$, violating (2.7). Thus, $A_{i+4} \mathrel{\delta} A_i$ is excluded.

Second, let $A_i \mathrel{\delta} A_{i+4}$. Then (2.6) gives that $B_{i+1} < A_i$, whereby Lemma 3.5 gives that $B_{i+1} \mathrel{\delta} A_{i+4}$. We claim that $B_{i+1} \mathrel{\delta} A_{i+4}$ leads to contradiction. Indeed, $B_{i+1} \mathrel{\delta} A_{i+4}$ together with Lemma 3.4 give that ${\downharpoonleft}^{\prec} A_{i+4} \cap {\Downarrow}B_{i+4-3} = {\downharpoonleft}^{\prec} A_{i+4} \cap {\Downarrow}B_{i+1} = \emptyset$, violating (2.7). Hence, $A_i \mathrel{\delta} A_{i+4}$ is excluded.

Third, let $A_i \mathrel{\lambda} A_{i+4}$. Then $A_i \mathrel{\lambda} A_{i+4} \mathrel{\lambda} A_{i+2}$ and, by (2.6), $B_{i+1} \prec A_i$ and $B_{i+1} \prec A_{i+2}$. Hence, Lemma 3.6 implies that $B_{i+1} \mathrel{\delta} A_{i+4}$. We have seen in the previous case that this leads to contradiction. Thus, $A_i \mathrel{\lambda} A_{i+4}$ is excluded.

Fourth, let $A_{i+4} \mathrel{\lambda} A_i$. Then $A_{i+4} \mathrel{\lambda} A_i \mathrel{\lambda} A_{i+2}$, and we know from (2.6) that $B_{i+3} \prec A_{i+4}$ and $B_{i+3} \prec A_{i+2}$. Using Lemma 3.6, we obtain that $B_{i+3} \mathrel{\delta} A_i$. We know from the first case that this leads to contradiction. Therefore, $A_{i+4} \mathrel{\lambda} A_i$ is excluded.

Now that each of the four possibilities have been excluded, we conclude (3.12).

Finally, using (3.9) and reflecting the diagram across a vertical axis if necessary, we can assume that $A_0 \mathrel{\lambda} A_2$. Then, using (3.12) repeatedly, we obtain that

$$A_0 \mathrel{\lambda} A_2 \mathrel{\lambda} A_4 \mathrel{\lambda} A_6 \mathrel{\lambda} \dots \mathrel{\lambda} A_{2n-2} \mathrel{\lambda} A_0.$$

By the transitivity of λ, see (3.1), it follows that $A_0 \mathrel{\lambda} A_0$, which is a contradiction since λ is irreflexive by (3.1). This completes the proof of Theorem 2.9. ☐

4. Concluding remarks

We know from Grätzer [6, Corollary 108] that every finite poset is isomorphic to $(\mathrm{J}(D); \leq)$ for a finite distributive lattice D, which is unique up to isomorphism. This allows us to define D simply by defining $(\mathrm{J}(D); \leq)$; we will do so without further explanation. Let Con(SPS≥3) denote the class of the congruence lattices of at least 3-element slim semimodular lattices. Next, we compare the new properties.

Remark 4.1. In general, the SCTF(n)-property is stronger then the CTF(n) property. That is, if a finite distributive lattice D satisfies the SCTF(n)-property, then it satisfies the CTF(n)-property, but not conversely in general. The reason is two-fold. First, (2.3) is a weaker assumption than requiring that c_j and c_{j+1} have a common lower cover, and similarly for (2.4). Second, say, the CTF(28)-property allows that a crown with two fences is four-fold "coiled up" by stipulating that $a_{i+7} = a_i$, $b_{i+7} = b_i$, $c_{i+7} = c_i$, and $d_{i+7} = d_i$ for all $i \in \mathbb{Z}_{28}$, and $x_{j+7} = x_j$ and $y_{j+7} = y_j$ for all $j \in \mathbb{Z}_{28} \setminus \{20, 21, \dots, 27\}$. As opposed to the CTF(28)-property, the SCTF(28)-property excludes this situation.

