FINITE CONVEX GEOMETRIES OF CIRCLES

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ABSTRACT. Let F be a finite set of circles in the plane. The usual convex closure restricted to F yields a convex geometry, which is a combinatorial structure introduced by P. H. Edelman in 1980 under the name "anti-exchange closure system". We prove that if the circles are collinear and they are arranged in a "concave way", then they determine a convex geometry of convex dimension at most 2, and each finite convex geometry of convex dimension at most 2 can be represented this way. The proof uses some recent results from lattice theory, and some of the auxiliary statements on lattices or convex geometries could be of separate interest. The paper concludes with some open problems.

1. INTRODUCTION

The concept of convex geometries was introduced by Edelman [25, 26]; see also Edelman and Jamison [27], Adaricheva, Gorbunov, and Tumanov [6], and Armstrong [7]. *Convex geometries* are combinatorial structures: finite sets with anti-exchange closures such that the empty set is closed. They are equivalent to antimatroids, which are particular greedoids, and also to meet-distributive lattices. Actually, the concept of convex geometries has many equivalent variants. The first of these variants is due to Dilworth [23], and the early ones were surveyed in Monjardet [37]. Since it would wander too far if we described the rest (more than a dozen approaches), we only mention Adaricheva [3], Abels [1], Caspard and Monjardet [11], Avann [8], Jamison-Waldner [30], and Ward [42] for additional sources, and Stern [41], Adaricheva and Czédli [4], and Czédli [13] for some recent surveys. In fact, we need only a small part of the theory of convex geometries, and the present paper is intended to be self-contained for those who know the rudiments of lattice theory up to, say, the concept of semimodularity.

From a combinatorial point of view, the finite convex geometries are the interesting ones. The tools we use are elaborated only for the finite case. Hence, except for $\langle S, \operatorname{Hull}_{S}^{(n)} \rangle$ in the next paragraph, the present paper is restricted to *finite* convex geometries. Postponing the exact definition of convex geometries to Section 2, we present an important finite example as follows. Let $n \in \mathbb{N} = \{1, 2, 3, \ldots\}$, and let E be a finite subset of the *n*-dimensional space \mathbb{R}^{n} . The set of all subsets of E is denoted by $\mathcal{P}(E)$. For $Y \subseteq E$, we define $\operatorname{Hull}_{E}^{(n)}(Y) = E \cap \operatorname{Conv}(Y)$, where $\operatorname{Conv}(Y)$ denotes the usual convex hull of Y in \mathbb{R}^{n} . The map $\operatorname{Hull}_{E}^{(n)} : \mathcal{P}(E) \to \mathcal{P}(E)$ is a closure operator, and $\langle E, \operatorname{Hull}_{E}^{(n)} \rangle$ is a finite convex geometry. Convex geometries

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of this form are called *geometries of relatively convex sets*, and they (not only the finite ones) were studied by Adaricheva [2] and [3], Bergman [9], and Huhn [29].

For a convex geometry G, let Lat G denote the lattice of closed sets of G. Based on Bergman [9] and Pudlák and Tůma [39], the introduction of Adaricheva [3] points out that Lat G can be embedded into Lat $\langle S, \operatorname{Hull}_{S}^{(n)} \rangle$ for some n and $S \subseteq \mathbb{R}^{n}$. However, even if a finite convex geometry is of convex dimension 2, it is not necessarily isomorphic to some $\langle E, \operatorname{Hull}_{E}^{(n)} \rangle$.

Besides geometries of relatively convex sets, there exists a more complicated way to define a convex geometry on a subset $B \subseteq \mathbb{R}^n$ by means of the usual convex hull operator Conv. For definition, let B and A be finite subsets, acting as a base set and an auxiliary set, of \mathbb{R}^n such that $\operatorname{Conv}(A) \cap B = \emptyset$. For $X \subseteq B$, let $\operatorname{Hull}_B^{(n,A)}(X) = B \cap \operatorname{Conv}(X \cup A)$. By Kashiwabara, Nakamura, and Okamoto [34], $\langle B, \operatorname{Hull}_B^{(n,A)} \rangle$ is a convex geometry, and each finite convex geometry is *isomorphic* to an appropriate $\langle B, \operatorname{Hull}_B^{(n,A)} \rangle$. Note that $\operatorname{Lat}\langle B, \operatorname{Hull}_B^{(n,A)} \rangle$ is isomorphic to an interval of $\operatorname{Lat}\langle A \cup B, \operatorname{Hull}_{A \cup B}^{(n)} \rangle$.

Motivated by the results of [9] and [34] mentioned above, the present paper introduces another kind of "concrete" finite convex geometry that is still based on the usual concept Conv of convexity. However, our primary purpose is to represent finite convex geometries in a visual, conceptually simple way. In particular, we look for a representation theorem that leads to readable figures, at least in case of small size, because figures are generally useful in understanding a subject. (This is well exemplified by the role that Hasse diagrams play, even if the present paper cannot compete with their importance.) The space \mathbb{R}^n for $n \geq 3$ can hardly offer comprehendible figures. The real line $\mathbb{R}^1 = \mathbb{R}$ is too "narrow" to hold overlapping objects in a readable way. Therefore, with the exception of Subsection 4.2, we will only work in the plane \mathbb{R}^2 . We will see that the plane is general enough to represent all finite convex geometries of convex dimension 2, and also some additional ones. Although Adaricheva [5] observed that our result has a counterpart based on 1dimensional circles, see Corollary 4.6 later, an additional advantage of our approach with planar circles is that it raises interesting problems, see (4.5) and (4.6).

To accomplish our goal, we start from a finite set F of circles in the plane \mathbb{R}^2 , and we define a convex geometry $\langle F, \operatorname{Hull}_F^o \rangle$ with the help of forming usual convex hulls in the plane, analogously to the geometries of relative convex sets. The structures $\langle F, \operatorname{Hull}_F^o \rangle$, called *convex geometries of circles*, are very close to the usual closure $\operatorname{Conv}: \mathbb{R}^2 \to \mathbb{R}^2$. As opposed to geometries of relatively convex sets, we can prove that each finite convex geometry of convex dimension at most 2 is isomorphic to some convex geometry $\langle F, \operatorname{Hull}_F^o \rangle$ of circles. Actually, our representation theorem, the main result of the paper, will assert more by imposing some conditions on F; see Figure 1 (without C'_3 , D, the grey-colored plane shape H, and the dotted curves) for a first impression.

This paper uses some lattices as auxiliary tools in proving the main result. In fact, the theory of slim semimodular lattices has recently developed rapidly, as witnessed by Czédli [12, 14, 15], Czédli, Dékány, Ozsvárt, Szakács, and Udvari [16], Czédli and Grätzer [17], Czédli, L. Ozsvárt, and Udvari [18], Czédli and Schmidt [19, 20, 21, 22], Grätzer and Knapp [31, 32, 33], and Schmidt [40]. Even if only a part of the results in these papers are eventually needed here, this progress provides the background of the present work. On the other hand, some auxiliary statements



FIGURE 1. A concave set of collinear circles (disregard the greycolored shape H until the proof of Lemma 3.1)

we prove here, namely some of the propositions, seem to be of some interest in the theory of slim semimodular lattices, while some other propositions could be interesting in the theory of convex geometries.

The paper is structured as follows. Section 2 gives the basic concepts and formulates two results. Proposition 2.1 asserts that, for every finite set F of circles in the plane, $\langle F, \operatorname{Hull}_F^o \rangle$ is a convex geometry, while our main result, Theorem 2.2, is the converse statement for the case of convex dimension ≤ 2 . The results of Section 2 are proved in Section 3, where several auxiliary statements are cited or proved, and some more concepts are recalled. Section 4 contains examples, statements, and open problems to indicate that although Theorem 2.2 gives a satisfactory representation of finite convex geometries of convex dimension at most 2, we are far both from settling the case of higher convex dimensions and from understanding what the abstract class of our convex geometries of circles is.

As mentioned already, we only assume a little knowledge of lattices. Besides the first few pages of any book on lattices or particular lattices, including Grätzer [28], Nation [38], and Stern [41], even the first chapter of Burris and Sankappanavar [10], which does not even focus on lattice theory, is sufficient. Note that [10] and [38] are freely downloadable.

2. Convex geometries and our results

Assume that we are given a set U and a map $\Phi: \mathcal{P}(U) \to \mathcal{P}(U)$. If $X \subseteq \Phi(X) = \Phi(\Phi(X)) \subseteq \Phi(Y)$ holds for all $X \subseteq Y \subseteq U$, then Φ is a *closure operator* on U. The pair $\langle U, \Phi \rangle$ is a *convex geometry* if Φ is a closure operator on U, $\Phi(\emptyset) = \emptyset$, and Φ satisfies the so-called *anti-exchange property*

(2.1)
$$if \Phi(X) = X \in \mathcal{P}(U), \, x, y \in U \setminus X, \, x \neq y, \\ and \, x \in \Phi(X \cup \{y\}), \, then \, y \notin \Phi(X \cup \{x\}).$$

Given a convex geometry $\langle U, \Phi \rangle$, we use the notation

$$Lat\langle U, \Phi \rangle = \{ X \in \mathcal{P}(U) : X = \Phi(X) \}$$

to denote the set of closed sets of $\langle U, \Phi \rangle$. Actually, the structure $\operatorname{Lat}\langle U, \Phi \rangle = \langle \operatorname{Lat}\langle U, \Phi \rangle, \subseteq \rangle$ is a *lattice*, and it is a complete meet-subsemilattice of the powerset lattice $\mathcal{P}(U) = \langle \mathcal{P}(U), \cup, \cap \rangle$. It is well-known that $\operatorname{Lat}\langle U, \Phi \rangle$ determines $\langle U, \Phi \rangle$, since we have $\Phi(X) = \bigcap \{Y \in \operatorname{Lat}\langle U, \Phi \rangle : X \subseteq Y\}$. Hence, it is natural to say

that $\langle U, \Phi \rangle$ can be *embedded* into or *isomorphic* to a convex geometry $\langle V, \Psi \rangle$, if the lattice Lat $\langle U, \Phi \rangle$ can be embedded into or is isomorphic to Lat $\langle V, \Psi \rangle$, respectively.

