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Four-generated large equivalence lattices

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Dedicated to E. Tamás Schmidt on his 60th birthday

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Abstract. Strietz [4, 5] has shown that Equ(A), the lattice of all equivalences of a finite set A, has a four-element generating set. We extend this result for many infinite sets A; even for all sets if there are no inaccessible cardinals. Namely, we prove that if A is a set consisting of at least four elements and there is no inaccessible cardinal $\leq |A|$, then the *complete* lattice Equ(A) can be generated by four elements. This result is sharp in the sense that Equ(A) cannot be generated by three elements.

1. The main result

Given a set A, let Equ(A) denote the (complete) lattice of all equivalences of A. If A is finite, then Equ(A) can be generated by four elements, but three elements are insufficient for $|A| \geq 4$, cf. Strietz [4, 5] and Zádori [6]. Our aim is to extend this result for some infinite sets A. Then, of course, we have to consider Equ(A) as a complete lattice, for otherwise it would not be finitely generated. An analogous result (for some infinite cardinalities) was proved for the (involution) lattice of all quasiorders of A in [1]. The present paper benefits a lot from the ideas developed in [1] and Zádori [6].

As usual, $\aleph_0$ denotes the smallest infinite cardinal. A cardinal m is called *inaccessible* if it satisfies the following three conditions: (i) $m > \aleph_0$; (ii) $n < m$

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implies $2^n < m$; and (iii) if I is a set of cardinals such that $|I| < m$ and $n < m$ for all $n \in I$, then $\sup\{n\colon n \in I\} < m$. Note that $\sup\{n\colon n \in I\}$ in (iii) can be replaced by $\sum_{n \in I} n$. For details on inaccessible cardinals the reader can resort to standard textbooks, e.g., to Levy [3, pp. 138–141]. By Kuratowski's result [2] (cf. also [3]), ZFC has a model without inaccessible cardinals. Hence the existence of inaccessible cardinals cannot be proved from ZFC, and the scope of the following theorem includes all sets in an appropriate model of set theory.

Theorem 1. *Let A be a set with at least four elements, and suppose that there is no inaccessible cardinal m such that $m \le |A|$. Then the complete lattice* $\mathrm{Equ}(A)$ *of all equivalences of A has a four-element generating set.*

The rest of the paper is devoted to the proof of this theorem. But first of all we notice that $\mathrm{Equ}(A)$ in the theorem cannot be generated by less than four elements. This observation for finite sets was proved by Zádori [6, Lemma 2 and page 580], and — luckily enough — his proof is valid for infinite sets without any essential change.

II. Boxes and their extensions

By Strietz [4, 5], it is sufficient to prove the result only for infinite sets. So even if we start the proof at some finite sets, we do not have to (and will not) deal with all finite sets. Basically, the proof is an induction on $|A|$. However, the mere assumption of the statement for a given cardinal is far from being a suitable induction hypothesis. Therefore we have to build a structure on A and study these structures in the necessary extent. Before developing the necessary terminology, we give an example.

Let $L_0 = \{a_0, a_1, \dots, a_{16}, b_0, b_1, \dots, b_{15}\}$. For $p, q \in L_0$ (or p, q in any set), let $\langle p, q\rangle$ denote the smallest equivalence collapsing p and q. Note that $\langle p, q\rangle = \langle q, p\rangle$ is an atom in $\mathrm{Equ}(L_0)$ if $p \ne q$, and $\langle p, p\rangle = 0 \in \mathrm{Equ}(L_0)$. If $x \in L_0$ and $\Theta \in \mathrm{Equ}(L_0)$, then the Θ-class containing x will be denoted by $[x]\Theta$. Denoting the lattice operations by $\sum$ or $+$ (join) and $\prod$ or $\cdot$ (meet), we let

$$\alpha_0 = \sum_{i=0}^{15} \langle a_i, a_{i+1}\rangle + \sum_{i=0}^{14} \langle b_i, b_{i+1}\rangle, \qquad \beta_0 = \sum_{i=0}^{15} \langle a_i, b_i\rangle,$$

$$\gamma_0 = \sum_{i=0}^{15} \langle b_i, a_{i+1}\rangle, \qquad \delta_0 = \langle a_0, b_0\rangle + \langle a_{16}, b_{15}\rangle.$$

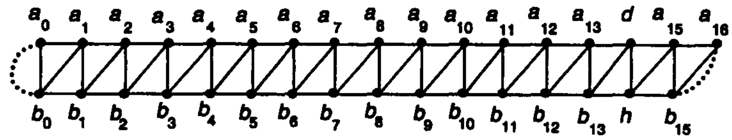

Figure 1

These equivalences are represented by horizontal, vertical, southwest—northeast, and dotted lines in Figure 1, respectively.

We will soon show that $\{\alpha_0, \beta_0, \gamma_0, \delta_0\}$ generates $\mathrm{Equ}(L_0)$. The elements $d = d_0 = a_{14}$ and $h = h_0 = b_{14}$ will be treated as constants. For $i = 1, 3, 5, 7, 9$, the quadruplet $e = (b_i, a_{i+3}, b_{i+1}, a_{i+4})$ is called an *edge pair*. Let E_0 denote the set of edge pairs. The edge pairs of L_0 are represented by parallelograms in Figure 2. Associated with an $e \in E_0$ we will use the notation $e = (b_e, a_e, b'_e, a'_e)$.

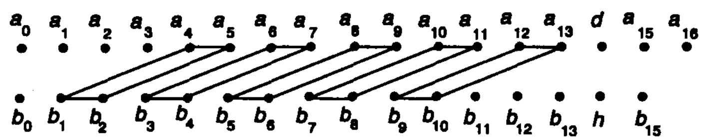

Figure 2

What we have depicted in Figures 1 and 2 is just a particular case of a more general structure, which we introduce under the name "box".

Definition 2. By a *box* we mean a structure

$$A = (A, d, h, E, \alpha, \beta, \gamma, \delta)$$

provided

(b1) A is a set with $|A| \geq 6$, and $d, h \in A$ are distinct constants;

(b2) $\alpha, \beta, \gamma, \delta \in \mathrm{Equ}(A)$ such that $\alpha\delta = 0$;

(b3) $E \subseteq (A \setminus \{d, h\})^4$, each $e = (b_e, a_e, b'_e, a'_e) \in E$ (called an edge pair) has four distinct components, and $\{b_e, a_e, b'_e, a'_e\} \cap \{b_f, a_f, b'_f, a'_f\} = \emptyset$ for all distinct $e, f \in E$;

(b4) For all $e \in E$, the restriction of α to $\{b_e, a_e, b'_e, a'_e\}$ is $\langle a_e, a'_e \rangle + \langle b_e, b'_e \rangle$ in $\mathrm{Equ}(\{b_e, a_e, b'_e, a'_e\})$, while the restrictions of β, γ and δ to $\{b_e, a_e, b'_e, a'_e\}$ are 0;

(b5) $(d, h) \in \beta$, but $(d, h) \notin \alpha$, $(d, h) \notin \gamma$, $(d, h) \notin \delta$;

(b6) the δ-classes $[d]\delta$, $[h]\delta$ and, for every $e \in E$, $[b_e]\delta$, $[a_e]\delta$, $[b'_e]\delta$, $[a'_e]\delta$ are singletons; and (b7) for all $e \in E$, none of (b_e, d), (b'_e, d), (a_e, h), (a'_e, h) belongs to α.

The conditions defining a box are not quite independent; (b6) implies the δ-part of (b4) and (b5). The idea behind this notion is that $\{\alpha, \beta, \gamma, \delta\}$ will, by the end of the proof, generate $\mathrm{Equ}(A)$. Let $\mathcal{A}_i = (A_i, d_i, h_i, E_i, \alpha_i, \beta_i, \gamma_i, \delta_i)$ be boxes for $i = 1, 2$. A bijective map $\varphi\colon A_1 \to A_2$ is called an *isomorphism* if $\varphi(d_1) = d_2$, $\varphi(h_1) = h_2$, $\varphi(\alpha_1) = \{(\varphi(x), \varphi(y))\colon (x, y) \in \alpha_1\} = \alpha_2$, $\varphi(\beta_1) = \beta_2$, $\varphi(\gamma_1) = \gamma_2$, $\varphi(\delta_1) = \delta_2$, and $\varphi(E_1) = \{(\varphi(x), \varphi(y), \varphi(z), \varphi(t))\colon (x, y, z, t) \in E_1\} = E_2$. Since we want to create large boxes from smaller ones, we introduce a concept that expresses how the small boxes can be put together.

Definition 3. Suppose $\mathcal{A}_0 = (A_0, d_0, h_0, E_0, \alpha_0, \beta_0, \gamma_0, \delta_0)$ and $\mathcal{B} = (B, d, h, E, \alpha, \beta, \gamma, \delta)$ are boxes and Γ is a partition on the set B. Let A_i $(i \in I)$ denote the classes of this partition. We assume that $0 \in I$, so the support of the box $\mathcal{A}_0$ is one of the classes. For $i \in I \setminus \{0\}$, let $\varphi_i\colon A_0 \to A_i$ be a bijection, and define $d_i = \varphi_i(d_0)$, $h_i = \varphi_i(h_0)$, $E_i = \varphi_i(E_0) = \{(\varphi_i(x), \varphi_i(y), \varphi_i(z), \varphi_i(t))\colon (x, y, z, t) \in E_0\}$, $\alpha_i = \varphi_i(\alpha_0)$, $\beta_i = \varphi_i(\beta_0)$, $\gamma_i = \varphi_i(\gamma_0)$, and $\delta_i = \varphi_i(\delta_0)$. Let φ_0 denote the identical automorphism of A_0. For $p \in A_0$ or $e \in E_0$, we use the notation $p_i = \varphi_i(p)$ or $e_i = \varphi_i(e)$, respectively. Then $\mathcal{A}_i = (A_i, d_i, h_i, E_i, \alpha_i, \beta_i, \gamma_i, \delta_i)$ is a box isomorphic to $\mathcal{A}_0$. Let us assume that

(e1) $\beta = \bigcup_{i\in I} \beta_i, \qquad \gamma = \bigcup_{i \in I} \gamma_i$;

(e2) $\alpha = \bigcup_{i\in I} \alpha_i + \sum_{i,j\in I} \langle d_i, d_j\rangle$;

(e3) there exist $F, G \subseteq E_0 \times I$ and $H \subseteq E_0 \times I \times I$ such that $(e, i) \in F \cup G$ implies $\{(e, i, j), (e, j, i)\} \cap H = \emptyset$ for all $j \in I$, $F \cap G = \emptyset$, $(e, i, j) \in H$ implies $i \neq j$, $(e, i, j) \in H$ implies $(e, \ell, i) \notin H$ for all $\ell \in I$, $(e, i, j) \in H$ and $(e, \ell, j) \in H$ imply $i = \ell$, and

$$\delta = \bigcup_{i\in I} \delta_i + \sum_{(e,i)\in F} \langle b_{e_i}, a_{e_i}\rangle + \sum_{(e,i)\in G} \langle b'_{e_i}, a'_{e_i}\rangle + \\ + \sum_{(e,i,j)\in H} \Big(\langle a_{e_i}, b_{e_j}\rangle + \langle b'_{e_i} + b'_{e_j}\rangle\Big);$$

(e4) $d = d_0$, $h = h_0$; and

(e5) $E \subseteq \bigcup_{i\in I} E_i$.

