This is a preprint of August 13, 2016. The final version appeared later in *Communications in Algebra*, **45/9** (2017) 4037–4049;
http://www.tandfonline.com/10.1080/00927872.2016.1257710

FOUR-GENERATED QUASIORDER LATTICES AND THEIR ATOMS IN A FOUR-GENERATED SUBLATTICE

GÁBOR CZÉDLI

Abstract. Quasiorders, also known as preorders, on a set A form a lattice $\mathrm{Quo}(A)$. We prove that if A is a finite set consisting of 2, 3, 5, 7, 9, or more than 10 elements, then $\mathrm{Quo}(A)$ is four-generated but not three-generated. Also, if A is countably infinite, then a four-generated sublattice contains all atoms of $\mathrm{Quo}(A)$. These statements improve Ivan Chajda and the present author's 1996 result, where six generators were constructed, and Tamás Dolgos and Júlia Kulin's recent results, where five generators were given.

1. Introduction and the main result

A lattice is *k-generated* if it has a k-element generating set. A *quasiorder*, also known as a preorder, is a reflexive, transitive relation on a set. Quasiorders are frequently used in many fields of algebra and mathematics. Equipped with the subset relation, the collection of all quasiorders on a set A is a lattice $\mathrm{Quo}(A) = (\mathrm{Quo}(A); \subseteq)$. The size of this lattice as a function of $|A|$ grows quite fast; we know from [2] that $|\mathrm{Quo}(A)|$ is 1, 4, 29, 355, and 6942 for $|A|$ being 1, 2, 3, 4, and 5, respectively. Hence, it was a surprise in [2, Page 416] that these lattices are six-generated for all finite sets A. Recently, Dolgos [6] and Kulin [7] have given generating sets consisting only of five elements. Besides [2], it is their results that motivated our research leading to the present paper. Further historical remarks will be given soon after formulating the main result.

As usual, the least infinite cardinal is denoted by $\aleph_0$. Given a set A, $|A|$ stands for the cardinality of A. The least element of $\mathrm{Quo}(A)$ is the equality relation $\Delta_A = \{(x, x) : x \in A\}$. Quasiorders of the form $\{(p, q)\} \cup \Delta_A$ with $p \neq q$ are the *atoms* of $\mathrm{Quo}(A)$. Our goal is to prove the following theorem.

Theorem 1.1. *If A is a set with $|A| \in \{n \in \mathbb{N} : n \geq 11\} \cup \{2, 3, 5, 7, 9, \aleph_0\}$, then*

(i) *the quasiorder lattice $\mathrm{Quo}(A)$ has a four-generated sublattice that contains all atoms of $\mathrm{Quo}(A)$, and*

(ii) *if, in addition, $|A|$ is finite, then $\mathrm{Quo}(A)$ is a four-generated lattice.*

Furthermore, for every set A with at least three elements, no three-generated sublattice of $\mathrm{Quo}(A)$ contains all atoms of $\mathrm{Quo}(A)$.

Date: Submitted on November 6, 2015, revised on August 13, 2016

2000 *Mathematics Subject Classification.* Primary 06B99.

Key words and phrases. Quasiorder lattice, lattice of preorders, minimum-sized generating set, four-generated lattice.

This research was supported by NFSR of Hungary (OTKA), grant number K 115518.

Remark 1.2. Part (i) cannot hold for $|A| > \aleph_0$ by cardinality reasons. We do not know whether the theorem holds for $|A| \in \{4, 6, 8, 10\}$.

1.1. **Prerequisite and notation.** Practically, there is no prerequisite and the paper is self-contained for most algebraists. If the reader knows how the join in $\mathrm{Quo}(A)$ is described, then he can read the paper without difficulties. We usually write $+$ and $\cdot$ (or concatenation) rather than $\vee$ and $\wedge$ for lattice joins and meets. This allows us to use the convention that $\cdot$ takes precedence over $+$; for example, $\alpha\beta + \alpha\gamma$ stands for $(\alpha \wedge \beta) \vee (\alpha \wedge \gamma)$. Without this convention, several displayed equations would split into two lines.

1.2. **Outline.** Subsections 1.3 and 1.4 below give a historical overview and the initial idea. The rest of the paper is devoted to the proof of Theorem 1.1. Section 2 deals with the case where $|A| \geq 5$ is an odd number. If $|A| \geq 12$ is an even number or $|A| = \aleph_0$, then we need a different construction, which is presented in Section 3. Finally, in Section 4, we prove that $\mathrm{Quo}(A)$ cannot be generated by three elements, and we complete the proof of Theorem 1.1

1.3. **Historical overview.** The lattice $\mathrm{Equ}(A)$ of all equivalence relations on A and $\mathrm{Quo}(A)$ are closely related to each other. In particular, as we point out below and also by means of (1.1), this is so when the minimum size of possible generating sets are considered. In order to avoid trivial exceptions, this subsection assumes that A consists of at least three elements even if this is not always mentioned in what follows. A lattice is $(1+1+2)$-*generated* if it has a generating set $\{a_0, a_1, b, c\}$ such that $a_0 < a_1$ and both $\{a_0, b, c\}$ and $\{a_1, b, c\}$ are antichains. The search for small generating sets of equivalence lattices began with Strietz [8, 9]. Besides proving that $\mathrm{Equ}(A)$ is four-generated for all finite A, he also proved that it is $(1 + 1 + 2)$-generated for every finite set A with at least 10 elements. Also, he proved that $\mathrm{Equ}(A)$ has many non-isomorphic four-element generating sets for large $|A| < \aleph_0$ but no three-element generating set for $4 \leq |A| < \aleph_0$.

The next step is due to Zádori [12]. He reduced 10 in Strietz's result by showing that $\mathrm{Equ}(A)$ is $(1+1+2)$-generated for $7 \leq |A| < \aleph_0$; we do not know whether this holds for $|A| \in \{4, 5, 6\}$, but it trivially fails for $|A| = 3$ since $\mathrm{Equ}(A)$ is the five-element non-distributive modular lattice in that case. He also proved that $\mathrm{Equ}(A)$ is three-generated in the following "congruence lattice sense": there exist $\alpha_1, \alpha_2, \alpha_3 \in \mathrm{Equ}(A)$ such that whenever F is a set of operations on A and α_1, α_2, and α_3 are congruences of the algebra $\langle A; F\rangle$, then all equivalences on A are congruences of this algebra. However, what is most interesting in Zádori's paper from our perspective is that he gave an *easy* construction of a four-element generating set. Actually, with the exception of Dolgos [6] and Kulin [7], all subsequent papers, that is, Chajda and Czédli [2], Czédli [3, 4, 5], and Takách [10] rely on the key idea of Zádori's construct in some extent, and so does the present paper.

Next, while dealing with quasiorder lattices in [1] and motivated by Zádori's approach, Chajda and Czédli [2] got interested in small generating sets of these lattices. However, [2] considered the *complete involution lattice* $\mathrm{Q}^{\mathrm{inv}}_{\mathrm{compl}}(A)$ of quasiorders on A; that is, besides arbitrary joins and meets, the period-two lattice automorphism which maps each quasiorder to its inverse is also an operation of $\mathrm{Q}^{\mathrm{inv}}_{\mathrm{compl}}(A)$. It is proved in [2] that $\mathrm{Q}^{\mathrm{inv}}_{\mathrm{compl}}(A)$ is three-generated. Although brute computer force shows that $\mathrm{Q}^{\mathrm{inv}}_{\mathrm{compl}}(A)$ is not two-generated if $|A| \in \{3, 4\}$, the conjecture of [2] that $\mathrm{Q}^{\mathrm{inv}}_{\mathrm{compl}}(A)$ is not two-generated for $3 \leq |A|$ is still open. Since the involution operation commutes with the lattice operations, the result of [2] trivially implies that $\mathrm{Quo}(A)$ is six-generated, because one can take the three generators from [2] and their inverses. Note that $\mathrm{Equ}(A)$ is the subalgebra of $\mathrm{Q}^{\mathrm{inv}}_{\mathrm{compl}}(A)$ formed by the fixed points of the involution, whereby the relation between studying small generating sets for quasiorders and that for equivalences is not surprising in itself. The real surprise for us in [2] was that, without any intention or effort, the construction worked even for $|A| = \aleph_0$. Then, with some extra effort, $|A|$ in [2] were pushed somewhat higher than $\aleph_0$. Note at this point that, in view of Remark 1.2, neither the quasiorder lattice $\mathrm{Quo}(A)$, nor its enrichment with involution can be generated by finitely many elements. This is why it is reasonable to emphasize here that $\mathrm{Q}^{\mathrm{inv}}_{\mathrm{compl}}(A)$ is a *complete* involution lattice in which arbitrary (not only finitary) joins and meets are considered.