As usual, a *circle* is a set $\{\langle x, y \rangle \in \mathbb{R}^2 : (x-u)^2 + (y-v)^2 = r\}$, where $u, v, r \in \mathbb{R}$ and $r \geq 0$. A circle of radius 0 consists of a single point. Since Conv, the operator of forming convex hulls, is defined for subsets of \mathbb{R}^2 rather than for sets of circles, we introduce a shorthand notation for "points of" (or "point set of") as follows. For a set X of circles in \mathbb{R}^2 , the *set of points* belonging to some member of X is denoted by X^{ps} . In other words,

(2.2)
$$X^{\rm ps} = \bigcup_{C \in X} C.$$

For a set F of circles in \mathbb{R}^2 and $X \subseteq F$, we define

(2.3)
$$\operatorname{Hull}_{F}^{o}(X) = \left\{ C \in F : C \subseteq \operatorname{Conv}(X^{\operatorname{ps}}) \right\}.$$

(The superscript circle in the notation will remind us that $\operatorname{Hull}_{F}^{o}$ is defined on a set of circles.) The structure $\langle F, \operatorname{Hull}_{F}^{o} \rangle$ will be called the *convex geometry* of F, and we call it a *convex geometry of circles* if F is not specified. (We shall soon prove that it is a convex geometry.) Note that if all circles of F are of radius 0 and $E \subseteq \mathbb{R}^{2}$ is the set of their centers, then $\langle F, \operatorname{Hull}_{F}^{o} \rangle$ is obviously isomorphic to $\langle E, \operatorname{Hull}_{F}^{(2)} \rangle$.

We are now in the position to state our first observation; the statements of this section will be proved in Section 3.

Proposition 2.1. For every finite set F of circles in \mathbb{R}^2 , $\langle F, \operatorname{Hull}_F^o \rangle$ is a convex geometry.

If there is a line containing the centers of all members of a finite set F of circles in \mathbb{R}^2 , then F is a set of collinear circles. By symmetry, we will always assume that the line in question is the x axis. That is, in case of a set of collinear circles, all the centers are of the form $\langle u, 0 \rangle$. For example, F in Figure 1 is a set of collinear circles; note that the dotted curves and the x-axis do not belong to F. Note that a set of collinear circles can always be given by a set of intervals of the real line \mathbb{R} ; this comment will be expanded in Subsection 4.2. However, circles lead to a stronger result and more readable figures than intervals. The label of a circle in our figures is either below the center (inside or outside but close to the circle), or we use an arrow. The radius of a circle C is denoted by $\operatorname{rad}(C)$. If the center of Cis $\langle u, 0 \rangle$, then

$$\operatorname{LPt}(C) = u - \operatorname{rad}(C) \text{ and } \operatorname{RPt}(C) = u + \operatorname{rad}(C)$$

will denote the *leftmost point* and the *rightmost point* of C, respectively. Since we allow that two distinct circles have the same leftmost point or the same rightmost point, we also need the following concept. Although circles are usually treated as endless figures, in case the center of a circle C lies on the x axis, we define the *left end* and the *right end* of C as follows:

 $\operatorname{LEnd}(C) = \langle \operatorname{LPt}(C), -\operatorname{rad}(C) \rangle$ and $\operatorname{REnd}(C) = \langle \operatorname{RPt}(C), \operatorname{rad}(C) \rangle$.

Left and right ends are ordered lexicographically; this order is denoted by \Box . Thus

 $\operatorname{LEnd}(C) \sqsubset \operatorname{LEnd}(D) \iff \operatorname{LPt}(C) < \operatorname{LPt}(D), \text{ or}$ $\operatorname{LPt}(C) = \operatorname{LPt}(D) \text{ and } \operatorname{rad}(C) > \operatorname{rad}(D),$

$$\operatorname{REnd}(C) \sqsubset \operatorname{REnd}(D) \iff \operatorname{RPt}(C) < \operatorname{RPt}(D), \text{ or}$$
$$\operatorname{RPt}(C) = \operatorname{RPt}(D) \text{ and } \operatorname{rad}(C) < \operatorname{rad}(D).$$

For later reference, note the obvious rules:

(2.4)
$$\operatorname{LPt}(C) < \operatorname{LPt}(D) \Rightarrow \operatorname{LEnd}(C) \sqsubset \operatorname{LEnd}(D) \text{ and}$$

$$\operatorname{RPt}(C) < \operatorname{RPt}(D) \Rightarrow \operatorname{REnd}(C) \sqsubset \operatorname{REnd}(D).$$

Let F be a set of collinear circles. We say that F is a concave set of collinear circles if for all $C_1, C_2, C_3 \in F$,

(2.5) whenever
$$\operatorname{LEnd}(C_1) \sqsubset \operatorname{LEnd}(C_2)$$
 and $\operatorname{REnd}(C_2) \sqsubset \operatorname{REnd}(C_3)$.

then
$$C_2 \subseteq \operatorname{Conv}(C_1 \cup C_3).$$

For illustration, see C_1 , C_2 , and C_3 , or C'_1 , C'_2 , and C'_3 , in Figure 1. Note that since each $C \in F$ is uniquely determined by LEnd(C) and also by REnd(C), " \Box " in (2.5) can be replaced by " \sqsubseteq ". If $|\{\text{LPt}(C_1), \text{LPt}(C_2), \text{RPt}(C_1), \text{RPt}(C_2)\}| = 4$ for any two distinct C_1 and C_2 in a set F of collinear circles, then F is separated. For example, $F \setminus \{C'_3, D\}$ in Figure 1 is a separated, concave set of collinear circles. Clearly, if F is a separated set of collinear circles, then F is concave if and only if

whenever
$$LPt(C_1) < LPt(C_2)$$
 and $RPt(C_2) < RPt(C_3)$,

(2.6) then
$$C_2 \subseteq \operatorname{Conv}(C_1 \cup C_3)$$
.

For a finite lattice L, the set of elements with exactly one lower cover (the set of non-zero join-irreducible elements) is denoted by Ji L. Dually, Mi L stands for the set of elements with exactly one cover. The *convex dimension* of a finite convex geometry $\langle U, \Phi \rangle$ is the maximum size of an antichain in Mi(Lat $\langle U, \Phi \rangle$). In other words, the convex dimension is the *width* of the poset Mi(Lat $\langle U, \Phi \rangle$).

We can now formulate our main result, which characterizes finite convex geometries of convex dimension at most 2 as the convex geometries of finite (separated or not necessarily separated) concave sets of collinear circles.

Theorem 2.2.

- (A) If F is a finite, concave set of collinear circles in the plane, then $\langle F, \operatorname{Hull}_{F}^{o} \rangle$ is a convex geometry of convex dimension at most 2.
- (B) For each finite convex geometry (U, Φ) of convex dimension at most 2, there exists a finite, separated, concave set F of collinear circles in the plane such that (U, Φ) is isomorphic to (F, Hull^o_F).

3. Proofs and auxiliary statements

3.1. Not necessarily collinear circles.

Proof of Proposition 2.1. For every $X \subseteq F$, we have

(3.1)
$$\operatorname{Conv}(X^{\operatorname{ps}}) = \operatorname{Conv}((\operatorname{Hull}_F^{\operatorname{o}}(X))^{\operatorname{ps}}),$$

since the " \subseteq " inclusion follows from $X \subseteq \operatorname{Hull}_F^o(X)$ while the converse inclusion comes from the obvious $\operatorname{Conv}(X^{\operatorname{ps}}) \supseteq (\operatorname{ps}\operatorname{Hull}_F^o(X))$.

Assume that $X \subseteq F$, and let $Y = \operatorname{Hull}_{F}^{o}(X)$. Since $C \subseteq \operatorname{Conv}(X^{\operatorname{ps}})$ holds for all $C \in Y$, that is $Y^{\operatorname{ps}} \subseteq \operatorname{Conv}(X^{\operatorname{ps}})$, we obtain

$$\operatorname{Conv}(Y^{\operatorname{ps}}) \subseteq \operatorname{Conv}(X^{\operatorname{ps}}).$$



FIGURE 2. Illustrating the proof of Proposition 2.1

We have equality here, since $Y \supseteq X$. This implies $\operatorname{Hull}_{F}^{o}(Y) = \operatorname{Hull}_{F}^{o}(X)$, and it follows that $\operatorname{Hull}_{F}^{o}$ is a closure operator with the property $\operatorname{Hull}_{F}^{o}(\emptyset) = \emptyset$. Observe that a closure operator Φ on U satisfies the anti-exchange property (2.1) if and only if

(3.2)
if
$$\Phi(X) = X \in \mathcal{P}(U), x_0, x_1 \in U \setminus X,$$

and $\Phi(X \cup \{x_0\}) = \Phi(X \cup \{x_1\}),$ then $x_0 = x_1$

Therefore, tailoring (3.2) to our situation, we assume $X = \operatorname{Hull}_{F}^{o}(X) \in \mathcal{P}(F)$, $\{C_{0}, C_{1} \subseteq F \setminus X\}$, and $\operatorname{Hull}_{F}^{o}(X \cup \{C_{0}\}) = \operatorname{Hull}_{F}^{o}(X \cup \{C_{1}\})$. We have to show $C_{0} = C_{1}$. Combining our assumption with (3.1), we obtain

(3.3)
$$\operatorname{Conv}(C_0 \cup X^{\operatorname{ps}}) = \operatorname{Conv}(C_1 \cup X^{\operatorname{ps}}).$$

Let Γ and Δ be the boundary of $\operatorname{Conv}(X^{\operatorname{ps}})$ and the boundary of the set given in (3.3), respectively; see the thick closed curves in Figure 2. In the figure, X is depicted twice: in itself on the left and with C_i on the right. We can imagine Γ and Δ as tight resilient rubber nooses around the members of X and $X \cup \{C_i\}$, respectively. Pick an $i \in \{0, 1\}$. Observe that $C_i \notin X = \operatorname{Hull}_F^o(X)$ implies $C_i \not\subseteq$ $\operatorname{Conv}(X^{\operatorname{ps}}) = \operatorname{Conv}(\Gamma)$. Clearly, Δ can be decomposed into finitely many segments $K_0, K_1, \ldots, K_{m-1}$, listed anti-clockwise, such that these segments are of positive length and the following properties hold for all $j \in \{0, \ldots, m-1\}$ and $i \in \{0, 1\}$:

- (i) The endpoint of K_j is the first point of K_{j+1} , where j + 1 is understood modulo m.
- (ii) Either $K_j \subseteq \Gamma$, or no inner point of K_j belongs to $\text{Conv}(\Gamma)$.
- (iii) If no inner point of K_j belongs to $\text{Conv}(\Gamma)$, then either K_j is a straight line segment, or K_j is an arc of C_i .
- (iv) There exists a $t \in \{0, ..., m-1\}$ such that either
 - (a) K_t is not a straight line segment and none of its inner points belongs to $\text{Conv}(\Gamma)$; or
 - (b) both K_t and K_{t+1} are straight line segments and their common point is outside Conv(Γ).