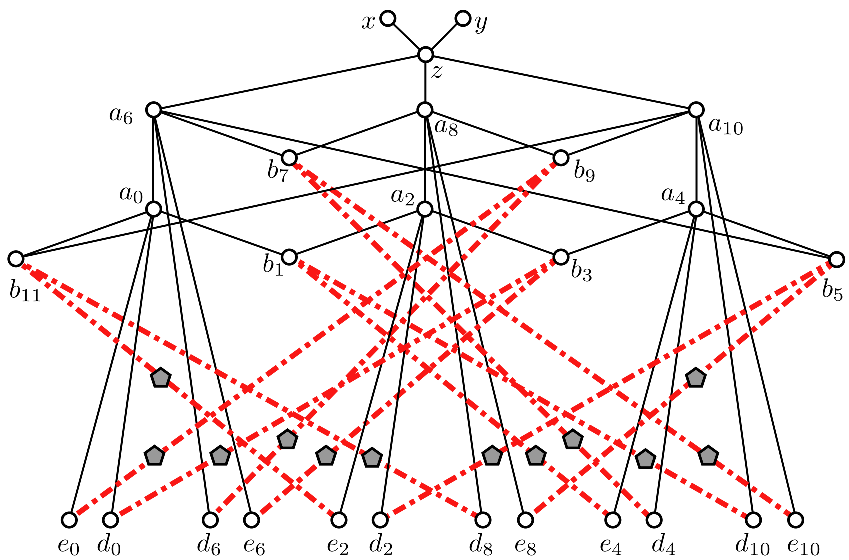

FIGURE 6. CDE(6) is "coiled up spirally"

Remark 4.2. For $n \geq 3$, the SCDE(n)-property is stronger than the CDE(n)-property. In addition to other obvious reasons, some of which are mentioned in Remark 4.1, we illustrate this by Figure 6. As opposed to the previous remark, where maximal elements had to remain maximal, the figure shows how to coil up CDE(6) "spirally". Let D be the finite distributive lattice such that J(D) is the poset drawn in Figure 6. Then CDE(6) holds but SCDE(6) fails in D. The same is true if one or two out of the $a_i \prec a_{6+i}$ edges are omitted or changed to $a_i = a_{6+i}$, or some of the grey-filled pentagon-shaped elements of Figure 6 are omitted. (The elements x, y, and z play no role here but their presence allows us to say that the seven old properties hold in D.)

Remark 4.3 (Old properties that are strengthened)**.** This paper strengthens the following three old properties of Con(SPS≥3).

(a) The *Two-Pendant Four-crown Property* from Czédli [2, Definition 4.1], *2P4C-property* in short, is the same as the CTF(2)-property; now the SCTF(2)-property is clearly a stronger property.

(b) The *Three-pendant Three-crown Property* from Czédli and Grätzer [3, Page 2], *3P3C-property* for short, is similar to the CDE(3)-property. Clearly, the SCDE(3)-property is stronger than the 3P3C-property and the CDE(3)-property.

(c) The SBMEP from Proposition 2.12 is stronger than its precursor, the *Bipartite Maximal Elements Property*; see Czédli [2, Corollary 3.4].

Observation 4.4. *For $3 \le k < n$ such that k divides n, if a finite distributive lattice D satisfies the* SCDE(n)*-property, then it also satisfies the* SCDE(k)*-property.*

Proof. We use the "coiling up" technique mentioned in Remark 4.1. Assume that the SCDE(k)-property fails in D. Then we can pick elements $a_i, b_{i+1} \in \mathrm{J}(D)$, $i \in 2\mathbb{Z}_{2k}$, satisfying (2.5)–(2.7). For $j = 2k, 2k+2, \dots, 2n-2 \in \mathbb{Z}_{2n}$, we define $a_j \in \mathrm{J}(D)$ by the rule

$$a_j = a_i \iff \Big(i \in \mathbb{Z}_{2k} \text{ and } i \equiv j \pmod{2k}\Big),$$

and we define $b_{j+1} \in \mathrm{J}(D)$ analogously. Now the elements $a_j, b_{j+1} \in \mathrm{J}(D)$, $j \in 2\mathbb{Z}_{2n}$, witness that the SCDE(n)-property fails in D. Therefore, Observation 4.4 holds. ☐

Next, as a counterpart of Observation 4.4, we present the following observation.

