Diagrams and rectangular extensions of planar semimodular lattices

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Dedicated to George Grätzer on his eightieth birthday

ABSTRACT. In 2009, G. Grätzer and E. Knapp proved that every planar semimodular lattice has a rectangular extension. We prove that, under reasonable additional conditions, this extension is unique. This theorem naturally leads to a hierarchy of special diagrams of planar semimodular lattices. These diagrams are unique in a strong sense; we also explore many of their additional properties. We demonstrate the power of our new classes of diagrams in two ways. First, we prove a simplified version of our earlier Trajectory Coloring Theorem, which describes the inclusion $\text{con}(\mathfrak{p}) \supseteq \text{con}(\mathfrak{q})$ for prime intervals $\mathfrak{p}$ and $\mathfrak{q}$ in slim rectangular lattices. Second, we prove G. Grätzer's Swing Lemma for the same class of lattices, which describes the same inclusion more simply.

1. Introduction

A *planar lattice* is a finite lattice that has a planar (Hasse) diagram. All lattices in this paper are assumed to be *finite*. With the appearance of G. Grätzer and E. Knapp [19] in 2007, the theory of planar semimodular lattices became a very intensively studied branch of lattice theory. This activity is witnessed by more than two dozen papers; some of them are listed in the References section while some others are discussed in the book chapter G. Czédli and G. Grätzer [10].

The study of planar semimodular lattices and, in particular, slim planar semimodular lattices is motivated by three factors.

First, these lattices are *general enough*; for example G. Grätzer, H. Lakser, and E. T. Schmidt [21] proved that every finite distributive lattice can be represented as the congruence lattice of a planar semimodular lattice L. In addition, one can also stipulate that every congruence of L is principal, see G. Grätzer and E. T. Schmidt [23]. Even certain maps between two finite distributive lattices can be represented; see G. Czédli [3] for the latest results in this direction, and see its bibliography for many earlier results.

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Second, these lattices offer *useful links* between lattice theory and the rest of mathematics. For example, G. Grätzer and J. B. Nation [22] and, by adding a uniqueness part to it, G. Czédli and E. T. Schmidt [13], improve the classical Jordan–Hölder theorem for groups from the nineteenth century. Also, these lattices are connected with combinatorial structures, see G. Czédli [6] and [7], and they raise interesting combinatorial problems, see G. Czédli, T. Dékány, L. Ozsvárt, N. Szakács, and B. Udvari [8] and its bibliography.

Third, there are *lots of tools* to deal with these lattices; see, for example, G. Czédli [2], [5], [6], G. Czédli and G. Grätzer [9], G. Czédli and E. T. Schmidt [14], [15], and [16], and G. Grätzer and E. Knapp [19] and [20]; see also G. Czédli and G. Grätzer [10], where most of these tools are discussed; many of them are needed in this paper. Note at this point that some of the available tools make it easy to see that our diagrams, with the exception of Figure 5, define *semimodular lattices*; see, for example, G. Grätzer and E. Knapp [19] or G. Czédli and E. T. Schmidt [14, Propositions 9 and 10 and Theorems 11 and 12].

Target. The first goal is to extend a planar semimodular lattice to a unique rectangular lattice. Definitions will be given soon. For a first impression on our result, let D_2 be the third lattice diagram given in Figure 1, consisting of 34 empty-filled elements; if we add the three black-filled pentagon-shaped elements together with the dotted edges to D_2, then we obtain its rectangular extension. While the existence of such an extension is known from G. Grätzer and E. Knapp [20], its uniqueness needs some natural additional assumptions and a nontrivial proof.

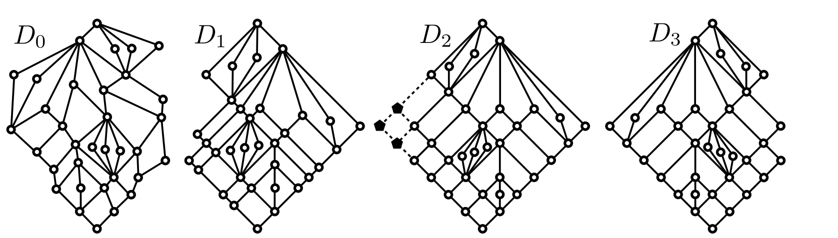

FIGURE 1. $D_0 \in \mathcal{C}_0 \setminus \mathcal{C}_1$, $D_1 \in \mathcal{C}_1 \setminus \mathcal{C}_2$, $D_2 \in \mathcal{C}_2 \setminus \mathcal{C}_3$, and $D_3 \in \mathcal{C}_3$

The second goal, motivated by the first one, is to associate a special diagram with each planar semimodular lattice L. Besides the class $\mathcal{C}_0$ of planar diagrams of slim semimodular lattices, we define a hierarchy $\mathcal{C}_0 \supset \mathcal{C}_1 \supset \mathcal{C}_2 \supset \mathcal{C}_3$ of (classes of) diagrams. For a first impression, we present Figure 1, where the black-filled pentagon-shaped elements do not belong to D_2 and each of the four diagrams determines the same planar semimodular lattice. Also, we list some of the diagrams or lattices whose diagrams are depicted in the paper:

(i) In $\mathcal{C}_1 \setminus \mathcal{C}_2$, we have L and R of Figure 2 and Figures 6, 11, and 12.
(ii) In $\mathcal{C}_2 \setminus \mathcal{C}_3$, we have $\widehat{R}$ in Figure 2 and L_1 and R_1 in Figure 4.

(iii) In $\mathcal{C}_3$, we have L_2 and R_2 in Figure 4, D and E in Figure 8, and Figures 3 and 10.

Although the systematic study and several statements on $\mathcal{C}_2$, $\mathcal{C}_3$, even on $\mathcal{C}_0$ and, mainly, on $\mathcal{C}_1$ are new, note that we often used diagrams from $\mathcal{C}_1$ and $\mathcal{C}_2$ previously. Choosing a smaller hierarchy class, the diagrams of L become unique in a stronger sense. For example, in the plane of complex numbers (with $0, 1 \in \mathbb{C}$ fixed), a planar semimodular lattice has exactly one diagram that belongs to $\mathcal{C}_3$. Besides introducing new diagrams, we prove several useful properties for them. While $\mathcal{C}_2$ and $\mathcal{C}_3$ seem to have only some aesthetic advantage over $\mathcal{C}_1$, the passage from $\mathcal{C}_0$ to $\mathcal{C}_1$ gives some extra insight into the theory of planar semimodular lattices.

Finally, to demonstrate that our diagrams and the toolkit we elaborate are useful, we improve the Trajectory Coloring Theorem from G. Czédli [5, Theorem 7.3.(i)], which describes the ordered set of join-irreducible congruences of a slim rectangular lattice. The improved version is based on $\mathcal{C}_1$; it is easier to understand and apply the new version than the original one. As a nontrivial joint application of the improved Trajectory Coloring Theorem and our toolkit for $\mathcal{C}_1$, we prove G. Grätzer's Swing Lemma for slim rectangular lattices. The Swing Lemma gives a particularly elegant condition for $\mathrm{con}(\mathfrak{p}) \geq \mathrm{con}(\mathfrak{q})$, where $\mathfrak{p}$ and $\mathfrak{q}$ are prime intervals. Although we know from G. Grätzer [18] that this lemma also holds for a larger class of lattices, the slim semimodular ones, the lion's share of the difficulty is to conquer the slim rectangular case.

Outline. The present section is introductory. In Section 2, we introduce the concept of a normal rectangular extension of a slim semimodular lattice, and state its uniqueness in Theorem 2.2. Also, this section contains some analysis of this theorem and the way we prove it in subsequent sections. To make the paper easier to read, some concepts and results are surveyed in Section 3. Section 4 is devoted to the proof of Theorem 2.2, but many of the auxiliary statements are of further interest. Namely, Lemma 4.1 on cover-preserving sublattices of slim semimodular lattices, Lemmas 4.4 and 4.6 on join-coordinates, Lemma 4.7 on the explicit description of normal rectangular extensions, and Lemma 4.9 on the categorical properties of the antislimming procedure deserve separate mentioning here. In Section 5, a hierarchy $\mathcal{C}_0 \supset \mathcal{C}_1 \supset \mathcal{C}_2 \supset \mathcal{C}_3$ of classes of diagrams of planar semimodular lattices is introduced and appropriate uniqueness statements are proved. Here we only mention Proposition 5.1 on $\mathcal{C}_0$, which extends the scope of a known result from "slim semimodular" to "planar semimodular", and Theorem 5.5 on $\mathcal{C}_1$. Section 6 proves several easy statements on diagrams in $\mathcal{C}_1$ and their trajectories. The rest of the paper demonstrates the usefulness of $\mathcal{C}_1$ and the toolkit presented in Section 6. Section 7 improves the Trajectory Coloring Theorem, while Section 8 proves G. Grätzer's Swing Lemma for slim rectangular lattices.

Method. Our lattices are planar and they are easy to visualize. However, instead of relying too much on geometric intuition, we give rigorous proofs for many auxiliary statements. Fortunately, we can use a rich toolkit available in the papers we reference, including D. Kelly and I. Rival [24] and G. Grätzer and E. Knapp [19] and [20].

In order to prove Theorem 2.2 on normal rectangular extensions, we coordinatize our lattices. Although our terminology is different, the coordinates we use are essentially the largest homomorphic preimages with respect to the 2-dimensional case of M. Stern's join-homomorphisms in [26], which were rediscovered in G. Czédli and E. T. Schmidt [12, Corollary 2].

By a *grid* we mean the direct product of two finite nontrivial (that is, non-singleton) chains. Once we have coordinatization, it is natural to position the elements in a grid according to their coordinates. This leads to a hierarchy of planar diagrams with useful properties. The emphasis is put on the properties of trajectories, because they are powerful tools to understand slim rectangular lattices and their congruences.

Although we mostly deal with slim rectangular lattices in this paper, many of our statements can be extended to slim semimodular lattices in a straightforward but sometimes a bit technical way. Namely, one can follow G. Czédli [5, Remark 8.5] or use Theorem 2.2. Because of space considerations, we do not undertake this task.

2. Normal rectangular extensions

Following G. Czédli and E. T. Schmidt [15], a *glued sum indecomposable lattice* is a finite non-chain lattice L such that each $x \in L \setminus \{0,1\}$ is incomparable with some element of L. Such a lattice consists of at least 4 elements. Following G. Grätzer and E. Knapp [20], a *rectangular lattice* is a planar semimodular lattice R such that R has a planar diagram D with the following properties:

(i) $D \setminus \{0,1\}$ has exactly one double irreducible element on the left boundary chain of D; this element, called *left corner*, is denoted by $\mathrm{lc}(D)$.
(ii) $D \setminus \{0,1\}$ has exactly one double irreducible element, $\mathrm{rc}(D)$, on the right boundary chain of D. It is called the *right corner* of D.
(iii) These two elements are complementary, that is, $\mathrm{lc}(D) \wedge \mathrm{rc}(D) = 0$ and $\mathrm{lc}(D) \vee \mathrm{rc}(D) = 1$.

Note that a rectangular lattice has at least four elements. Following G. Czédli and E. T. Schmidt [13], a lattice L is *slim*, if it is finite and $\mathrm{Ji}(L)$, the (ordered) set of (non-zero) join-irreducible elements of L, is the union of two chains. It follows from G. Czédli and E. T. Schmidt [15, page 693] that, for a *slim* semimodular lattice L,

$$L \text{ is rectangular iff } \mathrm{Ji}(L) \text{ is the union of two chains, } W_1 \text{ and } W_2, \text{ such that } w_1 \wedge w_2 = 0 \text{ for all } \langle w_1, w_2\rangle \in W_1 \times W_2. \tag{2.1}$$

We know from G. Czédli and G. Grätzer [10, Exercise 3.55] (which follows from (2.1), [15, Lemma 6.1(ii)], and G. Czédli and G. Grätzer [10, Theorem 3-4.5]) that

$$\text{if one planar diagram of a semimodular lattice } L \text{ satisfies (i)–(iii) above, then so do all planar diagrams of } L. \tag{2.2}$$

Let us emphasize that slim lattices, planar lattices, and rectangular lattices are finite by definition. Since a slim lattice is necessarily planar by G. Czédli and E. T. Schmidt [13, Lemma 2.2], we usually say "slim" rather than "slim planar".

Definition 2.1. Let L be a planar semimodular lattice. We say that a lattice R is a *normal rectangular extension* of L if the following hold.

(i) R is a rectangular lattice.
(ii) L is a cover-preserving $\{0,1\}$-sublattice of R.
(iii) For every $x \in R$, if x has a lower cover outside L, then x has at most two lower covers in R.

In Figure 2, R is a normal rectangular extension of L but $\widehat{R}$ is not; no matter if we consider the pentagon-shaped grey-filled elements with the dotted edges or we omit them. This example witnesses that a normal rectangular extension of L need not be a minimum-sized rectangular, cover-preserving extension of L.

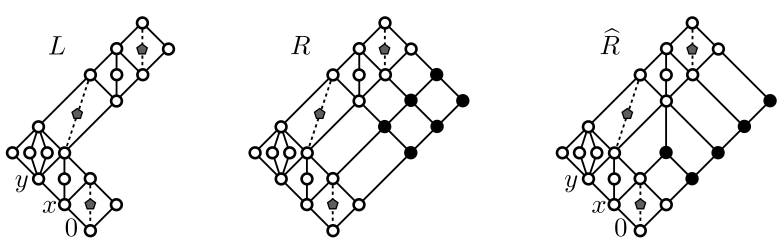

FIGURE 2. R is the normal rectangular extension but $|\widehat{R}| < |R|$

If R_1 and R_2 are extensions of a lattice L and $\varphi\colon R_1 \to R_2$ is a lattice isomorphism whose restriction $\varphi]_L$ to L is the identity map, then φ is a *relative isomorphism over* L.

Theorem 2.2. *If L is a planar semimodular lattice with more than two elements, then the following two statements hold.*

(i) *L has a normal rectangular extension.*
(ii) *L is slim iff it has a slim normal rectangular extension iff all normal rectangular extensions of L are slim.*

Moreover, if L is a glued sum indecomposable planar semimodular lattice, then the following three statements also hold.

(iii) *The normal rectangular extension of L is unique up to isomorphisms.*

(iv) *If, in addition, L is slim, then its normal rectangular extension is unique up to relative isomorphisms over L.*

(v) *Furthermore, if L is slim and $\psi\colon L \to L'$ is a lattice isomorphism, R is a normal rectangular extension of L, and R' is a normal rectangular extension of L', then ψ extends to a lattice isomorphism $R \to R'$.*

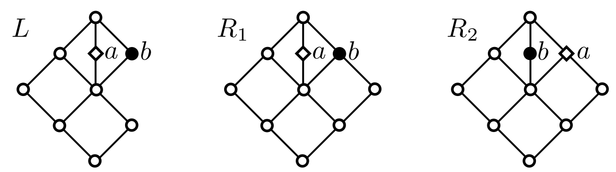

FIGURE 3. Isomorphic but not relatively over L

For a variant of this theorem in terms of diagrams, see Proposition 5.10. The two-element lattice cannot have a normal rectangular extension. Although a finite chain C has a normal rectangular extension if $|C| \geq 3$, it has non-isomorphic normal rectangular extensions in case $|C| \geq 5$. Figure 3, where both R_1 and R_2 are normal rectangular extensions of L, shows that slimness cannot be removed from part (iv). Figure 4 shows that glued sum indecomposability is also inevitable. In this figure, $L_1 \cong L_2$ are isomorphic slim semimodular lattices but they are not glued sum indecomposable. Their diagrams are similar in the sense of D. Kelly and I. Rival [24], so they are the same in $\mathcal{C}_0$-sense, to be defined in Section 5. For $i \in \{1,2\}$, R_i is a normal rectangular extension of L_i. However, $R_1 \ncong R_2$ since $|R_1| \neq |R_2|$.