In the rest of this subsection, let $\mathrm{E}_{\mathrm{compl}}(A)$ and $\mathrm{Q}_{\mathrm{compl}}(A)$ denote the *complete* lattice of equivalences and that of quasiorders on A, respectively. Clearly, if A is finite, then $\mathrm{Equ}(A)$ and $\mathrm{E}_{\mathrm{compl}}(A)$ have exactly the same generating sets, and the same holds for $\mathrm{Quo}(A)$ and $\mathrm{Q}_{\mathrm{compl}}(A)$.

As opposed to the case of quasiorders in [2], the passage from finite to $\aleph_0$ is not so easy for equivalences. The first steps in this direction were made by Czédli [3] and [4]. For $|A| = \aleph_0$, [3] constructs a $(1+1+2)$-generated sublattice containing all atoms of $\mathrm{Equ}(A)$. Note that part (i) of Theorem 1.1 is a counterpart of this result for quasiorders. In [4], an involved construction and a long technical proof lead to the result stating that $\mathrm{E}_{\mathrm{compl}}(A)$ is four-generated if $|A|$ is an accessible cardinal. Instead of a definition, here we note only that ZFC has a model in which all cardinals are accessible. Next, motivated by [4] and improving its technique, Takách [10] proved that $\mathrm{Q}^{\mathrm{inv}}_{\mathrm{compl}}(A)$ is three-generated for infinite sets A with accessible cardinalities. Finally, Czédli [5] constructed a $(1+1+2)$-generating set for every $\mathrm{E}_{\mathrm{compl}}(A)$ such that $7 \leq |A|$ is accessible. It is still an open problem whether the assumption of accessibility can be removed from the above-mentioned results.

In 2015, Dolgos [6] proved that for $3 \leq |A| \leq \aleph_0$, $\mathrm{Q}_{\mathrm{compl}}(A)$ is five-generated. (His construction, which is entirely different from [2, 3, 4, 5, 12], is outlined between Figures 1 and 2 in Kulin [7].) Soon after [6], Kulin [7] extended Dolgos' result by proving that $\mathrm{Q}_{\mathrm{compl}}(A)$ is five-generated for all sets A such that $3 \leq |A|$ and $|A|$ is accessible. Actually, her paper contains two proofs. First, and this is of high importance from our perspective, she gives a short proof; see [7, page 61]. This short approach is, in essence, based on the 4-generability of $\mathrm{E}_{\mathrm{compl}}(A)$ and the following statement; see also Lemma 3.2 here for another variant.

(1.1) If $3 \leq |A|$ and S is a complete sublattice of $\mathrm{Q}_{\mathrm{compl}}(A)$ such that $\mathrm{E}_{\mathrm{compl}}(A) \subset S$, then $S = \mathrm{Q}_{\mathrm{compl}}(A)$.

Here, as it is usual in lattice theory, "$\subset$" stands for proper inclusion, which excludes equality. The proof of (1.1) can easily be extracted from [7, page 61]. Armed with (1.1), she could simply take the four equivalences from [4] that generate $\mathrm{E}_{\mathrm{compl}}(A)$ and, as the fifth generator for $\mathrm{Q}_{\mathrm{compl}}(A)$, an arbitrary quasiorder outside $\mathrm{E}_{\mathrm{compl}}(A)$. Second, in order to give a self-contained approach that avoids the long technicalities of [4] in connection with accessible cardinals, she modified Dolgos' construction in an involved way to obtain five generators when A has the power of continuum.

1.4. **Method.** The initial idea came from Kulin's (1.1), which raised the possibility of modifying one (or some) of the four generators of $\mathrm{Equ}(A)$ in order to obtain a generating set of $\mathrm{Quo}(A)$. The main issue was to find appropriate constructs; the experience with [2, 4, 5] and, mainly, with [3] helped a lot. The technique of [2, 3] and a variant of (1.1) stated in Lemma 3.2 are used here in the proof.

2. An odd-sized construct

This section is devoted to the case when $|A| \geq 5$ is an odd natural number. (The case $|A| = 3$ is easy and will be settled in Section 4.) We are going to define quasiorders by means of graphs. Our considerations will frequently refer to geometric properties like "vertical" or "of slope 45°". Therefore,

(2.1) a (*directed*) *graph* in this paper is always understood as a *concrete graph diagram* in the plane.

This means that each vertex of a graph has a given abscissa and ordinate and each edge is a line segment or a concrete planar curve. Vertices are drawn as small circles. We do not assume planarity in graph theoretical sense, so two edges may cross not only at a vertex; however, unpleasant crosses of this kind will rarely occur. Note that an edge can be *directed* or *non-directed*; a non-directed edge is just an abbreviation for two oppositely directed parallel edges. A *colored graph* is a concrete graph diagram whose edges are colored by α, β, γ, and δ. Parallel edges with different colors may occur but loops are not allowed. Note that a non-directed edge colored by $\varepsilon \in \{\alpha, \beta, \gamma, \delta\}$ represents two ε-colored directed edges that are oppositely oriented. In absence of room, the colors of edges are often determined by the following convention rather than labels:

(2.2)
the *horizontal edges* are α-colored,
the *vertical* edges are β-colored,
the *slanted edges with slope* 45° are γ-colored, and
the *dotted curves* are δ-colored.

Our figures include a reminder to (2.2). Let $\mathbb{N} = \{1, 2, 3, \dots\}$ and $\mathbb{N}_0 = \{0, 1, 2, \dots\}$. For $2 \leq n \in \mathbb{N}$, we define a graph F_n as follows. See also Figure 1, which motivates that this graph will be called a *fence of rank n*. The vertex set of F_n, which is also denoted by F_n, is the disjoint union of $A_n = \{a_0, a_1, \dots, a_n\}$ and $B_n = \{b_0, b_1, \dots, b_{n-1}\}$. These subsets of F_n are called *rows*; A_n is the *upper row* and B_n is the *lower row* of F_n. The edges of F_n are the *non-directed* α-colored (a_0, a_1), …, (a_{n-1}, a_n) and (b_0, b_1), …, (b_{n-2}, b_{n-1}), the β-colored (a_0, b_0), …, (a_{n-1}, b_{n-1}), and the γ-colored (a_1, b_0), …, (a_n, b_{n-1}), and the *directed* δ-colored (a_0, b_0), (b_1, a_0), (b_{n-1}, a_n), and (a_n, b_{n-2}). Except for the δ-colored curves, each edge is non-directed and represents two directed edges of the same color. For $\varepsilon \in \{\alpha, \beta, \gamma, \delta\}$, we define a quasiorder ε on the set F_n as follows:

(2.3) for vertices x, y, we let $(x, y) \in \varepsilon$ if there is an ε-colored directed path from x to y.

For example, $(a_0, a_n) \in \alpha$ but $(a_0, a_n) \notin \beta$. We make no notational distinction between the four colors and the quasiorders they determine. Note that the undirected edges of our figures can be used in both directions in a directed path. Note also that $\alpha, \beta, \gamma \in \mathrm{Equ}(A)$. On the other hand, $\delta \notin \mathrm{Equ}(A)$, because $\delta\delta^{-1} = \delta \cap \delta^{-1} = \Delta_{F_n}$.

A pair $(x, y) \in A^2$ is *nontrivial* if $x \neq y$. For a nontrivial pair $(x, y) \in A^2$, the quasiorder $\{(x, y)\} \cup \Delta_A$ and the equivalence $\{(x, y), (y, x)\} \cup \Delta_A$ will be denoted by $\langle x, y\rangle^{\mathrm{q}}$ and $[x, y]^{\mathrm{e}}$, respectively. Actually, $\langle x, y\rangle^{\mathrm{q}}$ is a (partial) order, not only a quasiorder. Note that $[x, y]^{\mathrm{e}} = \langle x, y\rangle^{\mathrm{q}} + \langle y, x\rangle^{\mathrm{q}}$. Sometimes when $x \neq y$ is not guaranteed, $\langle x,y\rangle^{\mathrm{q}}$ and $[x,y]^{\mathrm{e}}$ may turn to $\langle x,x\rangle^{\mathrm{q}} = [x,x]^{\mathrm{e}} := \Delta_A$. The following straightforward lemma was used, explicitly or implicitly, in several earlier papers; see Chajda and Czédli [2, second display in page 423], Czédli [3, circle principle in page 12], [4, last display in page 55], and [5, first display in page 451], Kulin [7, Lemma 2.2], Takách [10, page 90], and Zádori [12, second display in page 583]. Its proof is trivial if one considers the nontrivial pairs of the quasiorders or equivalences in question; the details will be omitted.

