Idempotent Mal'cev conditions and 2-uniform congruences

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Dedicated to George Grätzer and E. Tamás Schmidt on their seventieth birthdays

Abstract. A congruence is called *2-uniform* if all of its blocks have exactly two elements. We prove that if A is a finite algebra that satisfies a nontrivial idempotent Mal'cev condition then any two 2-uniform congruences of A permute.

1. Introduction and the main result

A congruence α of an algebra A is called a *2-uniform congruence* if all the α-blocks (in other words, α-classes) are two-element. If all of its blocks have the same number of elements then α is called a *uniform congruence*. A finite algebra is said to be a *uniform algebra* if all of its congruences are uniform. A finite lattice is called *isoform* if all blocks of any of its congruences are isomorphic lattices. Clearly, isoform lattices are uniform. Grätzer, Quackenbush and Schmidt [4] raised the question if finite isoform lattices are congruence permutable. Recently Kaarli [6] has shown even more: every finite uniform lattice is congruence permutable. It is an open problem how far Kaarli's result can be generalized. In particular, we do not know if uniform algebras with a majority term are congruence permutable.

As a modest step towards the general problem, the following result was proved in [1]: if a finite algebra has a majority term then any two of its 2-uniform congruences permute. The assumption on finiteness is essential, for the lattice of integer numbers $(\mathbf{Z}; \leq)$ has exactly two 2-uniform congruences but they do not permute. There are examples in [1] showing that the result of [1] and that of [6] are independent even for lattices. The goal of the present paper is to replace the existence of a majority term by a much weaker condition. This will not make [1] superfluous, for we use its result in the present proof and the closure operator introduced in [1] is of separate interest.

The notion of a Mal'cev (also spelled Mal'tsev) condition was introduced by Grätzer [3]. Hence we extract the short definition we need directly from [3]: an

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idempotent Mal'cev condition for algebras is a condition of the form

“ (P_n): there exists a natural number n, and term symbols $p_0, \ldots, p_{m_n-1}$ satisfying a set Σ_n of identities ”,

where (P_{n+1}) is weaker than (P_n), the form of Σ_n is independent of the type of algebras considered, and Σ_n implies that all the $p_0, \ldots, p_{m_n-1}$ are idempotent. A Mal'cev condition is called *trivial* if it is satisfied by sets, i.e., by algebras with no operation. The condition to be tailored to our *finite* algebra A is that

(1) *A should satisfy a nontrivial idempotent Mal'cev condition.*

An equivalent condition is that

(2) *A has a so-called Taylor term;*

see Cor. 5.3 in Taylor [8] or Lemma 9.4 in Hobby and McKenzie [5] for more details and more exact definition of Mal'cev conditions. Another equivalent condition is that

(3) ***1** is not in the type set of the variety V generated by A,*

cf. Theorem 9.6 of [5]. Notice that if the class of congruence lattices of algebras in Vsatisfies a nontrivial lattice identity (cf. Thm. 9.18 in [5]) or a nontrivial lattice Horn sentence from [2], like meet or join semidistributivity, then (1) holds. Our result is the following.

Theorem 1. *Let A be a finite algebra satisfying a nontrivial idempotent Mal'cev condition. Then any two 2-uniform congruences of A permute.*

2. The proof of the theorem

Let A be a finite algebra satisfying an idempotent nontrivial Mal'cev condition M. Up to Figure 1 we can follow the ideas of Kaarli [6] the same way as in [1].