Then $\mathcal{B}$ is called an *extension* of $\mathcal{A}_0$, in notation $\mathcal{B} \mid \mathcal{A}_0$. Let $\Phi = (\Gamma, \{\varphi_i\colon i \in I\})$; it is called the *way of extension*. Sometimes, when Φ is relevant, we say that $\mathcal{B}$ *is an extension of* $\mathcal{A}_0$ *by* Φ, in notation $\mathcal{B} \mid_\Phi \mathcal{A}_0$. For $E'_0 \subseteq E_0$, $\bigcup_{i\in I} E'_i = \{e_i\colon i \in I, e \in E'_0\}$ is called the *extension* of E'_0 to $\mathcal{B}$ (by Φ). If the extension of E'_0 $(\subseteq E_0)$ is included in E, then the box extension $\mathcal{B} \mid \mathcal{A}_0$ is called E'_0*-preserving*. By the *degree* of the box extension $B \mid_{\Phi} A_0$ we mean $|I|$; the degree is denoted by $[B : A_0]$. (We will use this notation only when the meaning of Φ — at least implicitly — is already given. Note that $|B| = [B : A_0] \cdot |A_0|$.)

For example, if $A_0 = L_0$ is the box defined by Figures 1 and 2, $I = \{\emptyset, U_1, U_2, U_3\}$, $H = \{(b_1, a_4, b_2, a_5)\} \times \{\emptyset\} \times \{U_1, U_2, U_3\}$, $E = \{(b_7, a_{10}, b_8, a_{11}), (b_9, a_{12}, b_{10}, a_{13})\} \times I$, and F and G are appropriately chosen, then an extension B of A_0 is depicted in Figure 3. Notice that $A_0 \subseteq B$ but $E_0 \not\subseteq E$.

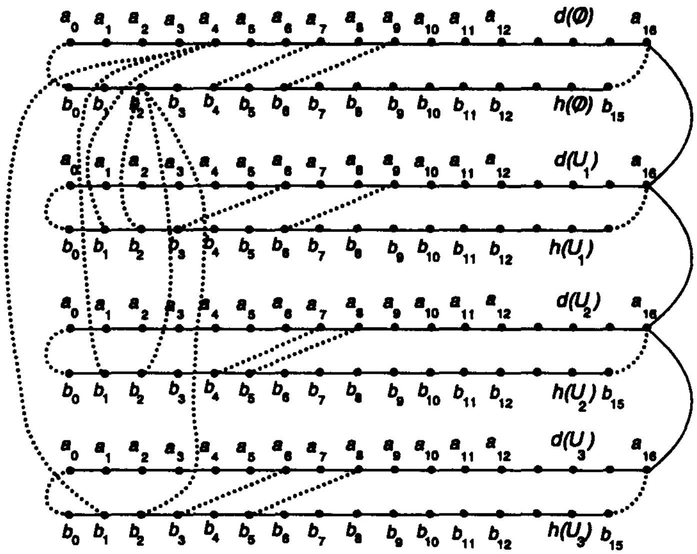

Figure 3

Now, for better understanding and later reference, we formulate (e3) less formally. For an edge pair e, the atoms $\langle b_e, a_e \rangle$ and $\langle b'_e, a'_e \rangle$ are called the *left atom* and *right atom* of e, respectively. For distinct edge pairs e and f, let $\langle a_e, b_f \rangle + \langle b'_e, b'_f \rangle$ be called the *connecting equivalence* from e to f. Now (e3) means

(e3') To obtain δ from $\bigcup_{i \in I} \delta_i$, we can add some left atoms, some right atoms, and some connecting equivalences but we have to follow the following rules. Every edge pair can be used for at most one of the following purposes: either to determine a left atom or to determine a right atom or (together with another edge) to determine a connecting equivalence. If e and f are edge pairs and we use (e, f) to obtain a connecting equivalence (from e to f), then f cannot be used at other connecting equivalences, and e can be used only as the first edge pair to obtain other connecting equivalences.

Now we formulate some kind of transitivity for extensions. Let $B_0 = (B_0, \bar{d}_0, \bar{h}_0, \bar{E}_0, \bar{\alpha}_0, \bar{\beta}_0, \bar{\gamma}_0, \bar{\delta}_0)$ be an extension of A_0 by $\Phi = (\Gamma, \{\varphi_i\colon i \in I\})$. Further, let $C = (C, d, h, E, \alpha, \beta, \gamma, \delta)$ be an extension of B_0 by $\Psi = (\Delta, \{\psi_j\colon j \in J\})$. I.e., $\Delta = \{B_j\colon j \in J\}$ is a partition of C, $\psi_j\colon B_0 \to B_j$ is a bijection ($j \in J$, ψ_0 is the identical map of B_0), $\bar{d}_j := \psi_j(\bar{d}_0)$, $\bar{h}_j := \psi_j(\bar{h}_0)$, $\bar{\alpha}_j := \psi_j(\bar{\alpha}_0)$, etc. For $i \in I$ and $j \in J$, let $A_{j,i} = \psi_j(\varphi_i(A_0)) = (\psi_j \circ \varphi_i)(A_0)$, and define $\varrho_{j,i} = \psi_j \circ \varphi_i$. Then the $A_{j,i}$ $((j,i) \in J \times I)$ form a partition of C, which we denote by $\Delta \circ \Gamma$. Let us identify $(0,0)$ with 0. Then we obtain $\Psi \circ \Phi := (\Delta \circ \Gamma, \{\varrho_{j,i}\colon (j,i) \in J \times I\})$, which we call the *composition of* Ψ *and* Φ. With the above notations we have

Claim 4. *Suppose A_0, B_0 and C are boxes, $B_0 \mid A_0$ and $C \mid B_0$. Then*

(i) *if $B_0 \mid_\Phi A_0$ and $C \mid_\Psi B_0$, then $C \mid_{\Psi\circ\Phi} A_0$;*

(ii) $[C : A_0] = [C : B_0] \cdot [B_0 : A_0]$;

(iii) *if $E_0' \subseteq E_0$, $B_0 \mid_\Phi A_0$ is an E_0'-preserving extension, the extension of E_0' to B_0 is denoted by $\bar{E}_0'$, and $C \mid_\Psi B_0$ is an $\bar{E}_0'$-preserving extension, then $C \mid_{\Psi\circ\Phi} A_0$ is an E_0'-preserving extension.*

We do not know if $\Psi \circ \Phi$ is the only way of extension $C \mid A_0$, but permitting other ways would cause trouble in the sequel.

Proof. It suffices to show (i); the rest will follow as evident consequence. Roughly speaking, the proof is based on the fact that the edge pairs that we used to obtain $\bar{\delta}_j$ from the $\delta_{j,i}$ according to (e3') are not edge pairs in B_j by (b6). Conditions (e1), (e4) and (e5) are obvious. Now (e2) for A_0 and C asserts that

$$\alpha = \bigcup_{(j,i)\in J\times I} \alpha_{j,i} + \sum_{(j,i),(j',i')\in J\times I} \langle d_{j,i}, d_{j',i'}\rangle, \tag{1}$$

where $\alpha_{j,i} = \varrho_{j,i}(\alpha_0) = \psi_j(\varphi_i(\alpha_0))$, $d_{j,i} = \varrho_{j,i}(d_0)$, etc. We know that

$$\alpha = \bigcup_{j\in J} \bar{\alpha}_j + \sum_{j,j'\in J} \langle \bar{d}_j, \bar{d}_{j'}\rangle, \tag{2}$$

and

$$\bar{\alpha}_j = \bigcup_{i\in I} \alpha_{j,i} + \sum_{i,i'\in I} \langle d_{j,i}, d_{j,i'}\rangle. \tag{3}$$

Suppose $(p,q) \in \alpha$, $p \in A_{j,i}$, $q \in A_{j',i'}$. If $j = j'$, then $p, q \in B_j$, so, by (2), $(p,q) \in \bar{\alpha}_j$, so (p,q) is in the righthand side of (3), which is included in the righthand side of (1). Assume $j \neq j'$. Then $p \; \alpha \; p_1 \; \langle \bar{d}_j, \bar{d}_{j'} \rangle \; q_1 \; \alpha \; q$ by (2), where $p_1 \in B_j$, $q_1 \in B_{j'}$. By the previous case, (p, p_1) and (q_1, q) are in the righthand side of (1). But so is $\langle \bar{d}_j, \bar{d}_{j'} \rangle = \langle d_{j,0}, d_{j',0} \rangle$, and therefore (p, q) as well. This proves the "$\subseteq$" part of (1).

By (2) and (3),

$$\bigcup_{(j,i) \in J \times I} \alpha_{j,i} = \bigcup_{j \in J} \bigcup_{i \in I} \alpha_{j,i} \subseteq \bigcup_{j \in J} \bar{\alpha}_j \subseteq \alpha.$$

On the other hand,

$$\langle d_{j,i}, d_{j',i'} \rangle \subseteq \langle d_{j,i}, d_{j,0} \rangle + \langle \bar{d}_j, \bar{d}_{j'} \rangle + \langle d_{j',0}, d_{j',i'} \rangle.$$

Since these summands belong to $\alpha_j \subseteq \alpha$, α and $\alpha_{j'} \subseteq \alpha$, respectively, we obtain the "$\supseteq$" part of (1). Hence (e2) for A_0 and C holds.