Lemma 2.1. *For an arbitrary set A and $j,k\in\mathbb{N}$, if $\{u,v\}$, $\{x_1,\dots,x_{j-1}\}$ and $\{y_1,\dots,y_{k-1}\}$ are pairwise disjoint subsets of A, $u=x_0=y_0$, and $v=x_j=y_k$, then*

$$\langle u,v\rangle^{\mathrm{q}} = \Big(\sum_{i=1}^{j}\langle x_{i-1},x_i\rangle^{\mathrm{q}}\Big)\cdot\Big(\sum_{i=1}^{k}\langle y_{i-1},y_i\rangle^{\mathrm{q}}\Big),\ \textit{and}$$

$$[u,v]^{\mathrm{e}} = \Big(\sum_{i=1}^{j}[x_{i-1},x_i]^{\mathrm{e}}\Big)\cdot\Big(\sum_{i=1}^{k}[y_{i-1},y_i]^{\mathrm{e}}\Big).$$

The following lemma settles Theorem 1.1 for $|A|\geq 5$ odd.

Lemma 2.2. *For $2\leq n\in\mathbb{N}$, the quasiorder lattice $\mathrm{Quo}(F_n)$ is generated by $\{\alpha,\beta,\gamma,\delta\}$.*

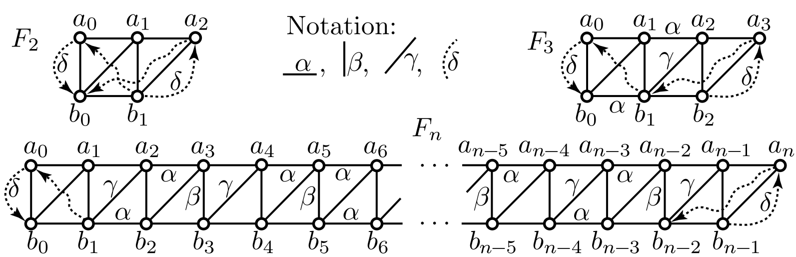

FIGURE 1. F_2, F_3, and F_n

Proof of Lemma 2.2. Let L_n denote the sublattice of $\mathrm{Quo}(F_n)$ generated by the four-element subset $\{\alpha,\beta,\gamma,\delta\}$. Clearly,

$$\langle a_0,b_0\rangle^{\mathrm{q}} = \beta\delta\in L_n \quad\text{and}\quad \langle b_{n-1},a_n\rangle^{\mathrm{q}} = \gamma\delta\in L_n. \tag{2.4}$$

In order to avoid extra line breaks, we will often drop the "$\in L_n$" part from displayed equalities. However, when these equalities are referenced, the meaning is that they belong to L_n. A quasiorder $\varepsilon\in\mathrm{Quo}(F_n)$ is *row-preserving* if for every nontrivial pair $(x,y)\in\varepsilon$, x and y belong to the same row of F_n. For example,

$$\alpha \text{ is row-preserving.} \tag{2.5}$$

Therefore, using that $\beta\in\mathrm{Equ}(F_n)$ with two-element "vertical" blocks and (2.4), we obtain that

$$\beta_* := \sum_{i=0}^{n-1}\langle a_i,b_i\rangle^{\mathrm{q}} = \beta(\alpha+\langle a_0,b_0\rangle^{\mathrm{q}})\in L_n \text{ and} \tag{2.6}$$

$$\beta^* := \sum_{i=0}^{n-1}\langle b_i,a_i\rangle^{\mathrm{q}} = \beta(\alpha+\langle b_{n-1},a_n\rangle^{\mathrm{q}})\in L_n. \tag{2.7}$$

By the same reasons,

$$\gamma_* := \sum_{i=1}^{n} \langle a_i, b_{i-1}\rangle^{\mathrm{q}} = \gamma(\alpha + \langle a_0, b_0\rangle^{\mathrm{q}}) \in L_n \text{ and} \tag{2.8}$$

$$\gamma^* := \sum_{i=1}^{n} \langle b_{i-1}, a_i\rangle^{\mathrm{q}} = \gamma(\alpha + \langle b_{n-1}, a_n\rangle^{\mathrm{q}}) \in L_n. \tag{2.9}$$

Note the role of $*$ in our notation: it indicates the orientation of the edges generating the quasiorder in question. The role of arrows is similar in our next two statements,

$$\overleftarrow{\alpha} := \sum_{i=1}^{n} \langle a_i, a_{i-1}\rangle^{\mathrm{q}} + \sum_{i=1}^{n-1} \langle b_i, b_{i-1}\rangle^{\mathrm{q}} = \alpha\big(\beta^* + \gamma_*\big) \in L_n \text{ and} \tag{2.10}$$

$$\overrightarrow{\alpha} := \sum_{i=1}^{n} \langle a_{i-1}, a_i\rangle^{\mathrm{q}} + \sum_{i=1}^{n-1} \langle b_{i-1}, b_i\rangle^{\mathrm{q}} = \alpha\big(\beta_* + \gamma^*\big) \in L_n, \tag{2.11}$$

which follow easily from (2.5), (2.6), (2.7), (2.8), and (2.9). Note that (2.5) gives that both $\overleftarrow{\alpha}$ and $\overrightarrow{\alpha}$ are row-preserving; (2.5) will also refer to these two facts. Next, we claim that

$$\langle a_0, a_1\rangle^{\mathrm{q}} = \overrightarrow{\alpha}\left(\langle a_0, b_0\rangle^{\mathrm{q}} + \gamma^*\right) \text{ and } \langle a_{n-1}, a_n\rangle^{\mathrm{q}} = \overrightarrow{\alpha}\left(\beta_* + \langle b_{n-1}, a_n\rangle^{\mathrm{q}}\right) \tag{2.12}$$

belong to L_n; this follows from definitions, (2.4), and (2.5) easily. For example, to see the first equality in (2.12), observe that for all $(x, y) \in \langle a_0, b_0\rangle^{\mathrm{q}} + \gamma^*$, we have either $(x, y) \in \gamma^*$, or $(x, y) \in \{(a_0, b_0), (b_0, a_1), (a_0, a_1)\}$. So if a nontrivial pair (x, y) belongs also to $\overrightarrow{\alpha}$, then $(x, y) = (a_0, a_1) \in \langle a_0, a_1\rangle^{\mathrm{q}}$. This proves the "$\geq$" (that is, the "$\supseteq$") part of the first equality in (2.4), while the converse inequality is obvious. So this equality holds and $\langle a_0, a_1\rangle^{\mathrm{q}} \in L_n$ by (2.4). In what follows, arguments of similar complexity will not be as detailed as above for (2.12); however, we point out that

$$\overleftarrow{\alpha} \leq \alpha, \quad \ldots, \quad \gamma^* \leq \gamma, \quad \alpha\beta = \alpha\gamma = \beta\gamma = \beta^*\delta = \gamma_*\delta = \Delta_{F_n},$$

and our figures can often be used in these cases. In particular, the straightforward argument showing that

$$\langle b_0, a_1\rangle^{\mathrm{q}} = \gamma^*\left(\beta^* + \langle a_0, a_1\rangle^{\mathrm{q}}\right) \text{ and } \langle a_{n-1}, b_{n-1}\rangle^{\mathrm{q}} = \beta_*\left(\langle a_{n-1}, a_n\rangle^{\mathrm{q}} + \gamma_*\right) \tag{2.13}$$

are in L_n follow from (2.6), (2.7), (2.8), (2.9), and (2.12). Next, we let

$$\mu_k^* := \sum_{i=0}^{k-1} \langle b_i, a_i\rangle^{\mathrm{q}}, \quad \nu_*^k := \sum_{i=0}^{k-1} \langle a_{n-i}, b_{n-1-i}\rangle^{\mathrm{q}}, \text{ for } 2 \leq k \leq n-1, \tag{2.14}$$

$$\kappa_k^* := \sum_{i=1}^{k} \langle b_{i-1}, a_i\rangle^{\mathrm{q}}, \text{ and } \lambda_*^k := \sum_{i=1}^{k} \langle a_{n-i}, b_{n-i}\rangle^{\mathrm{q}}, \text{ for } 1 \leq k \leq n-1. \tag{2.15}$$