First we can assume that A is an idempotent algebra, for otherwise we can replace it by its full idempotent reduct. If α and β are 2-uniform congruences then it suffices to show that for any block B of $\alpha \vee \beta$, the restriction of α and that of β to B permute. Since A is idempotent, B is a subalgebra, and B also satisfies M. Hence we can assume that A is this $\alpha \vee \beta$ block itself, i.e., $\alpha \vee \beta = 1_A$. Since any two 2-uniform equivalences on an at most five element set permute, we can assume that $|A| \geq 6$. Then 2-uniformity and $\alpha \vee \beta = 1_A$ gives $\alpha \wedge \beta = 0_A$. This leads to a subdirect decomposition

$$A \cong S \leq_{\text{sd}} A^{(0)} \times A^{(1)}$$

where $A^{(0)} = A/\alpha$ and $A^{(1)} = A/\beta$. We can assume that A equals S. Hence A is a subalgebra of the direct product $A^{(0)} \times A^{(1)}$, and as a relation between $A^{(0)}$ and $A^{(1)}$, it is also a bipartite graph. Clearly, $|A^{(0)}| = |A^{(1)}| = |A|/2 \geq 3$. Let $n = |A^{(0)}|$. Since each 2-regular bipartite finite graph has a Hamiltonian circle, we can choose the notation so that

$$A^{(0)} = \{a_0^{(0)}, a_1^{(0)}, \ldots, a_{n-1}^{(0)}\},$$

$$A^{(1)} = \{a_0^{(1)}, a_1^{(1)}, \ldots, a_{n-1}^{(1)}\},$$

$$A = \{(a_j^{(0)}, a_j^{(1)}) : j \in \mathbf{Z}_n\} \cup \{(a_j^{(0)}, a_{j-1}^{(1)}) : j \in \mathbf{Z}_n\}.$$

All subscript calculations were and will be done in $\mathbf{Z}_n$, i.e., modulo n. The situation is depicted in Figure 1 for $n = 3$, the extremal case, and for $n = 7$, illustrating the general case.

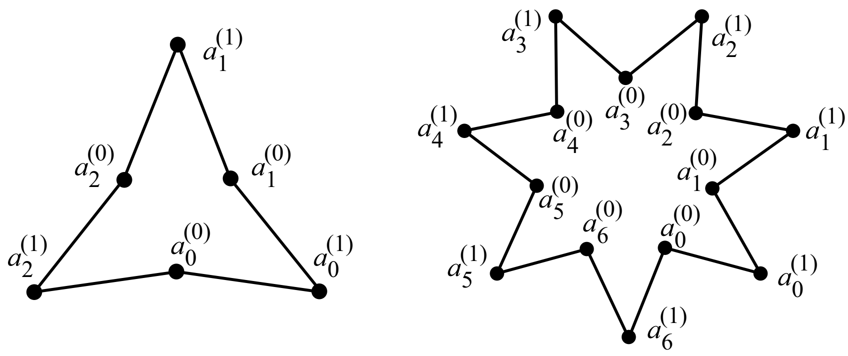

Figure 1

Here A is the set of edges. The inner vertices of the "circular saw" form the algebra $A^{(0)}$ while the outer vertices form $A^{(1)}$.

By an arc we mean a nonempty subset $X = \{a_m^{(i)}, a_{m+1}^{(i)}, \dots, a_{m+j-1}^{(i)}\}$ of $A^{(i)}$. Here $i \in \{0, 1\}$, and $j = |X| \leq n$ is the *length* of the arc. If $j < n$ then $a_m^{(i)}$ and $a_{m+j-1}^{(i)}$ are the *endpoints* of the arc, while the remaining points are said to be *inner points*. When $X = A^{(i)}$ then all of its points are inner points and the arc has no endpoints. For $X \subseteq A^{(0)}$ and $Y \subseteq A^{(1)}$ define

$$X' = \{y \in A^{(1)} : \text{ there is an } x \in X \text{ with } (x, y) \in A\},$$
$$Y' = \{x \in A^{(0)} : \text{ there is a } y \in Y \text{ with } (x, y) \in A\},$$
$$X^* := \{y \in A^{(1)} : \{y\}' \subseteq X\},$$

and

$$Y^* := \{x \in A^{(0)} : \{x\}' \subseteq Y\}.$$

For example, for $n = 7$, $\{a_1^{(0)}, a_2^{(0)}, a_3^{(0)}\}^* = \{a_1^{(1)}, a_2^{(1)}\}$. Clearly, if X is an arc with $2 \leq |X| < n$ then X^* is also an arc, $|X^*| = |X| - 1$ and $(X^*)' = X$.