To prove (e3) we work with (the equivalent) (e3'). We obtain δ from the δ_{ji} in two steps. In the first step, within every B_j, we add some left/right atoms and some connecting equivalences to $\bigcup_{i \in I} \delta_{j,i}$. I.e., we add these atoms and connecting equivalences to $\bigcup_{(j,i) \in J \times I} \delta_{j,i}$. Since the B_j are boxes, (b6) applies for them and implies that the edge pairs used in the first step cannot be edge pairs of B_j (i.e., they do not belong to $\bar{E}_j$). Therefore, when we add atoms and connecting equivalences to $\bigcup \bar{\delta}_j$ in the second step (to obtain δ from the $\bar{\delta}_j$), the edge pairs used in this second step are necessarily distinct from those used in the first step. We can, of course, replace these two steps by one in which we add all the atoms and connecting equivalences to $\bigcup_{(j,i) \in J \times I} \delta_{j,i}$ at the same time. Since different edge pairs were used in the two original steps, now every edge pair is still used at most once, and (e3') for A_0 and C holds. This proves (i) of Claim 4. ∎

Now let $B \mid_{\Phi} A_0$, and let us use the notations of Definition 3. For $p, q \in A_0$ we introduce the notation

$$\langle p, q \rangle^{(A_0, B)} = \langle p, q \rangle^{(A_0, B, \Phi)} = \sum_{i \in I} \langle p_i, q_i \rangle \in \mathrm{Equ}(B).$$

For $i \in I$, we will also use the notations $\langle p_i, q_i \rangle^{(A_0, B, \Phi)} := \langle p, q \rangle^{(A_0, B, \Phi)}$ and $\langle p_i, q_i \rangle^{(A_0, B)} := \langle p, q \rangle^{(A_0, B)}$. I.e., for $u, v \in A_i$ we define $\langle u, v \rangle^{(A_0, B, \Phi)}$ as $\langle \varphi_i^{-1}(u), \varphi_i^{-1}(v) \rangle^{(A_0, B, \Phi)}$. Usually we drop Φ from these notations but we must be careful: always a fixed Φ should be understood when it is not indicated.

Definition 5. A box A is called a *good box* if, for every extension $B = (B, d, h, E, \alpha, \beta, \gamma, \delta)$ of A by Φ and every complete sublattice Q of $\mathrm{Equ}(B)$ with $\{\alpha, \beta, \gamma, \delta\} \subseteq Q$, $\langle p, q\rangle^{(A,B,\Phi)} \in Q$ for all $p, q \in A$.

Notice that A is an extension of itself, and $\langle p, q\rangle^{(A,A)} = \langle p, q\rangle$ in this case. Since $\mathrm{Equ}(A)$ is clearly generated by its atoms $\langle p, q\rangle$ $(p \neq q)$, we conclude that $\mathrm{Equ}(A)$ is four-generated, provided A is a good box. This is why we want to find good boxes of any cardinality below the first inaccessible cardinal. To accomplish this task, first we show that L_0 (given in Figures 1 and 2) is a good box, then we give two methods to obtain larger good boxes from given good boxes, and finally we show that we can reach all infinite cardinals m (such that no inaccessible cardinal is $\leq m$) this way.

Claim 6. *L_0, the box defined in Figures 1 and 2, is a good box.*

Proof. Let $A = A_0 = L_0$, and let us consider an extension B of A; the notations from Definition 3 (except for p_i) will be in effect. Let Q be a complete sublattice of $\mathrm{Equ}(B)$ such that $\{\alpha, \beta, \gamma, \delta\} \subseteq Q$. For $X \subseteq B$ and $\Theta \in \mathrm{Equ}(B)$, X is said to be *closed* with respect to Θ if $[x]\Theta \subseteq X$ holds for every $x \in X$. E.g., each A_i is closed with respect to β and γ. So it is closed with respect to $g_0 := \beta\delta$ and $H_0 := \gamma\delta$, whence we easily infer that $g_0 = \langle a_0, b_0\rangle^{(A,B)}$ and $H_0 = \langle b_{15}, a_{16}\rangle^{(A,B)}$ belong to Q. Now, similarly to Zádori [6], we can define some further members of Q inductively:

$$h_{i+1} = \Big((g_i + \gamma)\alpha + g_i\Big)\gamma \qquad (i = 0, 1, \ldots, 15),$$
$$g_{i+1} = \Big((h_{i+1} + \beta)\alpha + h_{i+1}\Big)\beta \qquad (i = 0, 1, \ldots, 14),$$
$$G_{i+1} = \Big((H_i + \beta)\alpha + H_i\Big)\beta \qquad (i = 0, 1, \ldots, 15), \text{ and}$$
$$H_{i+1} = \Big((G_{i+1} + \gamma)\alpha + G_{i+1}\Big)\gamma \qquad (i = 0, 1, \ldots, 14).$$

Since δ does not occur in the above inductive equations and α occurs only in meets, all the A_j $(j \in J)$ are closed with respect to every g_i, h_i, G_i and H_i. Now an easy induction shows that

$$g_j = \sum_{i=0}^{j} \langle a_i, b_i\rangle^{(A,B)} \qquad (j = 0, 1, \ldots, 15),$$
$$h_j = \sum_{i=1}^{j} \langle b_{i-1}, a_i\rangle^{(A,B)} \qquad (j = 1, 2, \ldots, 16),$$

$$H_j = \sum_{i=0}^{j} \langle a_{16-i}, b_{15-i} \rangle^{(A,B)} \qquad (j = 0, 1, \dots, 15), \text{ and}$$

$$G_j = \sum_{i=1}^{j} \langle a_{16-i}, b_{16-i} \rangle^{(A,B)} \qquad (j = 1, 2, \dots, 16).$$

(Note that, for $B = A = L_0$, these formulas with notational changes occur in Zádori [6, p. 582].) Therefore the following elements

$$\begin{aligned}
\langle a_j, b_j \rangle^{(A,B)} &= g_j \cdot G_{16-j} \qquad (j = 0, 1, \dots, 15),\\
\langle b_{j-1}, a_j \rangle^{(A,B)} &= h_j \cdot H_{16-j} \qquad (j = 1, 2, \dots, 16),\\
\langle a_{j-1}, a_j \rangle^{(A,B)} &= \Big(\langle a_{j-1}, b_{j-1} \rangle^{(A,B)} + \langle b_{j-1}, a_j \rangle^{(A,B)} \Big) \alpha \quad (j = 1, 2, \dots, 16), \text{ and}\\
\langle b_{j-1}, b_j \rangle^{(A,B)} &= \Big(\langle b_{j-1}, a_j \rangle^{(A,B)} + \langle a_j, b_j \rangle^{(A,B)} \Big) \alpha \qquad (j = 1, 2, \dots, 15)
\end{aligned}$$

all belong to Q. Now let $p, q \in A = A_0$ be distinct elements. Then there is a circle $p = u_0, u_1, \dots, u_i = q, u_{i+1}, \dots, u_{i+j-1}, u_{i+j} = p$ in the graph depicted in Figure 1 such that $|\{u_0, u_1, \dots, u_{i+j-1}\}| = i + j$. Since the $\langle u_{\ell-1}, u_\ell \rangle^{(A,B)}$ already belong to Q,

$$\langle p, q \rangle^{(A,B)} = \left(\sum_{\ell=1}^{i} \langle u_{\ell-1}, u_\ell \rangle^{(A,B)} \right) \cdot \left(\sum_{\ell=i+1}^{i+j} \langle u_{\ell-1}, u_\ell \rangle^{(A,B)} \right) \in Q.$$

This proves Claim 6. ■

Given a cardinal m, the smallest cardinal that is greater than m will be denoted by m^+. For a finite set X, let $R(X) = P(X)$, the set of all subsets of X. When X is infinite, $R(X)$ will always denote a fixed subset of $P(X)$ such that $\emptyset \in R(X)$ and, unless explicitly otherwise stated, $|R(X)| = |X|^+$. $R^+(X)$ will always stand for $R(X) \setminus \{\emptyset\}$.

Definition 7. Suppose $A = A_0 = (A_0, d_0, h_0, E_0, \alpha_0, \beta_0, \gamma_0, \delta_0)$ is a box. Choose $c \in E_0$, $D \subseteq E_0$ and $F \subseteq E_0$ such that $E_0 = \{c\} \cup D \cup F$ and, further, the sets $\{c\}$, D and F are pairwise disjoint. For each $U \in R(D)$ we take an isomorphic copy $A(U) = (A(U), d_0(U), h_0(U), E_0(U), \alpha_0(U), \beta_0(U), \gamma_0(U), \delta_0(U))$ of A such that these copies are pairwise disjoint, and $A(\emptyset)$ coincides with $A = A_0$. Let φ_U: $A \to A(U)$ be a fixed isomorphism for $U \in R^+(D)$; $\varphi_\emptyset = \varphi_0$ will stand for the identity map on $A = A(\emptyset)$. Let

$$B = \bigcup_{U\in R(D)} A(U), \qquad \beta = \bigcup_{U\in R(D)} \beta_0(U), \qquad \gamma = \bigcup_{U\in R(D)} \gamma_0(U),$$

$$\alpha = \bigcup_{U\in R(D)} \alpha_0(U) + \sum_{U\in R^+(D)} \langle d_0(\emptyset), d_0(U)\rangle, \text{ and}$$

$$\delta = \bigcup_{U\in R(D)} \delta_0(U) + \sum_{U\in R(D)} \Big(\sum_{e\in U}\langle (b_e(U), a_e(U)\rangle + \sum_{e\in D\setminus U} \langle (b'_e(U), a'_e(U)\rangle\Big) + \\ + \sum_{U\in R^+(D)} \Big(\langle a_c(\emptyset), b_c(U)\rangle + \langle b'_c(\emptyset), b'_c(U)\rangle\Big).$$

Define $E = \bigcup_{U\in R(D)} F(U)$, $d = d_0(\emptyset)$, and $h = h_0(\emptyset)$. This way we obtain $B = (B, d, h, E, \alpha, \beta, \gamma, \delta)$, which we call a *successor* of A.

For example, if $A = L_0$ (cf. Claim 6), $D = \{(b_3, a_6, b_4, a_7), (b_5, a_8, b_6, a_9)\}$, $F = \{(b_7, a_{10}, b_8, a_{11}), (b_9, a_{12}, b_{10}, a_{13})\}$ and $c = (b_1, a_4, b_2, a_5)$, then the corresponding successor of A is depicted in Figure 3, where $U_1 = \{(b_3, a_6, b_4, a_7)\}$, $U_2 = D \setminus U_1$ and $U_3 = D$.

