Algebra Universalis, 30 (1993) 291–293 0002-5240/93/020291-03$01.50 + 0.20/0

Mailbox

How are diamond identities implied in congruence varieties?

GÁBOR CZÉDLI*

For a set Σ of lattice identities and a lattice identity λ, Σ is said to imply λ in congruence varieties, in notation $\Sigma \models_c \lambda$, if every congruence variety which satisfies all members of Σ also satisfies λ (cf. Jónsson [11]). In this note we prove three theorems on $\models_c$, including the following compactness result.

THEOREM 3. *If $\Sigma \models_c \lambda$ and λ is a diamond identity (to be defined later) then there exists a finite subset Σ' of Σ such that $\Sigma' \models_c \lambda$.*

For the special case where λ is the modular or distributive law our theorems have already been proved; cf. [1], [3] and the very deep Day and Freese [4] and Freese, Herrmann and Huhn [6, Cor. 14]. It is the results and/or methods of these papers, [10] and Day and Kiss [5] that makes our approach possible and relatively simple.

For $n \geq 2$, an n-diamond in a modular lattice L is defined to be an $(n+1)$-tuple $\vec{a} = (a_0, a_1, \ldots, a_n) \in L^{n+1}$ satisfying $\sum_{i \neq j}^{0,n} a_i = 1_{\vec{a}}$ and $a_l \sum_{i \neq k,l}^{0,n} a_i = 0_{\vec{a}}$ for all j and all $k \neq l$ where $1_{\vec{a}} = \sum_i^{0,n} a_i$ and $0_{\vec{a}} = \prod_i^{0,n} a_i$. This concept is due to András Huhn [9], [8] but occurs under several names in the literature (cf., e.g., Day and Kiss [5]). Let $\lambda : p(x_1, \ldots, x_t) = q(x_1, \ldots, x_t)$ be a lattice identity. We call λ a diamond identity if λ implies modularity and, in addition, there are $(n+1)$-ary lattice terms $c_1(y_0, y_1, \ldots, y_n), \ldots, c_t(y_0, y_1, \ldots, y_n)$ for some $n \geq 2$ such that for an arbitrary modular lattice L if $p(c_1(\vec{a}), \ldots, c_t(\vec{a})) = q(c_1(\vec{a}), \ldots, c_t(\vec{a}))$ for every n-diamond $\vec{a}$ in L then λ holds in L.

The conjunction of the modular law with any of the identities in Herrmann and Huhn [8] or Freese and McKenzie [7, XIII] is an interesting example for diamond identities. For further examples cf. [2].

Presented by R. Freese.
Received August 23, 1989; accepted in final form September 9, 1991.
*Research partially supported by Hungarian National Foundation for Scientific Research grant no. 1813.

For a variety $\mathscr{V}$ let $\mathbf{Con}(\mathscr{V}) = \mathbf{HSP}\{\operatorname{Con} A : A \in \mathscr{V}\}$ denote the congruence variety of $\mathscr{V}$. Given a ring R with 1, let R-**Mod** stand for the variety of (unitary left) R-modules. For integers $m \geq 0$ and $n \geq 1$ the ring sentence $(\exists r)\,(m \cdot r = n \cdot 1)$, denoted by $D(m, n)$, is called a divisibility condition. (Here $k \cdot x = x + x + \cdots + x$, k times.) In [10] an algorithm is described which associates two integers $m_\lambda \geq 0$ and $n_\lambda \geq 1$ with any given lattice identity λ such that

(1) for any ring R, λ holds in $\mathbf{Con}(R\text{-}\mathbf{Mod})$ iff $D(m_\lambda, n_\lambda)$ holds in R

(cf. [10, Theorems 2 and 3]). For an integer k and a prime p let $\operatorname{expt}(k, p)$ denote the largest integer $i \geq 0$ for which $p^i \mid k$; by $\operatorname{expt}(0, p)$ we mean that smallest infinite ordinal. By [10, Prop. 1]

(2) $D(m, n)$ holds in a ring R iff for any prime p with $\operatorname{expt}(m, p) > \operatorname{expt}(n, p)$ R satisfies $D(p^{\operatorname{expt}(n,p)+1}, p^{\operatorname{expt}(n,p)})$ and, in addition, $m = 0$ implies that the characteristic of R is 0. In case the characteristic of R is $k > 0$ then $D(m, n)$ holds in R iff $(k, m) \mid n$.

Let $\mathscr{V}(0) = \mathbf{Con}(\mathbf{Q}\text{-}\mathbf{Mod})$, i.e., the lattice variety generated by the rational projective geometries. For $k > 0$ let $\mathscr{V}(k) = \mathbf{Con}(\mathbf{Z}_k\text{-}\mathbf{Mod})$ where $\mathbf{Z}_k$ is the factor ring of integers modulo k. For technical reasons, in connection with (2), we define $K(m, n)$ as $\{p^{i+1} : p \text{ prime}, \ i = \operatorname{expt}(n, p) < \operatorname{expt}(m, p)\} \cup \{i : i = 0 = m\}$. Note that $\{i : i = 0 = m\}$ is $\{0\}$ or $\emptyset$. We have

THEOREM 1. *If a diamond identity λ does not hold in a modular congruence variety $\mathscr{U}$ then there is a $k \in K(m_\lambda, n_\lambda)$ such that $\mathscr{V}(k) \subseteq \mathscr{U}$.*

THEOREM 2. *Let λ be a diamond identity and Σ be a set of lattice identities. Then $\Sigma \models_c \lambda$ if and only if*

(i) *$\Sigma \models_c$ modularity,*

(ii) *$\{0\} \cap \{m_\lambda\} \subseteq \{m_\varepsilon : \varepsilon \in \Sigma\}$, and*