We claim that every arc $X \subseteq A^{(i)}$ is a subalgebra of $A^{(i)}$. This will be shown via induction on $|X|$. Since A and therefore its homomorphic images, $A^{(0)}$ and $A^{(1)}$ are idempotent algebras, one element arcs are subalgebras. Let $f = f(x_1, \dots, x_k)$ be an arbitrary term of A, let $X \subseteq A^{(0)}$ be an arc with length $1 < |X| < n$, and let $x_1, \dots, x_k \in X$. For each x_i we can choose an $y_i \in X^*$ such that $(x_i, y_i) \in A$. Since $|X^*| < |X|$, X^* is a subalgebra of $A^{(1)}$. Now from

$$\big(f(x_1, \dots, x_k), f(y_1, \dots, y_k)\big) = f\big((x_1, y_1), \dots, (x_k, y_k)\big) \in A$$

and $(X^*)' = X$ we obtain that $f(x_1, \dots, x_k) \in X$. The case $X = A^{(0)}$ is evident and the case $X \subseteq A^{(1)}$ is similar. We have seen that each arc is a subalgebra.

Since each subset of $A^{(i)}$ is the intersection of all arcs including it, we obtain that

Fact 1. For $i \in \{0,1\}$, every nonempty subset of $A^{(i)}$ is a subalgebra of $A^{(i)}$.

A k-ary term g is called a *semiprojection* if there is an $i \in \{1,\dots,k\}$ such that $|\{x_1,\dots,x_k\}| < k$ always implies $g(x_1,\dots,x_k) = x_i$. In this case g is said to be a k-ary i-th semiprojection. The next step in the proof is to show that A has a semiprojection term. Since A satisfies M, a nontrivial idempotent Mal'cev condition, A has terms distinct from projections. Let f be a term of minimal arity such that f is not a projection.

First assume that f is at least quaternary. By the minimality of its arity, every term arising from f by identification of some variables is a projection. Hence the well-known Świerczkowski Lemma, cf. [7], applies and we conclude that f is a semiprojection.

Now we assume that f is at most ternary. Since we can always increase the arity by adding redundant variables, we may treat f as a ternary term. (But then f is not necessarily of minimal arity, so we cannot automatically say that $f(x,x,y)$ is a projection.) The arc $\{a_0^{(0)}, a_1^{(0)}\}$ is a subalgebra of $A^{(0)}$, so either $f(a_0^{(0)}, a_0^{(0)}, a_1^{(0)}) = a_0^{(0)}$ or $f(a_0^{(0)}, a_0^{(0)}, a_1^{(0)}) = a_1^{(0)}$.

Suppose first that $f(a_0^{(0)}, a_0^{(0)}, a_1^{(0)}) = a_0^{(0)}$. We claim that

$$f(a_j^{(i)}, a_j^{(i)}, a_{j+\ell}^{(i)}) = a_j^{(i)} \quad \text{for all } i \in \{0,1\} \text{ and } j,\ell \in \mathbf{Z}_n. \tag{2}$$

From $f(a_0^{(0)}, a_0^{(0)}, a_1^{(0)}) = a_0^{(0)}$ we obtain

$$\big(a_0^{(0)}, f(a_0^{(1)}, a_0^{(1)}, a_1^{(1)})\big) = f\big((a_0^{(0)}, a_0^{(1)}), (a_0^{(0)}, a_0^{(1)}), (a_1^{(0)}, a_1^{(1)})\big) \in A.$$

