GEOMETRIC CONSTRUCTIBILITY OF THALESIAN POLYGONS

GÁBOR CZÉDLI

Abstract. A *cyclic polygon* is a convex n-gon inscribed in a circle. If, in addition, one of its sides is a diameter of the circle, then the polygon will be called *Thalesian*. Up to permutation, a Thalesian n-gon is determined by the *lengths* of its non-diametric sides. It is also determined by the *distances* of its non-diametric sides from the center of its circumscribed circle. We prove that the Thalesian n-gon in general can be constructed with straightedge and compass neither from these lengths if $n \geq 4$, nor from these distances if $n \geq 5$.

An analogous statement for the constructibility of cyclic n-gons from the side lengths was found by P. Schreiber in 1993; his statement was first proved by the present author and Á. Kunos in 2015. The 2015 paper could only prove the non-constructibility of cyclic n-gons from the distances for n even; here we extend this result for all $n \geq 5$.

1. Introduction and results

A *cyclic polygon* is a convex n-gon inscribed in a circle. *Constructibility* is always understood as the classical geometric constructibility with straightedge and compass. The concept of *constructibility in general* is not as obvious as one may think and the meaning "constructible for all (meaningful) data" does not lead to the usual algebraic characterization. For example, consider the task when we are given $0, 1, a, b \in \mathbb{R}$ as points on the real line such that $0 < a < b$ and we want to construct the unique $c \in \mathbb{R}$ defined by

$$c := \begin{cases} b + q, & \text{if } b/a = p/q \text{ for relatively prime } p, q \in \mathbb{N}, \\ a + b, & \text{otherwise.} \end{cases}$$

A rigorous definition of this concept is given in [2] and it is outlined in the introduction of [3]. However, in the present paper, the reader may safely assume that "constructible in general" means "*constructible for all meaningful data*". Here "meaningful" refers to the property that the cyclic polygon exists; we will see that in this case the polygon is determined by the data we consider up to permutation of its edges. Motivated by Thales' theorem, by a *Thalesian polygon* we mean a cyclic polygon with exactly one of its sides being a diameter of the circumscribed circle; see Figure 1 for an illustration.

Up to permutation, a cyclic n-gon is determined by its *side lengths*, that is, by n data. These data are not quite arbitrary. We know from Schreiber [4] that for

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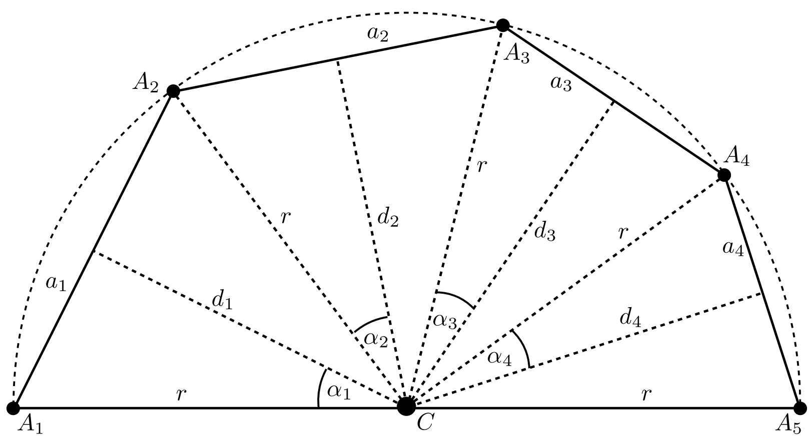

FIGURE 1. A Thalesian pentagon

positive real numbers $a_1, \dots, a_n$, there exists a (non-degenerate) cyclic n-gon with these side lengths iff each of these n numbers is less than the sum of the rest. By the *central distances* of a cyclic n-gon we mean the distances of its sides from the center C of its circumscribed circle. If C is outside the n-gon, then one of these distances is negative. An n-tuple $\vec{d} = \langle d_1, \dots, d_n \rangle$ is *representable by central distances* if there exists a cyclic n-gon whose system of central distances is $\vec{d}$. As opposed to side lengths, we have no characterization of the representable tuples $\vec{d}$. However, geometric intuition makes it clear that $\vec{d}$ is surely representable if all the ratios d_i/d_j are sufficiently close to 1; we will rely on this fact implicitly. The following statement, for all $n \geq 5$, was observed by Schreiber [4]; he proved it for $n = 5$; the first complete proof for $n > 5$ is due to [3].