Note that (2.14) is vacuous for $n = 2$; similar situations will frequently occur without further warning. We are going to show by induction on k that the quasiorders in (2.14) and (2.15) belong to L_n. In order to establish the base of the induction, we conclude from (2.13) that, for $n > 2$,

$$\mu_2^* := \beta^*\left(\overleftarrow{\alpha} + \langle b_0, a_1\rangle^{\mathrm{q}}\right) \text{ and } \nu_*^2 := \gamma_*\left(\overleftarrow{\alpha} + \langle a_{n-1}, b_{n-1}\rangle^{\mathrm{q}}\right) \tag{2.16}$$

are in L_n. Also, (2.13) gives that $\kappa_1^* = \langle b_0, a_1\rangle^{\mathrm{q}} \in L_n$ and $\lambda_*^1 = \langle a_{n-1}, b_{n-1}\rangle^{\mathrm{q}} \in L_n$. Hence (2.15) holds for $n = 2$. So does (2.14), vacuously. Thus, in the rest of the argument for (2.14) and (2.15), let $n > 2$. It follows easily that $\kappa_2^* = \gamma^*\big(\overrightarrow{\alpha} + \mu_2^*\big) \in L_n$ and $\lambda_*^2 = \beta_*\big(\overrightarrow{\alpha} + \nu_*^2\big) \in L_n$. Now, assume that $k < n-1$ such that both (2.14) and (2.15) define elements of L_n. It is straightforward and easy to see that

$$\begin{aligned} &\mu_{k+1}^* := \beta^*\big(\overleftarrow{\alpha} + \kappa_k^*\big), &&\nu_*^{k+1} := \gamma_*\big(\overleftarrow{\alpha} + \lambda_*^k\big),\\ &\kappa_{k+1}^* := \gamma^*\big(\overrightarrow{\alpha} + \mu_{k+1}^*\big), \text{ and } &&\lambda_*^{k+1} := \beta_*\big(\overrightarrow{\alpha} + \nu_*^{k+1}\big) \end{aligned}$$

are also in L_n. This completes the induction and proves that the quasiorders given in (2.14) and (2.15) belong to L_n. Consequently, it follows easily that

$$\begin{aligned} \overrightarrow{\rho}_k &:= \overrightarrow{\alpha}(\beta_* + \kappa_k^*) = \sum_{i=1}^{k}\big(\langle a_{i-1}, a_i\rangle^{\mathrm{q}} + \langle b_{i-1}, b_i\rangle^{\mathrm{q}}\big), \text{ for } 1 \le k \le n-1,\\ \overrightarrow{\tau}_k &:= \overrightarrow{\alpha}(\gamma^* + \lambda_*^k) = \sum_{i=1}^{k}\big(\langle a_{n-i}, a_{n+1-i}\rangle^{\mathrm{q}} + \langle b_{n-1-i}, b_{n-i}\rangle^{\mathrm{q}}\big),\\ &\qquad\qquad\qquad\qquad\qquad\qquad \text{for } 1 \le k \le n-1,\\ \overleftarrow{\rho}_k &:= \overleftarrow{\alpha}(\mu_k^* + \gamma_*) = \sum_{i=1}^{k}\langle a_i, a_{i-1}\rangle^{\mathrm{q}} + \sum_{i=1}^{k-1}\langle b_i, b_{i-1}\rangle^{\mathrm{q}}, \text{ for } 2 \le k \le n-1,\\ \overleftarrow{\tau}_k &:= \overleftarrow{\alpha}(\nu_*^k + \beta^*) = \sum_{i=1}^{k}\langle a_{n+1-i}, a_{n-i}\rangle^{\mathrm{q}} + \sum_{i=1}^{k-1}\langle b_{n-i}, b_{n-1-i}\rangle^{\mathrm{q}},\\ &\qquad\qquad\qquad\qquad\qquad\qquad \text{for } 2 \le k \le n-1, \end{aligned}$$

belong to L_n. Hence, we obtain that

$$\begin{aligned} \langle b_{k-1}, b_k\rangle^{\mathrm{q}} &= \overrightarrow{\rho}_k \overrightarrow{\tau}_{n-k}, \text{ for } 1 \le k \le n-1, \text{ and}\\ \langle a_k, a_{k-1}\rangle^{\mathrm{q}} &= \overleftarrow{\rho}_k \overleftarrow{\tau}_{n+1-k}, \text{ for } 2 \le k \le n-1, \end{aligned} \tag{2.17}$$

are in L_n. As a particular case of (2.17), $\langle b_0, b_1\rangle^{\mathrm{q}} \in L_n$ and $\langle b_{n-2}, b_{n-1}\rangle^{\mathrm{q}} \in L_n$. Hence,

$$\langle b_0, a_0\rangle^{\mathrm{q}} = \beta^*\big(\langle b_0, b_1\rangle^{\mathrm{q}} + \delta\big) \text{ and } \langle a_n, b_{n-1}\rangle^{\mathrm{q}} = \gamma_*\big(\delta + \langle b_{n-2}, b_{n-1}\rangle^{\mathrm{q}}\big) \tag{2.18}$$

belong to L_n. This implies that

$$\langle a_1, a_0\rangle^{\mathrm{q}} = \overleftarrow{\alpha}\big(\gamma_* + \langle b_0, a_0\rangle^{\mathrm{q}}\big) \text{ and } \langle a_n, a_{n-1}\rangle^{\mathrm{q}} = \overleftarrow{\alpha}\big(\langle a_n, b_{n-1}\rangle^{\mathrm{q}} + \beta^*\big) \tag{2.19}$$

are also in L_n. Observe that

$$\langle a_1, b_0\rangle^{\mathrm{q}} = \gamma_*\big(\langle a_1, a_0\rangle^{\mathrm{q}} + \langle a_0, b_0\rangle^{\mathrm{q}}\big) \text{ and } \langle a_1, b_1\rangle^{\mathrm{q}} = \beta_*\big(\langle a_1, b_0\rangle^{\mathrm{q}} + \langle b_0, b_1\rangle^{\mathrm{q}}\big) \tag{2.20}$$

are in L_n by (2.4), (2.17), and (2.19). Hence, using (2.17) (and (2.20) for $k = 1$), we obtain by induction on k that

$$\begin{aligned} \langle a_k, b_{k-1}\rangle^{\mathrm{q}} &= \gamma_*\big(\langle a_k, a_{k-1}\rangle^{\mathrm{q}} + \langle a_{k-1}, b_{k-1}\rangle^{\mathrm{q}}\big) \quad \text{and}\\ \langle a_k, b_k\rangle^{\mathrm{q}} &= \beta_*\big(\langle a_k, b_{k-1}\rangle^{\mathrm{q}} + \langle b_{k-1}, b_k\rangle^{\mathrm{q}}\big), \quad \text{for } 1 \le k \le n-1, \end{aligned} \tag{2.21}$$

are in L_n. Using (2.4) and (2.19), it follows that

$$\langle b_{n-1}, a_{n-1}\rangle^{\mathrm{q}} = \beta^*\big(\langle b_{n-1}, a_n\rangle^{\mathrm{q}} + \langle a_n, a_{n-1}\rangle^{\mathrm{q}}\big) \tag{2.22}$$

is in L_n. Starting from (2.22) and using (2.17), an obvious induction on k yields that for $2 \le k \le n-1$,

$$\begin{aligned} \langle b_{n-k}, a_{n+1-k}\rangle^{\mathrm{q}} &= \gamma^*\big(\langle b_{n-k}, b_{n+1-k}\rangle^{\mathrm{q}} + \langle b_{n-(k-1)}, a_{n-(k-1)}\rangle^{\mathrm{q}}\big) \text{ and}\\ \langle b_{n-k}, a_{n-k}\rangle^{\mathrm{q}} &= \beta^*\big(\langle b_{n-k}, a_{n+1-k}\rangle^{\mathrm{q}} + \langle a_{n+1-k}, a_{n-k}\rangle^{\mathrm{q}}\big) \end{aligned} \tag{2.23}$$

are in L_n. Combining (2.4), (2.22), and (2.23), we obtain that

$$\langle b_{n-k}, a_{n+1-k}\rangle^{\mathrm{q}},\ \langle b_{n-k}, a_{n-k}\rangle^{\mathrm{q}} \in L_n \text{ for } 1 \le k \le n-1. \tag{2.24}$$

Now that every non-horizontal directed edge in the graph generates an atom in L_n by (2.4), (2.13), (2.18), (2.20), (2.21), and (2.24), it is trivial to see that the same holds for the horizontal directed edges; for example,

$$\langle a_{k-1}, a_k\rangle^{\mathrm{q}} = \vec{\alpha}\left(\langle a_{k-1}, b_{k-1}\rangle^{\mathrm{q}} + \langle b_{k-1}, a_k\rangle^{\mathrm{q}}\right), \text{ for } 1 \le k \le n.$$

That is, for every undirected edge (x, y) of the graph, $\langle x, y\rangle^{\mathrm{q}}, \langle y, x\rangle^{\mathrm{q}} \in L_n$. Furthermore, for all $x \neq y \in F_n$, there are two vertex-disjoint (non-directed) paths from x to y. Thus, by Lemma 2.1, L_n contains all atoms of $\mathrm{Quo}(F_n)$. Since every element of $\mathrm{Quo}(F_n)$ is the join of some atoms, it follows that $L_n = \mathrm{Quo}(F_n)$. This completes the proof of Lemma 2.2 □