Observation 4.5. *If $3 \le k < n$, then the* SCDE(k)*-property does not imply the* SCDE(n)*-property. Similarly, for $2 \le k < n$, the* SCTF(k)*-property does not imply the* SCTF(n)*-property.*

Proof. Since crowns of different sizes cannot be embedded into each other, the poset CTF(n) satisfies the SCTF(k)-property but not the SCTF(n)-property. Using the same reasoning, (3.10), and (3.11), we obtain that the poset CDE(n) satisfies the SCDE(k)-property but not the SCDE(n)-property. ☐

In spite of Observation 4.5, we can easily present an independent infinite set of properties of Con(SPS≥3). To do so, the two properties proved in Grätzer [7] and [8] and recalled in Theorem 1.3(i) and Theorem 1.3(ii) of Czédli and Grätzer [3] will be denoted by P_{3i} and P_{3ii}, respectively. Similarly, two properties given by Czédli [2] and recalled in [3, Theorem 1.4 (iii) and (iv)] will be denoted by P_{4iii} and P_{4iv}, respectively. For the rest of previously known properties, see Remark 4.3. As usual, $\mathbb{N}^+ = \{1, 2, 3, \dots\}$ is the set of positive integers.

Observation 4.6. *Let*

$\Gamma_{\text{old}} := \{P_{3i}, P_{3ii}, P_{4iii}, P_{4iv}$, *SBMEP, 2P4C-property, 3P3C-property*$\}$ *and*

$\Gamma := \Gamma_{\text{old}} \cup \{\text{CTF}(n)\textit{-property} : 3 \le n \in \mathbb{N}^+\} \cup \{\text{CDE}(n)\textit{-property} : 4 \le n \in \mathbb{N}^+\}$.

Then Γ is an independent set of properties of finite distributive lattices. That is, for each property $\pi \in \Gamma$, there is a finite distributive lattice D_π such that D_π satisfies all properties belonging to $\Gamma \setminus \{\pi\}$ but π fails in D_π.

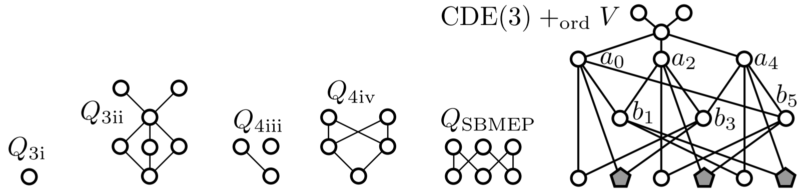

FIGURE 7. Posets for the proof of Observation 4.6

From Observation 4.6, the following statement follows trivially.

Corollary 4.7. *The set Γ_{old} of the previously known seven properties, see Observation 4.6 and the paragraph before it, is an independent set.*

Proof of Observation 4.6. First, let π be a CDE(n)-property for some $n \geq 4$ or let π be the 3P3C-property and $n = 3$. Denote by V the three-element poset that has a least element and two maximal elements. (Its diagram is V-shaped.) Let P_π be the ordinal sum CDE(n) $+_{\mathrm{ord}}$ V; see on the right of Figure 7 for $n = 3$. (For the current purpose, the three grey-filled pentagon-shaped elements can be omitted but only if π is the 3P3C-property.) Using that no crown can be embedded into a crown of another size, it is easy to see that D_π defined by $\mathrm{J}(D_\pi) \cong P_\pi$ does the job. Similarly, if π is a CTF(m)-property for some $m \geq 3$ or π is the 4P4C-property and $m = 2$, then we let $P_\pi = \mathrm{CTF}(m) +_{\mathrm{ord}} V$ and we define D_π by $\mathrm{J}(D_\pi) \cong P_\pi$. We define D_π analogously when π is one of the first five properties listed in Γ_{old} but then P_π is the corresponding poset in Figure 7. □