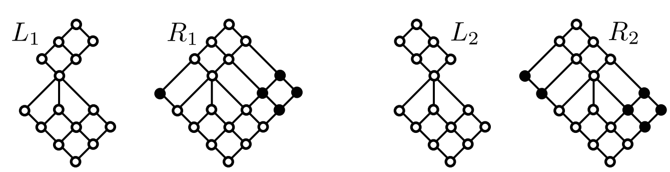

FIGURE 4. L_1 and L_2 are isomorphic but R_1 and R_1 are not

Let L_1 be a sublattice of another lattice, L_2. We say that L_2 is a *congruence-preserving extension* of L_1 if the restriction map $\operatorname{Con} L_2 \to \operatorname{Con} L_1$ from the congruence lattice of L_2 to the congruence lattice of L_1, defined by $\boldsymbol{\alpha} \mapsto \boldsymbol{\alpha} \cap (L_1 \times L_1)$, is a lattice isomorphism. We know from G. Grätzer and E. Knapp [20, Theorem 7] that every planar semimodular lattice has a rectangular congruence-preserving extension. Analyzing their proof, it appears that they construct a normal rectangular extension. Hence, using the uniqueness granted by Theorem 2.2, we obtain the following statement; note that it also follows from G. Czédli [3, Lemmas 5.4 and 6.4].

Corollary 2.3 (compare with G. Grätzer and E. Knapp [20, Theorem 7]). *If L is a glued sum indecomposable planar semimodular lattice, then its normal rectangular extension is a congruence-preserving extension of L.*

Remark 2.4. Omit the dotted edges and the three pentagon-shaped grey-filled elements from Figure 2. Then, as opposed to the normal rectangular extension R of L, $\widehat{R}$ is a rectangular extension but not a congruence-preserving extension of L, because $\langle x, y\rangle \in \mathrm{con}(0, x)$ holds in $\widehat{R}$ but fails in L. Also, if we omit 1 and the rightmost coatom from this L, then the remaining planar semimodular lattice has two non-isomorphic normal rectangular extensions but only one of them is a congruence-preserving extension.

Remark 2.5. Consider the lattices in Figure 2 together with the pentagon-shaped grey-filled elements and the dotted edges. Then R is a normal rectangular extension of L, $|\widehat{R}| < |R|$, and $\widehat{R}$ is a congruence-preserving extension of L since both L and $\widehat{R}$ are simple lattices.

These two remarks explain why we deal with normal rectangular extensions rather than with minimum-sized ones or with congruence-preserving ones. Note that the construct in G. Grätzer and E. Knapp [20, Theorem 7] turns out to be a normal rectangular extension of L, but this fact does not imply Theorem 2.2.

For a given n, up to isomorphism, there are finitely many slim semimodular lattices of length n; their number is determined in G. Czédli, L. Ozsvárt, and B. Udvari [11]. With the notation $f(n) = \max\{|L| : L \text{ is a slim semimodular lattice of length } n\}$, one may have the idea of proving Theorem 2.2(iii) by induction on $f(\mathrm{length}(L)) - |L|$. Although such a proof seems to be possible and, probably, it would be somewhat shorter than the proof we are going to present here, our approach has two advantages. First, it gives an explicit formula for the normal rectangular extension rather than a recursive one; see Lemmas 4.6 and 4.7. Second, it is the present approach that leads us directly to a better understanding of slim semimodular lattices; see Sections 5 and 7. In particular, the explicit description of a normal rectangular extension is heavily used in the proof of Theorem 5.5.

3. Preparations for the proof of Theorem 2.2

For the reader's convenience, this section collects briefly the most important conventions, concepts, and tools needed in our proofs. Note that, with much more details, the majority of this section is covered by the book chapter G. Czédli and G. Grätzer [10]. This paper is on *planar* semimodular lattices. Unless otherwise stated, we always assume that a fixed planar diagram of the lattice under consideration is given. Some concepts, such as "left" of "right", may depend on the diagram. However, the choice of the diagram is irrelevant in the statements and proofs. Later in Sections 5, 7, and 8, we focus explicitly on diagrams rather than lattices, and we apply lattice adjectives, like "slim" or "semimodular", to the corresponding diagrams as well. Also, if D_i is a planar diagram of L_i for $i \in \{1, 2\}$, then we do not make a distinction between a map from L_1 to L_2 and the corresponding map from D_1 to D_2. This allows us to speak of lattice isomorphisms between diagrams. Similarly, we can use the statements and concepts that are introduced in Section 4 both for lattices and for diagrams.

For a maximal chain C of a planar lattice L, the set of elements $x \in L$ that are on the left of C is the *left side* of C, and it is denoted by $\mathrm{LS}(C)$. The *right side* of C, $\mathrm{RS}(C)$, is defined similarly. Note that $C = \mathrm{LS}(C) \cap \mathrm{RS}(C)$. If $x \in \mathrm{LS}(C) \setminus C$, then x is *strictly on the left* of C; "*strictly on the right*" is defined analogously. Let us emphasize that, for an element x and a maximal chain C, "left" and "right" is always understood in the wider sense that allows $x \in C$. We need some results from D. Kelly and I. Rival [24]; the most frequently used result is the following.

Lemma 3.1 (D. Kelly and I. Rival [24, Lemma 1.2])**.** *Let L be a finite planar lattice, and let $x \leq y$ in L. If x and y are on different sides of a maximal chain C in L, then there exists an element $z \in C$ such that $x \leq z \leq y$.*

Next, let x and y be elements of a finite planar lattice L, and assume that they are incomparable, in formula, $x \parallel y$. If $x \vee y$ has lower covers x_1 and y_1 such that $x \leq x_1 \prec x \vee y$, $y \leq y_1 \prec x \vee y$, and x_1 is on the left of y_1, then the element x is *on the left* of the element y. In notation, $x \mathrel{\lambda} y$. If $x \mathrel{\lambda} y$, then we also say that y is *on the right* of x. Let us emphasize that whenever λ, that is "left", or "right" are used for two elements, then the two elements are incomparable. That is, the notation $x \mathrel{\lambda} y$ implies that $x \parallel y$. Note the difference; while λ is an irreflexive relation for elements, "left" and "right" are used in the wider sense if an element and a maximal chain are considered.

Lemma 3.2 (D. Kelly and I. Rival [24, Propositions 1.6 and 1.7])**.** *Let L be finite planar lattice. If $x, y \in L$ and $x \parallel y$, then the following hold.*

- (i) *$x \mathrel{\lambda} y$ if and only if x is on the left of some maximal chain through y if and only if x is on the left of all maximal chains through y.*
- (ii) *Exactly one of $x \mathrel{\lambda} y$ and $y \mathrel{\lambda} x$ holds.*
- (iii) *If $z \in L$, $x \mathrel{\lambda} y$, and $y \mathrel{\lambda} z$, then $x \mathrel{\lambda} z$.*

Let L be a slim semimodular lattice. According to the general convention in the paper, a planar diagram of L is fixed. Let $\mathfrak{p}_i = [x_i, y_i]$ be prime intervals, that is, edges in the diagram, for $i \in \{1, 2\}$. These two edges are *consecutive* if they are opposite sides of a covering square, that is, of a 4-cell in the diagram. Following G. Czédli and E. T. Schmidt [13], an equivalence class of the transitive reflexive closure of the "consecutive" relation is called a *trajectory.*

Recall from [13] that

$$\text{A trajectory begins with an edge on the left boundary chain } \mathrm{C_l}(L)\text{, it goes from left to right, it cannot branch out, and it terminates at an edge on the right boundary chain, } \mathrm{C_r}(L). \tag{3.1}$$

These boundary chains are also important because of

$$\mathrm{Ji}(L) \subseteq \mathrm{C_l}(L) \cup \mathrm{C_r}(L); \tag{3.2}$$

see G. Czédli and E. T. Schmidt [14, Lemma 6]. The boundary chains are important for planar lattices K (with fixed planar diagrams) in general. It follows from D. Kelly and I. Rival [24] that $\mathrm{C_l}(K)$ is the unique maximal chain I of K such that $\mathrm{RS}_K(I) = K$, and analogously for $\mathrm{C_r}(K)$. Hence, by Lemma 3.2 and its left-right dual, for arbitrary $x \parallel y \in K$,

$$\text{if} \quad x \in \mathrm{C_l}(K) \quad \text{or} \quad y \in \mathrm{C_r}(K), \quad \text{then} \qquad x \,\lambda\, y. \tag{3.3}$$

According to G. Czédli and G. Grätzer [9] or [10], there are three types of trajectories: an *up-trajectory*, which goes up (possibly, in zero steps), a *down-trajectory*, which goes down (possibly, in zero steps), and a *hat-trajectory*, which goes up (at least one step), then turns to the lower right, and finally it goes down (at least one step). Let $\mathfrak{p}_1 = [x_1, y_1]$, $\mathfrak{p}_2 = [x_2, y_2]$, and $\mathfrak{p}_3 = [x_3, y_3]$ be three consecutive edges of a trajectory T, listed from left to right. If $y_1 < y_2 < y_3$, then T *goes upwards* at $\mathfrak{p}_2$. Similarly, T *goes downwards* at $\mathfrak{p}_2$ if $y_1 > y_2 > y_3$. The third possibility is that $y_1 < y_2 > y_3$; then T is a hat-trajectory and $\mathfrak{p}_2$ is called its *top edge*. If x_1 and y_1 are on the left boundary chain, then we say that the trajectory containing $\mathfrak{p}_1 = [x_1, y_1]$ and $\mathfrak{p}_2 = [x_2, y_2]$ goes upwards or downwards at $\mathfrak{p}_1$ if $y_1 < y_2$ or $y_1 > y_2$, respectively. Since there are only three types of trajectories, if $\mathfrak{p}_1$ is on the left of $\mathfrak{p}_2$ in a trajectory T of L, then

$$\begin{aligned} &\text{if } T \text{ goes upwards at } \mathfrak{p}_2 \text{ then so it does at } \mathfrak{p}_1\text{, and} \\ &\text{if } T \text{ goes downwards at } \mathfrak{p}_1 \text{ then so it does at } \mathfrak{p}_2. \end{aligned} \tag{3.4}$$

4. Proving some lemmas and Theorem 2.2

If C_1 and C_2 are maximal chains of planar lattice L such that $C_1 \subseteq \mathrm{LS}(C_2)$, then $\mathrm{RS}(C_1) \cap \mathrm{LS}(C_2)$ is called a *region* of L. For a subset X of a slim semimodular lattice L, it follows from G. Czédli and G. Grätzer [10, Exercise 3.12 and Theorems 3-4.5 and 3-4.6] or G. Czédli and E. T. Schmidt [15, Lemma 4.7], see also Proposition 4.11 in this paper, that the predicate "X is a region of L" does not depend on the choice of the planar diagram. The following lemma is of separate interest.

Lemma 4.1. *If K is a cover-preserving $\{0, 1\}$-sublattice of a slim semimodular lattice L, then K is also a slim semimodular lattice, it is a region of L, and $K = \mathrm{RS}_L(\mathrm{C_l}(K)) \cap \mathrm{LS}_L(\mathrm{C_r}(K))$.*

Proof. For $x \in L$, the *left support* and the *right support* of x, denoted by $\mathrm{lsp}(x) = \mathrm{lsp}_L(x)$ and $\mathrm{rsp}(x) = \mathrm{rsp}_L(x)$, are the largest element of $\mathrm{C_l}(L) \cap \downarrow x$ and that of $\mathrm{C_r}(L) \cap \downarrow x$, respectively. Since $\mathrm{Ji}(L) \subseteq \mathrm{C_l}(L) \cup \mathrm{C_r}(L)$ and $\mathrm{C_l}(L)$ and $\mathrm{C_r}(L)$ are chains, it is straightforward to see that, for every $x \in L$, $y \in [\mathrm{lsp}_L(x), x]$, and $z \in [\mathrm{rsp}_L(x), x]$,

$$x = \mathrm{lsp}_L(x) \vee \mathrm{rsp}_L(x),\ [\mathrm{lsp}_L(x), x] \text{ and } [\mathrm{rsp}_L(x), x] \text{ are chains, } \mathrm{lsp}_L(y) = \mathrm{lsp}_L(x), \text{ and } \mathrm{rsp}_L(z) = \mathrm{rsp}_L(x). \tag{4.1}$$

Let $H = \mathrm{RS}_L(\mathrm{C_l}(K)) \cap \mathrm{LS}_L(\mathrm{C_r}(K))$; it is the smallest region of L that includes K. Consider an arbitrary element $x \in H$. Applying Lemma 3.1 to $\mathrm{lsp}_L(x) \le x$ and $\mathrm{C_l}(K)$, we obtain an element $y \in \mathrm{C_l}(K)$ such that $\mathrm{lsp}_L(x) \le y \le x$. Similarly, there is an element $z \in \mathrm{C_r}(K)$ such that $\mathrm{rsp}_L(x) \le z \le x$. Hence, $x = y \vee z \in K$ by (4.1). This shows that $K = H$ is a region, and it is a slim lattice since $\mathrm{Ji}(K) \subseteq \mathrm{C_l}(K) \cup \mathrm{C_r}(K)$. As a cover-preserving sublattice, K inherits semimodularity. □

In the rest of this section, unless otherwise stated, we always assume that

$$L \text{ is a planar semimodular lattice of length } n \ge 2 \text{ and } R \text{ is a normal rectangular extension of } L. \tag{4.2}$$

A planar diagram of R, denoted by D, is fixed; it determines the diagram of L as a subdiagram. Sometimes, we stipulate additional assumptions, including

$$L \text{ is a glued sum indecomposable.} \tag{4.3}$$

Sometimes, for emphasis, we repeat (4.2) and (4.3). By Lemma 4.1,

$$L = \mathrm{RS}_R(\mathrm{C_l}(L)) \cap \mathrm{LS}_R(\mathrm{C_r}(L)). \tag{4.4}$$

We know from G. Grätzer and E. Knapp [20, Lemmas 3 and 4] (see also G. Czédli and G. Grätzer [10, Lemma 3-7.1]) that

$$\text{The intervals } [0, \mathrm{lc}(R)] \text{ and } [0, \mathrm{rc}(R)] \text{ are chains.} \tag{4.5}$$

If R is slim, then we also know from (2.1), (3.2), and G. Grätzer and E. Knapp [20, Lemma 3], see also G. Czédli and G. Grätzer [10, Exercises 3.51 and 3.52], that

$$\mathrm{Ji}(R) = \big(C_{\mathrm{ll}}(R) \cup C_{\mathrm{lr}}(R)\big) \setminus \{0\} = \{c_1, \dots, c_{m_\mathrm{l}}, d_1, \dots, d_{m_\mathrm{r}}\}, \tag{4.6}$$

where $C_{\mathrm{ll}}(R)$ and $C_{\mathrm{lr}}(R)$ are defined in Definition 4.2.