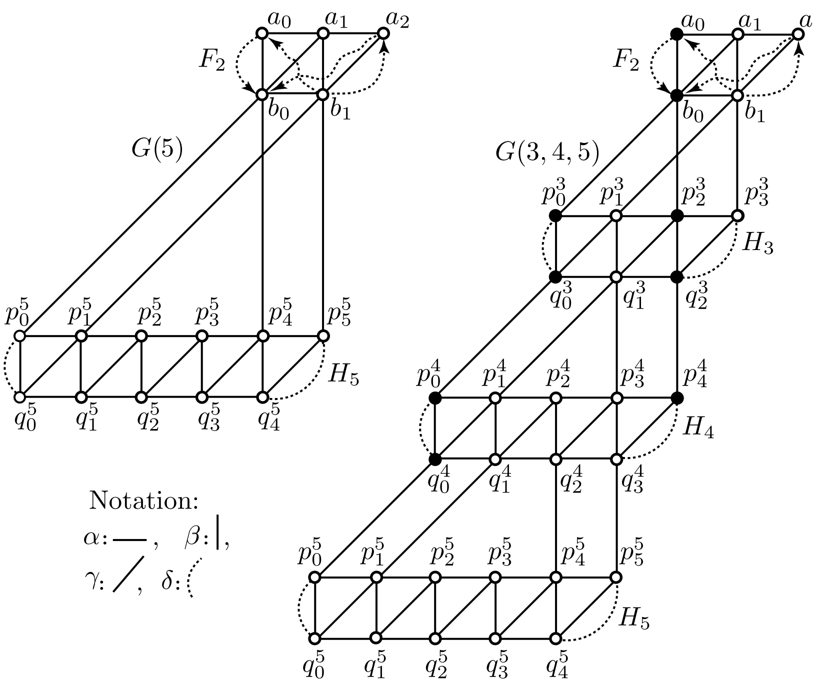

FIGURE 2. $G(5)$ and $G(3,4,5)$

3. EVEN-SIZED AND COUNTABLE CONSTRUCTS

In this section, we investigate sets A with $|A| \in \{12, 14, 16, 18, \dots\} \cup \{\aleph_0\}$. Let $\vec{m} = (n + i : i < g)$ denote a sequence of consecutive positive integers such that $n \ge 3$ and $g \le \aleph_0$. Here g is the *length* of the sequence, so "$i < g$" always stands for "$0 \le i < g$". The case $g = 1$, where $\vec{m} = (n)$ is the singleton sequence, and the case $g = \aleph_0$, where $\vec{m} = (3, 4, 5, 6, \dots)$ is the sequence of all natural numbers above 2, will be of a particular importance. Associated with $\vec{m}$, we define a colored graph $G(\vec{m})$ as follows; remember that (2.1) is still valid. First, modifying the set of δ-colored edges in F_k, we define a new graph H_k according to Figure 2 as follows. (In this figure, H_3, H_4, and H_5 are given as full subgraphs of larger graphs.) Instead of a_i and b_j, the vertices of H_k are denoted by p_i^k and q_j^k, respectively. Furthermore, H_k has exactly two δ-colored edges, (p_0^k, q_0^k) and (p_k^k, q_{k-1}^k), and both are undirected. Next, to obtain $G(\vec{m})$, form the (disjoint) union of F_2 and the fences H_{n+i}, $i < g$, both for the vertex sets and the edge sets, and add the following undirected edges:

$$\begin{aligned}&\text{the } \beta\text{-colored } (b_0, p_{n-1}^n),\ (b_1, p_n^n),\ (q_{n+i-3}^{n+i-1}, p_{n+i-1}^{n+i}), \text{ and}\\ &\qquad (q_{n+i-2}^{n+i-1}, p_{n+i}^{n+i}), \text{ for } 1 \le i < g, \quad \text{and}\\ &\text{the } \gamma\text{-colored } (b_0, p_0^n),\ (b_1, p_1^n),\ (q_0^{n+i-1}, p_0^{n+i}), \text{ and}\\ &\qquad (q_1^{n+i-1}, p_1^{n+i}), \text{ for } 1 \le i < g.\end{aligned}$$

In this way, we obtain the graph $G(\vec{m})$. For $G(5)$ and $G(3,4,5)$, see Figure 2. Our set is now $A = G(\vec{m})$, the vertex set of the graph, and we define our quasiorders on A according to (2.3). Note that with the exception of the four δ-colored edges in F_2, all edges of $G(\vec{m})$ are undirected. The aim of the present section is to prove the following statement.

Lemma 3.1. *In* $\mathrm{Quo}(G(\vec{m}))$*, the sublattice* L *generated by* $\{\alpha, \beta, \gamma, \delta\}$ *contains all atoms of* $\mathrm{Quo}(G(\vec{m}))$*.*

The proof of this statement is heavily based on the following lemma, which one can easily extract from Kulin [7, Proof of Theorem 2.1(i)]. For the reader's convenience, we outline the proof below.

Lemma 3.2 (Kulin [7])**.** *Assume that* A *is a set with at least three elements and* L *is a sublattice of* $\mathrm{Quo}(A)$ *such that* L *contains all atoms of* $\mathrm{Equ}(A)$*. If* L *is not a sublattice of* $\mathrm{Equ}(A)$*, then it contains all atoms of* $\mathrm{Quo}(A)$*.*

Proof of Lemma 3.2. Take a pair $(a, b) \in \varepsilon$ with $(b, a) \notin \varepsilon$ and $\varepsilon \in L \setminus \mathrm{Equ}(A)$. Then $\langle a, b\rangle^{\mathrm{q}} = \varepsilon \cdot [a, b]^{\mathrm{e}} \in L$. Observe that for pairwise distinct $x, y, z \in A$,

$$\langle x, z\rangle^{\mathrm{q}} = [x, z]^{\mathrm{e}} \cdot (\langle x, y\rangle^{\mathrm{q}} + [y, z]^{\mathrm{e}}) \text{ and } \langle z, y\rangle^{\mathrm{q}} = [z, y]^{\mathrm{e}} \cdot ([z, x]^{\mathrm{e}} + \langle x, y\rangle^{\mathrm{q}}).$$

For every nontrivial $(c, d) \in A^2$, using the rule above at most three times, it follows from $\langle a, b\rangle^{\mathrm{q}} \in L$ that $\langle c, d\rangle^{\mathrm{q}} \in L$. □

Proof of Lemma 3.1. Let $A = G(\vec{m})$. The subgraphs H_{n+i} for $i < g$ and F_2 will be called the *fences* of $G(\vec{m})$. They are of *rank* $n+i$ and 2, respectively. Observe that $\alpha+\delta \in \mathrm{Equ}(A)$ and the fences are exactly the $(\alpha+\delta)$-blocks. We let $\beta' := \beta(\alpha+\delta)$, $\gamma' := \gamma(\alpha+\delta)$, $\alpha' := \alpha(\alpha+\delta) = \alpha$, and $\delta' := \delta(\alpha+\delta) = \delta$. Clearly, each fence of $G(\vec{m})$ is ε'*-closed* for every $\varepsilon \in \{\alpha, \beta, \gamma, \delta\} \subseteq L$. That is, whenever x belongs to a fence and $(x, y) \in \varepsilon'$, then y also belongs to the same fence. Hence, at the beginning of the proof, we can work with $\alpha', \dots, \delta' \in L$ more comfortably than with $\alpha, \dots, \delta$. We will often use, sometimes without explicit mentioning, that

$$\alpha\beta = \alpha\gamma = \beta\gamma = \Delta_A \text{ and similarly for } \alpha',\ \beta',\ \text{and } \gamma'. \tag{3.1}$$

Our first task is to show that

$$\text{for all } x, y \in F_2 \text{ with } x \ne y,\ \langle x, y\rangle^{\mathrm{q}} \in L. \tag{3.2}$$

Clearly, $\langle b_1, b_0\rangle^{\mathrm{q}} = \alpha\delta \in L$. Thus, using the earlier containments and (3.1) in every step below, we obtain that the following quasiorders are in L:

$$\begin{aligned}&\langle a_1, b_0\rangle^{\mathrm{q}} = \gamma'(\beta' + \langle b_1, b_0\rangle^{\mathrm{q}}),\\ &\qquad \beta'_{2*} := \langle a_0, b_0\rangle^{\mathrm{q}} + \langle a_1, b_1\rangle^{\mathrm{q}} = \beta'(\alpha + \langle a_1, b_0\rangle^{\mathrm{q}}),\end{aligned}$$