(iii) *for any prime p if $\operatorname{expt}(m_\lambda, p) > \operatorname{expt}(n_\lambda, p)$ then $\operatorname{expt}(n_\lambda, p) \geq \operatorname{exp}(n_\varepsilon, p) < \operatorname{expt}(m_\varepsilon, p)$ holds for some $\varepsilon \in \Sigma$.*

If Σ is finite then (i) *is decidable (cf. Day and Freese* [4] *and* [3]) *and Theorem* 2 *offers an algorithm to check whether $\Sigma \models_c \lambda$.*

Proof of Theorem 1. Let $\mathscr{U} = \mathbf{Con}(\mathscr{V})$ be a modular congruence variety in which λ fails. There is an n-diamond $\vec{a}$ in the congruence lattice $\operatorname{Con} A$ of some algebra A in $\mathscr{V}$ such that λ fails in the interval $L = [0_{\vec{a}}, 1_{\vec{a}}]$ of $\operatorname{Con} A$. We can assume that $0_{\vec{a}} = 0_{\operatorname{Con} A}$ as otherwise A could be replaced by $A/0_{\vec{a}}$. By Lemma 3.1 in Day and Kiss [5], $1_L = 1_{\vec{a}}$ is an Abelian congruence of A. Therefore Lemma 7.1 and Theorem 7.2 of Day and Kiss [5] yield the existence of a ring S such that

$L \in \mathbf{Con}\,(S\text{-}\mathbf{Mod}) \subseteq \mathbf{Con}\,(\mathscr{V}) = \mathscr{U}$. This $\mathbf{Con}\,(S\text{-}\mathbf{Mod})$ fails λ. Now a routine calculation based on (1), (2) and the description of the inclusion relation amongst all $\mathbf{Con}\,(R\text{-}\mathbf{Mod})$ (cf. [10, Theorem 5]) completes the proof of Theorem 1.

Proof of Theorem 2. Assume that $\Sigma \models_c \lambda$. Then (i) is obvious. If (ii) or (iii) failed then, by (1) and (2), Σ would hold but λ would fail in $\mathscr{V}(k)$ for some $k \in K(m_\lambda, n_\lambda)$. Conversely, assume that in spite of (i), (ii) and (iii) $\Sigma \not\models_c \lambda$. Then Σ holds but λ fails in some modular congruence variety $\mathscr{U}$. By Theorem 1, there is a $k \in K(m_\lambda, n_\lambda)$ such that λ fails in $\mathscr{V}(k)$. But Σ holds in $\mathscr{V}(k) \subseteq \mathscr{U}$, which is a contradiction by (1) and (2).

Proof of Theorem 3. Assume that $\Sigma \models_c \lambda$. By a deep result of Day and Freese [4, Thm. 6.4] there is a $\kappa \in \Sigma$ such that $\kappa \models_c$ modularity. If $m_\lambda = 0$ then, by Theorem 2, there is an $\eta \in \Sigma$ with $m_\eta = 0$. This η can serve (iii) for all primes not dividing n_η. Hence there is a finite set Σ_1 such that $\eta \in \Sigma_1 \subseteq \Sigma$ and (iii) is fulfilled by Σ_1. Clearly, $\{\kappa\} \cup \Sigma_1 \models_c \lambda$. If $m_\lambda \neq 0$ then (iii) requires the existence of an $\varepsilon = \varepsilon_p$ for finitely many p only. These ε_p constitute a finite set Σ_2 and $\{\kappa\} \cup \Sigma_2 \models_c \lambda$.

REMARK. A lengthier proof shows that our results would remain true if $\models_c$ were understood as implication in $\{\mathrm{Con}\,A : A \in \mathscr{V}\}$ classes where $\mathscr{V}$ is a class of algebras closed under finite subdirect powers.

REFERENCES

[1] CZÉDLI, G., *A note on the compactness of the consequence relation for congruence varieties*, Algebra Universalis *15* (1982), 142–143.
[2] CZÉDLI, G., *Some nontrivial implications in congruence varieties*, Acta Sci. Math. (Szeged) *56* (1992), 15–18.
[3] CZÉDLI, G. and FREESE, R., *On congruence distributivity and modularity*, Algebra Universalis *17* (1983), 216–219.
[4] DAY, A. and FREESE, R., *A characterization of identities implying congruence modularity I*, Canadian J. Math. *32* (1980), 1140–1167.
[5] DAY, A. and KISS, E. W., *Frames and rings in congruence modular varieties*, J. Algebra *109* (1987), 479–507.
[6] FREESE, R., HERRMANN, C. and HUHN, A. P., *On some identities valid in modular congruence varieties*, Algebra Universalis *12* (1982), 322–334.
[7] FREESE, R. and MCKENZIE, R., *Commutator theory for congruence modular varieties*, London Math. Soc. Lecture Notes Series *125*, Cambridge University Press, Cambridge, New York, New Rochelle, Melbourne, Sidney, 1987.
[8] HERRMANN, C. and HUHN, A. P., *Zum Begriff der Charakteristik modularer Verbände*, Math. Zeitschrift *144* (1975), 188–194.
[9] HUHN, A. P., *Schwach distributive Verbände I*, Acta Sci. Math. Szeged *33* (1972), 297–305.
[10] HUTCHINSON, G. and CZÉDLI, G., *A test for identities satisfied in lattices of submodules*, Algebra Univeralis *8* (1978), 269–309.
[11] JÓNSSON, B., *Congruence varieties*, Algebra Universalis *10* (1980), 355–394.

Bolyai Institute
Szeged
Hungary