Definition 4.2. If condition (4.2) holds and L is slim, then we agree in the following.

(i) Let

$$C_{\mathrm{ll}}(D) = [0, \mathrm{lc}(D)]_R = \{0 = c_0 \prec c_1 \prec \dots \prec c_{m_\mathrm{l}}\}$$

(lower left boundary) and

$$C_{\mathrm{lr}}(D) = [0, \mathrm{rc}(D)]_R = \{0 = d_0 \prec d_1 \prec \dots \prec d_{m_\mathrm{r}}\}$$

(lower right boundary). Note that $c_{m_\mathrm{l}} = \mathrm{lc}(D)$ and $c_{m_\mathrm{r}} = \mathrm{rc}(D)$.

(ii) For $x \in L$, the left and right *join-coordinates* of x are defined by $\mathrm{ljc}_L(x) = |\mathrm{C_l}(L) \cap \mathrm{Ji}(L) \cap {\downarrow}x|$ and $\mathrm{rjc}_L(x) = |\mathrm{C_r}(L) \cap \mathrm{Ji}(L) \cap {\downarrow}x|$. It follows from (3.2) that x is determined by the pair $\langle \mathrm{ljc}_L(x), \mathrm{rjc}_L(x)\rangle$ of its join coordinates.

(iii) For $x \in R$, we obtain $\mathrm{ljc}_R(x)$ and $\mathrm{rjc}_R(x)$ by substituting R to L above. Combining (i) and (3.2), we obtain that

$$x = c_{\mathrm{ljc}_R(x)} \vee d_{\mathrm{rjc}_R(x)}. \tag{4.7}$$

By (4.6), understanding $\wedge$ in $\langle \mathbb{N}_0; \le\rangle$, we have that

$$\mathrm{ljc}_R(x) = m_\mathrm{l} \wedge \mathrm{height}(\mathrm{lsp}_R(x)), \quad \mathrm{rjc}_R(x) = m_\mathrm{r} \wedge \mathrm{height}(\mathrm{rsp}_R(x)).$$

Note that, for $x, x', y, y' \in R$ with $x \not\geq c_{m_\mathrm{l}}$ and $y \not\geq d_{m_\mathrm{r}}$,

$$\mathrm{lsp}_R(x) = c_{\mathrm{ljc}_R(x)}, \quad \mathrm{rsp}_R(y) = d_{\mathrm{rjc}_R(y)}, \tag{4.8}$$

$$\begin{aligned} \mathrm{ljc}_R(x') < \mathrm{ljc}_R(y') &\Rightarrow \mathrm{lsp}_R(x') < \mathrm{lsp}_R(y'),\\ \mathrm{rjc}_R(x') < \mathrm{rjc}_R(y') &\Rightarrow \mathrm{rsp}_R(x') < \mathrm{rsp}_R(y'). \end{aligned} \tag{4.9}$$

The conditions $x \not\geq c_{m_\mathrm{l}}$ and $y \not\geq d_{m_\mathrm{r}}$ right before (4.8) could be inconvenient at later applications. Hence, we are going to formulate a related condition, (4.12) below. As a preparation to do so, the set of *meet-irreducible elements* of R distinct from 1 is denoted by $\mathrm{Mi}(R)$. For $x \in R$, $x \in \mathrm{Mi}(R)$ iff x has exactly one cover. G. Grätzer and E. Knapp [20, Lemma 3] or G. Czédli and G. Grätzer [10, Exercise 3.52] yields that

$$\text{if } 1 \neq x \in (\mathrm{C_l}(R) \setminus C_{\mathrm{ll}}(R)) \cup (\mathrm{C_r}(R) \setminus C_{\mathrm{lr}}(R)), \text{ then } x \in \mathrm{Mi}(R). \tag{4.10}$$

This implies, see also G. Grätzer and E. Knapp [20, Lemma 4], that

$$[\mathrm{lc}(R), 1] = {\uparrow}c_{m_\mathrm{l}} \text{ and } [\mathrm{rc}(R), 1] = {\uparrow}d_{m_\mathrm{r}} \text{ are chains.} \tag{4.11}$$

By (4.7), for every $x \in R$, $c_{\mathrm{ljc}_R(x)} \le \mathrm{lsp}_R(x)$ and $d_{\mathrm{rjc}_R(x)} \le \mathrm{rsp}_R(x)$. Thus,

$$[\mathrm{lsp}_R(x), x] \subseteq [c_{\mathrm{ljc}_R(x)}, x] \quad \text{and} \quad [\mathrm{rsp}_R(x), x] \subseteq [d_{\mathrm{rjc}_R(x)}, x].$$

We conclude from (4.1), (4.8), and (4.11) that for all $x \in R$, if $y \in [c_{\mathrm{ljc}_R(x)}, x]$ and $z \in [d_{\mathrm{rjc}_R(x)}, x]$, then

$$\mathrm{ljc}_R(y) = \mathrm{ljc}_R(x) \quad \text{and} \quad \mathrm{rjc}_R(z) = \mathrm{rjc}_R(x). \tag{4.12}$$

The elements of R on the left of $\mathrm{C_l}(L)$ form a region

$$\mathrm{RS}_R\big(\mathrm{C_l}(R)\big) \cap \mathrm{LS}_R\big(\mathrm{C_l}(L)\big) = \mathrm{LS}_R\big(\mathrm{C_l}(L)\big);$$

it is called *the region on the left of* L, and we denote it by S.

Lemma 4.3. *Assume* (4.2).

(i) *The region on the left of L, denoted by $S = \mathrm{LS}_R\big(\mathrm{C_l}(L)\big)$, is a cover-preserving $\{0,1\}$-sublattice of R and it is distributive.*

(ii) *R is slim iff L is slim.*

Proof. As a region of R, S is a cover-preserving sublattice of R by D. Kelly and I. Rival [24, Proposition 1.4]. The inclusion $\{0_R, 1_R\} \subseteq S$ is obvious. As a cover-preserving sublattice, S is semimodular. As a region of a planar diagram, S is a planar lattice. We know from G. Czédli and G. Grätzer [10, Theorem 3-4.3], see also G. Czédli and E. T. Schmidt [13, Lemma 2.3], that

$$\text{a planar semimodular lattice is slim if it contains no cover-preserving diamond sublattice } M_3. \tag{4.13}$$

This property holds for S by Definition 2.1(iii), so S is slim. Recall from G. Czédli and E. T. Schmidt [14, Lemmas 14 and 15] or G. Czédli and G. Grätzer [10, Exercises 3.30 and 3.31] that

$$\text{if no element of a slim semimodular lattice covers more than 2 elements, then the lattice is distributive.} \tag{4.14}$$

By Definition 2.1(iii) again, no element of S covers more than two elements. Hence, S is distributive by (4.14) . This proves part (ii) .

By (4.13), if R is slim, then so is L. Suppose, for a contradiction, that L is slim but R is not. By (4.13), some element $x \in R$ is the top of a cover-preserving diamond. By Definition 2.1(iii), none of the coatoms (that is, the atoms) of this diamond are outside L. Hence, they are in L, the whole diamond is L, which contradicts the slimness of L by (4.13). This proves part (ii) □

In the following statement, R is slim by Lemma 4.3(ii).

Lemma 4.4. *If condition* (4.2) *holds, L is slim, and $x, y \in L$, then*

$$x \mathrel{\lambda} y \iff \big(\mathrm{ljc}_L(x) > \mathrm{ljc}_L(y) \text{ and } \mathrm{rjc}_L(x) < \mathrm{rjc}_L(y)\big), \tag{4.15}$$

$$x \le y \iff \big(\mathrm{ljc}_L(x) \le \mathrm{ljc}_L(y) \text{ and } \mathrm{rjc}_L(x) \le \mathrm{rjc}_L(y)\big). \tag{4.16}$$

If we substitute R to L, then (4.15) *and* (4.16) *still hold.*

Proof. The $\Rightarrow$ part of (4.16) is evident. To show the converse implication, assume that $\mathrm{ljc}_L(x) \le \mathrm{ljc}_L(y)$ and $\mathrm{rjc}_L(x) \le \mathrm{rjc}_L(y)$. For $z \in L$, let z' and z'' be the largest element of $\mathrm{Ji}(L) \cap \mathrm{C_l}(L) \cap {\downarrow} z$ and that of $\mathrm{Ji}(L) \cap \mathrm{C_r}(L) \cap {\downarrow} z$, respectively. By the inequalities we have assumed, $x' \le y'$ and $x'' \le y''$. Since $x = x' \vee x''$ and $y = y' \vee y''$ by (3.2), we obtain that $x \le y$. Thus, (4.16) holds.

In order to prove (4.15), recall from G. Czédli [7, Lemma 3.15] that

$$x \mathrel{\lambda} y \iff \big(\mathrm{lsp}_L(x) > \mathrm{lsp}_L(y) \text{ and } \mathrm{rsp}_L(x) < \mathrm{rsp}_L(y)\big). \tag{4.17}$$

Assume that $x \mathrel{\lambda} y$. Then $\mathrm{lsp}_L(x) > \mathrm{lsp}_L(y)$ by (4.17), and we obtain that $\mathrm{ljc}_L(x) \ge \mathrm{ljc}_L(y)$. Similarly, $\mathrm{rjc}_L(x) \le \mathrm{rjc}_L(y)$. Both inequalities must be sharp, because otherwise (4.16) would imply that $x \not\parallel y$. Therefore, the $\Rightarrow$ implication of (4.15) follows. Conversely, assume that $\mathrm{ljc}_L(x) > \mathrm{ljc}_L(y)$ and $\mathrm{rjc}_L(x) < \mathrm{rjc}_L(y)$. Clearly, $\mathrm{lsp}_L(x) > \mathrm{lsp}_L(x)$ and $\mathrm{rsp}_L(x) < \mathrm{rsp}_L(x)$. Hence, $x \mathrel{\lambda} y$ by (4.17), which gives the desired converse implication of (4.15). □

In the following lemma, the subscripts come from "left" and "right" and so they are not numbers. Hence, we write x_{l} and x_{r} rather than x_l and x_r.

Lemma 4.5. *Assume that condition* (4.2) *holds, L is slim, T is a trajectory of R, and $[x,y] \in T$. Let $[x_{\mathrm{l}}, y_{\mathrm{l}}]$ and $[x_{\mathrm{r}}, y_{\mathrm{r}}]$ be the leftmost* (*that is, the first*) *and the rightmost edge of T, respectively. If T goes upwards at $[x,y]$, then* $\mathrm{lsp}_R(x) = x_{\mathrm{l}} < y_{\mathrm{l}} \le \mathrm{lsp}_R(y)$. *Similarly, if T goes downwards at $[x,y]$, then* $\mathrm{rsp}_R(x) = x_{\mathrm{r}} < y_{\mathrm{r}} \le \mathrm{rsp}_R(y)$.

Proof. By left-right symmetry, we can assume that T goes upwards at $[x,y]$. The segment of T from $[x_{\mathrm{l}}, y_{\mathrm{l}}]$ to $[x,y]$ goes up by (3.4). Combining this fact with (4.10), it follows that the edge $[x_{\mathrm{l}}, y_{\mathrm{l}}]$ belongs to $C_{\mathrm{ll}}(R)$ and that $y = y_{\mathrm{l}} \vee x$. Hence, $y_{\mathrm{l}} \not\le x$. Thus, we obtain that $x_{\mathrm{l}} = \mathrm{lsp}_R(x)$ and $y_{\mathrm{l}} \le \mathrm{lsp}_R(y)$. ☐

The following lemma is of separate interest.

Lemma 4.6. *If conditions* (4.2) *and* (4.3) *hold, L is slim, and $x \in L$, then the pair of join-coordinates of x is the same in L as in R.*

Note that this lemma would fail without assuming that L is glued sum indecomposable; this is witnessed by $R = \{0, a, b, 1\}$, the 4-element Boolean lattice, and $L = \{0, a, 1\}$.

Proof. Since L is glued sum indecomposable,

$$\mathrm{C_l}(L) \cap \mathrm{C_r}(L) = \{0, 1\} \text{ and } |L| \ge 4. \tag{4.18}$$

By semimodularity, $|\mathrm{C_l}(L)| = \mathrm{length}(L) + 1 = n + 1 = |\mathrm{C_r}(L)|$. Let

$$\mathrm{C_l}(L) = \{0 = e_0 \prec e_1 \prec \cdots \prec e_n = 1\} \text{ and}$$
$$\mathrm{C_r}(L) = \{0 = f_0 \prec f_1 \prec \cdots \prec f_n = 1\}.$$

We claim that, for $i \in \{1, \dots, n\}$,

$$e_i \in \mathrm{Ji}(L) \iff \langle \mathrm{ljc}_R(e_i), \mathrm{rjc}_R(e_i)\rangle = \langle 1 + \mathrm{ljc}_R(e_{i-1}), \mathrm{rjc}_R(e_{i-1})\rangle. \tag{4.19}$$

First, to prove the "$\Leftarrow$" direction of (4.19), assume that $\mathrm{ljc}_R(e_i) = 1 + \mathrm{ljc}_R(e_{i-1})$ and $\mathrm{rjc}_R(e_i) = \mathrm{rjc}_R(e_{i-1})$. Suppose, for a contradiction, that $e_i \notin \mathrm{Ji}(L)$, and let $y \in L \setminus \{e_{i-1}\}$ be a lower cover of e_i. Since e_{i-1} is on the left boundary of L, $e_{i-1} \mathrel{\lambda} y$. Hence, we obtain from (4.15) that $\mathrm{rjc}_R(e_{i-1}) < \mathrm{rjc}_R(y)$. On the other hand, $e_i > y$ and (4.16) yield that $\mathrm{rjc}_R(e_{i-1}) = \mathrm{rjc}_R(e_i) \ge \mathrm{rjc}_R(y)$, which contradicts the previous inequality. Thus, the "$\Leftarrow$" part of (4.19) follows.

In order to prove the converse direction of (4.19), assume that $e_i \in \mathrm{Ji}(L)$. By (4.16),

$$\langle \mathrm{ljc}_R(e_i), \mathrm{rjc}_R(e_i)\rangle > \langle \mathrm{ljc}_R(e_{i-1}), \mathrm{rjc}_R(e_{i-1})\rangle \tag{4.20}$$

in the usual componentwise ordering "$\le$" of $\{0, 1, \dots, n\}^2$. We claim that

$$\mathrm{rjc}_R(e_i) = \mathrm{rjc}_R(e_{i-1}). \tag{4.21}$$

In order to prove this by contradiction, suppose $\mathrm{rjc}_R(e_i) > \mathrm{rjc}_R(e_{i-1})$. Applying Lemma 3.1 in R to $\mathrm{rsp}_R(e_i) \le e_i$ and the maximal chain $\mathrm{C_r}(L)$, we obtain an element $z \in \mathrm{C_r}(L) \subseteq L$ such that $\mathrm{rsp}_R(e_i) \le z \le e_i$. Combining (4.9) and $\mathrm{rjc}_R(e_i) > \mathrm{rjc}_R(e_{i-1})$, we have that $z \not\le e_{i-1}$. Hence, $e_{i-1} \prec e_i$ gives that $e_{i-1} \vee z = e_i \in \mathrm{Ji}(L)$. So we conclude that $z = e_i$, that is,

$0 \neq e_i \in \mathrm{C_l}(L) \cap \mathrm{C_r}(L)$. From (4.18), we obtain that $1 = e_i = f_i$ and $i = n$. Since $1 = e_i = e_n \in \mathrm{Ji}(L)$ has only one lower cover, we obtain that $e_{n-1} = f_{n-1} \in \mathrm{C_l}(L) \cap \mathrm{C_r}(L)$, which contradicts (4.18). This proves (4.21).