$$\begin{aligned}
&\gamma'_{2*} := \langle a_1, b_0\rangle^{\mathrm{q}} + \langle a_2, b_1\rangle^{\mathrm{q}} = \gamma'(\alpha + \langle a_1, b_0\rangle^{\mathrm{q}}),\\
&\langle a_0, b_0\rangle^{\mathrm{q}} = \delta\beta'_{2*}, \qquad \langle a_0, a_1\rangle^{\mathrm{q}} = \alpha(\langle a_0, b_0\rangle^{\mathrm{q}} + \gamma'),\\
&\langle b_0, a_1\rangle^{\mathrm{q}} = \gamma'(\beta' + \langle a_0, a_1\rangle^{\mathrm{q}}), \qquad \langle b_1, a_1\rangle^{\mathrm{q}} = \beta'(\langle b_1, b_0\rangle^{\mathrm{q}} + \langle b_0, a_1\rangle^{\mathrm{q}}),\\
&\beta'^{*}_{2} := \langle b_0, a_0\rangle^{\mathrm{q}} + \langle b_1, a_1\rangle^{\mathrm{q}} = \beta'(\alpha + \langle b_0, a_1\rangle^{\mathrm{q}}),\\
&\gamma'^{*}_{2} := \langle b_0, a_1\rangle^{\mathrm{q}} + \langle b_1, a_2\rangle^{\mathrm{q}} = \gamma'(\alpha + \langle b_0, a_1\rangle^{\mathrm{q}}),\\
&\langle b_1, a_2\rangle^{\mathrm{q}} = \gamma'^{*}_{2}\cdot\delta, \qquad \langle a_1, a_2\rangle^{\mathrm{q}} = \alpha(\beta'_{2*} + \langle b_1, a_2\rangle^{\mathrm{q}}),\\
&\langle a_1, b_1\rangle^{\mathrm{q}} = \beta'_{2*}(\langle a_1, a_2\rangle^{\mathrm{q}} + \gamma'_{2*}), \qquad \langle b_0, b_1\rangle^{\mathrm{q}} = \alpha(\langle b_0, a_1\rangle^{\mathrm{q}} + \langle a_1, b_1\rangle^{\mathrm{q}}),\\
&\langle a_2, b_1\rangle^{\mathrm{q}} = \gamma'_{2*}(\delta + \langle b_0, b_1\rangle^{\mathrm{q}}), \qquad \langle a_2, a_1\rangle^{\mathrm{q}} = \alpha(\langle a_2, b_1\rangle^{\mathrm{q}} + \langle b_1, a_1\rangle^{\mathrm{q}}),\\
&\langle b_0, a_0\rangle^{\mathrm{q}} = \beta'^{*}_{2}(\langle b_0, b_1\rangle^{\mathrm{q}} + \delta), \text{ and } \langle a_1, a_0\rangle^{\mathrm{q}} = \alpha(\langle a_1, b_0\rangle^{\mathrm{q}} + \langle b_0, a_0\rangle^{\mathrm{q}}).
\end{aligned}$$

That is, for each undirected edge (x, y) of F_2, both $\langle x, y\rangle^{\mathrm{q}}$ and $\langle y, x\rangle^{\mathrm{q}}$ are in L. Thus, (3.2) follows by Lemma 2.1.

Next, assume that k belongs to $\vec{m}$. In other words, we assume that H_k is a fence of $G(\vec{m})$. We are going to show that

(3.3) if $[p^k_0, q^k_0]^{\mathrm{e}}$ and $[p^k_k, q^k_{k-1}]^{\mathrm{e}}$ belong to L, then for all $x, y \in H_k$ with $x \neq y$, $[x, y]^{\mathrm{e}}$ also belongs to L.

We will follow the "symmetrized" (and easier) variant of some computations used in Section 2; note that this technique goes back to Zádori [12]. Temporarily, consider the following equivalences; their dependence on k will not be indicated.

$$\begin{aligned}
\overrightarrow{\alpha}_j &:= \sum_{i=1}^{j}[p^k_i, p^k_{i-1}]^{\mathrm{e}} + \sum_{i=1}^{j-1}[q^k_i, q^k_{i-1}]^{\mathrm{e}} \quad \text{for } j \in \{1, \dots, k\},\\
\overrightarrow{\beta}_j &:= \sum_{i=0}^{j}[p^k_i, q^k_i]^{\mathrm{e}} \quad \text{for } j \in \{0, \dots, k-1\},\\
\overrightarrow{\gamma}_j &:= \sum_{i=1}^{j}[p^k_i, q^k_{i-1}]^{\mathrm{e}} \quad \text{for } j \in \{1, \dots, k\},\\
\overleftarrow{\alpha}_j &:= \sum_{i=1}^{j}[p^k_{k-i}, p^k_{k-i+1}]^{\mathrm{e}} + \sum_{i=1}^{j-1}[q^k_{k-1-i}, q^k_{k-i}]^{\mathrm{e}} \quad \text{for } j \in \{1, \dots, k\},\\
\overleftarrow{\beta}_j &:= \sum_{i=1}^{j}[p^k_{k-i}, q^k_{k-i}]^{\mathrm{e}} \quad \text{for } j \in \{1, \dots, k\},\\
\overleftarrow{\gamma}_j &:= \sum_{i=0}^{j}[p^k_{k-i}, q^k_{k-1-i}]^{\mathrm{e}} \quad \text{for } j \in \{0, \dots, k-1\}.
\end{aligned}$$

By induction, we are going to show that all of them belong to L. Using (3.1), we obtain easily that for $j \in \{0, 1, \dots, k-1\}$,

$$\overrightarrow{\alpha}_{j+1} = \alpha'(\overrightarrow{\beta}_j + \gamma'), \quad \overrightarrow{\gamma}_{j+1} = \gamma'(\overrightarrow{\alpha}_{j+1} + \beta'), \quad \text{and, if}$$
$$j < k-1, \quad \overrightarrow{\beta}_{j+1} = ((\overrightarrow{\gamma}_{j+1} + \beta')\alpha' + \gamma')\beta'.$$

Hence, since $\overrightarrow{\beta}_0 = [p^k_0, q^k_0]^{\mathrm{e}} \in L$ by assumption, we obtain that the "right-going" equivalences $\overrightarrow{\alpha}_j$, $\overrightarrow{\beta}_j$, and $\overrightarrow{\gamma}_j$ belong to L for all permitted values of their subscripts.

A similar induction based on $\overleftarrow{\gamma}_0 = [p^k_k, q^k_{k-1}]^{\mathrm{e}} \in L$ and

$$\overleftarrow{\alpha}_{j+1} = \alpha'(\overleftarrow{\gamma}_j + \beta'),\quad \overleftarrow{\beta}_{j+1} = \beta'(\overleftarrow{\alpha}_{j+1} + \gamma'),\quad \text{and, if } j < k-1,\quad \overleftarrow{\gamma}_{j+1} = ((\overleftarrow{\beta}_{j+1} + \gamma')\alpha' + \beta')\gamma'$$

yield that each of $\overleftarrow{\alpha}_j$, $\overleftarrow{\beta}_j$, and $\overleftarrow{\gamma}_j$ is in L for all permitted values of its subscript. Hence,

$$\begin{aligned}
[p^k_{j-1}, p^k_j]^{\mathrm{e}} &= \overrightarrow{\alpha}_j \cdot \overleftarrow{\alpha}_{k+1-j} \in L \quad \text{for } j \in \{1, \dots, k\},\\
[p^k_j, q^k_j]^{\mathrm{e}} &= \overrightarrow{\beta}_j \cdot \overleftarrow{\beta}_{k-j} \in L \quad \text{for } j \in \{0, \dots, k-1\},\\
[p^k_j, q^k_{j-1}]^{\mathrm{e}} &= \overrightarrow{\gamma}_j \cdot \overleftarrow{\gamma}_{k-j} \in L \quad \text{for } j \in \{1, \dots, k\}, \text{ and}\\
[q^k_{j-1}, q^k_j]^{\mathrm{e}} &= \alpha'([p^k_j, q^k_{j-1}]^{\mathrm{e}} + [p^k_j, q^k_j]^{\mathrm{e}}) \quad \text{for } j \in \{1, \dots, k-1\}.
\end{aligned}$$

Hence, for every edge (x, y) of H_k, $[x, y]^{\mathrm{e}} \in L$, and we obtain the validity of (3.3) by Lemma 2.1.

Next, we claim that (in the fence H_n right below F_2)

$$(3.4)\qquad [p^n_0, q^n_0]^{\mathrm{e}} = \beta\delta(\gamma + [a_1, b_1]^{\mathrm{e}}) \in L \text{ and } [p^n_n, q^n_{n-1}]^{\mathrm{e}} = \gamma\delta(\beta + [a_1, b_0]^{\mathrm{e}}) \in L.$$

We know from (3.2) that $[a_1, b_1]^{\mathrm{e}} = \langle a_1, b_1\rangle^{\mathrm{q}} + \langle b_1, a_1\rangle^{\mathrm{q}} \in L$ and $[a_1, b_0]^{\mathrm{e}} = \langle a_1, b_0\rangle^{\mathrm{q}} + \langle b_0, a_1\rangle^{\mathrm{q}} \in L$. Hence, to prove (3.4), it suffices to prove the equalities in it. In case of the first equality, the "$\leq$" inequality is clear. In order to show the converse inequality, assume that $(x, y) \in \beta\delta(\gamma + [a_1, b_1]^{\mathrm{e}})$ and $x \neq y$. Since $(x, y) \in \beta\delta$,

$$(3.5)\qquad (x, y) \in \{(a_0, b_0)\} \cup \{(p^n_0, q^n_0)\} \cup \{(p^j_0, q^j_0) : n < j < n + g\},$$

or the same holds for (y, x) but this alternative can be neglected since x and y play symmetric roles. Furthermore, there is a shortest undirected $(\gamma \cup [a_1, b_1]^{\mathrm{e}})$-path P in the graph $G(\vec{m})$ from x to y. It is clear from (3.1) that P contains the edge $\{a_1, b_1\}$, and it contains it exactly once. Hence,

$$(3.6)\qquad \text{either } (x, a_1) \in \gamma \text{ and } (y, b_1) \in \gamma, \text{ or } (y, a_1) \in \gamma \text{ and } (x, b_1) \in \gamma.$$