This and Fact 1 yield

$$f(a_0^{(1)}, a_0^{(1)}, a_1^{(1)}) \in \{a_0^{(0)}\}' \cap \{a_0^{(1)}, a_1^{(1)}\} = \{a_0^{(1)}\},$$

so $f(a_0^{(1)}, a_0^{(1)}, a_1^{(1)}) = a_0^{(1)}$. Then similarly,

$$\big(f(a_1^{(0)}, a_1^{(0)}, a_2^{(0)}), a_0^{(1)}\big) = f\big((a_1^{(0)}, a_0^{(1)}), (a_1^{(0)}, a_0^{(1)}), (a_2^{(0)}, a_1^{(1)})\big) \in A$$

together with Fact 1 yield $f(a_1^{(0)}, a_1^{(0)}, a_2^{(0)}) = a_1^{(0)}$. Continuing the calculation around the "circular saw" anticlockwise we obtain that

$$f(a_j^{(i)}, a_j^{(i)}, a_{j+1}^{(i)}) = a_j^{(i)} \quad \text{for every } i \in \{0,1\} \text{ and } j \in \mathbf{Z}_n.$$

So (2) holds for $\ell = 1$. If it holds for $1 \le \ell < n-1$ then

$$\big(f(a_j^{(0)}, a_j^{(0)}, a_{j+\ell+1}^{(0)}), a_j^{(1)}\big) = f\big((a_j^{(0)}, a_j^{(1)}), (a_j^{(0)}, a_j^{(1)}), (a_{j+\ell+1}^{(0)}, a_{j+\ell}^{(1)})\big) \in A$$

together with Fact 1 gives $f(a_j^{(0)}, a_j^{(0)}, a_{j+\ell+1}^{(0)}) = a_j^{(0)}$, and

$$\big(a_{j+1}^{(0)}, f(a_j^{(1)}, a_j^{(1)}, a_{j+\ell+1}^{(1)})\big) = f\big((a_{j+1}^{(0)}, a_j^{(1)}), (a_{j+1}^{(0)}, a_j^{(1)}), (a_{(j+1)+\ell}^{(0)}, a_{j+\ell+1}^{(1)})\big) \in A$$

together with Fact 1 gives $f(a_j^{(1)}, a_j^{(1)}, a_{j+\ell+1}^{(1)}) = a_j^{(1)}$. This shows (2) for $\ell + 1$. Finally, when $\ell = n$, (2) is evident since f is idempotent.

When $f(a_0^{(0)}, a_0^{(0)}, a_1^{(0)}) = a_1^{(0)}$, a similar calculation (but going clockwise around the circular saw) shows that

$$f(a_j^{(i)}, a_j^{(i)}, a_{j+\ell}^{(i)}) = a_{j+\ell}^{(i)} \quad \text{for all } i \in \{0,1\} \text{ and } j, \ell \in \mathbf{Z}_n. \tag{3}$$

Combining (2) and (3) we conclude

Fact 2. $f(x, x, y)$ is a projection on A.

Clearly, the same argument shows that $f(x, y, x)$ and $f(y, x, x)$ are projections, too. Depending on which of x or y the projections $f(x, x, y)$, $f(x, y, x)$ and $f(y, x, x)$ are, we have, up to permutation of variables, three possibilities.

The first possibility is when $f(x, x, y) = f(x, y, x) = f(y, x, x) = x$, i.e., f is a majority term. This case has been excluded by [1]. The second possibility is that $f(x, x, y) = f(y, x, x) = y$, i.e., f is a Mal'cev term. Then any two congruences must permute, so this case is excluded either. So we are left with the third possibility when $f(x, x, y) = f(x, y, x) = x$ and $f(y, x, x) = y$, i.e., f is a semiprojection.

Now we know that A has a semiprojection term, say a first semiprojection $f = f(x_1, \ldots, x_k)$, which is not a projection. To obtain a contradiction, we will show that f is a projection. It suffices to show, via induction on $|X|$, that whenever $X \subseteq A^{(i)}$ is an arc then

$$a_{j_1}^{(i)}, \ldots, a_{j_k}^{(i)} \in X \text{ implies } f(a_{j_1}^{(i)}, \ldots, a_{j_k}^{(i)}) = a_{j_1}^{(i)}. \tag{4}$$

This is clear when $|X| < k$. In the sequel we can assume that $|\{a_{j_1}^{(i)}, \ldots, a_{j_k}^{(i)}\}| = k$, and we also assume that X is an arc of minimal length such that $\{a_{j_1}^{(i)}, \ldots, a_{j_k}^{(i)}\} \subseteq X$.