Claim 8. *Let B be a successor of a box A. Then B is a box. Moreover, $B \mid_\Phi A$ for the "canonical" $\Phi = (\Gamma, \{\varphi_U\colon U \in R^+(D)\})$, where the classes of Γ are the $A(U)$, $U \in R(D)$.*

Proof. Let as assume that A is a box, and denote by $(\mathrm{b}i)_A$ resp. $(\mathrm{b}i)_B$ the satisfaction of (bi) for A resp. B. First we show $(\mathrm{b2})_B$. Given $\Theta_j \in \mathrm{Equ}(B)$ for $j \in J$, we call $\sum_{j\in J} \Theta_j$ a *direct sum* of the Θ_j if for every choice of a non-singleton Θ_j-class C_j for each $j \in J$ the sets C_j, $j \in J$, are pairwise disjoint. By $(\mathrm{b6})_A$, δ is a direct sum of certain equivalences, and each of these direct summands is of the form

$$\Theta_1 = \bigcup_{U\in R(D)} \delta_0(U), \qquad \Theta_2 = \langle b_e(U), a_e(U)\rangle, \qquad \Theta_3 = \langle b'_e(U), a'_e(U)\rangle,$$

$$\Theta_4 = \sum_{U\in R^+(D)} \langle a_c(\emptyset), b_c(U)\rangle \quad \text{or} \quad \Theta_5 = \sum_{U\in R^+(D)} \langle b'_c(\emptyset), b'_c(U)\rangle.$$

Hence, to show $\alpha\delta = 0$, it suffices to show $\alpha\Theta_\ell = 0$ for $\ell = 1, 2, 3, 4, 5$. For $\ell = 1, 2, 3$ this is obvious from $\alpha|_{A(U)} = \alpha(U)$, $(\mathrm{b2})_A$ and $(\mathrm{b4})_A$. Suppose now that $(u, v) \in \alpha\Theta_4$ and $u \neq v$. If $\{u, v\} = \{a_c(\emptyset), b_c(U)\}$, say $u = a_c(\emptyset)$ and $v = b_c(U)$, then, by $U \neq \emptyset$ and the definition of α, we obtain

$(a_c(\emptyset), d(\emptyset)) \in \alpha(\emptyset)$ and $(d(U), b_c(U)) \in \alpha(U)$, contradicting $(\mathrm{b7})_A$. The other possibility, $\{u,v\} = \{b_c(V), b_c(U)\}$ for distinct U and V, and the case of Θ_5 (where $\{u,v\} = \{b'_c(V), b'_c(U)\}$) lead to $(b_c(U), d(U)) \in \alpha(U)$ or $(b'_c(U), d(U)) \in \alpha(U)$, which contradicts $(\mathrm{b7})_A$ again. Therefore $\alpha\delta = 0$.

We have seen $(\mathrm{b2})_B$; $(\mathrm{b1})_B$ and $(\mathrm{b3})_B$ are trivial. Since $\alpha|_{A(U)} = \alpha(U)$ and similarly for β and γ, the α, β and γ parts of $(\mathrm{b4})_B$ follow. We do not have to deal with the δ part of $(\mathrm{b4})_B$, for it will follow from $(\mathrm{b6})_B$. Since the edge pairs of $E = \bigcup_{U\in R(D)} F(U)$ were not used in the construction, $(\mathrm{b6})_B$ follows form $(\mathrm{b6})_A$. Since $\alpha|_{A(\emptyset)} = \alpha(\emptyset)$, $\beta|_{A(\emptyset)} = \beta(\emptyset)$ and $\gamma|_{A(\emptyset)} = \gamma(\emptyset)$, we obtain $(\mathrm{b5})_B$ from $(\mathrm{b5})_A$ and $(\mathrm{b6})_B$. Now let $e = e(U) \in E$. If $(b_e, d) = (b_e(U), d(\emptyset)) \in \alpha$, then $(d(\emptyset), d(U)) \in \alpha$ gives $(b_e(U), d_e(U)) \in \alpha$, which is impossible by $\alpha|_{A(U)} = \alpha(U)$ and $(\mathrm{b7})_A$. $(b'_e, d) \notin \alpha$ is obtained similarly. Suppose now that $(a_e, h) = (a_e(U), h(\emptyset)) \in \alpha$. If $U = \emptyset$ then $\alpha|_{A(\emptyset)} = \alpha(\emptyset)$ and $(\mathrm{b7})_A$ gives a contradiction. So let $U \neq \emptyset$. By the definition of α, we obtain $(a_e(U), d(U)) \in \alpha(U)$, $(d(U), d(\emptyset)) \in \alpha$ and $(d(\emptyset), h(\emptyset)) \in \alpha(\emptyset)$, contradicting $(\mathrm{b5})_A$. $(a'_e, h) \notin \alpha$ is derived similarly. Hence $(\mathrm{b7})_B$ holds, and B is a box. Finally, it is trivial that $B \mid_\Phi A$. ∎

Claim 9. *Let B be a successor of A. If A is a good box, then so is B.*

Proof. Let $A_0 = A$ and use the notations of Definition 7. We know that $B \mid_\Phi A$ with the canonical Φ. Let us consider an extension $C = (C, \bar{d}, \bar{h}, \bar{E}, \bar{\alpha}, \bar{\beta}, \bar{\gamma}, \bar{\delta})$ of B, say $C \mid_\Psi B$. Then C is an extension of A_0 by $\Psi \circ \Phi$, cf. Claim 4. So we can use the following, self-explaining, notations: $B_0 = B$, $\Psi = (\Delta, \{\psi_j\colon j \in J\})$, $B_j := \psi_j(B_0)$, $A(j,U) := \psi_j(A(U))$ for $U \in R^+(D)$ and $A(0,U) = A(U)$. Then $B_j := \bigcup_{U\in R(D)} A(j,U)$ and $C = \bigcup_{(j,U)\in J\times R(D)} A(j,U)$; both of these unions are disjoint ones. Similar notations (with obvious meaning) will be used for d, α, etc. The smallest complete sublattice of $\mathrm{Equ}(C)$ that contains $\bar{\alpha}$, $\bar{\beta}$, $\bar{\gamma}$, and $\bar{\delta}$ will be denoted by Q.

First we deal with the case $p = d(0,U) \in B = B_0$ and $q = h(0,U) \in B$. Then we assert

$$\begin{aligned}\langle p,q\rangle^{(B,C)} = \langle p,q\rangle^{(A,C)} &\cdot \prod_{e\in U}\Big(\langle p,b_e\rangle^{(A,C)} + \bar{\delta} + \langle a_e,q\rangle^{(A,C)}\Big)\cdot \\ &\cdot \prod_{e\in D\setminus U}\Big(\langle p,b'_e\rangle^{(A,C)} + \bar{\delta} + \langle a'_e,q\rangle^{(A,C)}\Big).\end{aligned} \tag{4}$$

Before proving (4) let us point out that it easily implies $\langle p,q\rangle^{(B,C)} \in Q$. Indeed, this follows from Claim 4 and the goodness of A. Similarly, all the subsequent equations will automatically imply that their lefthand sides belong to Q; we will rely on this fact implicitly.

The "$\subseteq$" inclusion in (4) is an obvious consequence of the definitions. To show the reverse inclusion, let us assume that (u,v) belongs to the righthand side of (4), $u,v \in C$ and $u \neq v$. From $(u,v) \in \langle p,q\rangle^{(A,C)}$ we infer that u and v are in the same copy of A; say they are in $A(j,V) \subseteq B_j$. It also follows from $(u,v) \in \langle p,q\rangle^{(A,C)} = \langle d(j,V), h(j,V)\rangle^{(A,C)}$ that $\{u,v\} = \{d(j,V), h(j,V)\}$. Hence, for any $e \in U$, (4) yields

$$(5)\qquad (d(j,V),h(j,V)) \in \langle d(j,V), b_e(j,V)\rangle^{(A,C)} + \bar{\delta} + \langle a_e(j,V), h(j,V)\rangle^{(A,C)}.$$

By the construction of B_j ($\cong B$), $\{a_e(j,V), b_e(j,V)\}$ is disjoint from the set of components of every edge pair of B_j. Let us observe that the restriction of $\bar{\delta}$ to $\{a_e(j,V), b_e(j,V)\}$ coincides with the restriction of δ_j. Indeed, (e3') says that we add some pairs (x,y) to δ_j and both components of these pairs are components of edge pairs. But formerly, by (b6), $|[x]\delta_j| = |[y]\delta_j| = 1$, so enlarging δ_j in B_j this way induces no change on

$$(6)\qquad Y = B_j \setminus \{x\colon x \text{ is a component of an edge pair}\}.$$

Since e was used when we obtained B_j from $A \cong A(j,V)$, e is not an edge pair of B_j. Therefore $a_e(j,V)$ and $b_e(j,V)$ belong to Y; and the above-mentioned two restrictions coincide, indeed.

Hence, from the construction of B, we obtain

$$(7)\qquad [b_e(j,V)]\bar{\delta} \subseteq \{b_e(j,V), a_e(j,V)\}, \text{ and}$$

$$(8)\qquad (b_e(j,V), a_e(j,V)) \in \bar{\delta} \Longrightarrow e \in V.$$

Since d and h belong to Y, defined in (6), we infer from (b6) for B_j that

$$(9)\qquad |[d(j,V)]\bar{\delta}| = |[h(j,V)]\bar{\delta}| = 1,$$

while

$$(10)\qquad \begin{aligned} &[d(j,V)]\langle d(j,V), b_e(j,V)\rangle^{(A,C)} = \{d(j,V), b_e(j,V)\},\\ &[h(j,V)]\langle a_e(j,V), h(j,V)\rangle^{(A,C)} = \{h(j,V), a_e(j,V)\}\end{aligned}$$

is trivial. Now (5), (7), (9) and (10) imply $(b_e(j,V), a_e(j,V)) \in \bar{\delta}$, and $e \in V$ follows from (8). This shows $U \subseteq V$. Using the $\prod_{e\in D\setminus U}$ part of (4), $D\setminus U \subseteq D\setminus V$ comes similarly. Hence $U = V$, and we obtain $(u,v) \in \langle p,q\rangle^{(B,C)}$, showing (4).