Combining (4.20) and (4.21), we obtain that $\mathrm{ljc}_R(e_i) > \mathrm{ljc}_R(e_{i-1})$. Let T denote the trajectory of R that contains $[e_{i-1}, e_i]$. In the moment, there are three possible ways how T can be related to $[e_{i-1}, e_i]$ but we want to exclude two of them. First, suppose that $[e_{i-1}, e_i]$ is the top edge of a hat-trajectory. Then e_i has a lower cover to the left of $e_{i-1} \in \mathrm{C_l}(L)$, so outside L, and e_i has at least three lower covers. This possibility is excluded by Definition 2.1(iii). Hence, $[e_{i-1}, e_i]$ cannot be the top edge of a hat-trajectory. (Note, however, that T can be a hat-trajectory whose top is above $[e_{i-1}, e_i]$ in a straightforward sense.) Second, suppose that T goes downwards at $[e_{i-1}, e_i]$. Then $\mathrm{rsp}_R(e_{i-1})$ is meet-reducible, because it is the bottom of the last edge of T by Lemma 4.5 and T arrives downwards at this last edge by (3.4). So (4.10) yields that $\mathrm{rsp}_R(e_{i-1}) < d_{m_\mathrm{r}}$, and we have that $e_{i-1} \not\geq d_{m_\mathrm{r}}$. Hence, there is a unique $j < m_\mathrm{r}$ such that $\mathrm{rsp}_R(e_{i-1}) = d_j$, and (4.8) gives that $j = \mathrm{rjc}_R(e_{i-1})$. Since $\mathrm{rjc}_R(e_i)$ is also j by (4.21), $d_{j+1} \not\leq e_i$, and we obtain that $\mathrm{rsp}_R(e_i) = d_j = \mathrm{rsp}_R(e_{i-1})$. This contradicts Lemma 4.5 and excludes the possibility that T goes downwards at $[e_{i-1}, e_i]$.

Therefore, T goes upwards at $[e_{i-1}, e_i]$. Let $[u_\mathrm{l}, v_\mathrm{l}]$ be the first edge of T. We know from Lemma 4.5 that $u_\mathrm{l} = \mathrm{lsp}_R(e_{i-1})$. The left-right dual of the argument used in the excluded previous case yields that $u_\mathrm{l} = \mathrm{lsp}_R(e_{i-1}) = c_{\mathrm{ljc}_R(e_{i-1})}$ where $\mathrm{ljc}_R(e_{i-1}) < m_\mathrm{l}$. If $\mathrm{ljc}_R(e_{i-1}) = m_\mathrm{l} - 1$, then the required equality $\mathrm{ljc}_R(e_i) = 1 + \mathrm{ljc}_R(e_{i-1})$ follows from $\mathrm{ljc}_R(e_i) > \mathrm{ljc}_R(e_{i-1})$ and from the fact that $\mathrm{ljc}_R(x) \leq m_\mathrm{l}$ for all $x \in R$. Thus, we can assume that $\mathrm{ljc}_R(e_{i-1}) \leq m_\mathrm{l} - 2$. Hence, by the first of the two displayed equalities below (4.7), $\mathrm{height}(u_\mathrm{l}) = \mathrm{height}(\mathrm{lsp}_R(e_{i-1})) = \mathrm{ljc}_R(e_{i-1}) \leq m_\mathrm{l} - 2$.

We need to show that $e_i \not> c_{m_\mathrm{l}}$. Suppose to the contrary that $e_i > c_{m_\mathrm{l}}$, and list the edges of T from $[u_\mathrm{l}, v_\mathrm{l}]$ to $[e_{i-1}, e_i]$ from left to right as follows: $\mathfrak{r}_0 = [u_\mathrm{l}, v_\mathrm{l}], \mathfrak{r}_1, \dots, \mathfrak{r}_s = [e_{i-1}, e_i]$. These edges form an initial section of T; we denote this initial section by T_0. Note that T_0 goes upwards by (3.4). Since $\mathrm{height}(v_\mathrm{l}) = \mathrm{height}(u_\mathrm{l}) + 1 \leq (m_\mathrm{l} - 2) + 1 = m_\mathrm{l} - 1 = \mathrm{height}(c_{m_\mathrm{l}}) - 1$, we have that $1_{\mathfrak{r}_0} = v_\mathrm{l} \not> c_{m_\mathrm{l}}$. On the other hand, $1_{\mathfrak{r}_s} = e_i > c_{m_\mathrm{l}}$. Thus, there is a smallest $k \in \{0, 1, \dots, s-1\}$ such that $1_{\mathfrak{r}_k} \not> c_{m_\mathrm{l}}$ but $1_{\mathfrak{r}_{k+1}} > c_{m_\mathrm{l}}$. Clearly, there is an $x \in R$ such that $c_{m_\mathrm{l}} \leq x \prec 1_{\mathfrak{r}_{k+1}}$. (Note that x is unique by (4.11), but we do not need this fact.) Also, $1_{\mathfrak{r}_k}$ and $0_{\mathfrak{r}_{k+1}}$ are two distinct lower covers of $1_{\mathfrak{r}_{k+1}}$. If $c_{m_\mathrm{l}} < x$, then x is distinct from $1_{\mathfrak{r}_k}$ by the definition of k. If we had $1_{\mathfrak{r}_k} = x = c_{m_\mathrm{l}}$, then $\mathrm{height}(0_{\mathfrak{r}_k}) = \mathrm{height}(1_{\mathfrak{r}_k}) - 1 = \mathrm{height}(c_{m_\mathrm{l}}) - 1 = m_\mathrm{l} - 1 > m_\mathrm{l} - 2 \geq \mathrm{height}(u_\mathrm{l}) = \mathrm{height}(0_{\mathfrak{r}_0})$ would give that $k \neq 0$ and, since T_0 goes upwards, $c_{m_\mathrm{l}} = 1_{\mathfrak{r}_k}$ would be join-reducible in R, contradicting (4.6). Hence, x is distinct from $1_{\mathfrak{r}_k}$. If we had $x = 0_{\mathfrak{r}_{k+1}}$, then $c_{m_\mathrm{l}} \leq x = 0_{\mathfrak{r}_{k+1}} \leq 0_{\mathfrak{r}_s} = e_{i-1}$ would give $\mathrm{ljc}_R(e_{i-1}) = m_\mathrm{l}$, contradicting $\mathrm{ljc}_R(e_{i-1}) \leq m_\mathrm{l} - 2$. Consequently, x, $1_{\mathfrak{r}_k}$, and $0_{\mathfrak{r}_{k+1}}$ are three distinct lower covers of $1_{\mathfrak{r}_{k+1}}$. By Definition 2.1(iii), all the three belong to L. In particular, $1_{\mathfrak{r}_k} \in L$. Using that T_0 goes upwards, it follows that $e_i = 1_{\mathfrak{r}_s} = 1_{\mathfrak{r}_k} \vee 0_{\mathfrak{r}_s} = 1_{\mathfrak{r}_k} \vee e_{i-1}$ is a nontrivial join in L. This contradicts $e_i \in \mathrm{Ji}(L)$ and shows that $e_i \not\geq c_{m_{\mathrm{l}}}$. Therefore, by $u_{\mathrm{l}} = \mathrm{lsp}_R(e_{i-1}) = c_{\mathrm{ljc}_R(e_{i-1})}$ and (4.8), the desired equation $\mathrm{ljc}_R(e_i) = 1 + \mathrm{ljc}_R(e_{i-1})$ and (4.19) will follow if we show that $v_{\mathrm{l}} = \mathrm{lsp}_R(e_i)$.

Suppose, for a contradiction, that $v_{\mathrm{l}} \neq \mathrm{lsp}_R(e_i)$. We have that $v_{\mathrm{l}} < \mathrm{lsp}_R(e_i)$, since $v_{\mathrm{l}} \leq \mathrm{lsp}_R(e_i)$ is clear by $v_{\mathrm{l}} \leq e_i$. Also, $\mathrm{lsp}_R(e_i) \not\leq e_{i-1}$, since $\mathrm{lsp}_R(e_i) \geq v_l > u_{\mathrm{l}} = \mathrm{lsp}_R(e_{i-1})$ and $\mathrm{lsp}_R(e_{i-1})$ is the largest element of $\mathrm{C_l}(R) \cap {\downarrow} e_{i-1}$. Since u_{l}, v_{l}, and $\mathrm{lsp}_R(e_i)$ are on the leftmost chain $\mathrm{C_l}(R)$ of R, these elements belong to S, the region on the left of L, defined before Lemma 4.3. By Lemma 4.3(ii), R is a slim rectangular lattice. Observe that $\mathrm{lsp}_R(e_i) \neq 0_S$, since $\mathrm{lsp}_R(e_i) > v_{\mathrm{l}} > u_{\mathrm{l}}$ in S. We obtain from $e_i \not\geq c_{m_{\mathrm{l}}}$ that $\mathrm{lsp}_R(e_i) \in C_{\mathrm{ll}}(R)$. Hence, (4.6) yields that $\mathrm{lsp}_R(e_i) \in \mathrm{Ji}(R)$, and we conclude that $\mathrm{lsp}_R(e_i) \in \mathrm{Ji}(S)$. Using $e_{i-1} \prec e_i$ and $e_i \geq v_{\mathrm{l}} \not\leq e_{i-1}$, we conclude that $\mathrm{lsp}_R(e_i) \leq e_i = v_{\mathrm{l}} \vee e_{i-1}$. Since S is distributive by Lemma 4.3 and the elements in the previous inequality belong to S, we have that

$$\mathrm{lsp}_R(e_i) = \mathrm{lsp}_R(e_i) \wedge (v_{\mathrm{l}} \vee e_{i-1}) = (\mathrm{lsp}_R(e_i) \wedge v_{\mathrm{l}}) \vee (\mathrm{lsp}_R(e_i) \wedge e_{i-1}). \tag{4.22}$$

Since $\mathrm{lsp}_R(e_i) \in \mathrm{Ji}(S)$ equals one of the two joinands above and $\mathrm{lsp}_R(e_i) \not\leq e_{i-1}$, we obtain that $\mathrm{lsp}_R(e_i) \leq v_{\mathrm{l}}$. This contradicts $v_{\mathrm{l}} < \mathrm{lsp}_R(e_i)$. In this way, we have shown that $v_{\mathrm{l}} = \mathrm{lsp}_R(e_i)$. This proves (4.19).

Next, with reference to the notation in Definition 4.2(i), we claim that

$$(\forall j \in \{0, \dots, m_{\mathrm{l}}\}) \bigl(\exists e_i \in \mathrm{C_l}(L)\bigr) \bigl(\mathrm{ljc}_R(e_i) = j\bigr). \tag{4.23}$$

In order to prove (4.23), let $j \in \{0, \dots, m_{\mathrm{l}}\}$. We can assume that $j < m_{\mathrm{l}}$, since otherwise we can let $e_i := e_n = 1 \in \mathrm{C_l}(L)$. Due to (4.8), it suffices to find an $e_i \in \mathrm{C_l}(L)$ such that $\mathrm{lsp}_R(e_i) = c_j$. If $c_j \in L$, then $c_j \in \mathrm{C_l}(R)$ implies $c_j \in \mathrm{C_l}(L)$, and we have that $c_j = \mathrm{lsp}_R(e_i)$ with $e_i := c_j$. Hence, we can assume that $c_j \notin L$. Consider the trajectory T that contains $\mathfrak{p}_0 = [x_0, y_0] := [c_j, c_{j+1}]$. Let $\mathfrak{p}_0, \mathfrak{p}_1 = [x_1, y_1], \mathfrak{p}_2 = [x_2, y_2], \dots, \mathfrak{p}_s = [x_s, y_s]$ be the edges that constitute T in R, listed from left to right. Since $\mathfrak{p}_0$ lies on $\mathrm{C_l}(R)$ and $j < m_{\mathrm{l}}$, $y_0 = c_{j+1} \in C_{\mathrm{ll}}(R)$; see Definition 4.2(i). We conclude from $y_s \in \mathrm{C_r}(R)$ that y_s is on the right of $\mathrm{C_l}(L)$; in notation, $y_s \in \mathrm{RS}_R(\mathrm{C_l}(L))$. Since $y_0 \in \mathrm{C_l}(R)$, $y_0 \in \mathrm{LS}_R(\mathrm{C_r}(L))$. Thus, as opposed to y_s, $y_0 = c_{j+1} \notin \mathrm{RS}_R(\mathrm{C_l}(L))$, because otherwise it would belong to $\mathrm{RS}_R(\mathrm{C_l}(L)) \cap \mathrm{LS}_R(\mathrm{C_r}(L))$, which is L by Lemma 4.1. Therefore, there exists a unique integer $t \in \{1, \dots, s\}$ such that $y_0, \dots, y_{t-1}$ are strictly on the left of $\mathrm{C_l}(L)$ but $y_t \in \mathrm{RS}(\mathrm{C_l}(L))$. Since $y_0 = c_{j+1} \in \mathrm{Ji}(R)$ by (4.6), T departs in upwards direction and $y_0 \prec y_1$. None of $y_0, \dots, y_{t-1}$ belongs to L, so none of $y_0, \dots, y_t$ can have more than two lower covers by Definition 2.1(iii). Hence, none of $\mathfrak{p}_1, \dots, \mathfrak{p}_t$ is the top edge of a hat trajectory, and the section of T from $\mathfrak{p}_0$ to $\mathfrak{p}_t$ goes upwards. That is, T goes upwards at $\mathfrak{p}_0, \dots, \mathfrak{p}_t$. Thus, $y_0 \prec y_1 \prec \cdots \prec y_t$. Applying Lemma 3.1 to the maximal chain $\mathrm{C_l}(L)$ of R and to the elements $y_{t-1} \prec y_t$, we obtain that $y_t \in \mathrm{C_l}(L)$. Therefore, since $c_j < c_{j+1} = y_0 < y_t$ excludes that $y_t = 0$, y_t is of the form $y_t = e_{i+1}$ for some $i \in \{0, 1, \dots, n-1\}$. Observe that $y_{t-1} \notin L$, x_t, and

$e_i \in L$ are lower covers of y_t. However, by Definition 2.1(iii), y_t has at most two lower covers. This implies that $x_t = e_i$, that is, $\mathfrak{p}_t = [x_t, y_t] = [e_i, e_{i+1}]$. Since T is also the trajectory through $\mathfrak{p}_t$, Lemma 4.5 implies that $c_j = x_0 = \mathrm{lsp}_R(e_i)$. This proves (4.23).