(1.1) If $n \geq 5$, then the cyclic n-gon is in general not constructible from its side lengths.

Also, [3] contains the following statement:

(1.2) If $n \geq 5$ is an even number, then the cyclic n-gon is in general not constructible from its central distances.

As a by-product of our approach here, the restricting stipulation "even" will be removed from (1.2). For a *Thalesian* n-gon, we only consider $n-1$ data: either the $(n-1)$-tuple $\langle a_1, \dots, a_{n-1} \rangle$ of the *side lengths* of its non-diametric sides, or the $(n-1)$-tuple $\langle d_1, \dots, d_{n-1} \rangle$ of the *central distances* of its non-diametric sides. For example, the task for the Thalesian pentagon in Figure 1 is either to construct it from $\langle a_1, \dots, a_4 \rangle$, or to construct it from $\langle d_1, \dots, d_4 \rangle$. Our goal is to prove the following theorem, which implies that the Thalesian n-gon is in general non-constructible from its side lengths for $n \geq 4$, and the same holds for its central distances if $n \geq 5$.

Theorem 1.1. *Let $n \geq 5$ be a natural number.*

(i) *There exists an $(n-1)$-tuple $\vec{a} = \langle a_1, \dots, a_{n-1} \rangle$ of positive real numbers such that $|\{a_1, \dots, a_{n-1}\}| \leq 2$, there is a Thalesian n-gon with these side lengths, but this Thalesian n-gon is not constructible from $a_1, \dots, a_{n-1}$.*

(ii) *There exists an* $(n-1)$*-tuple* $\vec{d} = \langle d_1, \dots, d_{n-1}\rangle$ *of positive real numbers such these numbers as central distances determine a Thalesian* n*-gon, this Thalesian* n*-gon is not constructible from* $d_1, \dots, d_{n-1}$*, and* $|\{d_1, \dots, d_{n-1}\}| \le 2$.

(iii) *In* (i) *and* (ii)*, we can also stipulate that the components of* $\vec{a}$ *and those of* $\vec{d}$ *are* positive integers.

(iv) *The Thalesian triangle is in general constructible from its side lengths and also from its central distances. The Thalesian quadrangle is in general constructible from its central distances.*

(v) *The Thalesian quadrangle with side lengths* $\langle 1, 2, 3\rangle$ *is not constructible from its side lengths.*

Note that a Thalesian triangle is simply a right-angled triangle. For many n, we can even say $|\{a_1, \dots, a_{n-1}\}| = 1$ and $|\{d_1, \dots, d_{n-1}\}| = 1$ in parts (i) and (ii) above. To see this, we call a Thalesian polygon *regular* if all its non-diametric sides are of the same length. In the constructibility problem of regular (Thalesian) polygons, the value of a in $\vec{a} = \langle a, \dots, a\rangle$ and that of d in $\vec{d} = \langle d, \dots, d\rangle$ are irrelevant. By reflecting a regular Thalesian n-gon across its diametric side, we obtain a regular $2(n-1)$-gon. The latter is constructible iff the regular $(n-1)$-gon is constructible; these $n-1$ are characterized by the well-known Gauss–Wantzel theorem; see [5]. Therefore, the following statement is trivial.

Remark 1.2. The regular Thalesian triangle is, of course, constructible. For $n \ge 4$, the regular Thalesian n-gon is constructible if and only if the regular $(n-1)$-gon is constructible.

In a sense, if $d_n = 0$ is allowed, Theorem 1.1(ii) extends the validity of (1.2) from "even" to "all". The following corollary does the same with *positive* central distances.