If $p, q \in A(0,U) \subseteq B = B_0$ such that $|\{p, q, d(0,U), h(0,U)\}| = 4$, then we easily obtain

$$\langle p,q\rangle^{(B,C)} = \langle p,q\rangle^{(A,C)} \cdot \Big(\langle p, d(0,U)\rangle^{(A,C)} + \langle d(0,U), h(0,U)\rangle^{(B,C)} + \\ + \langle h(0,U), q\rangle^{(A,C)}\Big).$$

Now let $p, q \in A(0,U) \subseteq B_0$ be arbitrary distinct elements. Since $|A(0,U)| \geq 6$ by (b1), we can choose distinct $p_1, q_1 \in A(0,U) \setminus \{d(0,U),\ h(0,U), p, q\}$. The previous formula applies for (p_1, q_1), and we obtain

$$\langle p,q\rangle^{(B,C)} = \langle p,q\rangle^{(A,C)} \cdot \Big(\langle p, p_1\rangle^{(A,C)} + \langle p_1, q_1\rangle^{(B,C)} + \langle q_1, q\rangle^{(A,C)}\Big).$$

The next step is to deal with the case $p = d(0,\emptyset) \in A(0,\emptyset)$ and $q = d(0,U) \in A(0,U)$ for $U \in R^+(D)$. We assert that

$$\langle d(0,\emptyset), d(0,U)\rangle^{(B,C)} \\ = \bar{\alpha} \cdot \Big(\langle d(0,\emptyset), b'_c(0,\emptyset)\rangle^{(B,C)} + \bar{\delta} + \langle b'_c(0,U), d(0,U)\rangle^{(B,C)}\Big). \tag{11}$$

The "$\subseteq$" part is evident. Before proving the converse inclusion we show that

$$(d(j,V), b'_c(j,W)) \notin \bar{\alpha} \quad \text{for all } j \in J, \text{ and } V, W \in R(D), \text{ and} \tag{12}$$

$$[b'_c(j,\emptyset)]\bar{\delta} = \{b'_c(j,W)\colon\ W \in R(D)\}. \tag{13}$$

If $V = W$, then (12) follows from Claim 4, $\bar{\alpha}|_{A(j,V)} = \alpha(j,V)$ and (b7) for $A(j,V)$. If $V \neq W$, then $(d(j,V), b'_c(j,W)) \in \bar{\alpha}$ would imply $(d(j,W), b'_c(j,W)) \in \bar{\alpha}$, but this case has already been excluded. This proves (12). If we are within B_j, i.e. if we write δ_j instead of $\bar{\delta}$, then (13) is clear by the construction of B_j from the $A(j,U)$ and (b6) for A. With the notation of (6) we have $\{b'_c(j,W)\colon\ W \in R(D)\} \subseteq Y$, so the validity of (13) for δ_j implies (13) for $\bar{\delta}$.

Suppose now that $u \neq v \in C$, and (u,v) belongs to the righthand side of (11). Let, say, $u \in B_j$. Then there is a shortest sequence $w_0 = u$, $w_1, \ldots, w_{t-1}$, $w_t = v$ such that every (w_{i-1}, w_i) belongs to

$$\langle d(0,\emptyset), b'_c(0,\emptyset)\rangle^{(B,C)} \cup \bar{\delta} \cup \langle b'_c(0,U), d(0,U)\rangle^{(B,C)} \\ = \langle d(j,\emptyset), b'_c(j,\emptyset)\rangle^{(B,C)} \cup \bar{\delta} \cup \langle b'_c(j,U), d(j,U)\rangle^{(B,C)}.$$

Since $\bar{\alpha}\,\bar{\delta} = 0$ by (b2), not all the (w_{i-1}, w_i) belong to $\bar{\delta}$. Hence there is an ℓ with

$$(w_{\ell-1}, w_\ell) \in \langle d(j,\emptyset), b'_c(j,\emptyset)\rangle^{(B,C)} = \Theta_1$$

or there is an r with

$$(w_{r-1}, w_r) \in \langle b'_c(j,U), d(j,U)\rangle^{(B,C)} = \Theta_2.$$

We can assume $\ell < r$ if both exist, for otherwise we could use (v,u) instead of (u,v). By (12) and the meaning of Θ_1 and Θ_2, some (w_{i-1}, w_i) must belong to $\bar{\delta}$. Based on (9), (13) and the meaning of Θ_1 and Θ_2, all the w_i belong to B_j, and only the following sequences are possible:

(i) $w_0 = d(j,\emptyset) \quad \Theta_1 \quad w_1 = b'_c(j,\emptyset) \quad \bar{\delta} \quad b'_c(j,W) = w_t$,
(ii) $w_0 = b'_c(j,W) \quad \bar{\delta} \quad w_1 = b'_c(j,\emptyset) \quad \Theta_1 \quad d(j,\emptyset) = w_t$,
(iii) $w_0 = d(j,\emptyset) \quad \Theta_1 \quad w_1 = b'_c(j,\emptyset) \quad \bar{\delta} \quad w_2 = b'_c(j,U) \quad \Theta_2 \quad d(j,U) = w_t$,
(iv) $w_0 = d(j,U) \quad \Theta_2 \quad w_1 = b'_c(j,U) \quad \bar{\delta} \quad b'_c(j,W) = w_t$, and
(v) $w_0 = b'_c(j,W) \quad \bar{\delta} \quad w_1 = b'_c(j,U) \quad \Theta_2 \quad d(j,U) = w_t$.

But (12) together with $(w_0, w_t) = (u,w) \in \bar{\alpha}$ exclude (i), (ii), (iv) and (v). Therefore $(u,v) = (d(j,\emptyset), d(j,U)) \in \langle d(0,\emptyset), d(0,U)\rangle^{(B,C)}$, proving (11).

Now we assert that, for $U \in R^+(D)$,

(14)
$$\langle b'_c(0,\emptyset), b'_c(0,U)\rangle^{(B,C)} = \\ = \bar{\delta}\cdot\Big(\langle b'_c(0,\emptyset), d(0,\emptyset)\rangle^{(B,C)} + \langle d(0,\emptyset), d(0,U)\rangle^{(B,C)} + \langle d(0,U), b'_c(0,U)\rangle^{(B,C)}\Big).$$

The "$\subseteq$" part is clear. Conversely, suppose $u \neq v$ and (u,v) is in the righthand side of (14). Then u and v are in the same B_j, and $u,v \in \{b'_c(j,\emptyset),\ d(j,\emptyset),\ d(j,U),\ b'_c(j,U)\}$. Applying (9) we obtain $u,v \in \{b'_c(j,\emptyset), b'_c(j,U)\}$, whence $(u,v) \in \langle b'_c(0,\emptyset), b'_c(0,U)\rangle^{(B,C)}$, indeed.

Based on (11) and (14), we can handle the case $p = p(0,\emptyset) \in A(0,\emptyset)$, $q = q(0,U) \in A(0,U)$, $U \neq \emptyset$, as follows

(15)
$$\langle p,q\rangle^{(B,C)} = \Big(\langle p, d(0,\emptyset)\rangle^{(B,C)} + \langle d(0,\emptyset), d(0,U)\rangle^{(B,C)} + \langle d(0,U), q\rangle^{(B,C)}\Big)\cdot \\ \cdot\Big(\langle p, b'_c(0,\emptyset)\rangle^{(B,C)} + \langle b'_c(0,\emptyset), b'_c(0,U)\rangle^{(B,C)} + \langle b'_c(0,U), q\rangle^{(B,C)}\Big).$$

The "$\subseteq$" part is clear. To show the converse inclusion in (15), suppose $u \neq v$ and (u,v) is in the righthand side of the formula. Any factor of the righthand side implies that u and v are in the same B_j. Moreover, we obtain

$$u,v \in \{p(j,\emptyset), d(j,\emptyset), d(j,U), q(j,U)\}$$

from the first factor, and

$$u,v \in \{p(j,\emptyset), b'_c(j,\emptyset), b'_c(j,U), q(j,U)\}$$

from the second one. Hence $\{u,v\} = \{p(j,\emptyset), q(j,U)\}$, and $(u,v) \in \langle p,q\rangle^{(B,C)}$ follows. This proves (15).

Finally, for $p = p(0,U) \in A(0,U)$ and $q = q(0,V) \in A(0,V)$ with distinct $U, V \in R^+(D)$ we assert

$$\text{(16)}\qquad \begin{aligned} \langle p,q\rangle^{(B,C)} =& \Big(\langle p, d(0,\emptyset)\rangle^{(B,C)} + \langle d(0,\emptyset), q\rangle^{(B,C)}\Big)\cdot \\ &\cdot\Big(\langle p, h(0,\emptyset)\rangle^{(B,C)} + \langle h(0,\emptyset), q\rangle^{(B,C)}\Big). \end{aligned}$$

The "$\subseteq$" part in (16) is clear. Conversely, suppose $u \neq v$ and (u,v) is in the righthand side of (16). Any factor of the righthand side implies that u and v are in the same B_j. Moreover, we obtain

$$u, v \in \{p(j,U), d(j,\emptyset), q(j,V)\}$$

from the first factor, and

$$u, v \in \{p(j,U), h(j,\emptyset), q(j,V)\}$$

from the second one. Hence $\{u,v\} = \{p(j,U), q(j,V)\}$, and $(u,v) \in \langle p,q\rangle^{(B,C)}$ follows.

Finally, (16) yields $\langle p,q\rangle^{(B,C)} \in Q$ for all $p,q \in B$, proving Claim 9. ∎

Definition 10. Let μ be an ordinal number. For $\nu < \mu$ let $A_\nu = (A_\nu,\ d_\nu,\ h_\nu,\ E_\nu,\ \alpha_\nu,\ \beta_\nu,\ \gamma_\nu,\ \delta_\nu)$ be a box. Suppose that $A_\nu \mid_{\Phi_{\lambda\nu}} A_\lambda$ for $\lambda < \nu < \mu$ such that $\Phi_{\kappa\nu} = \Phi_{\lambda\nu} \circ \Phi_{\kappa\lambda}$ for all $\kappa < \lambda < \nu < \mu$. (This condition will be referred to as "the ways of extensions are compatible".) Then we say that the A_ν $(\nu < \mu)$ together with the $\Phi_{\lambda\nu}$ $(\lambda < \nu < \mu)$ form a *directed system of boxes*. Associated with this directed system we define

$$A = \bigcup_{\nu<\mu} A_\nu, \quad \alpha = \bigcup_{\nu<\mu} \alpha_\nu, \quad \beta = \bigcup_{\nu<\mu} \beta_\nu, \quad \gamma = \bigcup_{\nu<\mu} \gamma_\nu, \quad \text{and } \delta = \bigcup_{\nu<\mu} \delta_\nu.$$

We let $d = d_0$ and $h = h_0$; note that $d = d_\nu$ and $h = h_\nu$ for all $\nu < \mu$. Let us choose a subset $E \subseteq \bigcup_{\nu<\mu} E_\nu$ such that

$$\text{(17)}\qquad E \subseteq \bigcup_{\nu<\mu} \ \bigcap_{\substack{\lambda \\ \nu\leq\lambda<\mu}} E_\lambda\,.$$

Then $A = (A,\ d,\ h,\ E,\ \alpha,\ \beta,\ \gamma,\ \delta)$ is called a *limit* of the A_ν $(\nu < \mu)$.