As opposed to Figure 2, note that t in (iv) need not be unique. Clearly, (iv)(b) holds if and only if the radius of C_i is zero; in this case C_i consists of the common point of K_t and K_{t+1} , which is the only angle point of Δ outside Conv(Γ), and we conclude $C_0 = C_1$. Otherwise, if (iv)(a) holds, then (iii) implies that K_t is a

common arc of C_0 and C_1 , whence we conclude $C_0 = C_1$ again. Therefore, $\operatorname{Hull}_F^{\circ}$ is an anti-exchange closure operator, and $\langle F, \operatorname{Hull}_F^{\circ} \rangle$ is a convex geometry.

3.2. Collinear circles. For a set F of collinear circles and $A, B \in F$, we define the *horizontal interval*

 $\operatorname{HInt}_{F}(A, B) = \{ C \in F : \operatorname{LEnd}(A) \sqsubseteq \operatorname{LEnd}(C), \operatorname{REnd}(C) \sqsubseteq \operatorname{REnd}(B) \}.$

Note that $\operatorname{HInt}_F(A, B)$ can be empty. If $A \neq B$ and A is inside the circle B, then $A \in \operatorname{HInt}_F(A, B) \cap \operatorname{HInt}_F(B, A)$ but $B \notin \operatorname{HInt}_F(A, B) \cup \operatorname{HInt}_F(B, A)$. Note also that, for $A \in F$, the horizontal interval $\operatorname{HInt}_F(A, A)$ is the set of circles of F that are *inside* A, including A itself.

Lemma 3.1. If F is a finite set of collinear circles in the plane, then the following three statements hold.

- (i) $\{\emptyset\} \cup \{\operatorname{HInt}_F(A, B) : A, B \in F\} \subseteq \operatorname{Lat}\langle F, \operatorname{Hull}_F^{\circ} \rangle$. If F is concave, then also the equality $\{\emptyset\} \cup \{\operatorname{HInt}_F(A, B) : A, B \in F\} = \operatorname{Lat}\langle F, \operatorname{Hull}_F^{\circ} \rangle$ holds.
- (ii) For Ø ≠ X ∈ Lat⟨F, Hull^o_F⟩, let A and B denote the circles in X with least left end and greatest right end, respectively. If F is concave, then the equality X = HInt_F(A, B) = HInt_F(A, A) ∨ HInt_F(B, B) holds in Lat⟨F, Hull^o_F⟩.
- (iii) If $X \in \text{Lat}\langle F, \text{Hull}_F^o \rangle$, then $X = \bigvee \{ \text{HInt}_F(C, C) : C \in X \}$.

Proof. We may assume $F \neq \emptyset$, since otherwise the lemma is trivial. Assume $\operatorname{HInt}_F(A, B) \neq \emptyset$. As indicated in Figure 1 by the grey area, there exists a convex subset H of \mathbb{R}^2 such that $(\operatorname{HInt}_F(A, B))^{\operatorname{ps}} \subseteq H$ but H includes no $C \in F \setminus \operatorname{HInt}_F(A, B)$ as a subset. (As in the figure, it is always possible to find an appropriate H whose boundary is the union of an arc of A, that of B, and four line segments. Note that if $\operatorname{rad}(A) = 0$, then the only arc of A is itself.) This implies that $\operatorname{HInt}_F(A, B) = \operatorname{Hull}_F^\circ(\operatorname{HInt}_F(A, B)) \in \operatorname{Lat}\langle F, \operatorname{Hull}_F^\circ\rangle$. Thus $\{\emptyset\} \cup \{\operatorname{HInt}_F(A, B) : A, B \in F\} \subseteq \operatorname{Lat}\langle F, \operatorname{Hull}_F^\circ\rangle$.

To prove the converse inclusion under the additional assumption that F is concave, let $\emptyset \neq X \in \text{Lat}\langle F, \text{Hull}_F^{\circ} \rangle$. Since F is finite, there are a unique $A \in X$ with least left end LEnd(A) and a unique $B \in X$ with largest right end REnd(B). Using that F is concave, we obtain for $C \in \text{HInt}_F(A, B)$ that $C \subseteq \text{Conv}(A \cup B) \subseteq$ $\text{Conv}(X^{\text{ps}})$, that is, $C \in \text{Hull}_F^{\circ}(X)$. This implies $\text{HInt}_F(A, B) \subseteq \text{Hull}_F^{\circ}(X) = X$. On the other hand, if $C \in X$, then the choice of A and B gives $\text{LEnd}(A) \sqsubseteq \text{LEnd}(C)$ and $\text{REnd}(C) \sqsubseteq \text{REnd}(B)$, that is, $C \in \text{HInt}_F(A, B)$. Hence, $X = \text{HInt}_F(A, B)$, and we obtain $\text{Lat}\langle F, \text{Hull}_F^{\circ} \rangle \subseteq \{\emptyset\} \cup \{\text{HInt}_F(A, B) : A, B \in F\}$. This proves (i) and the first equality in (ii).

Next, we do not assume that F is concave. If $C \in X \in \text{Lat}\langle F, \text{Hull}_F^{\text{o}} \rangle$, then $D \subseteq \text{Conv}(C) \subseteq \text{Conv}(X^{\text{ps}})$ holds for all $D \in \text{HInt}_F(C, C)$. That is, $D \in \text{Hull}_F^{\text{o}}(X) = X$ for all $D \in \text{HInt}_F(C, C)$. Thus, for all $C \in X$, we have $\{C\} \subseteq \text{HInt}_F(C, C) \subseteq X$, which clearly implies (iii).

Finally, assume again that F is concave. From (iii) and the first equality in (ii), we obtain $X = \operatorname{HInt}_F(A, B) \supseteq \operatorname{HInt}_F(A, A) \lor \operatorname{HInt}_F(B, B)$. The reverse inclusion follows from the assumption that F is concave. Hence, (ii) holds.

Lemma 3.2. If F is a finite set of collinear circles in the plane, then

$$\operatorname{Hi}(\operatorname{Lat}\langle F, \operatorname{Hull}_{F}^{\mathrm{o}}\rangle) = \{\operatorname{HInt}_{F}(A, A) : A \in F\}.$$

Proof. Although the statement is intuitively more or less clear, we give an exact proof. First, we show for $A \in F$ that $\operatorname{HInt}_F(A, A) \in \operatorname{Ji}(\operatorname{Lat}\langle F, \operatorname{Hull}_F^o \rangle)$. This is

obvious if $|\operatorname{HInt}_F(A, A)| = 1$, that is, if $\operatorname{HInt}_F(A, A)$ is an atom in $\operatorname{Lat}\langle F, \operatorname{Hull}_F^o \rangle$. Assume $|\operatorname{HInt}_F(A, A)| \geq 2$. Let B and C be the circles with least left end and greatest right end in $\operatorname{HInt}_F(A, A) \setminus \{A\}$, respectively. (They need not be distinct.) Since the circles of F are determined by their left ends, $\operatorname{LEnd}(A) \neq \operatorname{LEnd}(B)$, and we obtain $\operatorname{LEnd}(A) \sqsubset \operatorname{LEnd}(B)$ by the definition of $\operatorname{HInt}_F(A, A)$. Similarly, $\operatorname{REnd}(C) \sqsubset \operatorname{REnd}(A)$. Now if $D \in \operatorname{HInt}_F(A, A) \setminus A$, then the choice of B and C gives $D \in \operatorname{HInt}_F(B, C)$. On the other hand, if $D \in \operatorname{HInt}_F(B, C)$, then

 $\operatorname{LEnd}(A) \sqsubset \operatorname{LEnd}(B) \sqsubseteq \operatorname{LEnd}(D)$ and $\operatorname{REnd}(D) \sqsubseteq \operatorname{REnd}(C) \sqsubset \operatorname{REnd}(A)$

yield $D \in \operatorname{HInt}_F(A, A) \setminus A$. Thus $\operatorname{HInt}_F(A, A) \setminus A = \operatorname{HInt}_F(B, C) \in \operatorname{Lat}\langle F, \operatorname{Hull}_F^o \rangle$ is a unique lower cover of $\operatorname{HInt}_F(A, A)$, because $A \in X$ implies $\operatorname{HInt}_F(A, A) \subseteq X$ for all $X \in \operatorname{Lat}\langle F, \operatorname{Hull}_F^o \rangle$. Hence, $\operatorname{HInt}_F(A, A) \in \operatorname{Ji}(\operatorname{Lat}\langle F, \operatorname{Hull}_F^o \rangle)$. This proves $\operatorname{Ji}(\operatorname{Lat}\langle F, \operatorname{Hull}_F^o \rangle) \supseteq \{\operatorname{HInt}_F(A, A) : A \in F\}.$

To prove the converse inclusion, let $X \in \text{Ji}(\text{Lat}\langle F, \text{Hull}_F^o \rangle)$. We obtain from Lemma 3.1(iii) that $X = \text{HInt}_F(A, A)$ for some $A \in F$. Thus $\text{Ji}(\text{Lat}\langle F, \text{Hull}_F^o \rangle) \subseteq$ {HInt} $_F(A, A) : A \in F$ }.

Lemma 3.3. Assume that F is a finite, concave set of collinear circles in the plane, and $|F| \ge 1$. If the circle in F with least left end and that with largest right end are denoted by K_l and K_r , respectively, then

 $\{\emptyset\} \cup \{\operatorname{HInt}_F(K_l, B) : B \in F\}$ and $\{\emptyset\} \cup \{\operatorname{HInt}_F(A, K_r) : A \in F\}$

are chains in $\operatorname{Lat}\langle F, \operatorname{Hull}_{F}^{o} \rangle$, and each nonempty $X \in \operatorname{Lat}\langle F, \operatorname{Hull}_{F}^{o} \rangle$ is of the form

 $X = \operatorname{HInt}_F(K_l, B) \wedge \operatorname{HInt}_F(A, K_r), \text{ that is, } \operatorname{HInt}_F(K_l, B) \cap \operatorname{HInt}_F(A, K_r),$

where $A, B \in X$ are defined in Lemma 3.1(ii).