In order to make a distinction from what comes next, a (straight) line in the Euclidean plane will be called a *geometric line*. On the other hand, a $G(\vec{m})$*-line* is a maximal subset S of (the vertex set of) $G(\vec{m})$ such that S is a line (a sequence of adjacent vertices) in the graph and the members of S lie on the same geometric line. For $u \neq v$ in $G(\vec{m})$, let $\overline{uv}$ denote the unique $G(\vec{m})$-line that contains u and v; note that it need not exist. In order to avoid confusion, there will be no notation for the geometric line through u and v. For example, in Figure 2, $\overline{p^5_0 p^4_1} = \{p^5_0, q^4_0, p^4_1, q^3_1, p^3_2\}$, $q^4_2 \notin \overline{q^5_1 p^5_2}$ though q^4_2 lies on the geometric line through q^5_1 and p^5_2, and $\overline{p^5_2 q^4_2}$ does not exist. We obtain from (3.6) that

$$\text{either both } \overline{xa_1} \text{ and } \overline{yb_1}, \text{ or both } \overline{xb_1} \text{ and } \overline{ya_1} \text{ are of slope } 45^\circ.$$

Hence, the $G(\vec{m})$-lines through a_1 and b_1 of slopes 45° contain x and y. Combining this with (3.5), it follows that $\{x, y\} = \{p^n_0, q^n_0\}$, proving the first equality in (3.4). The second equality follows in an analogous way, but we need to consider $G(\vec{m})$-lines of slope 90°, that is, vertical $G(\vec{m})$-lines. This completes the proof of (3.4).

Next, we claim that

$$(3.7)\qquad \text{if } x, y \in G(\vec{m}) \text{ belong to the same fence and } x \neq y, \text{ then } [x, y]^{\mathrm{e}} \in L.$$

We prove this by induction on the rank of the fence containing x and y. If this rank is 2, then $[x, y]^{\mathrm{e}} \in L$ by (3.2). If the rank in question is n, the next one after

2, then $[x,y]^{\mathrm{e}} \in L$ by (3.3) and (3.4). So assume that $0 < i < g$ and (3.7) holds whenever the rank is $n+i-1$, that is, when $x, y \in H_{n+i-1}$. By this induction hypothesis, $[p_1^{n+i-1}, q_1^{n+i-1}]^{\mathrm{e}} \in L$. Repeating the argument with $G(\vec{m})$-lines of slopes 45° that was used in the proof of (3.4), we obtain easily that $[p_0^{n+i}, q_0^{n+i}]^{\mathrm{e}} = \beta\delta(\gamma + [p_1^{n+i-1}, q_1^{n+i-1}]^{\mathrm{e}}) \in L$. Similarly, working with vertical $G(\vec{m})$-lines, we obtain that $[p_{n+i}^{n+i}, q_{n+i-1}^{n+i}]^{\mathrm{e}} = \gamma\delta(\beta + [p_{n+i-2}^{n+i-1}, q_{n+i-3}^{n+i-1}]^{\mathrm{e}}) \in L$. Thus, $[x,y]^{\mathrm{e}} \in L$ by (3.3). This completes the induction step and proves (3.7).

We need to take care also of some edges that connect neighboring fences. There are two cases. First, we assert that, denoting $n+i$ by j for $1 \le i < g$,

$$
\begin{aligned}
[p_0^j, q_0^{j-1}]^{\mathrm{e}} = \gamma \cdot ([p_0^j, p_j^j]^{\mathrm{e}} + \beta + [q_{j-2}^{j-1}, q_0^{j-1}]^{\mathrm{e}})& \\
\times ([p_0^j, p_{j-1}^j]^{\mathrm{e}} + \beta + [q_{j-3}^{j-1}, q_0^{j-1}]^{\mathrm{e}}) \in L&,
\end{aligned}
\tag{3.8}
$$

and, analogously,

$$
\begin{aligned}
[p_j^j, q_{j-2}^{j-1}]^{\mathrm{e}} = \beta \cdot ([p_j^j, p_0^j]^{\mathrm{e}} + \gamma + [q_0^{j-1}, q_{j-2}^{j-1}]^{\mathrm{e}})& \\
\times ([p_j^j, p_1^j]^{\mathrm{e}} + \gamma + [q_1^{j-1}, q_{j-2}^{j-1}]^{\mathrm{e}}) \in L&.
\end{aligned}
\tag{3.9}
$$

The arguments for (3.8) and (3.9) are basically the same; apart from notation, the only difference is that $(\beta, \gamma, 90°, 45°)$ corresponds to $(\gamma, \beta, 45°, 90°)$. (In order to emphasize this duality, we will say "of slope 90°" rather than "vertical".) Hence, we give the details only for (3.8). Assume that a nontrivial pair (x,y) belongs to the right-hand side of the equality in (3.8). Then $(x,y) \in \gamma$ gives that the $G(\vec{m})$-line $\overline{xy}$ is of slope 45°. The intersection of this $G(\vec{m})$-line with some block B_1 of the second meetand contains x and y. Since this intersection contains two vertices while the β-blocks are $G(\vec{m})$-lines of slope 90°, B_1 cannot be a β-block. Hence, B_1 is the only block of $[p_0^j, p_j^j]^{\mathrm{e}} + \beta + [q_{j-2}^{j-1}, q_0^{j-1}]^{\mathrm{e}}$ not lying on a $G(\vec{m})$-line of slope 90°. Thus,

$$
\begin{aligned}
B_1 = \{p_0^j, q_0^j, q_0^{j-1}, p_0^{j-1}\} \cup \{&\text{the vertices of the } G(\vec{m})\text{-line} \\
&\text{of slope } 90° \text{ that contains } p_j^j\}.
\end{aligned}
$$

For $j = 4$ on the right of Figure 2, B_1 consists of the black-filled elements. It follows that $x, y \in B_1$. The third meetand, $[p_0^j, p_{j-1}^j]^{\mathrm{e}} + \beta + [q_{j-3}^{j-1}, q_0^{j-1}]^{\mathrm{e}}$, has also only one block not lying on a $G(\vec{m})$-line of slope 90°; this block is

$$
\begin{aligned}
B_2 = \{p_0^j, q_0^j, q_0^{j-1}, p_0^{j-1}\} \cup \{&\text{the vertices of the } G(\vec{m})\text{-line} \\
&\text{of slope } 90° \text{ that contains } p_{j-1}^j\}.
\end{aligned}
$$

Since the role of B_1 and that of B_2 are similar, $x, y \in B_2$. Since the geometrical lines occurring in the description of B_1 and B_2 are parallel, they have no vertex in common. Hence, $x, y \in B_1 \cap B_2 = \{p_0^j, q_0^j, q_0^{j-1}, p_0^{j-1}\}$. Since $\overline{xy}$ is of slope 45°, it follows that $\{x, y\} = \{p_0^j, q_0^{j-1}\}$. This yields the "$\ge$" inequality in (3.8). The converse inequality "$\le$" is trivial. Thus, (3.8) holds. So does (3.9).

Second, we assert that

$$
[p_0^n, b_0]^{\mathrm{e}} = \gamma([p_0^n, p_{n-1}^n]^{\mathrm{e}} + \beta) \in L \text{ and } [p_n^n, b_1]^{\mathrm{e}} = \beta([p_1^n, p_n^n]^{\mathrm{e}} + \gamma) \in L. \tag{3.10}
$$

Again, it suffices to show the inequality "$\ge$" in place of the first equality. For every nontrivial pair $(x,y) \in \gamma$, the only non-vertical $([p_0^n, p_{n-1}^n]^{\mathrm{e}} + \beta)$-block that can intersect $\overline{xy}$ in more than one vertex is $\{p_0^n, q_0^n\} \cup \{$the vertices of the vertical $G(\vec{m})$-line through $p_{n-1}^n\}$. Hence, $\{x,y\} = \{p_0^n, b_0\}$ and (3.10) holds.

Now, we are in the position to complete the proof. Although we have not considered all connecting edges between two adjacent fences, (3.7), (3.8), (3.9), (3.10), and Lemma 2.1 allow us to conclude that L contains $[x,y]^{\mathrm{e}}$ for all $x \neq y \in G(\vec{m})$. Therefore, since $\delta \notin \mathrm{Equ}(G(\vec{m}))$, Lemma 3.2 completes the proof of Lemma 3.1. $\square$

4. The rest of the proof

A nonempty subset P of a lattice L is a *complete prime ideal* if it is closed with respect to arbitrary joins, $L \setminus P$ is nonempty and closed with respect to arbitrary meets, and for all $x \leq y$ in L, $y \in P \Rightarrow x \in P$. Using this concept, we can reformulate the Wille's D_2 from [13] as follows. Note that the idea of using D_2-lemma in a similar context for *equivalence* lattices goes back to Strietz [8, 9]. A sublattice S of a lattice L is *proper* if $S \neq L$.