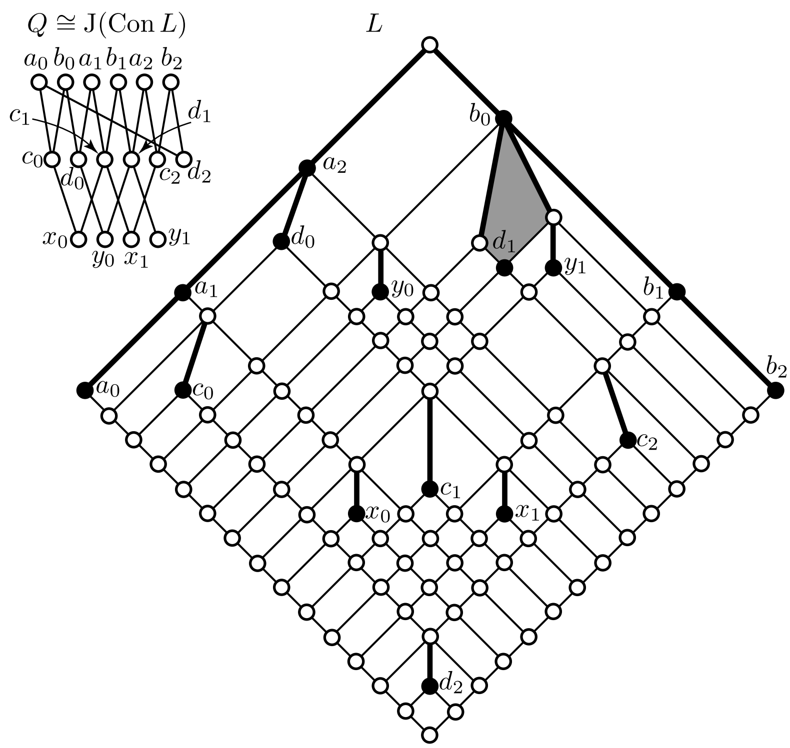

FIGURE 8. When the edge $y_1 \prec d_2$ of CTF(3) is removed

One may try to alter CTF(n) or CDE(n) by adding or removing an edge in order to find new (but less elegant) properties of Con(SPS≥3). Even though we have not investigated all possibilities of this kind, we have some comments. First, if we add a new edge, then the property we obtain is often a consequence Grätzer's "at most two covers" property; see [3, Theorem 1.3(ii)] where this property is cited. On the other hand, if we omit an edge, then the property we obtain is likely to fail in Con(SPS≥3); this is exemplified by the following two remarks.

Note in advance that Figures 8 and 9 follow the convention of Czédli [2]: the neon tubes are the thick edges, and lamps are labelled at their feet, which are black-filled. Furthermore, if a lamp has more than one neon tube, then its body is grey-filled. Based on [2, Lemma 2.12] (originally Czédli [1, Theorem 3.7]) and [2, Lemma 2.11], it is easy to see that each of Figures 8 and 9 presents a slim rectangular lattice L and $\mathrm{J}(\mathrm{Con}\,L)$ is correctly drawn in the figure. (Note that Czédli [1] offers another but less convenient way to determine $\mathrm{J}(\mathrm{Con}\,L)$ in Figures 8 and 9.)