Proof. Let $B_1, B_2 \in F$. We can assume $\operatorname{REnd}(B_1) \sqsubseteq \operatorname{REnd}(B_2)$, since " \sqsubseteq " is a linear order. By transitivity, $\operatorname{HInt}_F(K_l, B_1) \subseteq \operatorname{HInt}_F(K_l, B_2)$. Therefore, the first set in the lemma is a chain. By left-right duality, so is the second one.

Next, let $\emptyset \neq X \in \text{Lat}\langle F, \text{Hull}_F^o \rangle$. We have $X = \text{HInt}_F(A, B)$ by Lemma 3.1(ii), and the obvious equality $\text{HInt}_F(A, B) = \text{HInt}_F(K_l, B) \cap \text{HInt}_F(A, K_r)$ completes the proof.

3.3. Lattices associated with convex geometries. One of the many equivalent ways, actually the first way, of defining convex geometries was to use meetdistributive lattices; see Dilworth [23]. Now we recall some concepts from lattice theory. Let \prec denote the "is covered by" relation. A finite lattice *L* is *lower semimodular* if $a \prec a \lor b$ implies $a \land b \prec b$ for all $a, b \in L$. An equivalent condition is that $a \prec b$ implies $a \land c \preceq b \land c$ for all $a, b, c \in L$. We will often use the trivial fact that this property is inherited by intervals and, more generally, by cover-preserving sublattices. For $u \neq 0$ in a finite lattice *L*, let u_* denote the meet of all lower covers of *u*. A finite lattice *L* is *meet-distributive* if the interval $[u_*, u]$ is a distributive lattice for all $u \in L \setminus \{0\}$. For other definitions, see Adaricheva [3], Adaricheva, Gorbunov and Tumanov [6], and Caspard and Monjardet [11]; see also Czédli [13, Proposition 2.1 and Remark 2.2] and Adaricheva and Czédli [4] for recent surveys and developments.

The study of meet-distributive lattices (and their duals) goes back to Dilworth [23], 1940. There were a lot of discoveries and rediscoveries of these lattices and equivalent combinatorial structures (including convex geometries); see [6], [13], Monjardet [37], and Stern [41] for surveys. We recall the following statement;

its origin is the combination of Ward [42] (see also Dilworth [23, page 771], where [42] is cited) and Avann [8] (see also Edelman [26, Theorem 1.1(E,H)], where [8] is recalled).

Claim 3.4. Every finite meet-distributive lattice is lower semimodular.

The width of a partially ordered set P, denoted by width(P), is the smallest k such that P is the union of some k chains of L. By the famous theorem of Dilworth [24], width(P) equals the maximum size of an antichain in P. For a finite lattice L, we are interested in the width of Mi L. Note that, clearly, width(Mi L) is the smallest k such that the union of k maximal chains of L includes Mi L. Following Grätzer and Knapp [31] and, in the present form, Czédli and Schmidt [19], finite lattices L with width(Mi L) ≤ 2 are called *dually slim*. Finite lattices L with width(Ji L) ≤ 2 are, of course, called *slim*.

If L is a lattice and $x \in L$, then the principal ideal $\{y \in L : y \leq x\}$ is denoted by $\downarrow x$. We have already mentioned that a finite convex geometry $G = \langle U, \Phi \rangle$ determines a lattice, the lattice $\text{Lat}(G) = \text{Lat}\langle U, \Phi \rangle$ of its closed sets. Conversely, if L is a finite meet-distributive lattice, then we can take the combinatorial structure $\text{Geom}(L) = \langle \text{Ji } L, \{\text{Ji } L \cap \downarrow x : x \in L\} \rangle$. Part of the following lemma, which asserts that finite convex geometries and finite meet-distributive lattices are essentially the same, was proved by Edelman [25, Theorem 3.3], see also Armstrong [7, Theorem 2.8]. The rest can be extracted from Adaricheva, Gorbunov, and Tumanov[6, proof of Theorem 1.9]; see also Czédli [13, Lemma 7.4] for more details.

Lemma 3.5. If L is a finite meet-distributive lattice and $G = \langle U, \Phi \rangle$ is a finite convex geometry, then the following three statements hold.

- (i) Lat(G) is a finite meet-distributive lattice.
- (ii) $\operatorname{Geom}(L)$ is a finite convex geometry.
- (iii) $\operatorname{Lat}(\operatorname{Geom}(L)) \cong L$ and $\operatorname{Geom}(\operatorname{Lat}(G)) \cong G$.

3.4. **Dually slim, lower semimodular lattices.** Finite, slim, semimodular lattices are more or less understood. Therefore, so are their duals, the dually slim, lower semimodular, finite lattices. The following lemma is practically known, but we will explain how to extract it from the literature. The notation introduced before Lemma 3.5 is still in effect.

Lemma 3.6. If Φ is a closure operator on U, then $G = \langle U, \Phi \rangle$ is a finite convex geometry of convex dimension at most 2 if and only if Lat(G) is a finite, dually slim, lower semimodular lattice.

Proof. In view of Lemma 3.5, all we have to show is that, for a finite lattice L, the following two conditions are equivalent:

- (i) L is meet-distributive and width(Mi L) ≤ 2 ;
- (ii) L is lower semimodular and dually slim.

With reference to Ward [42] and Avann [8], we have already mentioned that meet-distributivity implies lower semimodularity. Thus (i) implies (ii). Conversely, assume (ii). We conclude that L is meet-distributive by the dual¹ of Czédli, Ozsvárt, and Udvari [18, Corollary 2.2], and width(Mi L) \leq 2 by the definition of dual slimness. Thus (ii) implies (i).

¹In what follows, dual statements are often cited without pointing out that they are the duals of the original ones.

Next, motivated by Lemma 3.6, we will have a closer look at finite, dually slim, lower semimodular lattices. A finite lattice is *planar* if it has a planar diagram in the obvious sense; for more details see the next subsection. Planarity is a great help for us due to (B) of the following lemma. A cover-preserving M_3 sublattice is a 5-element sublattice $\{u, a_0, a_1, a_2, v\}$ such that $u \prec a_i \prec v$ for $i \in \{0, 1, 2\}$.

Lemma 3.7 (Czédli and Schmidt [19, Lemmas 2.2 and 2.3]). If L is a finite, lower semimodular lattice, then the following two statements hold.

- (A) L is dually slim if and only if it has no cover-preserving M_3 sublattice.
- (B) If L is dually slim, then it L is planar, and each of its elements has at most two lower covers.

3.5. Dual slimness and Carathéodory's condition. Following Libkin [36], a finite lattice K is said to satisfy Carathéodory's condition (CC_n) if for any $a, b_1, \ldots, b_k \in Ji K$ such that $a \leq b_1 \vee \cdots \vee b_k$, there are $i_1, \ldots, i_n \in \{1, \ldots, k\}$ such that $a \leq b_{i_1} \vee \cdots \vee b_{i_n}$. Carathéodory's classical theorem asserts that whenever p is a point and X is a subset of \mathbb{R}^{n-1} such that p belongs to the convex hull Conv(X)of X, there exists an at most n-element subset Y of X such that $p \in Conv(Y)$. This theorem implies that the lattices $Lat\langle E, Hull_E^{(n-1)} \rangle$ satisfy (CC_n) , since their join-irreducible elements are exactly the atoms. We say that a finite convex geometry $\langle U, \Phi \rangle$ satisfies (CC_n) if the corresponding lattice $Lat\langle U, \Phi \rangle$ satisfies (CC_n) . Lower semimodularity is not assumed in the following statement.

Proposition 3.8. Every finite, dually slim lattice satisfies (CC_2) . Also, every finite convex geometry of convex dimension at most 2 satisfies (CC_2) .

Proof. The lattices in question are planar by Czédli and Schmidt [19, Lemma 2.2], and planar lattices satisfy (CC_2) by Libkin [36, Corollary 4.7 and Theorem 3]. The rest follows from Lemma 3.6.

Proposition 3.9. Let $n \in \mathbb{N}$, and let L_1 and L_2 be finite lattices satisfying (CC_n) . If there exists a bijection φ : Ji $L_1 \rightarrow Ji L_2$ such that

(3.4) for all
$$a, b_1, \ldots b_n \in \operatorname{Ji} L_1$$
, $a \leq \bigvee_{i=1}^n b_i \iff \varphi(a) \leq \bigvee_{i=1}^n \varphi(b_i)$,

then φ can be extended to an isomorphism from L_1 onto L_2 .

Proof. First, we show that n is not relevant in (3.4), that is, (3.4) implies the following property of φ :

(3.5) for all
$$a \in \operatorname{Ji} L_1$$
 and $B \subseteq \operatorname{Ji} L_1$, $a \leq \bigvee B \iff \varphi(a) \leq \bigvee \varphi(B)$,

where $\varphi(B) = \{\varphi(b) : b \in B\}$. Assume that $a \leq \bigvee B$. By (CC_n) , there exists a subset $C \subseteq B$ such that $|C| \leq n$ and $a \leq \bigvee C$. Hence $\varphi(a) \leq \bigvee \varphi(C)$ by (3.4), and $\varphi(a) \leq \bigvee \varphi(B)$. The converse implication in (3.5) is obtained by using φ^{-1} . For the sake of Remark 3.10 coming soon, we note that the rest of the proof relies only on (3.5) and does not use any specific property of L_1 and L_2 .

Next, let ψ : Ji $L_2 \to$ Ji L_1 denote the inverse of φ . We define a map $\widehat{\varphi}$: $L_1 \to L_2$ by $\widehat{\varphi}(x) = \bigvee \{\varphi(a) : a \in \downarrow x \cap \text{Ji } L_1\}$. Similarly, let $\widehat{\psi}$: $L_2 \to L_1$ be defined by $\widehat{\psi}(y) = \bigvee \{\psi(b) : b \in \downarrow y \cap \text{Ji } L_2\}$. The choice $b_1 = \cdots = b_n$ in (3.4) shows that φ and ψ are order isomorphisms. This implies that $\widehat{\varphi}$ and $\widehat{\psi}$ are extensions of φ and ψ , respectively. Using the formula $x = \bigvee (\downarrow x \cap \text{Ji } L_i)$ for $x \in L_i$, it is routine to

check that $\widehat{\psi}$ and ψ are reciprocal bijections. Hence, they are lattice isomorphisms, since they are obviously order-preserving.