Corollary 1.3. *For every* $n \ge 5$ *or* $n = 3$*, there are* positive *integers numbers* $d_1, \dots, d_n$ *such that they determine a cyclic* n*-gon with central distances* $\vec{d} = \langle d_1, \dots, d_n\rangle$*, this* n*-gon is not constructible from* $\vec{d}$*, and* $|\{d_1, \dots, d_n\}| \le 3$*. However, the cyclic quadrangle is constructible from* $\vec{d}$*.*

For $n \in \{3, 4\}$, this corollary was proved in [2]; see also [3, Proposition 1.3] where this is cited. The rest of the corollary follows from Theorem 1.1, the Limit Theorem [3, Theorem 9.1], and the Rational Parameter Theorem [3, Theorem 11.1]; we will not recall these advanced tools and the straightforward details are left to the reader.

With the technique presented in this paper, one can disregard Thalesian polygons and give a direct proof that improves $|\{d_1, \dots, d_n\}| \le 3$ to $|\{d_1, \dots, d_n\}| \le 2$, provided $n \ge 5$; the details are again omitted. For small n, note the following difference between the cyclic and the Thalesian case. For all $n \in \{3, 4\}$, a cyclic n-gon is in general constructible from its side lengths, but this is not true for $n = 3$ with central distances. On the other hand, for all $n \in \{3, 4\}$, a Thalesian n-gon is in general constructible from its central distances, but this is not true for $n = 4$ with side lengths.

Besides cyclic, that is, inscribed polygons, we can consider *circumscribed* polygons. However, their study is easily reduced to that of cyclic polygons. The following corollary of Theorem 1.1 follows obviously with the help of [3, Remark 1.4]; no further detail will be given here.

Corollary 1.4. *For $n \geq 3$, a circumscribed n-gon is in general constructible from the distances of its vertices from the center of the inscribed circle if and only if $n = 4$.*

2. Proofs and auxiliary statements

The greatest common divisor will be denoted and abbreviated by gcd. A polynomial is *primitive* if the gcd of its coefficients is 1. The following well-known statement is due to C. F. Gauss; we cite parts (i) and (iii) from [1, Thm. 2.16 (page 90) and Prop. 7.24 (page 260)]; (ii) follows from (iii).

Lemma 2.1. *If R is a unique factorization domain with field of fractions F, then*

(i) *the polynomial ring $R[x]$ is also a unique factorization domain,*
(ii) *if a polynomial is irreducible in $R[x]$, then it is also irreducible in $F[x]$, and*
(iii) *a primitive polynomial is irreducible in $R[x]$ iff it is irreducible in $F[x]$.*

For the ring $\mathbb{Z}$ of integers, the field of fractions of $\mathbb{Z}[x]$ is $\mathbb{Q}(x)$, the field of rational functions over $\mathbb{Q}$; note that $\mathbb{Q}(x)$ is a transcendental extension of the field $\mathbb{Q}$ of rational numbers. We know that the polynomial rings $\mathbb{Z}[x, y]$, $\mathbb{Z}[x][y]$, and $\mathbb{Z}[y][x]$ are isomorphic. This fact allows us to write $f_x(y)$ and $f_y(x)$ instead of $f(x, y) \in \mathbb{Z}[x, y]$. That is, $f_x(y)$, $f_y(x)$, and $f(x, y)$ are essentially the same polynomials but we put an emphasis on $f_x(y) \in \mathbb{Z}[x][y] \subseteq \mathbb{Q}(x)[y]$ and $f_y(x) \in \mathbb{Z}[y][x] \subseteq \mathbb{Q}(y)[x]$. Therefore, the following convention applies in the paper:

(2.1) no matter which of $f(x, y) \in \mathbb{Z}[x, y]$, $f_x(y) \in \mathbb{Z}[x][y]$, and $f_y(x) \in \mathbb{Z}[y][x]$ is given first, we can also use the other two.