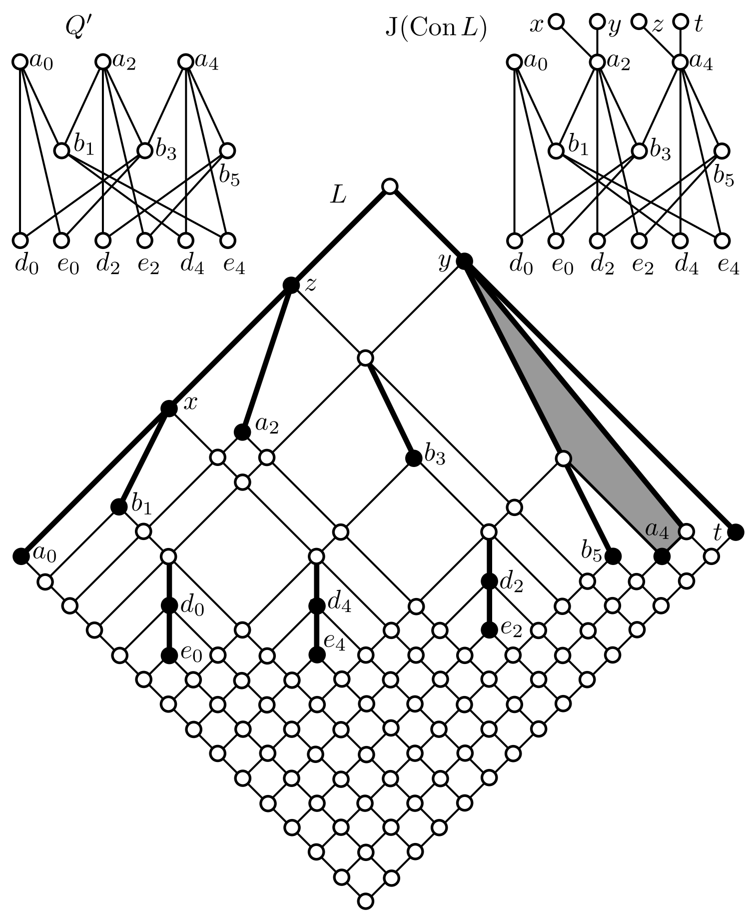

FIGURE 9. When the edge $b_5 \prec a_0$ of CDE(3) is removed

Remark 4.8. If we remove the edge $y_1 \prec d_2$ from CTF(3), then the poset Q we obtain, see at the top left of Figure 8, is (isomorphic to) $\mathrm{J}(\mathrm{Con}\,L)$ for the slim rectangular lattice L drawn in the same figure.

Remark 4.9. If we remove the edge $b_5 \prec a_0$ from CDE(3), then the poset Q' we obtain, see at the top left of Figure 9, has a cover-preserving embedding into $\mathrm{J}(\mathrm{Con}\,L)$ where L is the slim rectangular lattice at the bottom of the figure.

Statements and declarations

Funding. This research was supported by the National Research, Development and Innovation Fund of Hungary, under funding scheme K 134851.

Data availability. Not applicable.

Code availability. Not applicable.

Competing interests. Not applicable.

References

[1] Czédli, G.: Patch extensions and trajectory colorings of slim rectangular lattices. Algebra Universalis 72, 125–154 (2014)

[2] Czédli, G.: Lamps in slim rectangular planar semimodular lattices. Acta Sci. Math. (Szeged) 87, 381–413 (2021).

[3] Czédli, G., Grätzer, G.: A new property of congruence lattices of slim, planar, semimodular lattices. Categories and General Algebraic Structures with Applications 16, 1-28 (2022).

[4] Czédli, G., Kurusa, Á.: A convex combinatorial property of compact sets in the plane and its roots in lattice theory. Categories and General Algebraic Structures with Applications 11, 57–92 (2019) http://cgasa.sbu.ac.ir/article_82639.html

[5] Czédli, G., Schmidt, E. T.: The Jordan-Hölder theorem with uniqueness for groups and semimodular lattices. Algebra Universalis 66, 69–79 (2011)

[6] Grätzer, G.: Lattice Theory: Foundation. Birkhäuser, Basel (2011)

[7] Grätzer, G.: The Congruences of a Finite Lattice, A Proof-by-Picture Approach, second edition. Birkhäuser, 2016. xxxii+347. Part I is accessible at https://www.researchgate.net/publication/299594715

[8] Grätzer, G.: Congruences of fork extensions of slim, planar, semimodular lattices. Algebra Universalis 76, 139–154 (2016)

[9] Grätzer, G.: Notes on planar semimodular lattices. VIII. Congruence lattices of SPS lattices. Algebra Universalis 81 (2020), Paper No. 15, 3 pp.

[10] Grätzer, G., Knapp, E.: Notes on planar semimodular lattices. I. Construction. Acta Sci. Math. (Szeged) 73, 445–462 (2007)

[11] G. Grätzer and E. Knapp, Notes on planar semimodular lattices. III. Rectangular lattices. Acta Sci. Math. (Szeged) 75 (2009), 29–48.

E-mail address: czedli@math.u-szeged.hu
URL: http://www.math.u-szeged.hu/~czedli/

University of Szeged, Bolyai Institute. Szeged, Aradi vértanúk tere 1, HUNGARY 6720