Next, we claim that, for $x, y \in R$,

$$\begin{aligned}&\text{if } x \prec y,\ \mathrm{ljc}_R(x) < \mathrm{ljc}_R(y), \text{ and } \mathrm{rjc}_R(x) < \mathrm{rjc}_R(y), \text{ then}\\ &\text{there are } u, v \in R \text{ such that } u \prec y,\ v \prec y,\ u \mathrel{\lambda} x, \text{ and } x \mathrel{\lambda} v.\end{aligned} \tag{4.24}$$

In order to prove this, assume the first line of (4.24). We conclude from (4.9) that $\mathrm{lsp}_R(x) < \mathrm{lsp}_R(y)$ and $\mathrm{rsp}_R(x) < \mathrm{lsp}_R(y)$. We have that $c_{m_\mathrm{l}} \not\leq y$, because otherwise $c_{m_\mathrm{l}} \leq x$ by (4.10) and (4.11), and so $\mathrm{ljc}_R(x) = m_\mathrm{l} = \mathrm{ljc}_R(y)$ would contradict our assumption. Similarly, $d_{m_\mathrm{r}} \not\leq y$. First we show that $y \notin \mathrm{C_l}(R)$ and $y \notin \mathrm{C_r}(R)$. Suppose, for a contradiction, that $y \in \mathrm{C_l}(R)$. Then $y \in C_{\mathrm{ll}}(R)$ since $c_{m_\mathrm{l}} \not\leq y$. We know from (2.1), (4.6), and Definition 4.2(i) that

$$\text{for all } \langle i, j\rangle \in \{0, \dots, m_\mathrm{l}\} \times \{0, \dots, m_\mathrm{r}\}, \quad c_i \wedge d_j = 0. \tag{4.25}$$

In particular, $y \wedge d_j$ for all $j \in \{0, \dots, m_\mathrm{r}\}$. But $y \neq 0$, so $y \not\geq d_i$ for $i \in \{1, \dots, m_\mathrm{r}\}$, and we obtain that $\mathrm{rsp}_R(y) = 0$. This contradicts $\mathrm{rsp}_R(x) < \mathrm{rsp}_R(y)$, and we conclude that $y \notin \mathrm{C_l}(R)$. Similarly, $y \notin \mathrm{C_r}(R)$. We know from (4.1) that $[\mathrm{lsp}_R(y), y]$ is a chain. This chain is nontrivial, because $y \notin \mathrm{C_l}(R)$. Thus, we can pick a unique element u of this chain such that $\mathrm{lsp}_R(y) \leq u \prec y$. By (4.1), $\mathrm{lsp}_R(u) = \mathrm{lsp}_R(y) > \mathrm{lsp}_R(x)$. We claim that $\mathrm{rsp}_R(u) < \mathrm{rsp}_R(x)$. Suppose, for a contradiction, that $\mathrm{rsp}_R(u) \geq \mathrm{rsp}_R(x)$. Combining this inequality with $\mathrm{lsp}_R(u) > \mathrm{lsp}_R(x)$ and (4.16), we obtain that $x < u$. This is a contradiction since both x and u are lower covers of y. Hence, (4.17) applies and $u \mathrel{\lambda} x$. By left-right symmetry, y also has a lover cover $v \in R$ with $x \mathrel{\lambda} v$. This proves (4.24).

The next step is to show that, for $x \in L$,

$$\mathrm{lsp}_L(x) = \mathrm{lsp}_L\big(\mathrm{lsp}_L(x)\big) \quad \text{and} \quad \mathrm{lsp}_R(x) = \mathrm{lsp}_R\big(\mathrm{lsp}_L(x)\big). \tag{4.26}$$

The first equation is a consequence of (4.1). In order to prove the second, we can assume that $c_{m_\mathrm{l}} \not\leq x$, because otherwise $\mathrm{lsp}_R(x) = x = \mathrm{lsp}_L(x) = \mathrm{lsp}_R(\mathrm{lsp}_L(x))$ by (4.10) and (4.11). Let $u = \mathrm{lsp}_L(x)$, $v = \mathrm{lsp}_R(u)$, and $w = \mathrm{lsp}_R(x)$. Since $x \geq u \geq v \in \mathrm{C_l}(R)$, we have $v \leq w$. Applying Lemma 3.1 to $w \leq x$ and the maximal chain $\mathrm{C_l}(L)$, we obtain an element $t \in \mathrm{C_l}(L)$ such that $w \leq t \leq x$. By the definition of u, we have that $t \leq u$. By transitivity, $w \leq u$. Hence, the definition of v yields that $w \leq v$. Thus, $v = w$, proving (4.26).

Now, we are in the position to complete the proof of Lemma 4.6. Let $x \in L$. By left-right symmetry, it suffices to show that $\mathrm{ljc}_R(x) = \mathrm{ljc}_L(x)$. However, by (4.26), it is sufficient to show that

$$\text{for} \quad x = e_k \in \mathrm{C_l}(L), \quad \mathrm{ljc}_R(e_k) = \mathrm{ljc}_L(e_k). \tag{4.27}$$

First, we assume that $c_{m_\mathrm{l}} \not\leq e_k$. Let $t = \mathrm{ljc}_R(e_k)$; by (4.8), this means that $\mathrm{lsp}_R(e_k) = c_t$. Consider the chain $H := \mathrm{C_l}(L) \cap {\downarrow} e_k = \{e_k \succ e_{k-1} \succ \cdots \succ e_0 = 0\}$. When we walk down along this chain, at each step from e_i to e_{i-1}, (4.16) yields that at least one of the join-coordinates $\mathrm{ljc}_R(e_i)$ and $\mathrm{rjc}_R(e_i)$ decreases. By the definition of $\mathrm{ljc}_L(e_k)$, it suffices to show that $\mathrm{ljc}_R(e_i)$ decreases iff $e_i \in \mathrm{Ji}(L)$, and it can decrease by at most 1. Therefore, by (4.19), it suffices to show that, for $i \in \{1, \dots, k\}$,

$$\text{if } \mathrm{ljc}_R(e_i) > \mathrm{ljc}_R(e_{i-1}), \text{ then } \mathrm{ljc}_R(e_i) - \mathrm{ljc}_R(e_{i-1}) = 1, \text{ and} \tag{4.28}$$

$$\text{if } \mathrm{ljc}_R(e_i) > \mathrm{ljc}_R(e_{i-1}), \text{ then } \mathrm{rjc}_R(e_i) = \mathrm{rjc}_R(e_{i-1}). \tag{4.29}$$

Suppose, for a contradiction, that (4.29) fails. Since $\mathrm{rjc}_R(e_i) > \mathrm{rjc}_R(e_{i-1})$ by (4.16), (4.24) yields $u, v \in R$ such that $u \prec e_i$, $v \prec e_i$, $u \mathrel{\lambda} e_{i-1}$, and $e_{i-1} \mathrel{\lambda} v$. Since $e_{i-1} \in \mathrm{C_l}(L)$, u is strictly on the left of $\mathrm{C_l}(L)$, and so $u \notin L$. This contradicts Definition 2.1(iii), proving (4.29). The proof of (4.28) is even shorter. By (4.16), for each $e_j \in \mathrm{C_l}(L)$, either $e_j \geq e_i$ and $\mathrm{ljc}_R(e_j) \geq \mathrm{ljc}_R(e_i)$, or $e_j \leq e_{i-1}$ and $\mathrm{ljc}_R(e_j) \leq \mathrm{ljc}_R(e_{i-1})$. So if the gap $\mathrm{ljc}_R(e_i) - \mathrm{ljc}_R(e_{i-1}) > 1$, then (4.23) fails. Hence, (4.28) holds, and so does (4.27) if $c_{m_\mathrm{l}} \not< e_k$.

Second, we assume that $c_{m_\mathrm{l}} < e_k$. Let t be the smallest subscript such that $c_{m_\mathrm{l}} \leq e_t$; note that $0 < t \leq k$. Since c_{m_l}, e_t and e_{t-1} belongs to S, which is distributive by Lemma 4.3(i), so does $c_{m_\mathrm{l}} \wedge e_{t-1}$. By distributivity, $c_{m_\mathrm{l}} \wedge e_{t-1} \prec c_{m_\mathrm{l}}$. Hence (4.5) and (4.6) give that $c_{m_\mathrm{l}} \wedge e_{t-1} = c_{m_\mathrm{l}-1}$. So $c_{m_\mathrm{l}-1} \leq e_{t-1}$. By the definition of t, $c_{m_\mathrm{l}} \nleq e_{t-1}$. Hence, $\mathrm{ljc}_R(e_{t-1}) = m_\mathrm{l} - 1$. Since $c_{m_\mathrm{l}} \not< e_{t-1}$, (4.27) is applicable and we have that $\mathrm{ljc}_L(e_{t-1}) = \mathrm{ljc}_R(e_{t-1}) = m_\mathrm{l} - 1$. Hence, for the validity of (4.27) for e_k, we only have to show that $|\{e_t, \dots, e_k\} \cap \mathrm{Ji}(L)| = 1$. This will follow from the following observation:

$$e_t \in \mathrm{Ji}(L) \text{ but for all } s, \text{ if } t < s \leq k, \text{ then } e_s \notin \mathrm{Ji}(L). \tag{4.30}$$

To show that $e_t \in \mathrm{Ji}(L)$, we can assume that $e_t > c_{m_\mathrm{l}}$, because otherwise e_t belongs even to $\mathrm{Ji}(R)$ by (4.6) and (4.10). Hence, e_t has a lower cover y in $\uparrow c_{m_\mathrm{l}}$, which is a chain by (4.11). By the choice of t, $y \neq e_{t-1}$. We have that $y \parallel e_{t-1}$ since both are lower covers of e_t. Using (3.3) and Lemma 3.2, we obtain that $y \mathrel{\lambda} e_{t-1}$ and $y \notin \mathrm{RS}_R(\mathrm{C_l}(L))$. Since $y \notin L$ by (4.4), e_t has exactly two lower covers in R by Definition 2.1(iii). Exactly one of these lower covers, y and e_{t-1}, belongs to L. Thus, $e_t \in \mathrm{Ji}(L)$, as desired. Next, to prove the second half of (4.30), assume that $t < s \leq k$. We want to show that e_s is join-reducible in L. Since the join-reducibility of $1 = e_n$ in L follows promptly from (4.18), we can assume that $e_s \neq 1$. Since $c_{m_\mathrm{l}} \leq e_t < e_s$, we obtain from (4.6) that e_s is join-reducible in R. Let $z \in R$ be a lower cover of e_s such that $z \neq e_{s-1}$. Since $c_{m_\mathrm{l}} \leq e_t \leq e_{s-1}$, (4.11) gives that $e_{s-1} \in \mathrm{C_l}(R)$. By (3.3), $e_{s-1} \mathrel{\lambda} z$, so the left-right dual of Lemma 3.2(i) gives that $z \in \mathrm{RS}_R(\mathrm{C_l}(L))$. We claim that $z \in \mathrm{LS}_R(\mathrm{C_r}(L))$ and then, by (4.4), $z \in L$. Suppose, for a contradiction, that this is not the case and z is strictly on the right of $\mathrm{C_r}(L)$. Since $e_s \in \mathrm{C_l}(L)$ belongs to $\mathrm{LS}_R(\mathrm{C_r}(L))$ and $z \prec e_s$, Lemma 3.1 yields that $e_s \in \mathrm{C_r}(L)$. Hence, $e_s = 1$ by (4.18), but this possibility has previously been excluded. This shows that $z \in L$ is another lower cover of e_s. Therefore, e_s is join-reducible in L, as required. This proves (4.27) and Lemma 4.6. $\square$

Next, we still assume that (4.2) and (4.3) hold and L is slim. We define the following sets of coordinate pairs; the acronyms come from "Internal", "Left", "Right", and "All" Coordinate Pairs, respectively.

$$\begin{aligned}
\mathrm{ICP}_L(L) &:= \{\langle \mathrm{ljc}_L(x), \mathrm{rjc}_L(x)\rangle : x \in L\},\\
\mathrm{ICP}_R(L) &:= \{\langle \mathrm{ljc}_R(x), \mathrm{rjc}_R(x)\rangle : x \in L\},\\
\mathrm{LCP}_R(L) &:= \{\langle \mathrm{ljc}_R(x), \mathrm{rjc}_R(x)\rangle : x \in R \text{ is strictly on the left of } \mathrm{C_l}(L)\},\\
\mathrm{RCP}_R(L) &:= \{\langle \mathrm{ljc}_R(x), \mathrm{rjc}_R(x)\rangle : x \in R \text{ is strictly on the right of } \mathrm{C_r}(L)\},\\
\mathrm{ACP}_R(L) &:= \mathrm{ICP}_R(L) \cup \mathrm{LCP}_R(L) \cup \mathrm{RCP}_R(L).
\end{aligned}$$

For a simpler notation for these sets, see Remark 4.8 later. We know from (4.15) and (4.16) that

$$\text{these sets describe } R \text{ and, in an appropriate sense, its diagram.} \tag{4.31}$$

As an important step towards the uniqueness of R, the following lemma states that these sets do not depend on R. The following lemma would fail without assuming (4.3); for instance, it would fail if L is a chain.

Lemma 4.7. *Assume* (4.2), (4.3)*, and that L is slim. With the notation given in Definition 4.2 and $G := \{0, \dots, m_\mathrm{l}\} \times \{0, \dots, m_\mathrm{r}\}$, the following hold.*

$$m_\mathrm{l} = \max\{\mathrm{ljc}_L(x) : x \in L\}, \quad m_\mathrm{r} = \max\{\mathrm{rjc}_L(x) : x \in L\}, \tag{4.32}$$

$$\mathrm{ICP}_R(L) = \mathrm{ICP}_L(L), \tag{4.33}$$

$$\begin{aligned}\mathrm{LCP}_R(L) = \{\langle i,j\rangle : \langle i,j\rangle \in G \setminus \mathrm{ICP}_L(L) \textit{ and } \exists x \in \mathrm{C_l}(L)\\ \textit{such that } i > \mathrm{ljc}_L(x) \textit{ and } j = \mathrm{rjc}_L(x)\},\end{aligned} \tag{4.34}$$

$$\begin{aligned}\mathrm{RCP}_R(L) = \{\langle i,j\rangle : \langle i,j\rangle \in G \setminus \mathrm{ICP}_L(L) \textit{ and } \exists x \in \mathrm{C_r}(L)\\ \textit{such that } j > \mathrm{rjc}_L(x) \textit{ and } i = \mathrm{ljc}_L(x)\}.\end{aligned} \tag{4.35}$$

Also, $\mathrm{LCP}_R(L)$ *and* $\mathrm{RCP}_R(L)$ *are given in terms of* $\mathrm{ICP}_L(L)$ *as follows:*

$$\begin{aligned}\mathrm{LCP}_R(L) = \{\langle i,j\rangle \in G \setminus \mathrm{ICP}_L(L) : \exists i' \textit{ such that } \langle i',j\rangle \in \mathrm{ICP}_L(L),\\ i > i', \textit{ and for every } \langle i'',j''\rangle \in \mathrm{ICP}_L(L),\ i'' > i' \Rightarrow j'' \geq j\},\end{aligned} \tag{4.36}$$

$$\begin{aligned}\mathrm{RCP}_R(L) = \{\langle i,j\rangle \in G \setminus \mathrm{ICP}_L(L) : \exists j' \textit{ such that } \langle i,j'\rangle \in \mathrm{ICP}_L(L),\\ j > j', \textit{ and for every } \langle i'',j''\rangle \in \mathrm{ICP}_L(L),\ j'' > j' \Rightarrow i'' \geq i\}.\end{aligned} \tag{4.37}$$

The componentwise ordering, $\langle i_1, j_1\rangle \leq \langle i_2, j_2\rangle$ iff $i_1 \leq i_2$ and $j_1 \leq j_2$, turns $\mathrm{ACP}_R(L) = \mathrm{ICP}_R(L) \cup \mathrm{LCP}_R(L) \cup \mathrm{RCP}_R(L)$ *into a lattice, which depends only on the fixed diagram of L. Actually,*

$$\mathrm{ACP}_R(L) \textit{ only depends on } \mathrm{ICP}_L(L). \tag{4.38}$$

Furthermore, the "coordinatization maps" *$\gamma\colon L \to \mathrm{ICP}_L(L)$ defined by $x \mapsto \langle \mathrm{ljc}_L(x), \mathrm{rjc}_L(x)\rangle$ and $\delta\colon R \to \mathrm{ACP}_R(L)$ defined by $x \mapsto \langle \mathrm{ljc}_R(x), \mathrm{rjc}_R(x)\rangle$ are lattice isomorphisms.*

Proof. Using Lemma 4.6 and $1 = 1_R \in L$, we obtain that $\max\{\mathrm{ljc}_L(x) : x \in L\} = \mathrm{ljc}_L(1) = \mathrm{ljc}_R(1) = m_{\mathrm{l}}$. The other half of (4.32) follows similarly. (4.33) also follows from Lemma 4.6. In the rest of the proof, (4.33) and Lemma 4.6 allow us to write $\mathrm{ICP}_R(L)$, ljc_R and rjc_R instead of $\mathrm{ICP}_L(L)$, ljc_L and rjc_L, and vice versa, respectively, without further warning.