Remark 3.10. If L_1 and L_2 are finite lattices and φ : Ji $L_1 \to$ Ji L_2 is a bijection satisfying (3.5), then $L_1 \cong L_2$ and φ extends to an isomorphism $L_1 \to L_2$.

3.6. More about dually slim, lower semimodular lattices. Even if φ in Proposition 3.9 is an order-isomorphism, (3.4) with n = 2 may fail; its satisfaction depends mainly on the case where $\{a, b_1, b_2\}$ is an antichain. This is one of the reasons why we are going to look more closely at dually slim, lower semimodular lattices. Since dually slim lattices are planar by Lemma 3.7(B), the propositions of this subsection may look intuitively clear. However, their exact proofs need some preparation. Fortunately, the theory of planar lattices is satisfactorily developed in Kelly and Rival [35] at a rigorous level, so we can often rely on results from [35] instead of going into painful rigor. Whenever we deal with a planar lattice, always a fixed planar diagram is assumed. Actually, most of the concepts, like left and right, depend on the planar diagram chosen (sometimes implicitly) at the beginning rather than on the lattice. This will not cause any trouble, since our arguments do not depend on the choice of planar diagrams.

Now, we recall some necessary concepts and statements for planar lattices; the reader may (but need not) look into [35] for more exact details. Let C be a maximal chain in a finite planar lattice L (with a fixed planar diagram). This chain cuts L into a left side and a right side, see Kelly and Rival [35, Lemma 1.2]. The intersection of these sides is C. If $x \in L$ is on the left side of C but not in C, then x is strictly on the left of C. Let D be another maximal chain of L. If all elements of D are on the left of C, then D is on the left of C. In this sense, we can speak of the leftmost maximal chain of L, called the *left boundary chain*, and the rightmost maximal chain, called the *right boundary chain*. The union of these two chains is the *boundary* of L. Also, if E is a (not necessarily maximal) chain of L, then the *leftmost maximal chain through* E (or extending E) and the rightmost one make sense. If $E = \{e_1, \ldots, e_n\}$ with $e_1 < \cdots < e_n$, then the leftmost maximal chain of L through E is the union of the left boundary chains of the intervals $[0, e_1], [e_1, e_2], \ldots, [e_{n-1}, e_n], \text{ and } [e_n, 1].$ (The diagrams of these intervals are the respective subdiagrams of the fixed diagram of L.) The most frequently used result of Kelly and Rival [35] is probably the following one.

Lemma 3.11 (Kelly and Rival [35, Lemma 1.2]). Let L be a finite planar lattice, and let $x \leq y \in L$. If x and y are on different sides of a maximal chain C in L, then there exists an element $z \in C$ such that $x \leq z \leq y$.

Next, let x and y be elements of a finite planar lattice L, and assume that they are incomparable, written as $x \parallel y$. If $x \lor y$ has lower covers x_1 and y_1 such that $x \le x_1 \prec x \lor y$, $y \le y_1 \prec x \lor y$, and x_1 is on the left of y_1 , then the element x is on the left of the element y. If x is on the left of y, then we say that y is on the right of x. Let us emphasize that whenever left or right is used for two elements, the elements in question are incomparable.

Lemma 3.12 (Kelly and Rival [35, Propositions 1.6 and 1.7]). Let L be finite planar lattice. If $x, y \in L$ are incomparable elements, then the following hold.

(A) x is on the left of y if and only if x is on the left of some maximal chain through y if and only if x is on the left of all maximal chains through y.

- (B) Either x is on the left of y, or x is on the right of y.
- (C) If $z \in L$, $x \parallel y$, $y \parallel z$, x is on the left of y, and y is on the left of z, then x is on the left of z.

If $\{x_0, x_1, y\}$ is a 3-element antichain such that x_i is on the left of y and y is on the left of x_{1-i} for some (necessarily unique) $i \in \{0, 1\}$, then y is horizontally between the elements x_0 and x_1 .

Proposition 3.13. Let L be a finite lattice. If $\{x_0, x_1, y\}$ is a 3-element antichain in L, then the following two statements hold.

- (A) If L is planar and y is horizontally between x_0 and x_1 , then $y \leq x_0 \lor x_1$.
- (B) If L is a dually slim, lower semimodular lattice and $y \le x_0 \lor x_1$, then y is horizontally between x_0 and x_1 .

Proof. To prove (A), pick a maximal chain C through y. Without loss of generality, we may assume that x_0 and $x_0 \vee x_1$ are on the left and x_1 is on the right of C. Applying Lemma 3.11, there exists an element $z \in C$ such that $x_1 \leq z \leq x_0 \vee x_1$. Belonging to the same chain, y and z are comparable. Since $z \leq y$ contradicts $x_1 \leq y$, we have $y < z \leq x_0 \vee x_1$, proving (A).

Next, to prove (B) by contradiction, suppose that L is a dually slim, lower semimodular lattice, $y \leq x_0 \vee x_1$, but y is not between x_0 and x_1 . Let, say, x_0 be on the left of x_1 and x_1 be on the left of y. Let $I = [x_0 \wedge y, x_0 \vee x_1]$. This interval contains x_0, x_1 , and y, since $y \leq x_0 \vee x_1$ and, by the dual of part (A), $x_0 \wedge y \leq x_1$. As an interval of L, it is lower semimodular, and it follows from Lemma 3.7(A) that this interval is dually slim. There are two cases.

First, suppose $y \vee x_1 = x_0 \vee x_1$. Let *E* be a maximal chain of *I* through $\{x_1, x_0 \vee x_1\}$. By Lemma 3.12(A), x_0 is on the left of *E* and *y* is on the right of *E*. Note that the left side of *E* (including *E* itself) is a cover-preserving sublattice by Kelly and Rival [35, Proposition 1.4]; this can also be derived from Lemma 3.11 easily. Hence, we can pick lower covers x'_0, x'_1 of $x_0 \vee x_1$ on the left of *E* such that $x_0 \leq x'_0$ and $x_1 \leq x'_1$. Similarly, let $y' \in I$ be a lower cover of $x_0 \vee x_1$ such that $y \leq y'$ and y' is on the right of *E*. If $x'_0 = y'$, then *E* contains an element *z* by Lemma 3.11, necessarily strictly above x_1 since $y \not\leq x_1$, such that $y \leq z \leq x'_0$; this yields the following contradiction:

$$y \lor x_1 \le z \le x'_0 \prec x_0 \lor x_1 = y \lor x_1.$$

Hence, $x'_0 \neq y'$. If $x'_0 = x'_1$ or $x'_1 = y'$, then $x_0 \lor x_1 \leq x'_1 \prec x_0 \lor x_1$ or $y \lor x_1 \leq x'_1 \prec x_0 \lor x_1 = y \lor x_1$, respectively, again a contradiction. Hence x'_0, x'_1 , and y' are three distinct lower covers of $x_0 \lor x_1$, which contradicts Lemma 3.7(B).

Second, suppose $y \vee x_1 \neq x_0 \vee x_1$. Since these elements are in I, we have $y < y \vee x_1 < x_0 \vee x_1$. Take a maximal chain G of I through $\{y, y \vee x_1\}$, and let J be the left side of G. By Lemma 3.12(A), $x_0, x_1 \in J$. In J, take a maximal chain F through x_0 , and let K be the right side of F in J. By Lemma 3.12(A) again, $x_1 \in K$. Clearly, K is a cover-preserving sublattice of I, again by Kelly and Rival [35, Propositions 1.4]. Now F and G are the left and right boundary chains of K, respectively. Like we obtained it for I, we conclude that K, which is a cover-preserving sublattice of I (and also of L), is a dually slim, lower semimodular lattice. By Czédli and Schmidt [20, Lemma 6], Mi $K \subseteq F \cup G$. Therefore, there exist $f \in F$ and $g \in G$ such that $x_1 = f \wedge g$. Since $f \leq x_0$ is excluded by $x_1 \not\leq x_0$ and F is a chain, $x_0 < f$. Similarly, using that G is a chain and $y \parallel x_1$, we obtain

y < g. Thus $y \lor x_1 \le g$, and we conclude $x_1 = f \land (y \lor x_1)$. Clearly, $f \ne x_0 \lor x_1$, since otherwise $x_1 = (x_0 \lor x_1) \land (y \lor x_1) = y \lor x_1 \ge y$ contradicts $x_1 \parallel y$. Hence $f < x_0 \lor x_1$, which leads to the contradiction $x_0 \lor x_1 \le f < x_0 \lor x_1$.

Note that it would be unreasonable to have the same condition on L in parts (A) and (B) of Proposition 3.13. The 5-element, modular, non-distributive lattice M_3 indicates that planarity in itself would not be sufficient in part (B). On the other hand, although dual slimness (with or without lower semimodularity) would be sufficient in part (A) by Lemma 3.7, in this case the statement would be weaker and we could not have used the dual of (A) in the proof of (B).

We also need the following statement.

Proposition 3.14. If L is a finite lower semimodular lattice, $a \in \text{Ji } L$, $b, c \in L$, c < a, and $a \leq b \lor c$, then $a \leq b$.

Proof. To prove the statement by contradiction, suppose that in spite of the assumptions, $a \not\leq b$. If b < a, then $a \geq b \lor c$ together with $a \leq b \lor c$ contradicts $a \in \text{Ji } L$. Hence $b \parallel a$. Let $d = b \lor c$; we know that $a \leq d$. Since $b \not\parallel c$ together with $b \lor c \geq a$ contradicts $\{b, c\} \cap \uparrow a = \emptyset$, we have $b \parallel c$. Hence b < d, and we can pick an element $e \in [b, d]$ such that $e \prec d$. Denoting the unique lower cover of a by a_* , we conclude

$$d = e \lor a = e \lor a_*, \quad e \parallel a, \quad \text{and} \quad e \parallel a_*,$$

because of the following reasons: $c \leq a_*$ yields $d = b \lor c \leq e \lor a_* \leq e \lor a \leq d$, $e \leq a_*$ or $e \leq a$ contradicts $b \parallel a$, and $e \geq a_*$ or $e \geq a$ leads to the contradiction $e \prec d = e \lor a_* = e$ or $e \prec d = e \lor a = e$.