The first part of the following lemma will play an important role in the paper; the second part will not be used. The *degree* of a polynomial $g(x)$ will be denoted by $\deg(g(x)) = \deg_x(g(x))$

Lemma 2.2. *Assume that $g_1(x)$ and $g_2(x)$ are non-zero polynomials in $\mathbb{Z}[x]$ such that $\gcd\big(g_1(x), g_2(x)\big) = 1$. Let $f(x, y) = g_1(x)y + g_2(x)$. Then $f_y(x)$ is an irreducible polynomial in $\mathbb{Z}[y][x]$ and, consequently, also in $\mathbb{Q}(y)[x]$. Furthermore, $f_y(x)$ is a primitive polynomial in $\mathbb{Z}(y)[x]$.*

Proof. . Assume that

$$u_y(x), v_y(x) \in \mathbb{Z}[y][x] \text{ such that } f_y(x) = u_y(x) \cdot v_y(x). \tag{2.2}$$

Then $g_1(x)y + g_2(x) = f_x(y) = u_x(y) \cdot v_x(y)$. Since $f_x(y) \in \mathbb{Z}[x][y]$ is of degree 1, one of $u_x(y) \in \mathbb{Z}[x][y]$ and $v_x(y) \in \mathbb{Z}[x][y]$ is of degree 0. Let, say, $u_x(y)$ be of degree 0; then $v_x(y) \in \mathbb{Z}[x][y]$ is of degree 1. Hence, there are $u(x), v_1(x), v_2(x) \in \mathbb{Z}[x]$ such that $u_x(y) = u(x)$ and $v_x(y) = v_1(x)y + v_2(x)$. Comparing the coefficients of y in the factorization

$$g_1(x)y + g_2(x) = u_x(y) \cdot v_x(y) = u(x) \cdot \big(v_1(x)y + v_2(x)\big),$$

we obtain that $u(x) \mid g_1(x)$ and, consequently, $u(x) \mid g_2(x)$. Hence, using the assumption $\gcd(g_1(x), g_2(x)) = 1$, we conclude that $u(x) \in \{-1, 1\}$. Thus, $f_y(x)$ is irreducible in $\mathbb{Z}[y][x]$. By Lemma 2.1(ii), it is also irreducible in $\mathbb{Q}(y)[x]$.

Finally, since (2.2) also implies $u(x) \in \{-1, 1\}$ in the particular case when $u(x)$ is an element of $\mathbb{Z}$, it follows that $f_y(x)$ is a primitive polynomial in $\mathbb{Z}[y][x]$. $\square$

The following statement is well-known and often taught for students; see [2, Theorem V.3.6]; see also the list of references right before [3, Proposition 3.1]. As usual, the field extension of $\mathbb{Q}$ by a transcendental number c is denoted by $\mathbb{Q}(c)$; it is isomorphic to the field $\mathbb{Q}(x)$ of fractions of $\mathbb{Z}[x]$.

Proposition 2.3. *Let c be a real transcendental number and let $u \in \mathbb{R}$. If there exists an irreducible polynomial $h(x) \in \mathbb{Q}(c)[x]$ such that $h(u) = 0$ and $\deg_x(h(x))$ is not a power of 2, then u is not constructible from $\mathbb{Q} \cup \{c\}$.*

For all $k \in \mathbb{N} := \{1, 2, \dots\}$ and $\gamma \in \mathbb{R}$, the following two identities are well-known, see, for example, [3, Section 3]; $2 \mid j = 0$ will mean that j is even and runs from 0 while $2 \not| \, j = 1$ refers to indices running through odd numbers.

$$\cos(k\gamma) = \sum_{2|j=0}^{k} (-1)^{j/2} \binom{k}{j} (\cos\gamma)^{k-j} \cdot (\sin\gamma)^j \tag{2.3}$$

$$\sin(k\gamma) = \sum_{2 \not| j=1}^{k} (-1)^{(j-1)/2} \binom{k}{j} (\cos\gamma)^{k-j} \cdot (\sin\gamma)^j. \tag{2.4}$$