Assume that $\langle i, j\rangle \in \mathrm{LCP}_R(L)$, and that $\langle i, j\rangle = \langle \mathrm{ljc}_R(y), \mathrm{rjc}_R(y)\rangle$ for some $y \in R$ strictly on the left of $\mathrm{C}_{\mathrm{l}}(L)$. Applying Lemma 3.1 to $d_j = d_{\mathrm{rjc}_R(y)} \le y$ and $\mathrm{C}_{\mathrm{l}}(L)$, we obtain an element $x \in \mathrm{C}_{\mathrm{l}}(L) \cap [d_j, y]$. By (4.12), $\mathrm{rjc}_R(x) = \mathrm{rjc}_R(y) = j$. We know that $x \ne y$, because $x \in L$ but $y \notin L$. Hence, $x < y$, and (4.16) gives that $i = \mathrm{ljc}_R(y) > \mathrm{ljc}_R(x)$. This proves the "$\subseteq$" part of (4.34).

In order to prove the converse inclusion, assume that $x \in \mathrm{C}_{\mathrm{l}}(L)$, $\langle i, j\rangle \in G$, $\langle i, j\rangle \notin \mathrm{ICP}_R(L) = \mathrm{ICP}_L(L)$, $i > \mathrm{ljc}_R(x)$, and $j = \mathrm{rjc}_R(x)$. Let $k = \mathrm{ljc}_R(x)$. In the distributive lattice S from Lemma 4.3, let $y = c_i \vee x \in S$. Next, we show that

$$\mathrm{ljc}_R(y) = i \text{ and, if } i < m_{\mathrm{l}},\ \mathrm{lsp}_R(y) = c_i. \tag{4.39}$$

Since (4.39) is obvious if $i = m_{\mathrm{l}}$, we can assume that $i < m_{\mathrm{l}}$. Since $c_i \le y$, we obtain that $\mathrm{lsp}_R(y) \ge c_i$. Suppose, for a contradiction, that $\mathrm{lsp}_R(y) > c_i$. Then $c_{i+1} \le y$. Since c_{i+1} is join-irreducible in R by (4.6) and $c_{i+1} \ne 0_S = 0_R$, we have that $c_{i+1} \in \mathrm{Ji}(S)$. Using distributivity in the standard way as above (4.22) and taking $c_{i+1} \not\le c_i$ and $c_{i+1} \le y = c_i \vee x$ into account, we obtain that $c_{i+1} \le x$. By (4.9), $i + 1 \le \mathrm{ljc}_R(x) = k$. This contradicts $i > k$, proving the second equation in (4.39). The first equation in (4.39) follows from (4.8).

Observe that, for every $z \in R$,

$$\text{the intervals } [c_{\mathrm{ljc}_R(z)}, z] \text{ and } [d_{\mathrm{rjc}_R(z)}, z] \text{ are chains.} \tag{4.40}$$

If $\mathrm{ljc}_R(z) = m_{\mathrm{l}}$, then $z \ge c_{m_{\mathrm{l}}} = \mathrm{lc}(R)$, and the first interval is a chain by (4.11). If $\mathrm{ljc}_R(z) \ne m_{\mathrm{l}}$, then $z \not\ge c_{m_{\mathrm{l}}}$, and the first interval is a chain by (4.1) and (4.8). Similarly, the second interval is also a chain, proving (4.40).

Next, we prove that

$$\mathrm{rjc}_R(y) = j. \tag{4.41}$$

If $\mathrm{rjc}_R(x) = j = m_{\mathrm{r}}$, then $y \ge x \ge d_{m_{\mathrm{r}}}$ yields that $j = m_{\mathrm{r}} = \mathrm{rjc}_R(y)$ as required. Hence, we can assume that $j < m_{\mathrm{r}}$. From $y \ge x$ and (4.16), we obtain that $\mathrm{rjc}_R(y) \ge j$. Suppose, for a contradiction, that $t := \mathrm{rjc}_R(y) > j$. By (4.40), we can let $[d_t, y] = \{y_0 := y \succ y_1 \succ \cdots \succ y_s = d_t\}$. Since $y = y_0 \in S$, there is a largest $q \in \{0, \dots, s\}$ such that $\{y_0, \dots, y_q\} \subseteq S$. The situation is roughly visualized in Figure 5, where only a part of R is depicted and the black-filled elements belong to $\mathrm{C}_{\mathrm{l}}(L)$. (Note, however, that a targeted contradiction cannot be satisfactorily depicted.)

We claim that $q < s$. Suppose, for a new contradiction, that $q = s$. We know that $0 < d_t$, since $j < t$. Since $d_t = 1$ would imply by (4.25) that $m_{\mathrm{l}} = 0$ and $\mathrm{lc}(R) = c_{m_{\mathrm{l}}} = 0$, which would contradict the rectangularity of R, we conclude that $d_t \notin \{0, 1\}$. Hence, (4.18) yields that $d_t \notin \mathrm{C}_{\mathrm{r}}(L)$. Since $\mathrm{C}_{\mathrm{r}}(L)$ is a maximal chain, we can pick an element $u \in \mathrm{C}_{\mathrm{r}}(L)$ such that $d_t \parallel u$. Using

$d_t = y_s = y_q \in \mathrm{C_l}(L) \subseteq L \subseteq \mathrm{LS}(\mathrm{C_r}(L))$ and Lemma 3.2, we have that $d_t \mathrel{\lambda} u$. This is a contradiction, because $u \mathrel{\lambda} d_t$ by (3.3). Thus, $q < s$.

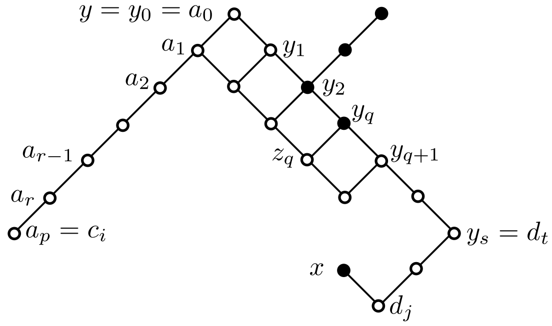

FIGURE 5. If (4.41) fails

Note that $q < s$ implies that $s \geq 1$ and that y_1 and y_{q+1} will make sense later. Since $\mathrm{ljc}_R(y) = i$ by (4.39), (4.40) yields that $[c_i, y]$ is a chain. Since $y \geq d_t > 0$ but, by (4.25), $c_i \not\geq d_t$, we obtain that $[c_i, y]$ is a nontrivial chain. Denote its element as follows:

$$[c_i, y] = \{y = a_0 \succ a_1 \succ \dots \succ a_p = c_i\},$$

where $p \geq 1$. By (4.12), $\mathrm{ljc}_R(a_1) = \mathrm{ljc}_R(y) = i$. By (4.7), $y = c_i \vee d_t$. Since $y = c_i \vee d_t \leq a_1 \vee y_1 \leq y$, we obtain that $a_1 \neq y_1$. Thus, as two distinct lower covers of y, a_1 and y_1 are incomparable. Observe that $c_i \leq y_1$ is impossible because otherwise $y = c_i \vee d_t \leq y_1 < y$. Hence, $\mathrm{ljc}_R(y_1) < i = \mathrm{ljc}_R(a_1)$. Combining this inequality with $a_1 \parallel y_1$ and (4.15) and using Lemma 3.2(ii), we obtain that $a_1 \mathrel{\lambda} y_1$.

Next, we assert that $a_1 \in S$. Suppose, for a new contradiction, that $a_1 \notin S$. This means that a_1 is strictly on the right of $\mathrm{C_l}(L)$. Since $a_1 \prec y$, Lemma 3.1 excludes that y is strictly on the left of $\mathrm{C_l}(L)$. However, $y \in S$ is on the left of $\mathrm{C_l}(L)$, so $y \in \mathrm{C_l}(L)$. We know that $a_p = c_i \in \mathrm{C_l}(R)$ belongs to S, whence there exists a smallest $r \in \{2, \dots, r\}$ such that $\{a_p, a_{p-1}, \dots, a_r\} \subseteq S$. Since a_{r-1} is not in S, it strictly is on the right of $\mathrm{C_l}(L)$. But a_r is on the left of $\mathrm{C_l}(L)$ and $a_r \prec a_{r-1}$. Lemma 3.1 implies easily that $a_r \in \mathrm{C_l}(L)$. The interval $[a_r, a_0]$ is a chain since so is $[a_p, a_0] = [c_i, y]$ by (4.40). Since $a_r, a_0 \in \mathrm{C_l}(L)$ and $\mathrm{C_l}(L) \cap [a_r, a_0]$ is a maximal chain in the interval $[a_r, a_0] = \{a_r \prec \dots \prec a_1 \prec a_0\}$, it follows that $\{a_r, \dots, a_1, a_0\} \subseteq \mathrm{C_l}(L) \subseteq S$. This contradicts $a_1 \notin S$. Thus, $a_1 \in S$.

Since S is a sublattice by Lemma 4.3, $z_q := a_1 \wedge y_q \in S$. The distributivity of S, see Lemma 4.3, yields that $z_q \preceq y_q$. If $z_q = y_q$, then we have $y > a_1 = a_1 \vee z_q = a_1 \vee y_q \geq c_i \vee d_t$, which is a contradiction since $y = c_i \vee d_t$ by (4.7). Hence, $z_q \prec y_q$. We also know that $y_{q+1} \prec y_q$. By the choice of q, $y_{q+1} \notin S$, so y_{q+1} is strictly on the right of $\mathrm{C_l}(L)$. But the element $y_q \in S$ is on the left of $\mathrm{C_l}(L)$, and we conclude from Lemma 3.1 easily again that $y_q \in \mathrm{C_l}(L)$. Since $q < s$ and

$y_q > y_s$, we know that $y_q \neq 0$. Therefore, $\mathrm{C_l}(L)$ contains a unique element e such that $e \prec y_q$. Since $y_{q+1} \notin S$, we obtain that $e \neq y_{q+1} \neq z_q$. Suppose, for a new contradiction, that $e = z_q$. Since x and $z_q = e$ both belong to $\mathrm{C_l}(L)$, they are comparable. If $x > z_q$, then $x \geq y_q \in \mathrm{C_l}(L)$, so (4.12) and (4.16) imply that $j = \mathrm{rjc}_R(x) \geq \mathrm{rjc}_R(y_q) = \mathrm{rjc}_R(y) = t$, which is a contradiction excluding that $e > z_q$. If $x \leq z_q$, then $y = c_i \vee x \leq a_1 \vee z_q = a_1 \prec y$ is a contradiction again. Thus, $e \neq z_q$. Consequently, the set $\{z_q, e, y_{q+1}\}$, which consists of distinct lower covers of y_q, is a three-element antichain. Hence, as opposed to e, z_q does not belong to the chain $\mathrm{C_l}(L)$. If $z_q \mathrel{\lambda} e$, then z_q is strictly on the left of $\mathrm{C_l}(L)$ by Lemma 3.2, so $z_q \notin L$, which contradicts Definition 2.1(iii). Hence, by Lemma 3.2(ii), $e \mathrel{\lambda} z_q$. However, then z_q is strictly on the right of $\mathrm{C_l}(L)$ by the left-right dual of Lemma 3.2(i), which contradicts $z_q \in S$. That is, $e \neq z_q$ also leads to a contradiction. This proves (4.41).

It follows from (4.39) and (4.41) that $\langle \mathrm{ljc}_R(y), \mathrm{rjc}_R(y)\rangle = \langle i, j\rangle \notin \mathrm{ICP}_L(L)$. Hence, $y \notin L$, which gives that $y \notin \mathrm{C_l}(L)$. Combining this with $y \in S$, we obtain that y is strictly on the left of $\mathrm{C_l}(L)$. Thus, $\langle i, j\rangle = \langle \mathrm{ljc}_R(y), \mathrm{rjc}_R(y)\rangle \in \mathrm{LCP}_R(L)$. This implies the "$\supseteq$" part of (4.34). Thus, (4.34) holds, and so does (4.35) by left-right symmetry.

Next, we deal with (4.36). The pair $\langle i', j\rangle$ in (4.36) corresponds to the coordinate pair $\langle \mathrm{ljc}_L(x), \mathrm{rjc}_L(x)\rangle$ for some element $x \in L$. By (4.15), the condition that for every $\langle i'', j''\rangle \in \mathrm{ICP}_L(L)$, $i'' > i' \Rightarrow j'' \geq j$ says that no element of L is to the left of x, that is, this x belongs to $\mathrm{C_l}(L)$. Therefore, the right-hand side of the equation in (4.36) is the same as that in (4.34). Hence, (4.36) follows from (4.34). Similarly, (4.35) implies (4.37). In this way, we have proved the equations (4.32)–(4.37).