Note that A, α, β, γ and δ are unions of ascending chains. If μ is a successor ordinal, say $\mu = \varrho + 1$, then the limit is A_ϱ (with less edge pairs, perhaps). Hence the limit of boxes is interesting (and will be used) for limit ordinals μ only. Unfortunately, the union of the righthand side of (17) (and therefore E) can be empty. This phenomenon is responsible for a lot of work put in the present paper.

Claim 11. *The limit A defined above is a box. There are canonical $\Phi_{\nu\mu}$ such that $A \mid_{\Phi_{\nu\mu}} A_\nu$ for all $\nu < \mu$. Moreover, denoting A by A_μ, the A_ν ($\nu < \mu + 1$) with the $\Phi_{\nu\lambda}$ ($\nu < \lambda < \mu + 1$) form a directed system of boxes.*

Proof. We can assume that μ is a limit ordinal. First we show that A is a box. (b1) is evident. The definition of an extension ensures that $\alpha_\lambda \subseteq \alpha_\nu$ for $\lambda \le \nu$, whence α is an equivalence; so are β, γ and δ by the same reason. Since $\alpha_\nu\delta_\nu = 0$ for all ν, $\alpha\delta = 0$ follows. This shows (b2). By (e5), all edge pairs in E are edge pairs of copies of A_0; this gives (b3). Let $e \in E$, then $e \in E_\nu$ for some ν. Since $\alpha_\lambda|_{A_\nu} = \alpha_\nu$ for all $\nu \le \lambda < \mu$ by (e2), the restriction of α to $\{b_e, a_e, b'_e, a'_e\}$ is the same as that of α_ν. This implies the α part of (b4). The β and γ parts come similarly. The δ part follows from (b6) for A. This proves (b4). Since $d = d_\nu$ and $h = h_\nu$ for all $\nu < \mu$, (b5) holds for A. To show (b6), let $e \in E$. By (17), there is a ν such that $e \in E_\lambda$ for all $\nu \le \lambda < \mu$. Then the δ_λ class of each component of E is a singleton by (b6) for A_λ, and this property is inherited by $\bigcup_{\nu\le\lambda<\mu}\delta_\lambda = \delta$. If we had $|[d]\delta| > 1$ or $|[h]\delta| > 1$, then $|[d]\delta_\lambda| > 1$ or $|[h]\delta_\lambda| > 1$ for some $\lambda < \mu$, which would contradict (b6) for A_λ. Now, to show (b7), suppose that $(a'_e, h) \in \alpha$ for some $e \in E$. Then there is a $\nu_1 < \mu$ such that $(a'_e, h) \in \alpha_\lambda$ for all $\nu_1 \le \lambda < \mu$. Since $e \in E$, there is a $\nu_2 < \mu$ such that $e \in E_\lambda$ for all $\nu_2 \le \lambda < \mu$. For $\lambda = \max\{\nu_1, \nu_2\}$ this contradicts (b7) for A_λ. Hence $(a'_e, h) \notin \alpha$, and the rest of (b7) follows similarly.

Now as we already know that A is a box let us fix a $\nu < \mu$. For $\nu \le \lambda < \mu$, let $\Phi_{\nu\lambda} = (\Gamma_{\nu\lambda}, \{\varphi_i\colon i \in I_{\nu\lambda}\})$, where the classes of $\Gamma_{\nu\lambda}$ are denoted by $A_{\nu,i}$ ($i \in I_{\nu\lambda}$). By the compatibility of the $\Phi_{\kappa\varrho}$ we may assume that $I_{\nu\lambda} \subseteq I_{\nu\kappa}$ for $\lambda \le \kappa$, and φ_i and $A_{\nu,i}$ are the same for $i \in I_{\nu\lambda}$ as they are for $i \in I_{\nu\kappa}$. (This is clear from definition if we choose $I_{\varrho\lambda} = \Gamma_{\varrho\lambda}$, i.e. the set of classes, for all $\varrho < \lambda < \mu$.) Let $I = I_{\nu\mu} = \bigcup_{\nu\le\lambda<\mu} I_{\nu\lambda}$. Then $\{A_{\nu,i}\colon i \in I\}$ is a partition $\Gamma_{\nu\mu}$ on A, and (the collection of these) $\Psi_{\nu\mu} = (\Gamma_{\nu\mu}, \{\varphi_i\colon i \in I\})$ is compatible with all $\Phi_{\kappa\varrho}$ in the (original) directed system.

So all we have to show is that $\Phi_{\nu\mu}$ establishes an extension. Since $\beta_\lambda = \bigcup_{i\in I_{\nu\lambda}} \beta_{\nu,i}$ (where $\beta_{\nu,i} = \varphi_i(\beta_\nu)$), we obtain $\beta = \bigcup_{\nu\le\lambda<\mu}\beta_\nu = \bigcup_{i\in I}\beta_{\nu,i}$. The case of γ is similar, so (e1) holds. For $\nu \le \lambda < \mu$ we have

$$\alpha_\lambda = \bigcup_{i\in I_{\nu\lambda}} \alpha_{\nu,i} + \sum_{i,j\in I_{\nu\lambda}} \langle d_{\nu,i}, d_{\nu,j}\rangle \tag{18}$$

where $d_{\nu,i} = \varphi_i(d_\nu)$, and we want to show

$$\alpha = \bigcup_{i\in I} \alpha_{\nu,i} + \sum_{i,j\in I} \langle d_{\nu,i}, d_{\nu,j}\rangle\ . \tag{19}$$

Since the righthand side of (19) clearly includes the righthand side of (18), by forming the union of (18) for all λ $(\nu \le \lambda < \mu)$ and using $\alpha = \bigcup_{\nu\le\lambda<\mu}\alpha_\lambda$ we obtain the "$\subseteq$" part of (19). If $d_{\nu,i}, d_{\nu,j} \in A_\lambda$, then $\langle d_{\nu,i}, d_{\nu,j}\rangle \subseteq \alpha_\lambda \subseteq \alpha$ by (18), whence the "$\supseteq$" part of (19) is clear. This proves (e2).

For $\nu \le \lambda < \mu$, (e3) for $A_\lambda \mid A_\nu$ gives that

$$(20)\qquad \begin{aligned}\delta_\lambda = \bigcup_{i\in I_{\nu\lambda}} \delta_{\nu,i} + \sum_{(e,i)\in F_\lambda} \langle b_{e_i}, a_{e_i}\rangle + \sum_{(e,i)\in G_\lambda} \langle b'_{e_i}, a'_{e_i}\rangle + \\ + \sum_{(e,i,j)\in H_\lambda} \Big(\langle a_{e_i}, b_{e_j}\rangle + \langle b'_{e_i}, b'_{e_j}\rangle\Big).\end{aligned}$$

Using (b6) for A_ν it is not hard to observe that

$$(21)\qquad \begin{aligned}&F_\lambda = \{(e,i) \in E\times I_{\nu\lambda}\colon (b_{e_i}, a_{e_i}) \in \delta_\lambda\},\\ &G_\lambda = \{(e,i) \in E\times I_{\nu\lambda}\colon (b'_{e_i}, a'_{e_i}) \in \delta_\lambda\}, \text{ and}\\ &H_\lambda = \{(e,i,j) \in E\times I_{\nu\lambda}\times I_{\nu\lambda}\colon (a_{e_i}, b_{e_j}) \in \delta_\lambda\},\end{aligned}$$

Since $\delta_\lambda \subseteq \delta_\kappa$ for $\nu \le \lambda \le \kappa < \mu$, we infer $F_\lambda \subseteq F_\kappa$, $G_\lambda \subseteq G_\kappa$ and $H_\lambda \subseteq H_\kappa$. Let $F = \bigcup_{\nu\le\lambda<\mu} F_\lambda$, $G = \bigcup_{\nu\le\lambda<\mu} G_\lambda$ and $H = \bigcup_{\nu\le\lambda<\mu} H_\lambda$. Forming the union of (20) for all permitted λ we obtain the "$\subseteq$" part of

$$(22)\qquad \begin{aligned}\delta = \bigcup_{i\in I} \delta_{\nu,i} + \sum_{(e,i)\in F} \langle b_{e_i}, a_{e_i}\rangle + \sum_{(e,i)\in G} \langle b'_{e_i}, a'_{e_i}\rangle + \\ + \sum_{(e,i,j)\in H} \Big(\langle a_{e_i}, b_{e_j}\rangle + \langle b'_{e_i}, b'_{e_j}\rangle\Big),\end{aligned}$$

while the "$\supseteq$" part is clear from the fact that each of the $\delta_{\nu,i}$, the (left and right) atoms and the connecting equivalences occurring in the sum on the righthand side is smaller than some $\delta_\lambda \subseteq \delta$. Now (22) yields (e3) for A and A_ν, for all necessary conditions on F, G and H are implied by these conditions on all F_λ, G_λ and H_λ. (e4) for A and A_ν hardly needs any proof.