Since $e \parallel a$, we have $e \land a < a$. Lower semimodularity yields $e \land a \leq d \land a = a$, and we obtain $e \land a \prec a$. Since $a_* \parallel e$ and $e \land a \leq e$, the elements a_* and $e \land a$ are two distinct lower covers of a. This contradicts $a \in \text{Ji } L$.

The elements of Ji $L \cap$ Mi L are called *doubly irreducible elements*. A principal filter $\uparrow b$ of L is a *prime filter* if $\emptyset \neq L \setminus \uparrow b$ is closed with respect to joins or, equivalently, if $L \setminus \uparrow b$ is an ideal of L.

Proposition 3.15. Let L be a finite, dually slim, lower semimodular lattice. If $|L| \geq 3$, then the following three statements hold.

- (i) L has a maximal doubly irreducible element b, and this b belongs to the boundary of L.
- (ii) If $x \in L$ and x > b, then $x \in MiL$ but $x \notin JiL$. Furthermore, $\uparrow b$ is a chain.
- (iii) $\uparrow b$ is a prime filter of L.

Proof. By Lemma 3.7 (B), L is planar. We know from Kelly and Rival [35, Theorem 2.5] that each finite planar lattice L with at least three elements has a doubly irreducible element on its boundary. However, we only use this theorem to conclude Ji $L \cap \text{Mi } L \neq \emptyset$. Hence, Ji $L \cap \text{Mi } L$ contains a maximal element, b. Since Mi L is a subset of the boundary by the dual of Czédli and Schmidt [20, Lemma 6], b belongs to the boundary. This proves (i).

We may assume by symmetry that b belongs to the right boundary chain C_r of L. The dual of Czédli [15, Lemma 2.3] asserts $C_r \cap \uparrow b \subseteq \operatorname{Mi} L$. We claim $\uparrow b \subseteq C_r$. To prove this by contradiction, suppose $\uparrow b \not\subseteq C_r$. We obtain that $\uparrow b$ is not a chain, since $\uparrow b \cap C_r$ is a maximal chain in $\uparrow b$. Hence, there exist $u, v \in \uparrow b$ such that $u \parallel v$, and there exist chains $\{u_0, \ldots, u_t\}$ and $\{v_0, \ldots, v_s\}$ such that $b = u_0 \prec u_1 \prec \cdots \prec u_t = u$ and $b = v_0 \prec v_1 \prec \cdots \prec v_s = v$ in $\uparrow b$. Clearly, $s, t \ge 1$. Let *i* be the largest subscript such that $i \le t, i \le s$, and $u_i = v_i \in C_r$. This *i* exists, since $u_0 = v_0 = b \in C_r$. Since $u \parallel v$, we have i < s and i < t. There are two cases. First, if $u_{i+1} \neq v_{i+1}$, then $u_i = v_i$ has at least two distinct covers. Second, if $u_{i+1} = v_{i+1} \notin C_r$, then $u_i = v_i$ has at least two distinct covers again: $u_{i+1} = v_{i+1}$ and a cover belonging to C_r . Hence, in both cases, $u_i = v_i$ is meet-reducible and belongs to C_r , which contradicts $C_r \cap \uparrow b \subseteq MiL$. Therefore, $\uparrow b \subseteq C_r$ and, since subsets of chains are chains, $\uparrow b$ is a chain.

Now, assume x > b. Since $x \in \uparrow b = C_r \cap \uparrow b \subseteq \operatorname{Mi} L$ and b was a maximal doubly irreducible element, we conclude $x \notin \operatorname{Ji} L$. This proves (ii).

To prove (iii) by contradiction, consider $y \in \uparrow b$ such that there exist elements $u, v \in L \setminus \uparrow b$ with $y = u \lor v$. We have $u = u_1 \lor \cdots \lor u_s$ and $v = v_1 \lor \cdots \lor v_t$ for some $u_1, \ldots, u_s, v_1, \ldots, v_t \in \operatorname{Ji} L \setminus \uparrow b$. It follows from Proposition 3.8 that there are two elements in $\{u_1, \ldots, u_s, v_1, \ldots, v_t\}$ whose join belongs to $\uparrow b$. Therefore, there are incomparable elements $p, q \in \operatorname{Ji} L \setminus \uparrow b$ such that $b \leq p \lor q$. There are two cases. First, assume that $\{p, q, b\}$ is an antichain. Clearly, neither p nor q belongs to C_r , and Lemma 3.12(A) yields that both p and q are on the left of b. This is a contradiction, since Proposition 3.13(B) implies that b is horizontally between p and q.

Second, assume that $\{p, q, b\}$ is not an antichain. Since b is join-irreducible and $p, q \notin \uparrow b$, we cannot have $\{p, q\} \subseteq \downarrow b$. Hence, apart from symmetry between p and q, we have $p \parallel b$ and q < b. However, now Proposition 3.14 contradicts $b \leq p \lor q$. \Box

3.7. The rest of the proof. Before formulating the last auxiliary statement towards Theorem 2.2, remember that dual slimness implies planarity by Lemma 3.7(B).

Lemma 3.16. Let *L* be a finite, dually slim, semimodular lattice with a fixed planar diagram, and let *F* be a finite concave set of collinear circles in the plane. Assume a bijective map ψ : Ji $L \to F$ such that for any $u, v \in \text{Ji } L$,

- (i) $u \leq v$ if and only if $\psi(u) \subseteq \text{Conv}(\psi(v))$, and
- (ii) $u \parallel v$ and u is on the left of v if and only if $\text{LEnd}(\psi(u)) \sqsubset \text{LEnd}(\psi(v))$ and $\text{REnd}(\psi(u)) \sqsubset \text{REnd}(\psi(v))$.

These assumptions imply $L \cong \text{Lat}\langle F, \text{Hull}_F^{\text{o}} \rangle$.

Proof. Lemma 3.2 allows us to define a bijective map $\varphi \colon \text{Ji} L \to \text{Ji}(\text{Lat}\langle F, \text{Hull}_F^o\rangle)$ by $\varphi(u) = \text{HInt}_F(\psi(u), \psi(u))$. Clearly,

$$\psi(u) \subseteq \operatorname{Conv}(\psi(v)) \iff \operatorname{HInt}_F(\psi(u), \psi(u)) \subseteq \operatorname{HInt}_F(\psi(v), \psi(v)).$$

Therefore, by Assumption (i), φ is an order-isomorphism. Since we want to apply Propositions 3.8 and 3.9, we are going to show that φ satisfies Condition (3.4) with $\langle 2, a, b, c \rangle$ in place of $\langle n, a, b_1, \ldots, b_n \rangle$.

If $a \leq b$ or $a \leq c$, then $\varphi(a) \leq \varphi(b)$ or $\varphi(a) \leq \varphi(c)$, since φ is an orderisomorphism. The case $b \not\parallel c$ is even more evident. Thus, if $\{a, b, c\}$ is not an antichain, we have

(3.6)
$$a \le b \lor c \iff \varphi(a) \le \varphi(b) \lor \varphi(c).$$

Next, suppose that $\{a, b, c\}$ is an antichain. So is $\{\varphi(a), \varphi(b), \varphi(c)\}$, since φ is an order-isomorphism. There are two cases: either a is horizontally between b and c, or not.

In the first case, we may assume that b is on the left of a and a is on the left of c. Proposition 3.13 gives $a \leq b \lor c$, and (ii) yields

(3.7)
$$\operatorname{LEnd}(\psi(b)) \sqsubset \operatorname{LEnd}(\psi(a)) < \operatorname{LEnd}(\psi(c)) \text{ and}$$

$$\operatorname{REnd}(\psi(b)) < \operatorname{REnd}(\psi(a)) \sqsubset \operatorname{REnd}(\psi(c))$$

Since F is concave, (3.7) implies $\psi(a) \subseteq \operatorname{Conv}(\psi(b) \cup \psi(c))$, which gives $\varphi(a) \leq \varphi(b) \lor \varphi(c)$. Hence, (3.6) holds in this case.

In the second case, where a is not horizontally between b and c, we can assume that a is on the left of b and b is on the left of c. By Proposition 3.13, we have $a \not\leq b \lor c$, and (ii) gives

(3.8)
$$\operatorname{LEnd}(\psi(a)) \sqsubset \operatorname{LEnd}(\psi(b)) \sqsubset \operatorname{LEnd}(\psi(c)) \text{ and} \\ \operatorname{REnd}(\psi(a)) \sqsubset \operatorname{REnd}(\psi(b)) \sqsubset \operatorname{REnd}(\psi(c)).$$

Let $X = \operatorname{HInt}_F(\psi(b), \psi(c))$. We have $\{\psi(b), \psi(c)\} \subseteq X \in \operatorname{Lat}\langle F, \operatorname{Hull}_F^o \rangle$ by Lemma 3.1, and $\psi(a) \notin X$ by the definition of horizontal intervals. Since $\varphi(b) \lor \varphi(c) \subseteq X$, we conclude $\varphi(a) \nleq \varphi(b) \lor \varphi(c)$, and (3.6) holds in this case.

Next, suppose a > b and a > c, and let a_* stand for the unique lower cover of a. Now $a_* \ge b \lor c$, and we have $a \not\le b \lor c$. Since ψ is an order-isomorphism, $\varphi(a_*) \ge \varphi(b) \lor \varphi(c)$. This gives $\varphi(a) \not\le \varphi(b) \lor \varphi(c)$, and (3.6) is fulfilled again.

Finally, by the symmetry between b and c, we are left with the case where $b \parallel a$ and c < a. We may assume $b \parallel c$, since otherwise both sides of (3.6) are obviously false and (3.6) holds. Take a maximal chain C including $\{c, a\}$; it does not contain b. We may assume that b is on the left of C. Now, by Lemma 3.12, b is on the left of c and also on the left of a. Since $\psi(c) \subseteq \text{Conv}(\psi(a))$ by Assumption (i), we conclude REnd $(\psi(c)) \sqsubseteq \text{REnd}(\psi(a))$. Thus REnd $(\psi(c)) \sqsubset \text{REnd}(\psi(a))$, since our circles are determined by their left ends. This and (ii) yield

(3.9)
$$\begin{array}{c} \operatorname{LEnd}(\psi(b)) \sqsubset \operatorname{LEnd}(\psi(c)) \quad \text{and} \\ \operatorname{REnd}(\psi(b)) \sqsubset \operatorname{REnd}(\psi(c)) \sqsubset \operatorname{REnd}(\psi(a)) \end{array}$$

As previously, this gives $\{\psi(b), \psi(c)\} \subseteq \operatorname{HInt}_F(\psi(b), \psi(c)) \in \operatorname{Lat}\langle F, \operatorname{Hull}_F^o \rangle$ and $\psi(a) \notin \operatorname{HInt}_F(\psi(b), \psi(c))$, which implies $\varphi(a) \nleq \varphi(b) \lor \varphi(c)$. Since $a \nleq b \lor c$ by Proposition 3.14, (3.6) is satisfied again.