It follows from (4.32), (4.33), (4.36), and (4.37) that $\mathrm{ACP}_R(L)$ depends only on the fixed diagram of L, and it only depends on $\mathrm{ICP}_L(L)$. Thus, (4.38) holds. Clearly, $\mathrm{ACP}_L(L)$ and $\mathrm{ICP}_L(L)$ are ordered sets. It follows from (4.16) that γ and δ are isomorphisms. Hence, $\mathrm{ACP}_L(L)$ and $\mathrm{ICP}_L(L)$ are lattices and γ and δ are lattice isomorphisms. This completes the proof of Lemma 4.7. $\square$

Remark 4.8. By Lemma 4.7, the sets of coordinate pairs defined before the lemma depend only on the fixed diagram of L. Hence, we can also use the following notation:

$$\begin{aligned}
&\mathrm{ICP}(L) := \mathrm{ICP}_L(L) = \mathrm{ICP}_R(L),\\
&\mathrm{LCP}(L) := \mathrm{LCP}_R(L), \quad \mathrm{RCP}(L) := \mathrm{RCP}_R(L),\\
&\mathrm{ACP}(L) := \mathrm{ACP}_R(L).
\end{aligned}$$

Let L be a planar semimodular lattice. According to G. Grätzer and E. Knapp [19], a *full slimming* (sublattice) L' of L is obtained from a planar diagram of L by omitting all elements from the interiors of intervals of length 2 as long as there are elements to omit in this way. Note that L', as a sublattice of L, is not unique; this is witnessed by $L = M_3$. However, the full slimming sublattice becomes unique if the planar diagram of L is fixed. In [19], the elements we omit are called "eyes". Note that L' is a slim semimodular lattice. Note also that when we omit an eye from the lattice, then we also omit this eye (which is a doubly irreducible element) from the diagram with the two edges from the eye. The converse procedure, when we put the omitted elements back, is called an *anti-slimming.* An element $x \in L$ is *reducible* if it is join-reducible or meet-reducible, that is, if it is not doubly irreducible. It follows obviously from the slimming procedure that if L' is a full slimming sublattice of L, then

$$L' \text{ contains every reducible element of } L. \tag{4.42}$$

Although we know from G. Czédli and E. T. Schmidt [15, Lemma 4.1] that L determines L' up to isomorphisms, we need a stronger statement here. By G. Grätzer and E. Knapp [19, Lemma 8], an element in a slim semimodular lattice can have at most two covers. Therefore, every 4-cell can be described by its bottom element. To capture the situation that L' is a full slimming (sublattice) of a planar semimodular lattice L, we define *the numerical companion map* $f^{\mathrm{nc}} = f^{\mathrm{nc}}_{L'\subseteq L}$ associated with the full slimming sublattice L' as follows. It is the map $f^{\mathrm{nc}}_{L'\subseteq L}\colon L' \to \mathbb{N}_0 := \{0, 1, 2, \dots\}$ defined by

$$f^{\mathrm{nc}}_{L'\subseteq L}(x) = \begin{cases} n, & \text{if } x \text{ is the bottom of a 4-cell that has } n \text{ eyes in } L, \\ 0, & \text{otherwise.} \end{cases} \tag{4.43}$$

Let L'_i be a full slimming sublattice of a planar semimodular lattice L_i, for $i \in \{1, 2\}$, and let $\varphi\colon L'_1 \to L'_2$ be an isomorphism. We say that φ is an f^{nc}*-preserving isomorphism* if $f^{\mathrm{nc}}_{L'_1\subseteq L_1} = f^{\mathrm{nc}}_{L'_2\subseteq L_2} \circ \varphi$. (We compose maps from right to left.) The map $f^{\mathrm{nc}}_{L'\subseteq L}$ exactly describes how to get L back from L' by anti-slimming. Hence, obviously,

$$\begin{array}{l}\text{every } f^{\mathrm{nc}}\text{-preserving } L'_1 \to L'_2 \text{ isomorphism} \\ \text{extends to an } L_1 \to L_2 \text{ isomorphism.}\end{array} \tag{4.44}$$

The restriction of a map κ to a set A is denoted by $\kappa\rceil_A$.

Lemma 4.9. *For $i \in \{1, 2\}$, let L'_i be a full slimming sublattice of a planar semimodular lattice L_i.*

(i) *L_1 is glued sum indecomposable iff so is L'_1.*
(ii) *L_1 is rectangular iff so is L'_1. (This is Lemma 6.1(ii) in G. Czédli and E. T. Schmidt* [15].)
(iii) *If $\varphi\colon L_1 \to L_2$ is an isomorphism, then there exists an automorphism π of L_1 such that the restriction $(\varphi \circ \pi)\rceil_{L'_1}$ is a f^{nc}-preserving $L'_1 \to L'_2$ isomorphism and, in addition, $\pi(x) = x$ for every reducible $x \in L_1$.*
(iv) *Any two full slimming sublattices of a planar semimodular lattice are isomorphic.*

Proof. In order to prove part (i), assume that L_1 is glued sum indecomposable and that $x \in L'_1 \setminus \{0, 1\}$. There is an element $y \in L_1$ such that $x \parallel y$. We can assume that $y \notin L'_1$, since otherwise there is nothing to do. Then y is an "eye", so there are $a, b \in L_1'$ such that $\{a \wedge b, a, b, a \vee b\}$ is a covering square in L_1 and $y \in [a \wedge b, a \vee b]$ is to the right of a and to the left of b. If $x \nparallel a$ and $x \nparallel b$, then either $x \le a \wedge b \le y$, or $x \ge a \vee b \ge y$, because the rest of cases would contradict $a \parallel b$. But this contradicts $x \parallel y$, proving that L_1' is slim. The converse direction is trivial, because if L_1' is glued sum indecomposable, then its elements outside $\{0, 1\}$ are incomparable with appropriate elements of L_1' while the eyes are incomparable with the corners of the covering squares they were removed from in order to obtain L_1'. This proves (i).

Part (ii) has already been proved in G. Czédli and E. T. Schmidt [15].

We assume that, for $i \in \{1, 2\}$, a planar diagram of L_i is fixed and that we form the full slimming sublattice L_i' according to this diagram. We prove (iii) by induction on $|L_1|$. If L_1 is slim, then the statement is trivial, because $L_1' = L_1$, $L_2' = L_2$, π is the identity map on L_1, and both numerical companion maps are the constant zero maps. Assume that L_1 is not slim. Then there are $a_1 < b_1 \in L_1$ with images $a_2 = \varphi(a_1)$ and $b_2 = \varphi(b_1)$ such that, for $i \in \{1, 2\}$, $[a_i, b_i]$ is an interval of length two and it contains a doubly irreducible element x_i in its interior such that $x_i \notin L_i'$. Let $y_1 = \varphi^{-1}(x_2)$; it is a doubly irreducible element of L_1 in $[a_1, b_1]$. Clearly, there is an automorphism π_0 of L_1 such that $\pi_0(x_1) = y_1$, $\pi_0(y_1) = x_1$, and $\pi_0(z) = z$ for $z \notin \{x_1, y_1\}$. As we require in case of our automorphisms, every reducible element is a fixed point of π_0.

Observe that $(\varphi \circ \pi_0)(x_1) = \varphi(\pi_0(x_1)) = \varphi(y_1) = x_2$. Since x_i is doubly irreducible, $L_i^* := L_i \setminus \{x_i\}$ is a sublattice of L_i and $\varphi* := (\varphi \circ \pi_0)\rceil_{L_1^*}$ is an $L_1^* \to L_2^*$ isomorphism. Note that L_i' is also a full slimming sublattice of L_i^*. By the induction hypothesis, L_1^* has an automorphism π^* such that $(\varphi^* \circ \pi^*)\rceil_{L_1'}$ is an f^{nc}-preserving $L_1' \to L_2'$ isomorphism and, in addition, $\pi^*(z) = z$ for every reducible element z of L_1^*. In particular,

$$f^{\mathrm{nc}}_{L_2' \subseteq L_2^*} \circ ((\varphi^* \circ \pi^*)\rceil_{L_1'}) = f^{\mathrm{nc}}_{L_1' \subseteq L_1^*}. \tag{4.45}$$

Let $\pi^\bullet : L_1 \to L_1$ be the only automorphism that extends π^*. That is, $\pi^\bullet(x_1) = x_1$ and, for $z \ne x_1$, $\pi^\bullet(z) = \pi^*(z)$. We define $\pi := \pi_0 \circ \pi^\bullet$, and we claim that it has the properties required in Lemma 4.9(iii). If z is a reducible element of L_1, then $z \notin \{x_1, y_1\}$, since x_1 and y_1 are doubly irreducible. Hence, $z \in L_1^*$. Furthermore, z is also reducible in L_1^*, because it is only a_1 and $b_1 \in L_1$ that loose one of their upper and lower covers, respectively, when passing from L_1 to L_1^*, but they still have at least two upper and lower covers, respectively, in L_1^*. Hence, z is a fixed point of π^* by the induction hypothesis, and we obtain that $\pi(z) = (\pi_0 \circ \pi^\bullet)(z) = \pi_0(\pi^\bullet(z)) = \pi_0(\pi^*(z)) = \pi_0(z) = z$, as Lemma 4.9(iii) requires. Next, we show that

$$(\varphi \circ \pi)\rceil_{L_1'} = (\varphi^* \circ \pi^*)\rceil_{L_1'}. \tag{4.46}$$

Let $z \in L_1'$. Since $x_1 \notin L_1'$, $z \ne x_1$. We compute as follows.

$$\begin{aligned}(\varphi \circ \pi)(z) &= (\varphi \circ \pi_0 \circ \pi^\bullet)(z) = (\varphi \circ \pi_0)(\pi^\bullet(z)) = (\varphi \circ \pi_0)(\pi^*(z)) \\ &= (\varphi \circ \pi_0)\rceil_{L_1^*}(\pi^*(z)) = \varphi^*(\pi^*(z)) = (\varphi^* \circ \pi^*)(z).\end{aligned}$$

This proves (4.46). In particular, this also gives that $(\varphi\circ\pi)\rceil_{L_1'}$ is an isomorphism from L_1' to L_2'. We have to prove that it is $f^{\rm nc}$-preserving, that is,

$$f^{\rm nc}_{L_2'\subseteq L_2}\circ((\varphi\circ\pi)\rceil_{L_1'})\overset{?}{=}f^{\rm nc}_{L_1'\subseteq L_1}. \tag{4.47}$$

Before proving (4.47), observe that, for $z\in L_i'$ and $i\in\{1,2\}$,

$$f^{\rm nc}_{L_i'\subseteq L_i}(z)=\begin{cases}f^{\rm nc}_{L_i'\subseteq L_i^*}(z), & \text{if } z\neq a_i.\\ 1+f^{\rm nc}_{L_i'\subseteq L_i^*}(z), & \text{if } z=a_i.\end{cases} \tag{4.48}$$

Hence $z=a_1$, which is in L_1' by (4.42), and $z\in L_1'\setminus\{a_1\}$ need separate treatments. First, since a_1 is reducible and π^*, π_0, and π keep it fixed,

$$((\varphi\circ\pi)\rceil_{L_1'})(a_1)=(\varphi\circ\pi)(a_1)=\varphi(\pi(a_1))=\varphi(a_1)=a_2, \tag{4.49}$$

$$((\varphi^*\circ\pi^*)\rceil_{L_1'})(a_1)\overset{(4.46)}{=}((\varphi\circ\pi)\rceil_{L_1'})(a_1)\overset{(4.49)}{=}a_2. \tag{4.50}$$

Hence, we can compute as follows.

$$\begin{aligned}\big(f^{\rm nc}_{L_2'\subseteq L_2}\circ((\varphi\circ\pi)\rceil_{L_1'})\big)(a_1)&=f^{\rm nc}_{L_2'\subseteq L_2}\big(((\varphi\circ\pi)\rceil_{L_1'})(a_1)\big)\overset{(4.49)}{=}f^{\rm nc}_{L_2'\subseteq L_2}(a_2)\\ &\overset{(4.48)}{=}1+f^{\rm nc}_{L_2'\subseteq L_2^*}(a_2)\overset{(4.50)}{=}1+f^{\rm nc}_{L_2'\subseteq L_2^*}\big(((\varphi^*\circ\pi^*)\rceil_{L_1'})(a_1)\big)\\ &\overset{(4.45)}{=}1+f^{\rm nc}_{L_1'\subseteq L_1^*}(a_1)\overset{(4.48)}{=}f^{\rm nc}_{L_1'\subseteq L_1}(a_1).\end{aligned}$$

This shows that (4.47) holds for the element a_1. Second, assume that $z\in L_1'\setminus\{a_1\}$. Since the map in (4.50) is a bijection, $((\varphi^*\circ\pi^*)\rceil_{L_1'})(z)\neq a_2$, and we can compute as follows.

$$\begin{aligned}\big(f^{\rm nc}_{L_2'\subseteq L_2}\circ((\varphi\circ\pi)\rceil_{L_1'})\big)(z)&\overset{(4.46)}{=}f^{\rm nc}_{L_2'\subseteq L_2}\big(((\varphi^*\circ\pi^*)\rceil_{L_1'})(z)\big)\\ &\overset{(4.48)}{=}f^{\rm nc}_{L_2'\subseteq L_2^*}\big(((\varphi^*\circ\pi^*)\rceil_{L_1'})(z)\big)\overset{(4.45)}{=}f^{\rm nc}_{L_1'\subseteq L_1^*}(z)\overset{(4.48)}{=}f^{\rm nc}_{L_1'\subseteq L_1}(z).\end{aligned}$$

Therefore, (4.47) holds and $(\varphi\circ\pi)\rceil_{L_1'}$ is $f^{\rm nc}$-preserving. This completes the proof of Lemma 4.9. □

Definition 4.10 (D. Kelly and I. Rival [24, p. 640])**.** For planar lattice diagrams D_1 and D_2, a bijection $\varphi\colon D_1\to D_2$ is a *similarity map* if it is a lattice isomorphism and, for all $x,y,z\in D_1$ with $y\prec x$ and $z\prec x$, y is to the left of z iff $\varphi(y)$ is to the left of $\varphi(z)$. If there is such a map, then D_1 is *similar* to D_2.

Note that similarity turns out to be a self-dual condition; see G. Czédli and G. Grätzer [10, Exercise 3.9]. Furthermore, if D_1 and D_2 are planar diagrams of slim (but not necessarily semimodular) lattices and a bijective map $\varphi\colon D_1\to D_2$ is a lattice isomorphism, then

$$\varphi \text{ is a similarity map iff it preserves the left boundary chain,} \tag{4.51}$$

that is, $\varphi(\mathrm{C_l}(D_1))=\mathrm{C_l}(D_2)$; see [10, Theorem 3-4.6]. A map between two lattices can be considered as a map between (the vertex sets) of their diagrams. For a diagram D, its mirror image across a vertical axis is denoted by $D^{(\rm mi)}$. We say that the planar diagrams of a planar lattice L are unique up to *left-right similarity* if for any two diagrams D_1 and D_2 of L, D_2 is similar to D_1

or it is similar to $D_1^{(\mathrm{mi})}$. For a statement stronger than the following one, see G. Czédli and G. Grätzer [10, Theorem 3-4.5].

Proposition 4.11 (G. Czédli and E. T. Schmidt [15, Lemma 4.7])**.** *Assume that L_1 and L_2 are glued sum indecomposable slim semimodular lattices with planar diagrams D_1 and D_2, respectively. If $\varphi\colon L_1 \to L_2$ is a lattice isomorphism, then $\varphi\colon D_1 \to D_2$ or $\varphi\colon D_1 \to D_2^{(\mathrm{mi})}$ is a similarity map. Consequently, the planar diagram of a glued sum indecomposable slim semimodular lattice is unique up to left-right similarity.*

Now, we are in the position to complete this section briefly.

Proof of Theorem 2.2. Let L be a planar semimodular lattice. The existence of a normal rectangular extension R of L follows from G. Grätzer and E. Knapp [20, Proof of Theorem 7], and it also follows from G. Czédli [3, Lemma 6.4]. Thus, part (i) of the theorem holds.