Proof of Theorem 1.1. For $n \geq 5$, there are several cases depending on n and, furthermore, (i) and (ii) need different arguments. To avoid unnecessary repetitions, we give the common parts of these arguments. We choose the data from one of the following two possibilities where $c \in \mathbb{R}$ is a transcendental number:

$$\langle 1, \dots, 1, \sqrt{c}\, \rangle, \quad \text{with } k := n-2 \text{ units and a single } \sqrt{c}, \text{ or} \tag{2.5}$$

$$\langle 1, \dots 1, \sqrt{c}, \sqrt{c}\, \rangle, \quad \text{with } k := n-3 \text{ units and two copies of } \sqrt{c}. \tag{2.6}$$

The $(n-1)$-tuples above are the system $\langle a_1, \dots, a_{n-1} \rangle$ of side lengths or the system $\langle d_1, \dots, d_{n-1} \rangle$ of central distances. To ensure that the Thalesian n-gon exists, c is assumed to be sufficiently close to 1. Note that $\sqrt{c}$ is transcendental iff so is c. Since $\sqrt{c}$ and c are mutually constructible from each other, we can assume that c rather than $\sqrt{c}$ is given. That is, 1 and c are the input data. Let α and β denote the central half angles corresponding to the first k edges and to the rest of non-diametric edges, respectively. After modifying Figure 1 so that $\alpha = \alpha_1 = \dots = \alpha_3$ and $\beta = \alpha_4$, we could visualize (2.5) for $n = 5$. Let r denote the radius of the circumscribed circle. For (i), where the side lengths are given, we let $u := 1/(2r)$; then we have

$$\sin\alpha = u, \quad \cos\alpha = \sqrt{1-u^2}, \quad \sin\beta = u\sqrt{c}, \quad \cos\beta = \sqrt{1-cu^2}. \tag{2.7}$$

Similarly, for (ii), where the central distances are given, we let $u := 1/r$; then we have

$$\cos\alpha = u, \quad \sin\alpha = \sqrt{1-u^2}, \quad \cos\beta = u\sqrt{c}, \quad \sin\beta = \sqrt{1-cu^2}. \tag{2.8}$$

Note that the constructibility of u is equivalent to that of the Thalesian n-gon.

If (2.5), then $k\alpha + \beta = \pi/2$. Hence, $\cos(k\alpha) = \sin\beta$, $\sin(k\alpha) = \cos\beta$,

$$\cos^2(k\alpha) = \sin^2\beta, \quad \text{and} \quad \sin^2(k\alpha) = \cos^2\beta \tag{2.9}$$

hold. Similarly, if (2.6), then $k\alpha + 2\beta = \pi/2$, $\cos(2\beta) = 1 - 2\sin^2\beta$, so

$$\sin(k\alpha) = 1 - 2\sin^2\beta. \tag{2.10}$$

As it will be clear soon, (2.3)–(2.10) offer several ways to find a polynomial $f^{(0)}(x, y)$ in $\mathbb{Z}[x, y]$, that is $f^{(0)}_y(x) \in \mathbb{Z}[y][x]$, such that u is a root of $f^{(0)}_c(x) \in \mathbb{Q}(c)[x]$.

Sometimes, we simplify $f^{(0)}(x,y)$ to another polynomial $f(x,y)$ with the same properties; otherwise, we let $f(x,y) := f^{(0)}(x,y)$.

Case 1 ($n > 5$ is even and the side lengths are given)**.** We go after (2.4), (2.6), (2.7), and (2.10) to obtain that

$$\sum_{2\nmid j=1}^{k} (-1)^{(j-1)/2}\binom{k}{j}\Big(\sqrt{1-u^2}\Big)^{k-j}\cdot u^j = 1-2cu^2.$$

So we let

$$f(x,y) := 2x^2y + \Big(-1 + \sum_{2\nmid j=1}^{k} (-1)^{(j-1)/2}\binom{k}{j}(1-x^2)^{(k-j)/2}\cdot x^j\Big). \tag{2.11}$$