Lemma 4.1 (Wille's D_2-lemma reformulated)**.** *Assume that L is a complete lattice and X and Y are disjoint nonempty subsets of L. If L has no complete prime ideal and no proper complete sublattice of L includes $X \cup Y$, then $\bigwedge X \leq \bigvee Y$.*

Proof. Although the original statement is about simple lattices and finite sets X and Y, Wille's proof in [13] works in the present situation without any significant change. Namely, suppose for a contradiction that $u := \bigwedge X \not\leq \bigvee Y =: v$. Let $P = \{z \in L : z \leq v\}$ and $Q = \{z \in L : z \geq u\}$. These sets are disjoint since $u \not\leq v$. Since $P \cup Q$ is closed with respect to arbitrary meets and joins, it is a complete sublattice. Since this sublattice includes $X \cup Y$, $P \cup Q = L$. Clearly, P is a complete prime ideal, which is a contradiction proving the lemma. $\square$

Proof of Theorem 1.1. If $k \in \{5, 7, 9, 11, \dots\}$, then $F_{(k-1)/2}$ from Section 2 is of size k. So is $G((k-6)/2)$ if $k \geq 12$ is even. Also, $G(\vec{m})$ is of size $\aleph_0$ if $\vec{m} = (3, 4, 5, 6, \dots)$. Thus, the required existence of four generators follows from Lemmas 2.2 and 3.1, except for $|A| \leq 3$. For $|A| = 2$, $|\mathrm{Quo}(A)| = 4$ and so $\mathrm{Quo}(A)$ is four-generated. Finally, as it was pointed out in Kulin [7, page 61], the five-element $\mathrm{Equ}(\{1, 2, 3\})$ is three-generated and so $\mathrm{Quo}(\{1, 2, 3\})$ is four-generated by Lemma 3.2. Note that there is an overlapping between the scopes of Section 2 and 3. The smallest odd size settled by both sections is $|A| = 21 = |G(3, 4)| = |F_{10}|$.

Next, suppose for a contradiction that there is a three-generated sublattice of $\mathrm{Quo}(A)$ containing all atoms of $\mathrm{Quo}(A)$. Let ε_0, ε_1, and ε_2 generate such a sublattice. Since every quasiorder is the (not necessarily finite) join of all atoms below it, no proper complete sublattice of $\mathrm{Quo}(A)$ includes $\{\varepsilon_0, \varepsilon_1, \varepsilon_2\}$. Since we are going to apply Lemma 4.1, we need to show that $\mathrm{Quo}(A)$ has no complete prime ideal. Suppose the contrary, and let P be a complete prime ideal. Denote $\mathrm{Quo}(A) \setminus P$ by Q; note that Q is called a complete prime filter. With $\mu := \bigvee P$ and $\nu := \bigwedge Q$, we have that $P = \{\rho \in \mathrm{Quo}(A) : \rho \leq \mu\}$ and $Q = \{\rho \in \mathrm{Quo}(A) : \rho \geq \nu\}$. Consider an atom $\langle a, b\rangle^{\mathrm{q}} \in \mathrm{Quo}(A)$. Since $\mathrm{Quo}(A)$ is the disjoint union of P and Q, either $\langle a, b\rangle^{\mathrm{q}} \in P$, or $\langle a, b\rangle^{\mathrm{q}} \in Q$. However, we cannot have $\langle a, b\rangle^{\mathrm{q}} \in P$ for *every* nontrivial pair $(a, b) \in A^2$, because otherwise P would contain all atoms and all joins of atoms, that is, all elements of $\mathrm{Quo}(A)$, contradicting $Q = \mathrm{Quo}(A) \setminus P \neq \varnothing$. Therefore, there are $c \neq d \in A$ such that $\langle c, d\rangle^{\mathrm{q}} \in Q$. Thus, $\langle c, d\rangle^{\mathrm{q}} \geq \nu$. So either $\nu = \langle c, d\rangle^{\mathrm{q}}$, or $\nu = \Delta_A$. But if we had $\nu = \Delta_A$, then $Q = \mathrm{Quo}(A)$ would contradict $P \neq \varnothing$. Hence, $\nu = \langle c, d\rangle^{\mathrm{q}}$. Therefore, if $(a, b) \in A^2$ is a nontrivial pair such that $(a, b) \neq (c, d)$, then $\langle a, b\rangle^{\mathrm{q}} \not\geq \langle c, d\rangle^{\mathrm{q}}$ gives that $\langle a, b\rangle^{\mathrm{q}} \in P$. Now, since $|A| \geq 3$, we can pick an element $e \in A \setminus \{c, d\}$. Since $(c, d) \notin \{(c, e), (e, d), (d, c)\}$, we have that $\langle c, e\rangle^{\mathrm{q}}$, $\langle e, d\rangle^{\mathrm{q}}$, and $\langle d, c\rangle^{\mathrm{q}}$ are in P. Also, $\langle c, e\rangle^{\mathrm{q}} + \langle e, d\rangle^{\mathrm{q}} + \langle d, c\rangle^{\mathrm{q}} \in P$ since P is join-closed. But $\langle c, d\rangle^{\mathrm{q}} \leq \langle c, e\rangle^{\mathrm{q}} + \langle e, d\rangle^{\mathrm{q}} + \langle d, c\rangle^{\mathrm{q}} \in P$ and P is a (complete prime) ideal, so we obtain that $\langle c, d\rangle^{\mathrm{q}} \in P$. This contradicts $P \cap Q = \varnothing$ and proves that $\mathrm{Quo}(A)$ has no prime ideal.

Now, for $\{i, j, k\} = \{0, 1, 2\}$ we can conclude from Lemma 4.1 that $\varepsilon_i \le \varepsilon_j + \varepsilon_k$ and $\varepsilon_i \cdot \varepsilon_j \le \varepsilon_k$. So we obtain easily that the join of any two of the three generators is $\varepsilon_0 + \varepsilon_1 + \varepsilon_2$ and dually. That is, $\mathrm{Quo}(A)$ consists of only five elements (and it is M_3, the five-element modular non-distributive lattice). This contradicts $|\mathrm{Quo}(A)| \ge 29$ from Section 1 and completes the proof of our theorem. □

Added on August 13, 2016. The absence of prime ideals in the proof above follows also from Tůma [11, Theorem 1], which asserts that $\mathrm{Quo}(A)$ is simple for $|A| \ge 3$.

References

[1] Chajda, I., Czédli, G. (1996). Four notes on quasiorder lattices. *Math. Slovaca* **46**: 371–378.
[2] Chajda, I., Czédli, G. (1996). How to generate the involution lattice of quasiorders? *Studia Sci. Math. Hungar.* **32**: 415–427.
[3] Czédli, G. (1996). Lattice generation of small equivalences of a countable set. *Order* **13**: 11–16.
[4] Czédli, G. (1996). Four-generated large equivalence lattices. *Acta Sci. Math. (Szeged)* **62**: 47–69.
[5] Czédli, G. (1999). (1+1+2)-generated equivalence lattices. *J. Algebra* **221**: 439–462.
[6] Dolgos, T. (2015). Generating equivalence and quasiorder lattices over finite sets. *BSc Theses, University of Szeged* (in Hungarian).
[7] Kulin, J. (2016). Quasiorder lattices are five-generated. *Discuss. Math. Gen. Algebra Appl.* **36**: 59–70.
[8] Strietz H. (1975). Finite partition lattices are four-generated. In: *Proc. Lattice Th. Conf. Ulm, 1975*, pp. 257–259.
[9] Strietz H. (1977). Über Erzeugendenmengen endlicher Partitionverbände. *Studia Sci. Math. Hungarica* **12**:1–17.
[10] Takách, G. (1996): Three-generated quasiorder lattices. *Discuss. Math. Algebra Stochastic Methods* **16**: 81–98.
[11] Tůma, J. (2002): On the structure of quasi-ordering lattices. *Acta Univ. Carolin. Math. Phys.* **43**: 65–74.
[12] Zádori, L. (1986) Generation of finite partition lattices. In: *Lectures in universal algebra. (Proc. Colloq. Szeged, 1983)* Colloq. Math. Soc. János Bolyai, Vol. 43. Amsterdam: North-Holland, pp. 573–586.
[13] Wille, R. (1973). Free modular lattices generated by finite chains. *Algebra Universalis* **3**:131–138.

E-mail address: `czedli@math.u-szeged.hu`
URL: `http://www.math.u-szeged.hu/~czedli/`

University of Szeged, Bolyai Institute. Szeged, Aradi vértanúk tere 1, HUNGARY 6720