Finally, to show (e5) for A and A_ν, suppose $e \in E$. By (17), there is a ν_1 such that $e \in E_\lambda$ for all $\nu_1 \le \lambda < \mu$. Choose $\lambda = \max\{\nu, \nu_1\}$. Then $e \in E_\lambda$ and $A_\lambda \mid A_\nu$ gives that $e \in E_{\nu,i}$ for some $i \in I_{\nu\lambda} \subseteq I$, showing (e5). ∎

Claim 12. *With the notations of Definition* 10, *if all the* A_ν $(\nu < \mu)$ *are good boxes, then their limit,* A, *is a good box as well.*

Proof. Let $B = (B, \bar{d}, \bar{h}, \bar{E}, \bar{\alpha}, \bar{\beta}, \bar{\gamma}, \bar{\delta})$ be an extension of $A = A_\mu$ by Ψ. We know from Claim 11 that $A \,|_{\Phi_{\nu\mu}}\, A_\nu$ for $\nu < \mu$. Claim 4 yields that B is an extension of A_ν by $\Psi \circ \Phi_{\nu\mu}$. Now all the necessary ways of extensions are fixed and compatible, so the notations $\langle p,q\rangle^{(A,B)}$ and $\langle p,q\rangle^{(A_\nu,B)}$ will make sense later in the proof.

Let Ψ be of the form $(\Gamma, \{\varphi_i\colon i \in I\})$ where $\Gamma = \{A^{(i)}\colon i \in I\}$ and $A = A^{(0)}$. Let Q denote the smallest complete sublattice of $\mathrm{Equ}(B)$ that includes $\{\bar{\alpha}, \bar{\beta}, \bar{\gamma}, \bar{\delta}\}$. Denote $\varphi_i(A_\nu)$ by $A_\nu^{(i)}$. Suppose that $p, q \in A$ are distinct elements. Then there is a smallest λ such that $p, q \in A_\lambda$. The goodness of A_ν yields $\langle p,q\rangle^{(A_\nu,B)} \in Q$ for all $\lambda \le \nu < \mu$. We assert that

$$\langle p,q\rangle^{(A,B)} = \prod_{\lambda\le\nu<\mu} \langle p,q\rangle^{(A_\nu,B)} \in Q; \tag{23}$$

only the equality has to be checked. The "$\subseteq$" part follows from the fact that the ways of extensions are compatible. For the converse inclusion, suppose (u,v) belongs to the righthand side of (23), $u \ne v \in B$. From $(u,v) \in \langle p,q\rangle^{(A_\lambda,B)}$ we obtain that u and v are in the same copy of A_λ. Hence, by compatibility, they are in the same $A^{(i)}$. Choose a (sufficiently large) $\lambda \le \nu < \mu$ such that $\{\varphi_i(p), \varphi_i(q), u, v\} \subseteq A_\nu^{(i)}$. Then $(u,v) \in \langle p,q\rangle^{(A_\nu,B)} = \langle \varphi_i(p), \varphi_i(q)\rangle^{(A_\nu,B)}$ gives $\{u,v\} = \{\varphi_i(p), \varphi_i(q)\}$. Consequently, $(u,v) \in \langle \varphi_i(p), \varphi_i(q)\rangle^{(A,B)} = \langle p,q\rangle^{(A,B)}$, proving (23). ∎

III. Enlarging boxes at infinity

Starting from $L_0 = (L_0, d_0, h_0, E_0, \alpha_0, \beta_0, \gamma_0, \delta_0)$ (cf. Claim 6) we intend to define a directed system $L_i = (L_i, d_i, h_i, E_i, \alpha_i, \beta_i, \gamma_i, \delta_i)$ of boxes, $i \in \mathbb{N}_0 = \{0,1,2,\ldots\}$, together with ways Φ_{ij} of extension $(i < j)$ such that, for all $i \in \mathbb{N}_0$, L_{i+1} is a successor of L_i and $L_{i+1} \,|_{\Phi_{i,i+1}}\, L_i$ is the canonical extension associated with the successor construction. Denoting by c_i, D_i and F_i the parameters establishing that L_{i+1} is a successor of L_i (cf. Definition 7, note that $c_i \in E_i$ and $D_i, F_i \subseteq E_i$), we also want that $F_0 \subseteq F_1 \subseteq F_2 \subseteq F_3 \subseteq \ldots$. Let L_1 be the box defined by Figure 3; the meaning of Φ_{01} is obvious. Now suppose that $i \ge 1$ and $L_0, L_1, \ldots, L_i$ are already defined together with compatible Φ_{jk} $(j < k \le i)$. By the construction of L_i from L_{i-1} and $L_i \,|_{\Phi_{i-1,i}}\, L_{i-1}$ we obtain $F_{i-1} \subseteq E_i$. Choose $F_i \subseteq E_i$ such that $F_{i-1} \subseteq F_i$ and $|F_i| = \frac{1}{2}|E_i|$ (we will see that this is possible); let $c_i \in E_i \setminus F_i$ and let $D_i = E_i \setminus (\{c_i\} \cup F_i)$. These parameters determine a unique successor L_{i+1} of L_i and a unique (canonical) $\Phi_{i,i+1}$ with $L_{i+1} \,|_{\Phi_{i,i+1}}\, L_i$; for $j < i$ we set $\Phi_{j,i+1} = \Phi_{i,i+1} \circ \Phi_{j,i}$. The sequence $s_i = (|E_i|, |F_i|, |D_i|)$, $i = 1,2,3,\ldots$, clearly obeys the following rule:

$$s_1 = (8,4,3), \quad s_2 = (32,16,15), \quad s_3 = (16 \cdot 2^{15}, 16 \cdot 2^{14}, 16 \cdot 2^{14} - 1), \quad \ldots,$$
$$s_{i+1} = (|F_i| \cdot 2^{|D_i|}, |F_i| \cdot 2^{|D_i|-1}, |F_i| \cdot 2^{|D_i|-1} - 1), \quad \ldots$$

It is easy to see that $2 \cdot |F_{i-1}| \leq |E_i|$ for $i = 1, 2, 3, \ldots$, so the choice of F_i is always possible. Since $F_0 \subsetneq F_1 \subsetneq F_2 \subsetneq F_3 \subsetneq \ldots$ and $F_i \subsetneq E_i$, for all natural numbers n we obtain $\bigcap_{n \leq \ell} E_\ell \supseteq \bigcap_{n \leq \ell} F_\ell = F_n$. Hence, the choice $E = \bigcup_{n \in \mathbf{N}_0} F_n$ is in accordance with (17). Now let $A = (A, d, h, E, \alpha, \beta, \gamma, \delta)$ be the limit of the directed system we have just defined. The fast growing of the sequence $(s_i)_{i \in \mathbf{N}_0}$ makes it clear that $|E| = \aleph_0$. Therefore $|A| = |E|$, and this property will be so important in the sequel that it deserves a separate name.

Definition 13. A box $A = (A, d, h, E, \alpha, \beta, \gamma, \delta)$ is called a *perfect box* if A is good and $|A| = |E|$. A cardinal m will be called *small* if $\aleph_0 \leq m$ and there is no inaccessible cardinal $\leq m$.

In virtue of Claim 12, the box we have defined before Definition 13 is a countable perfect box. Clearly, every perfect box is necessarily infinite. We want to show that for each small cardinal m there is a perfect box of power m. However, we need an even stronger induction hypothesis.

Definition 14. Given a small cardinal m, we say that the condition $\mathrm{H}(m)$ holds if

(i) for each small cardinal $n \leq m$ there is a perfect box of cardinality n; and

(ii) for any two small cardinals $n < k \leq m$, for every perfect box $A = (A, d, h, E, \alpha, \beta, \gamma, \delta)$ with $|A| = n$, and for each $E' \subset E$ with $|E \setminus E'| = n$ there is a perfect box B of power k such that B is an E'-preserving extension of A.

Clearly, if a cardinal m is small, then so is m^+, the least cardinal that is greater than m. The existence of a countable perfect box trivially implies that $\mathrm{H}(\aleph_0)$.

Claim 15. *Suppose m is a small cardinal and* $\mathrm{H}(m)$ *holds. Then* $\mathrm{H}(m^+)$ *holds as well.*

Proof. It suffices to show H(m^+)(ii) for $n \leq m$ and $k = m^+$. Let us take a perfect box $A = (A, d, h, E, \alpha, \beta, \gamma, \delta)$ of power n (such a box exists by H(m)), and a subset $E' \subset E$ with $|E \setminus E'| = |E| = n$. We want to construct an E'-preserving extension B of A such that $|B| = m^+$. Since $n = n + n$ by the cardinal arithmetics, we can chose an E'' such that $E' \subset E'' \subset E$ and $|E \setminus E''| = |E'' \setminus E'| = n$. In virtue of H($m$) we obtain a perfect box $C = (C, \hat{d}, \hat{h}, \hat{E}, \hat{\alpha}, \hat{\beta}, \hat{\gamma}, \hat{\delta})$ of power m such that C is an E''-preserving extension of A. Let $\hat{E}'$ and $\hat{E}''$ denote the extension of E' and E'' to C, respectively. Then $m = |\hat{E}| \geq |\hat{E} \setminus \hat{E}'| \geq |\hat{E}'' \setminus \hat{E}'| = |E'' \setminus E'| \cdot [C : A] = |A| \cdot [C : A] = |C| = m$, i.e., $|\hat{E} \setminus \hat{E}'| = m$. Hence we can partition $\hat{E}$ into $\{\hat{c}\}$, $\hat{D}$ and $\hat{F}$ such that $\hat{E}' \subseteq \hat{F}$ and $|\hat{F}| = |\hat{D}| = m$. These parameters determine a successor B of C. By Claim 8 and Definition 7, B is an $\hat{E}'$-preserving extension of C. Therefore, by Claim 4, B is an E'-preserving extension of A. ■

Claim 16. *Suppose that k is a small limit cardinal (i.e., $k = m^+$ holds for no m) and* H(m) *holds for all $m < k$. Then* H(k) *holds as well.*

Proof. We can assume that $k > \aleph_0$. Since k is small, either

(∗) $2^m \geq k$ for some $m < k$, or

(∗∗) there is a set M of cardinals such that $|M| < k$, $m < k$ for all $m \in M$, and $\sup\{m\colon m \in M\} = k$.

The treatment of (∗) is very similar to that of Claim 15; the only difference is that to obtain B from C we use the successor construction with $|R(\hat{D})| = k$ instead of $|R(\hat{D})| = |\hat{D}|^+ = m^+$.