Since $k := n-3$ is odd, $f_y(x) = f(x,y)$ is a polynomial. Clearly, u is a root of $f_c(x)$. The coefficient of x^k above is

$$\sum_{2\nmid j=1}^{k} (-1)^{(j-1)/2}\binom{k}{j}(-1)^{(k-j)/2} = (-1)^{(k-1)/2}\sum_{2\nmid j=1}^{k}\binom{k}{j} = \pm 2^{k-1}, \tag{2.12}$$

and we obtain that $\deg_x(f_y(x)) = k$. Since k is odd and $k = n-3 \geq 6-3 = 3$, k is not a power of 2. In (2.11), every summand after -1 is divisible by x. Hence, $2x^2$ and the parenthesized polynomial in (2.11) are relatively prime in $\mathbb{Z}[x]$. Thus, Lemma 2.2 yields that $f_y(x)$ is irreducible in $\mathbb{Q}(y)[x]$. Since $c \in \mathbb{R}$ is transcendental, we have that $\mathbb{Q}(y)[x] \cong \mathbb{Q}(c)[x]$. Therefore, $f_c(x)$ is irreducible in $\mathbb{Q}(c)[x]$ and $\deg_x(f_c(x)) = \deg_x(f_y(x)) = k$. It follows from Proposition 2.3 that u and, consequently, the Thalesian n-gon are not constructible from the data.

Remark 2.4. The argument above makes it clear that, besides referencing earlier parts of the paper, only the verification of the following two properties of $f(x,y)$ required some work.

- We had to show that $f_y(x)$ is irreducible in $\mathbb{Q}(y)[x]$. By Lemma 2.1(ii), it would have sufficed to show that $f_y(x)$ is irreducible in $\mathbb{Z}[y][x]$.
- We had to show that $\deg_x(f_y(x))$ is not a power of 2.

Therefore, in the rest of the proof, we only concentrate on these two properties.

Case 2 ($n \geq 5$ is odd and the side lengths are given)**.** We go after (2.3), (2.5), (2.7), and the first equation of (2.9) to obtain $f(x,y) := f^{(0)}(x,y)$ as follows:

$$f(x,y) = x^2y - (1-x^2)\Big(\sum_{2|j=0}^{k} (-1)^{j/2}\binom{k}{j}(1-x^2)^{(k-1-j)/2}\cdot x^j\Big)^2.$$

Since k is odd, the subtrahend is a polynomial. Since

$$\sum_{2|j=0}^{k} (-1)^{j/2}\binom{k}{j}(-1)^{(k-1-j)/2} = (-1)^{(k-1)/2}\sum_{2|j=0}^{k}\binom{k}{j} = \pm 2^{k-1}, \tag{2.13}$$

we square a polynomial of degree $k-1$. Hence, the degree of the subtrahend and $\deg_x(f_y(x))$ are $2k$, which is not a power of 2, because k is odd and $k = n-2 \geq 3$. Since the constant in the subtrahend is 1, the subtrahend and x^2 are relatively prime polynomials. Thus, $f_y(x)$ is irreducible in $\mathbb{Q}(y)[x]$ by Lemma 2.2, and the Thalesian n-gon is non-constructible by Remark 2.4.

Case 3 ($n \geq 5$ is odd and the central distances are given)**.** Going after (2.3), (2.5), (2.8), and the first equation of (2.9), we obtain $f(x,y) = f^{(0)}(x,y)$ as follows:

$$f(x,y) = \Big(\sum_{2|j=0}^{k} (-1)^{j/2} \binom{k}{j} x^{k-j} \cdot (1-x^2)^{j/2} \Big)^2 - (1 - yx^2). \tag{2.14}$$

Using a variant of (2.13), we obtain that the minuend is of degree $2k$. Hence, $\deg_x(f_y(x)) = 2k$, which is not a power of 2 since neither is $k = n-2 \geq 3$, which is odd. The sum is divisible by x. Hence, the minuend minus 1 is relatively prime to x^2. Thus, Lemma 2.2 implies that $f_y(x)$ is irreducible in $\mathbb{Q}(y)[x]$. Therefore, the Thalesian n-gon is non-constructible by Remark 2.4.