Since (3.6) holds in all cases, φ satisfies (3.4). Thus Propositions 3.8 and 3.9 apply.

Proof of Theorem 2.2. Let F be a finite, concave set of collinear circles in the plane. Proposition 2.1 yields that $\langle F, \operatorname{Hull}_{F}^{o} \rangle$ is a convex geometry. Hence, by Lemma 3.5, Lat $\langle F, \operatorname{Hull}_{F}^{o} \rangle$ is a finite meet-distributive lattice, and it is lower semimodular by Claim 3.4. We obtain from Lemma 3.3 that this lattice is dually slim. Therefore, Lemma 3.6 implies that $\langle F, \operatorname{Hull}_{F}^{o} \rangle$ is a convex geometry of convex dimension at most 2. This proves part (A).

In view of Lemmas 3.5 and 3.6, part (B) is equivalent to the following statement:

(C) If L is a finite, dually slim, lower semimodular lattice, then there exists a finite, separated, concave set F of collinear circles in the plane such that L is isomorphic to $\text{Lat}\langle F, \text{Hull}_F^o \rangle$.

We prove (C) by induction. By Lemma 3.16, it suffices to construct a pair $\langle F, \psi \rangle$ such that F is a finite, separated, concave set of collinear circles in the plane and ψ : Ji $L \to F$ is a bijective map satisfying Conditions (i) and (ii) of Lemma 3.16.



FIGURE 3. A dually slim lower semimodular lattice L and the corresponding separated concave set F of collinear circles

A pair $\langle F, \psi \rangle$ with these properties is an *appropriate pair*. Since we are going to construct a separated F, (2.4) allows us to satisfy these two conditions with leftmost and rightmost endpoints rather than left and right ends.

First, assume that L is a chain or, equivalently, Ji L is a chain. We can let F and ψ be a set of concentric circles and the unique map satisfying Condition (i) of Lemma 3.16, respectively; clearly, $\langle F, \psi \rangle$ is an appropriate pair. More generally, not assuming that L is a chain, we can prove the existence of an appropriate pair $\langle F, \psi \rangle$ by induction on the size |L| of L. Since the case of chains has been settled, the induction starts at size 4.

Assume that $|L| \ge 4$ and for each finite, dually slim, lower semimodular lattice of smaller size, there exists and appropriate pair. Take a maximal doubly irreducible element $c \in L$ (like *b* in Proposition 3.15). By left-right symmetry, we may assume that *c* is on the right boundary chain of *L*; see Figure 3, on the left, for illustration. (The figure serves only as an illustration, so the reader need not check the properties of *L*. However, we note that *L* is obviously a dually slim, lower semimodular lattice by the dual of Czédli and Schmidt [20, Theorem 12].)

Since $\uparrow c$ is a prime filter by Proposition 3.15, $L' = L \setminus \uparrow c$ is an ideal of L. As an interval of L, L' is lower semimodular. It follows from Lemma 3.7(A) that L'is dually slim. Hence, by the induction hypothesis, there exists an appropriate pair $\langle F', \psi' \rangle$ for L'. We want to define $\langle F, \psi \rangle$ such that $F' \subset F$ and ψ is an extension of ψ' . By Proposition 3.15(ii), $\operatorname{Ji} L = \operatorname{Ji} L' \cup \{c\}$. Therefore, our aim is to find an appropriate circle C and to let ψ be the map from $\operatorname{Ji} L$ to $F = F' \cup \{C\}$ defined by $\psi(c) = C$ and $\psi(x) = \psi'(x)$ for $x \in \operatorname{Ji} L'$. In the figure, $\operatorname{Ji} L' =$ $\{x_1, x_2, x_3, y_1, y_2, y_3\}$, and $\operatorname{Ji} L = \operatorname{Ji} L' \cup \{c\}$ is the set of black-filled elements. On the right of the figure, we write C, X_i , and Y_i instead of $\psi(c), \psi'(x_i)$, and $\psi'(y_i)$, respectively; F is the collection of all circles, and $F' = F \setminus \{C\}$.

Since c is on the right boundary chain, Lemma 3.12(A) yields that each $x \in \text{Ji } L$ is either strictly on the left of c or comparable to c. This together with Proposition 3.15(ii) imply that Ji L' is the disjoint union of the following two sets:

 $J_{\text{below}} = \operatorname{Ji} L' \cap \downarrow c$ and $J_{\text{left}} = \{ y \in \operatorname{Ji} L' : y \text{ is on the left of } c \}.$

Note that one of these two sets can be empty, but their union, Ji L', is nonempty. In the figure, $J_{\text{below}} = \{x_1, x_2, x_3\}$ and $J_{\text{left}} = \{y_1, y_2, y_3\}$. Condition (i) of



FIGURE 4. Modifying C to make F concave

Lemma 3.16 will hold if and only if

(3.10) $\operatorname{LPt}(C) < \operatorname{LPt}(\psi'(x)) \text{ and } \operatorname{RPt}(\psi'(x)) < \operatorname{RPt}(C) \text{ for all } x \in J_{\operatorname{below}}.$

Similarly, Condition (ii) of Lemma 3.16 will hold if and only if

(3.11)
$$\operatorname{LPt}(\psi'(y)) < \operatorname{LPt}(C) \text{ and } \operatorname{RPt}(\psi'(y)) < \operatorname{RPt}(C) \text{ for all } y \in J_{\operatorname{left}}.$$

The only stipulation that (3.10) and (3.11) impose on RPt(C) is

(3.12) $\operatorname{RPt}(C) > \max\{\operatorname{RPt}(\psi'(z)) : z \in \operatorname{Ji} L'\};$

this can be satisfied easily. Therefore, to see that we can choose LPt(C) such that (3.10) and (3.11) hold, it suffices to show that

(3.13) for all
$$x \in J_{\text{below}}$$
 and $y \in J_{\text{left}}$, $\text{LPt}(\psi'(y)) < \text{LPt}(\psi'(x))$.

For $x \in J_{below}$ and $y \in J_{left}$, there are two cases. First, assume $x \not| y$. Clearly, $y \not\leq x$, thus x < y. Since Condition (i) of Lemma 3.16 holds for $\langle F', \psi' \rangle$, we have $\text{LEnd}(\psi(y)) \subset \text{LEnd}(\psi(x))$ and $\text{REnd}(\psi(x)) \subset \text{REnd}(\psi(y))$. This yields $\text{LPt}(\psi'(y)) < \text{LPt}(\psi'(x))$, since F' is separated. Second, assume $x \mid | y$. Let E be a maximal chain in L that extends $\{x, c\}$. Since y is on the left of c, y is on the left of E by Lemma 3.12(A). Hence, again by Lemma 3.12(A), y is on the left of x. Therefore, using Condition (ii) of Lemma 3.16 for the appropriate pair $\langle F', \psi' \rangle$, we conclude $\text{LEnd}(\psi'(y)) \subset \text{LEnd}(\psi'(x))$. This implies $\text{LPt}(\psi'(y)) < \text{LPt}(\psi'(x))$, since F is separated. Thus (3.13) holds, and so do (3.10) and (3.11).

Since (3.10) and (3.11) are strict inequalities, we can choose both LPt(C) and RPt(C) infinitely many ways. Therefore, we may choose C so that F is separated.

Finally, we have to show that F is concave. Since F' is concave and separated, the only case we have to consider is

(3.14) $\operatorname{LPt}(C_1) < \operatorname{LPt}(C_2)$ and, automatically, $\operatorname{RPt}(C_2) < \operatorname{RPt}(C)$,

where $C_1, C_2 \in F'$; we have to show

$$(3.15) C_2 \subseteq \operatorname{Conv}(C_1 \cup C).$$

Suppose that after choosing C, (3.15) fails for some $C_1, C_2 \in F'$; see Figure 4. The circle C is in the interior of the region between the two common tangent lines h_1 and h_2 of C_1 and C_2 . Let t be the tangent line of C through $\langle \text{LPt}(C), 0 \rangle$, and let P_i be the intersection point of h_i and t for $i \in \{1, 2\}$. If RPt(C) tends to infinity while LPt(C) is unchanged, then the arc of C between h_1 and h_2 with



FIGURE 5. Here $Lat\langle F, Hull_F^o \rangle$ is the 128-element boolean lattice



FIGURE 6. An $\langle F, \operatorname{Hull}_{F}^{o} \rangle$ that does not satisfy (CC₄)

middle point $\langle \operatorname{LPt}(C), 0 \rangle$ approaches the line segment P_1P_2 . Therefore, replacing C by C' such that $\operatorname{LPt}(C') = \operatorname{LPt}(C)$ and $\operatorname{RPt}(C')$ is sufficiently large, we have $C_2 \subseteq \operatorname{Conv}(C_1 \cup C')$ and (3.12). We can treat all pairs $\langle C_1, C_2 \rangle \in F' \times F'$ with $\operatorname{LPt}(C_1) < \operatorname{LPt}(C_2)$, one by one, because $\operatorname{RPt}(C)$ can always be enlarged. This proves that F is concave for some C.

4. Odds and ends

Remark 4.1. It is not hard to see that Theorem 2.2 remains valid if we consider closed discs or open discs instead of circles and modify the definitions accordingly. The advantage of circles is that they are easier to visualize and label in figures. In particular, open discs are less pleasant than circles, since they cannot be singletons. Note that we cannot use semicircles or half discs since, by the following example, the corresponding structure is not a convex geometry in general.