Now let $\ell \in \{k, k+1, \ldots, n-1\}$ and suppose (4) is valid for all arcs shorter than ℓ. Consider an arc $X \subseteq A^{(0)}$ with length ℓ; the case $X \subseteq A^{(1)}$ would be similar. If $a_{j_1}^{(0)}$ is an inner point of X then our task is easy: there are elements $y_2, \ldots, y_k$ in X^* such that $(a_{j_2}^{(0)}, y_2), \ldots, (a_{j_k}^{(0)}, y_k) \in A$. Moreover, $a_{j_1-1}^{(1)}, a_{j_1}^{(1)} \in X^*$ and $(a_{j_1}^{(0)}, a_{j_1-1}^{(1)}), (a_{j_1}^{(0)}, a_{j_1}^{(1)}) \in A$. Since $|X^*| = |X| - 1$, the induction hypothesis gives

$$\big(f(a_{j_1}^{(0)}, \ldots, a_{j_k}^{(0)}), a_{j_1}^{(1)}\big) = f\big((a_{j_1}^{(0)}, a_{j_1}^{(1)}), (a_{j_2}^{(0)}, y_2), \ldots, (a_{j_k}^{(0)}, y_k)\big) \in A \tag{5}$$

and

$$\big(f(a_{j_1}^{(0)}, \ldots, a_{j_k}^{(0)}), a_{j_1-1}^{(1)}\big) = f\big((a_{j_1}^{(0)}, a_{j_1-1}^{(1)}), (a_{j_2}^{(0)}, y_2), \ldots, (a_{j_k}^{(0)}, y_k)\big) \in A. \tag{6}$$

These two formulas easily imply $f(a_{j_1}^{(0)}, \ldots, a_{j_k}^{(0)}) = a_{j_1}^{(0)}$. The same argument works when $|X| = n$ but instead of X^* we have to take an arbitrary arc $Y \subseteq A^{(1)}$ with $|Y| = n - 1$ and $a_{j_1-1}^{(1)}, a_{j_1}^{(1)} \in Y$.

Now, again for $\ell \in \{k, k+1, \ldots, n-1\}$, we consider the case when $a_{j_1}^{(0)}$ is one of the endpoints of X, so we assume that

$$X = \{a_{j_1}^{(0)}, a_{j_1+1}^{(0)}, \ldots, a_{j_1+\ell-1}^{(0)}\}.$$

If $a^{(0)}_{j_1}$ is a "lonely endpoint" in the sense that $a^{(0)}_{j_1+1} \notin \{a^{(0)}_{j_2}, \dots, a^{(0)}_{j_k}\}$ then $\{a^{(1)}_{j_1}\}' \cap \{a^{(0)}_{j_1}, \dots, a^{(0)}_{j_k}\} = \{a^{(0)}_{j_1}\}$ and (5) yields $f(a^{(0)}_{j_1}, \dots, a^{(0)}_{j_k}) = a^{(0)}_{j_1}$. Otherwise, when $a^{(0)}_{j_1}$ is not lonely, for $u = 2, \dots, k$ we can choose a

$$y_u \in \{a^{(1)}_{j_1+1}, \dots, a^{(1)}_{j_1+\ell-2}\}$$

such that $(a_{j_u}, y_u) \in A$. Then $a^{(1)}_{j_1-1}$ is a lonely endpoint of the arc

$$\{a^{(1)}_{j_1-1}, a^{(1)}_{j_1}, a^{(1)}_{j_1+1}, \dots, a^{(1)}_{j_1+\ell-2}\}.$$

Hence the previously considered case makes formula (6) valid, and $f(a^{(0)}_{j_1}, \dots, a^{(0)}_{j_k}) = a^{(0)}_{j_1}$ follows easily.

We have seen that f is the first projection on $A^{(0)}$ and $A^{(1)}$, hence it is the first projection on A. This contradiction proves the Theorem.

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