Case 4 ($n > 5$ is even and the central distances are given)**.** We go after (2.4), (2.5), (2.8), and the second equation of (2.9). So we obtain

$$f^{(0)}(x,y) = x^2y - (1-x^2)\Big(\sum_{2\not| j=1}^{k} (-1)^{(j-1)/2} \binom{k}{j} x^{k-j} \cdot (1-x^2)^{(j-1)/2} \Big)^2.$$

Using a variant of (2.12), we obtain that the big sum is of degree $k-1$ in $\mathbb{Z}[x]$. Hence, $\deg_x(f_y^{(0)}(x)) = 2(k-1)+2 = 2k$. Since $k = n-2$ is even and the subscript j is odd, the big sum is divisible by x and the subtrahend by x^2. Divide the subtrahend by x^2 and let $g_2(x) \in \mathbb{Z}[x]$ denote the polynomial we obtain in this way. Then $f(x,y) := f^{(0)}(x,y)/x^2$ is of the form $1 \cdot y - g_2(x)$. This polynomial is irreducible in $\mathbb{Q}(y)[x]$ by Lemma 2.2. Its degree is $\deg_x(f_y(x)) = \deg_x(f_y^{(0)}(x)) - 2 = 2(k-1)$, which is not a power of 2 since $k-1 = n-3$ is odd and $k-1 \geq 2$. We know that u is a root of $f_c^{(0)}(x)$. Since u is distinct from zero, it is also a root of $f_c(x)$. Thus, Remark 2.4 yields that the Thalesian n-gon is non-constructible.

Case 5 ($n = 4$ and the side lengths are given)**.** Let a, b, and c be the given side lengths. The corresponding central angles are denoted by α, β, and γ. Let $u = 1/(2r)$. Since $\sin(\alpha/2) = au$, we have $\cos\alpha = \cos^2(\alpha/2) - \sin^2(\alpha/2) = 1 - 2\sin^2(\alpha/2) = 1 - 2a^2u^2$. Similarly, $\cos\beta = 1 - 2b^2u^2$ and $\cos\gamma = 1 - 2c^2u^2$. It is well-known that if $\alpha + \beta + \gamma = \pi$, then

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma + 2\cos\alpha\cos\beta\cos\gamma - 1 \tag{2.15}$$

is 0; see, for example, [2]. Therefore, substituting $\langle x, 1-2a^2x, 1-2b^2x, 1-2c^2x\rangle$ for $\langle u^2, \cos\alpha, \cos\beta, \cos\gamma\rangle$ in (2.15), we obtain a polynomial $g(a,b,c,x) \in \mathbb{Z}[a,b,c,x]$ such that $\deg_x(g(a,b,c,x)) = 3$ and u^2 is a root of this polynomial. Using computer algebra, we obtain that this polynomial is irreducible. Note that

$$g(1,2,3,x) = -576x^3 + 784x^2 - 112x + 4$$

is also irreducible in $\mathbb{Q}[x]$ and the Thalesian quadrangle with $\vec{a} = \langle 1,2,3\rangle$ exists. Hence, neither u^2, nor the Thalesian quadrangle with side lengths $\langle 1,2,3\rangle$ is constructible.

Cases (1)–(5) imply parts (i), (ii), and (v). Parts (i)–(ii) and the Rational Parameter Theorem, see [3, Theorem 11.1], imply part (iii).

Next, we focus our attention to part (iv). For $n = 3$, the statement is obvious, because the triangle is right-angled by Thales' theorem, and we are given either the lengths of its legs, or the central distances, which are the halves of these lengths. So let $n = 4$ and assume that the three central distances are given. Then we also know the fourth distance, since it is 0 by the definition of a Thalesian quadrangle. Thus, the constructibility of the Thalesian quadrangle is a particular case of the constructibility of a cyclic quadrangle from its central distances in general; see [2] or, for a secondary source, [3, Proposition 1.3]. ☐

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E-mail address: `czedli@math.u-szeged.hu`
URL: `http://www.math.u-szeged.hu/~czedli/`

University of Szeged, Bolyai Institute. Szeged, Aradi vértanúk tere 1, HUNGARY 6720