Example 4.2. Let

 $H_1 = \{ \langle x, y \rangle : x^2 + y^2 = 4 \text{ and } x \le 0 \}, \ H_2 = \{ \langle x, y \rangle : x^2 + y^2 = 1 \text{ and } x \ge 0 \}.$

Rotating H_2 around $\langle 0, 0 \rangle$ by angle $\pi/100$, we obtain a half circle H_3 . Since H_{5-i} belongs to $\text{Conv}(H_1 \cup H_i)$ for $i \in \{2, 3\}$, the anti-exchange property fails, and we do not obtain a convex geometry from $\{H_1, H_2, H_3\}$.

The following example shows that the convex dimension of $\langle F, \operatorname{Hull}_{F}^{o} \rangle$ can be arbitrarily large even if the circles in F are collinear.



FIGURE 7. Illustrating the proof of Proposition 4.5

Example 4.3. Let F be an n-element set of collinear circles. Assume that there is an additional circle K such that every circle $C \in F$ is internally tangent to K; see Figure 5 for n = 7, where the dotted curve is an arc of K. Clearly, $\text{Lat}\langle F, \text{Hull}_F^o \rangle = \langle \mathcal{P}(F), \subseteq \rangle$ is the 2^n -element boolean lattice, and the convex dimension of $\text{Lat}\langle F, \text{Hull}_F^o \rangle$ is n.

Proof. The equality Lat $\langle F, \operatorname{Hull}_{F}^{\circ} \rangle = \langle \mathcal{P}(F), \subseteq \rangle$ is obvious. Since

$$\operatorname{Mi} \langle \mathcal{P}(F), \subseteq \rangle = \{F \setminus \{C\} : C \in F\}$$

is an n-element antichain, the convex dimension is n.

While $\langle E, \operatorname{Hull}_{E}^{(2)} \rangle$ satisfies Carathéodory's condition (CC₃) for every finite set E of points of the plane, the following example shows that circles are essentially different from points.

Example 4.4. For each natural number n, there exists an (n + 2)-element set F of circles in the plane such that $\langle F, \operatorname{Hull}_{F}^{\circ} \rangle$ does not satisfy (CC_{n}) . For example, we can take the inscribed circle of a regular (n + 1)-gon and n + 1 additional little circles whose centers are the vertices of the (n + 1)-gon; see Figure 6 for n = 4.

By the next proposition, Example 4.4 has no collinear counterpart.

Proposition 4.5. If F is a finite set of collinear circles, then $\langle F, \operatorname{Hull}_{F}^{o} \rangle$ satisfies Carathéodory's condition (CC₂).

Proof. In view of Lemma 3.2, we have to show the following: if $C, D_1, \ldots, D_k \in F$ such that $C \subseteq \operatorname{Conv}(D_1 \cup \cdots \cup D_k)$, then there exist $i, j \in \{1, \ldots, k\}$ such that $C \subseteq \operatorname{Conv}(D_i \cup D_j)$. Let G be the boundary of $\operatorname{Conv}(D_1 \cup \ldots D_k)$; see the thick closed curve in Figure 7, where k = 5 and F contains the solid circles and possibly some other circles not indicated. (The dotted circle need not belong to F.) Clearly, G can be divided into circular arcs and straight line segments of common tangent lines of some circles belonging to $\{D_1, \ldots, D_k\}$; these parts are separated by blackfilled points in the figure. Keeping its center fixed, we enlarge C to C' such that $C' \subseteq \operatorname{Conv}(D_1 \cup \ldots D_k)$ and C' is internally tangent to G at a point $T \in G \cap C'$. There are two cases.

In the first case, we assume that T belongs to a circular arc of G. In the figure, T in this case is not indicated; it can be any point of the closed circular arc V_1V_2 . This arc is also an arc of some member, D_m , of F. Clearly, $C' = D_m$. Thus D_m and C are concentric circles, and $C \subseteq \text{Conv}(D_m)$. Hence, we may let i = m and j = m. Observe that circular arcs of length 0 cause no problem, since then the radius of D_m is zero, the center of D_m belongs to G, and $C = D_m$.

In the second case, we assume that T belongs to a line segment PQ of G. Now C' is the dotted circle in the figure, and its center is U. We may assume that $T \notin \{P, Q\}$, since otherwise the previous case applies. Clearly, P is a point of a unique D_i with center U_i , and Q is on a unique D_j with center U_j . Since the radii PU_i , TU and QU_j are all perpendicular to the common tangent line PQ, it follows that U is between U_i and U_j , and $C' \subseteq \text{Conv}(D_i \cup D_j)$. Therefore, $C \subseteq \text{Conv}(D_i \cup D_j)$.

For a class \mathcal{U} of structures, let $I\mathcal{U}$ denote the class of structures that are isomorphic to some members of \mathcal{U} . We consider the following classes of finite convex geometries; by circles we mean circles in the plane.

$$\begin{split} \mathcal{K}_{\text{collinear}}^{\text{concave}} &= \mathbf{I}\left\{\langle F, \text{Hull}_{F}^{\text{o}} \rangle : F \text{ is a finite, concave set of collinear circles} \right\}, \\ \mathcal{K}_{\text{collinear}} &= \mathbf{I}\left\{\langle F, \text{Hull}_{F}^{\text{o}} \rangle : F \text{ is a finite set of collinear circles} \right\}, \\ \mathcal{K}_{\text{planar}} &= \mathbf{I}\left\{\langle F, \text{Hull}_{F}^{\text{o}} \rangle : F \text{ is a finite set of circles} \right\}, \\ \mathcal{R}_{\text{points}}^{\dim=n} &= \mathbf{I}\left\{\langle E, \text{Hull}_{E}^{(n)} \rangle : E \text{ is a finite subset of } \mathbb{R}^{n} \right\}, \\ \mathcal{G}_{\text{all}} &= \text{the class of all finite convex geometries.} \end{split}$$

Results by Adaricheva [3] and Bergman [9] show that

(4.1)
$$\mathcal{R}_{\text{points}}^{\dim=2} \subset \mathcal{R}_{\text{points}}^{\dim=3} \subset \mathcal{R}_{\text{points}}^{\dim=4} \subset \dots$$

We obtain from Examples 4.3 and 4.4, Theorem 2.2(A), and Proposition 4.5 that

(4.2)
$$\mathcal{K}_{\text{collinear}}^{\text{concave}} \subset \mathcal{K}_{\text{collinear}} \subset \mathcal{K}_{\text{planar}}$$
 and, clearly, $\mathcal{R}_{\text{points}}^{\dim=2} \subseteq \mathcal{K}_{\text{planar}}$

If $E \subseteq \mathbb{R}^2$ consists of three non-collinear points and their barycenter, then the convex geometry $\langle E, \operatorname{Hull}_{E}^{(2)} \rangle$ does not satisfy (CC₂), and we conclude from Proposition 4.5 that

(4.3)
$$\mathcal{R}_{\text{points}}^{\dim = 2} \not\subseteq \mathcal{K}_{\text{collinear}}$$

In the lattices associated with members of $\mathcal{R}_{points}^{\dim=n}$, all join-irreducible elements are atoms. This implies that

(4.4) for all
$$n \ge 2$$
, $\mathcal{K}_{\text{collinear}}^{\text{concave}} \not\subseteq \mathcal{R}_{\text{points}}^{\dim = n}$.

4.1. Some open problems. In spite of (4.1), (4.2), (4.3), and (4.4), we do not have a satisfactory description of the partially ordered set

$$(4.5) \qquad \langle \{\mathcal{K}_{\text{collinear}}^{\text{concave}}, \mathcal{K}_{\text{collinear}}, \mathcal{K}_{\text{planar}}, \mathcal{G}_{\text{all}}, \mathcal{R}_{\text{points}}^{\dim=1}, \mathcal{R}_{\text{points}}^{\dim=2}, \mathcal{R}_{\text{points}}^{\dim=3}, \ldots \}, \subseteq \rangle.$$

In particular, we do not know whether

(4.6)
$$\mathcal{R}_{\text{points}}^{\dim=3} \subseteq \mathcal{K}_{\text{planar}}$$
 or $\mathcal{K}_{\text{planar}} \stackrel{?}{=} \mathcal{G}_{\text{all}}$ holds.

If we augment the set (4.5) with convex geometries obtained from *n*-dimensional spheres or, say, coplanar three-dimensional spheres, then the problem becomes even more difficult. Finally, while Theorem 2.2 describes $\mathcal{K}_{\text{collinear}}^{\text{concave}}$ in an abstract way, we have no similar descriptions for $\mathcal{K}_{\text{collinear}}$ and $\mathcal{K}_{\text{planar}}$.

4.2. Representation by 1-dimensional circles. Let F be a finite subset of $\{\langle a, b \rangle : a, b \in \mathbb{R} \text{ and } a \leq b\}$. Its elements will be called 1-dimensional circles. If $C = \langle a, b \rangle \in F$, then a = LPt(C) and b = RPt(C) are the left and right endpoints of C, respectively. Formulas (2.2) and (2.3) still make sense, and we clearly obtain that $\langle F, \text{Hull}_{F}^{\circ} \rangle$ is a convex geometry. When reading the first version of the present paper, Adaricheva [5] observed that Theorem 2.2 has the following corollary.

Corollary 4.6 (Adaricheva [5]). Up to isomorphism, finite convex geometries of convex dimension at most 2 are characterized as the convex geometries $\langle F, \operatorname{Hull}_{F}^{\circ} \rangle$, where $F \subseteq \{\langle a, b \rangle : a, b \in \mathbb{R} \text{ and } a \leq b\}$ and F is finite.

Proof. Let $\langle U, \Phi \rangle$ be a convex geometry of dimension at most 2. Theorem 2.2 yields a finite, separated, concave set M of collinear circles such that $\langle U, \Phi \rangle \cong \langle M, \operatorname{Hull}_M^{\circ} \rangle$. For $C \in M$, let $\varphi(C) = \langle \operatorname{LPt}(C), \operatorname{RPt}(C) \rangle$. Let $F = \{\varphi(C) : C \in M\}$. By (2.5), $\varphi : \langle M, \operatorname{Hull}_M^{\circ} \rangle \to \langle F, \operatorname{Hull}_F^{\circ} \rangle$ is an isomorphism. This proves the non-trivial part; the trivial part has already been mentioned.

As opposed to the proof above, it is far less easy to derive Theorem 2.2 from Corollary 4.6, because a finite set F of 1-dimensional circles is rarely of the form $\{\varphi(C): C \in M\}$ for a set M of concave, collinear circles.

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