From now on we deal with (∗∗). Again, we have to prove only (ii) in a particular case. I.e., let $A = (A, d, h, E, \alpha, \beta, \gamma, \delta)$ be a perfect box of power $n < k$, and let $E' \subset E$ with $|E \setminus E'| = |A| = n$. By H($n$), A and E' exist. Our task is to give an E'-preserving extension of A to a perfect box of power k. We can assume $|M| \leq m$ and $n \leq m$ for all $m \in M$, for otherwise we can replace M by $\{m \in M\colon m \geq |M|,\ m \geq n\}$. Clearly, we can also suppose that $n \in M$. Since any set of cardinals is well-ordered, M is of the form $M = \{m_\xi\colon \xi < \mu\}$ where μ is a limit ordinal, $|\mu| = |M| < k$, $m_0 = n$, and $m_\xi < m_\eta$ for $\xi < \eta < \mu$. For convenience, define $m_\mu = k$; then $m_\mu = \sup\{m_\xi\colon \xi < \mu\}$.

Like in the previous proof, we can choose an E'' such that $E' \subset E'' \subset E$ and $|E \setminus E''| = |E'' \setminus E'| = n = m_0$. Since $n = 2n \cdot |\mu + 1|$, we can choose a partition $\{X_\xi\colon \xi \leq \mu\} \cup \{Y_\xi\colon \xi \leq \mu\}$ of $E'' \setminus E'$ such that $|X_\xi| = |Y_\xi| = n$ for all $\xi \leq \mu$. We define

$$E^{(\xi)} = E'' \setminus \Big(\bigcup_{\eta \leq \xi} X_\eta \cup \bigcup_{\eta < \xi} Y_\eta\Big)$$

and

$$T^{(\xi)} = E'' \setminus \Big(\bigcup_{\eta\le\xi} X_\eta \cup \bigcup_{\eta\le\xi} Y_\eta\Big) = E^{(\xi)} \setminus Y^{(\xi)}.$$

Then $T^{(\xi)}, E^{(\xi)} \subseteq E''$ for all $\xi \le \mu$, $T^{(\mu)} = E'$, and, for all $\xi < \eta \le \mu$,

$$E^{(\xi)} \supset T^{(\xi)} \supset E^{(\eta)} \supset T^{(\eta)} \quad \text{and}$$
$$|E^{(\xi)} \setminus T^{(\xi)}| = |T^{(\xi)} \setminus E^{(\eta)}| = |E^{(\eta)} \setminus T^{(\eta)}| = n.$$

Via induction on ν, for each $\nu \le \mu$ we want to define a directed system S_ν of perfect boxes $A_\xi = (A_\xi, d_\xi, h_\xi, E_\xi, \alpha_\xi, \beta_\xi, \gamma_\xi, \delta_\xi)$ ($\xi \le \nu$) together with compatible $\Phi_{\xi\eta}$ ($\xi < \eta \le \nu$) such that $|A_\xi| = m_\xi$, $A_\xi \mid_{\Phi_{0\xi}} A_0$ is a $T^{(\xi)}$-preserving extension, and $S_\lambda \subseteq S_\nu$ for all $\lambda \le \nu$. Let $I(\nu)$ denote this collection of conditions that we expect from S_ν.

Let S_0 consist only of $A_0 = A$; $I(0)$ is evident.

Now suppose that S_ν satisfying $I(\nu)$ is already constructed; we want to construct $S_{\nu+1}$. Since $A_\nu \mid_{\Phi_{0\nu}} A_0$ is $T^{(\nu)}$-preserving, we can extend $T^{(\nu)}$ and $T^{(\nu+1)}$ to A_ν; let $T_\nu^{(\nu)}$ and $T_\nu^{(\nu+1)}$ denote their extension, respectively. We have

$$\begin{aligned} m_\nu &= |E_\nu| \\ &\ge |E_\nu \setminus T_\nu^{(\nu+1)}| \ge |T_\nu^{(\nu)} \setminus T_\nu^{(\nu+1)}| \\ &= |T^{(\nu)} \setminus T^{(\nu+1)}| \cdot [A_\nu : A_0] = |A_0| \cdot [A_\nu : A_0] = |A_\nu| = m_\nu. \end{aligned}$$

I.e., $|E_\nu \setminus T_\nu^{(\nu+1)}| = m_\nu$. Hence, by $\mathrm{H}(m_{\nu+1})$, there exists a perfect box $A_{\nu+1} = (A_{\nu+1}, d_{\nu+1}, h_{\nu+1}, E_{\nu+1}, \alpha_{\nu+1}, \beta_{\nu+1}, \gamma_{\nu+1}, \delta_{\nu+1})$ of power $m_{\nu+1}$ such that $A_{\nu+1}|A_\nu$ is a $T_\nu^{(\nu+1)}$-extension. From Claim 4 we infer that $A_{\nu+1}$ is a $T^{(\nu+1)}$-preserving extension of A_0. Let $\Phi_{\nu,\nu+1}$ denote the way of $A_{\nu+1}|A_\nu$, and define $\Phi_{\xi,\nu+1} = \Phi_{\nu,\nu+1} \circ \Phi_{\xi\nu}$ for $\xi < \nu$. Hence, augmenting S_ν by $A_{\nu+1}$ and by the $\Phi_{\xi,\nu+1}$ ($\xi < \nu + 1$), we obtain a directed system $S_{\nu+1}$. $S_{\nu+1}$ clearly satisfies $\mathrm{I}(\nu + 1)$.

Now let ν ($\nu \le \mu$) be a limit ordinal, and suppose that the S_ξ satisfying $I(\xi)$ are already defined for all $\xi < \nu$. The union $\bigcup_{\xi<\nu} S_\xi$ is clearly a directed system again; let $B_\nu = (B_\nu, \hat{d}_\nu, \hat{h}_\nu, \hat{E}_\nu, \hat{\alpha}_\nu, \hat{\beta}_\nu, \hat{\gamma}_\nu, \hat{\delta}_\nu)$ be its limit. ($\hat{E}_\nu$ will be given soon.) By Claim 11, $B_\nu \mid_{\Psi_{\xi\nu}} A_\xi$ such that the $\Psi_{\xi\nu}$ ($\xi < \nu$) are compatible with the $\Phi_{\xi\varrho}$ ($\xi < \varrho < \nu$). For $\xi < \nu$, $A_\xi|A_0$ is a $T^{(\xi)}$-preserving extension, and $E^{(\nu)} \subset T^{(\xi)}$. So $E^{(\nu)}$ extends to A_ξ; let $E_\xi^{(\nu)}$ denote its extension. For $\xi < \varrho < \nu$ we have $E_\xi^{(\nu)} \subseteq E_\varrho^{(\nu)}$, for $\Phi_{0\varrho} = \Phi_{\xi\varrho} \circ \Phi_{0\xi}$. So we obtain

$$\bigcup_{\xi<\nu} \bigcap_{\xi\le\varrho<\nu} E_\varrho \supseteq \bigcup_{\xi<\nu} \bigcap_{\xi\le\varrho<\nu} E_\varrho^{(\nu)} = \bigcup_{\xi<\nu} E_\xi^{(\nu)}.$$

Hence, according to (17), we can let $\hat{E}_\nu = \bigcup_{\xi<\nu} E_\xi^{(\nu)}$, and B_ν becomes an $E^{(\nu)}$-preserving extension of A_0. Since $m_\xi = |E_\xi| \geq |E_\xi^{(\nu)}| = |E^{(\nu)}| \cdot [A_\xi : A_0] = |A_0| \cdot [A_\xi : A_0] = |A_\xi| = m_\xi$, we obtain $|\hat{E}_\nu| = \sup\{m_\xi\colon\ \xi < \nu\} = |B_\nu|$. Hence B_ν is a perfect box, and it is an $E^{(\nu)}$-preserving extension of A_0.

Now we have to distinguish two cases. First assume that $\nu < \mu$. Since $T^{(\nu)} \subset E^{(\nu)}$, $T^{(\nu)}$ extends to B_ν. Let $\hat{T}_\nu^{(\nu)}$ and $\hat{E}_\nu^{(\nu)}$ denote the extension of $T^{(\nu)}$ and $E^{(\nu)}$ to B_ν, respectively. Then

$$|\hat{E}_\nu \setminus \hat{T}_\nu^{(\nu)}| \geq |\hat{E}_\nu^{(\nu)} \setminus \hat{T}_\nu^{(\nu)}| = |E^{(\nu)} \setminus T^{(\nu)}| \cdot [B_\nu : A_0] = |B_\nu|.$$

On the other hand, $|B_\nu| = \sup\{m_\xi\colon\ \xi < \nu\} \leq m_\nu$. Thus $\mathrm{H}(m_\nu)$ applies and yields a perfect box $\mathcal{A}_\nu = (A_\nu, d_\nu, h_\nu, E_\nu, \alpha_\nu, \beta_\nu, \gamma_\nu, \delta_\nu)$ of power m_ν such that $A_\nu \mid_{\hat{\Psi}} B_\nu$ is a $\hat{T}_\nu^{(\nu)}$-preserving extension. Then A_ν is an extension of A_ξ by $\Phi_{\xi\nu} := \hat{\Psi} \circ \Psi_{\xi\nu}$ $(\xi < \nu)$, and A_ν is a $T^{(\nu)}$-preserving extension of A_0, cf. Claim 4. Further, for $\xi < \varrho < \nu$,

$$\Phi_{\varrho\nu} \circ \Phi_{\xi\varrho} = (\hat{\Psi} \circ \Psi_{\varrho\nu}) \circ \Phi_{\xi\varrho} = \hat{\Psi} \circ (\Psi_{\varrho\nu} \circ \Phi_{\xi\varrho}) = \hat{\Psi} \circ \Psi_{\xi\nu} = \Phi_{\xi\nu}.$$

This shows $\mathrm{I}(\nu)$.

The other case is $\nu = \mu$. Then $|B_\mu| = |B_\nu| = \sup\{m_\xi\colon\ \xi < \mu\} = k = m_\mu$. Hence we do not have to (and we are even not allowed to) apply $\mathrm{H}(m_\nu)$ to extend this B_ν to A_ν. We simply let $A_\mu := B_\mu$, $\Phi_{\xi\mu} := \Psi_{\xi\mu}$ $(\xi < \mu)$, and $\mathrm{I}(\mu)$ clearly holds.

From $\mathrm{I}(\mu)$ we obtain that A_μ is a $T^{(\mu)} = E'$ extension of $A = A_0$. This proves Claim 16.

∎

Finally, from the existence of a countable perfect box (i.e., $\mathrm{H}(\aleph_0)$) and Claims 15 and 16 we derive $\mathrm{H}(m)$ for all small cardinals m via induction. According to the remark after Definition 5, this proves Theorem 1.

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