In order to prove part (ii), let R be an arbitrary normal rectangular extension of L. Based on (4.13), it suffices to show that R has a cover-preserving diamond sublattice M_3 iff so has L. The "if" part is evident since L is a cover-preserving sublattice of R. Conversely, assume that M_3 is a cover-preserving sublattice of R. It follows from Definition 2.1(iii) that none of its three atoms is in $R \setminus L$. Hence, all atoms of M_3 belong to L. Since M_3 is generated by its atoms, M_3 is a cover-preserving sublattice of L. This proves part (ii).

In order to prove (v), assume that L and L' are glued sum indecomposable slim planar semimodular lattices with fixed planar diagrams and that $\psi\colon L \to L'$ is an isomorphism. Also, we assume that R and R' are normal rectangular extensions of L and L', respectively. By reflecting one of the diagrams over a vertical axis if necessary, Proposition 4.11 allows us to assume that ψ is a similarity map between the respective diagrams. Hence, $\psi(\mathrm{C_l}(L)) = \mathrm{C_l}(L')$ and $\psi(\mathrm{C_r}(L)) = \mathrm{C_r}(L')$. Also, $m_\mathrm{l} = m'_\mathrm{l}$ and $m_\mathrm{r} = m'_\mathrm{r}$; see Definition 4.2(i). We know from (the last sentence of) Lemma 4.7 that $\mathrm{ICP}(L)$ and $\mathrm{ICP}(L')$ are lattices with respect to the componentwise ordering; see also Remark 4.8 for the notation. The same lemma says that $\gamma\colon L \to \mathrm{ICP}(L)$ is a lattice isomorphism, and so is $\gamma'\colon L' \to \mathrm{ICP}(L')$, defined analogously by $x \mapsto \langle \mathrm{ljc}_{L'}(x), \mathrm{rjc}_{L'}(x)\rangle$. Since ψ is a similarity map, it preserves the left boundary chain and the right boundary chain. So we obtain from Definition 4.2(ii) that ψ preserves the left and right join-coordinates. Thus, for $x \in L$, $\gamma(x) = \gamma'(\psi(x))$, that is, $\gamma = \gamma' \circ \psi$, and we also conclude that $\mathrm{ICP}(L) = \mathrm{ICP}(L')$. Hence, (4.38) yields that $\mathrm{ACP}(L) = \mathrm{ACP}(L')$. Since γ, γ', and ψ are isomorphisms, the equality $\gamma = \gamma' \circ \psi$ implies that $\psi = \gamma'^{-1} \circ \gamma$. Consider the isomorphism $\delta\colon R \to \mathrm{ACP}(L)$ from Lemma 4.7 and the isomorphism $\delta'\colon R' \to \mathrm{ACP}(L')$ defined by $x \mapsto \langle \mathrm{ljc}_{R'}(x), \mathrm{rjc}_{R'}(x)\rangle$ analogously. It follows from Lemma 4.6 that δ and δ' extend γ and γ', respectively. Since δ' extends γ' and they are bijections, δ'^{-1} extends γ'^{-1}. The equality $\mathrm{ACP}(L) = \mathrm{ACP}(L')$ allows us to define a lattice isomorphism $\psi^*\colon R \to R'$ by $\psi^* := \delta'^{-1} \circ \delta$. Since δ and δ'^{-1}

extend γ and γ'^{-1}, we conclude that ψ^* extends $\gamma'^{-1} \circ \gamma$, which is ψ. This proves part (v) of the theorem.

Part (iv) follows from part (v) trivially.

Finally, in order to prove (iii), let L be a glued sum indecomposable planar semimodular lattice, and let R_1 and R_2 be normal rectangular extensions of L. Let R'_1, and R'_2 denote their full slimmings, respectively (with respect to their fixed planar diagrams, of course). These full slimmings are rectangular lattices by Lemma 4.9(ii). When we delete all eyes, one by one, from R_i to obtain R'_i, we also delete all eyes from its cover-preserving sublattice, L. So, this sublattice changes to a full slimming sublattice L'_i of L, for $i \in \{1, 2\}$. Since the deletion of eyes does not spoil the validity of Definition 2.1(iii), we conclude that R'_i is a normal rectangular extension of L'_i, for $i \in \{1, 2\}$.

Applying Lemma 4.9(iii) to the identity map $L \to L$, we obtain an automorphism π of L such that $\pi\rceil_{L'_1}$ is a f^{nc}-preserving isomorphism $\pi\rceil_{L'_1} : L'_1 \to L'_2$. Thus, $f^{\mathrm{nc}}_{L'_1 \subseteq L} = f^{\mathrm{nc}}_{L'_2 \subseteq L} \circ \pi\rceil_{L'_1}$. If $B \cong M_3$ is a cover-preserving diamond sublattice of R_i, then all the three coatoms of B belong to L by Definition 2.1(iii), and so do all elements of B, including 0_B. Hence, by the definition of antislimming, if $B \cong M_3$ is a cover-preserving diamond sublattice of R_i, then $0_B \in L'_i$. These facts imply that, for $x \in R'_i \setminus L'_i$ and $y \in L'_i$, we have that $f^{\mathrm{nc}}_{R'_i \subseteq R_i}(x) = 0$ and $f^{\mathrm{nc}}_{R'_i \subseteq R_i}(y) = f^{\mathrm{nc}}_{L'_i \subseteq L}(y)$. By the already proved part (v) of Theorem 2.2, $\pi\rceil_{L'_1}$ extends to an isomorphism $\varphi\colon R'_1 \to R'_2$. For $x \in R'_1 \setminus L'_1$, we have that $\varphi(x) \in R'_2 \setminus L'_2$, and the already established facts imply that

$$(f^{\mathrm{nc}}_{R'_2 \subseteq R_2} \circ \varphi)(x) = f^{\mathrm{nc}}_{R'_2 \subseteq R_2}(\varphi(x)) = 0 = f^{\mathrm{nc}}_{R'_1 \subseteq R_1}(x).$$

On the other hand, for $y \in L'_1$, we have that

$$\begin{aligned}(f^{\mathrm{nc}}_{R'_2 \subseteq R_2} \circ \varphi)(y) &= f^{\mathrm{nc}}_{R'_2 \subseteq R_2}(\varphi(y)) = f^{\mathrm{nc}}_{R'_2 \subseteq R_2}(\pi\rceil_{L'_1}(y)) = f^{\mathrm{nc}}_{L'_2 \subseteq L}(\pi\rceil_{L'_1}(y)) \\ &= (f^{\mathrm{nc}}_{L'_2 \subseteq L} \circ \pi\rceil_{L'_1})(y) = f^{\mathrm{nc}}_{L'_1 \subseteq L}(y) = f^{\mathrm{nc}}_{R'_1 \subseteq R_1}(y).\end{aligned}$$

The two displayed equations show that $f^{\mathrm{nc}}_{R'_2 \subseteq R_2} \circ \varphi = f^{\mathrm{nc}}_{R'_1 \subseteq R_1}$, which means that $\varphi\colon R'_1 \to R'_2$ is a f^{nc}-preserving isomorphism. By (4.44), it extends to an $R_1 \to R_2$ isomorphism. Consequently, the normal rectangular extension of L is unique up to isomorphism, which completes the proof of Theorem 2.2. $\square$

5. A hierarchy of planar semimodular lattice diagrams

Our experience with planar semimodular lattices makes it reasonable to develop a hierarchy of diagram classes for planar semimodular lattices. In this section, we do so. Several properties of our diagrams and their trajectories will be studied at various levels of this hierarchy. In particular, we are interested in what sense our diagrams are unique. The power of this approach is demonstrated in Section 8, where we give a proof of G. Grätzer's Swing Lemma. Let us repeat that, unless otherwise explicitly stated, our lattices are still assumed to be finite planar semimodular lattices and the diagrams are planar diagrams of these lattices. We are going to define diagram classes $\mathcal{C}_0$, $\mathcal{C}_1$, $\mathcal{C}_2$, and $\mathcal{C}_3$;

they form a "hierarchy" because of the inclusions $\mathcal{C}_0 \supset \mathcal{C}_1 \supset \mathcal{C}_2 \supset \mathcal{C}_3$. A small part of this section is just an overview of earlier results in the present setting.

5.1. Diagrams and uniqueness in Kelly and Rival's sense. Let $\mathcal{C}_0$ be the class of planar diagrams of planar semimodular lattices. We recall some well-known concepts from, say, G. Czédli and G. Grätzer [10, Definition 3-3.5 and Lemma 3-4.2]. An element x of a lattice L is a *narrows* if $x \nparallel y$ for all $y \in L$. The *glued sum* $L_1 +^{\mathrm{gl}} L_2$ of finite lattices L_1 and L_2 is a particular case of their (Hall–Dilworth) gluing: we put L_2 atop L_1 and identify the singleton filter $\{1_{L_1}\}$ with the singleton ideal $\{0_{L_2}\}$. Chains and lattices with at least two elements are called *nontrivial*. Remember that a glued sum indecomposable lattice consists of at least four elements by definition. A folklore result says that a finite lattice L and, consequently, any of its diagrams D can uniquely be decomposed as

$$L = L_1 +^{\mathrm{gl}} \dots +^{\mathrm{gl}} L_t \quad \text{and} \quad D = D_1 +^{\mathrm{gl}} \dots +^{\mathrm{gl}} D_t \tag{5.1}$$

where $t \in \mathbb{N}_0 := \{0, 1, 2, \dots\}$ and, for every $i \in \{1, \dots, t\}$, either L_i is a glued sum indecomposable lattice, or it is a maximal nontrivial (chain) interval that consists of narrows. By definition, the empty sum yields the one element lattice. This decomposition, called the *canonical glued sum decomposition*, makes it meaningful to speak of the *glued sum components* of L or D. Note that a glued sum component is either glued sum indecomposable, or it is a nontrivial chain. We say that the planar diagrams of a planar lattice L are *unique up to sectional left-right similarity* if for every L_i from the canonical decomposition (5.1), the planar diagram of L_i is unique up to left-right similarity. The uniqueness properties of $\mathcal{C}_0$, that is, the "natural isomorphism" concept in $\mathcal{C}_0$, are explored by the following statement.

Proposition 5.1. *If L is a planar semimodular lattice, then its planar diagrams are unique up to sectional left-right similarity. They are unique even up to left-right similarity if, in addition, L is glued sum indecomposable.*

Proof. First, assume that L is glued sum indecomposable. Let $D_1, D_2 \in \mathcal{C}_0$ be diagrams of L. For $i \in \{1, 2\}$, by deleting eyes as long as possible, we obtain a subdiagram D'_i of D_i such that D'_i determines a full slimming sublattice L'_i of L. By Lemma 4.9(i), the L'_i are glued sum indecomposable. Applying Lemma 4.9(iii) to the identity map $\mathrm{id}_L \colon L \to L$, we obtain an automorphism π of L such that $\pi\rceil_{L'_1} : L'_1 \to L'_2$ is an f^{nc}-preserving lattice isomorphism. We let $\kappa := \pi\rceil_{L'_1}$, and we consider it as a $D'_1 \to D'_2$ map. Also, let $\kappa^{(\mathrm{mi})} := \pi\rceil_{L'_1}$, which is treated as a $D'_1 \to D_2'^{(\mathrm{mi})}$ map. By Proposition 4.11, κ or $\kappa^{(\mathrm{mi})}$ is a similarity map. We can assume that $\kappa \colon D'_1 \to D'_2$ is a similarity map, because in the other case we could work with $D_2^{(\mathrm{mi})}$, whose full slimming subdiagram is $D_2'^{(\mathrm{mi})}$. Next, we define a map $\psi \colon D_1 \to D_2$ as follows. First, if $x \in D'_1$, then $\psi(x) := \kappa(x)$. Second, let $y \in D_1 \setminus D'_1$. By (4.42), y is a doubly irreducible element; its unique lower cover is denoted by y^-. It follows obviously from the slimming procedure that $y^- \in D_1'$ and that $f^{\rm nc}_{D_1'\subseteq D_1}(y^-) \geq 1$. Listing them from left to right, let y be the i-th cover of y^- in D_1; note that $1 < i < 2 + f^{\rm nc}_{D_1'\subseteq D_1}(y^-)$, because y^- has exactly $2 + f^{\rm nc}_{D_1'\subseteq D_1}(y^-)$ covers in D_1. Since κ is $f^{\rm nc}$-preserving, $f^{\rm nc}_{D_2'\subseteq D_2}(\kappa(y^-)) = f^{\rm nc}_{D_1'\subseteq D_1}(y^-)$. So, $\kappa(y^-)$ has the same number of covers as y^-. Hence, we can define $\psi(y)$ as the i-th cover of $\kappa(y^-)$, counting from left to right. To sum up, $\psi\colon D_1 \to D_2$ is defined by

$$\psi(z) = \begin{cases} \kappa(z), & \text{if } z \in D_1', \\ \text{the } i\text{-th cover of } \kappa(z^-), & \text{if } z \notin D_1' \text{ is the } i\text{-th cover of } z^-. \end{cases} \tag{5.2}$$

We claim that $\psi\colon D_1 \to D_2$ is a similarity map. Clearly, ψ is an order isomorphism, because so is κ. Hence, it is a lattice isomorphism. In order to prove that ψ is a similarity map, assume that $a, b, c \in D_1$, $a \prec b$, $a \prec c$, $b \neq c$, and b is to the left of c. By Definition 4.10 and the sentence following it, it suffices to show that $\psi(b)$ is to the left of $\psi(c)$. Having at least two covers, a belongs to D_1' by (4.42). If $b, c \in D_1'$, then $\psi(b) = \kappa(b)$ is to the left of $\psi(c) = \kappa(c)$, because κ is a similarity map. Hence, the second line of (5.2) implies that $\psi(b)$ is to the left of $\psi(c)$ even if $\{b, c\} \nsubseteq D_1'$. Therefore, ψ is a similarity map. This proves the second half of the proposition.

Based on (5.1), the first half follows from the second. □

As a preparation for later use, we formulate the following lemma.

Lemma 5.2. *Let L and L' be slim rectangular lattices with fixed diagrams $D, D' \in \mathcal{C}_0$, respectively, and let $\varphi\colon L \to L'$ be a lattice isomorphism. Then either $\varphi(\mathrm{C}_{\rm l}(D)) = \mathrm{C}_{\rm l}(D')$ and $\varphi(\mathrm{C}_{\rm r}(D)) = \mathrm{C}_{\rm r}(D')$, or $\varphi(\mathrm{C}_{\rm l}(D)) = \mathrm{C}_{\rm r}(D')$ and $\varphi(\mathrm{C}_{\rm r}(D)) = \mathrm{C}_{\rm l}(D')$.*

Although φ is also a $D \to D'$ map, it is not so obvious that it preserves the "to the left of" relation or its inverse. Hence, this lemma seems not to follow from Proposition 5.1 immediately.

Proof of Lemma 5.2. With self-explanatory notation, (4.6) yields that

$$\begin{aligned} \mathrm{Ji}(D) &= \big(C_{\rm ll}(D) \cup C_{\rm lr}(D)\big) \setminus \{0\} = \{c_1, \dots, c_{m_{\rm l}}, d_1, \dots, d_{m_{\rm r}}\}, \\ \mathrm{Ji}(D') &= \big(C_{\rm ll}(D') \cup C_{\rm lr}(D')\big) \setminus \{0\} = \{c_1', \dots, c_{m_{\rm l}'}', d_1', \dots, d_{m_{\rm r}'}'\}, \end{aligned} \tag{5.3}$$

where, as in Definition 4.2(i), $c_0 \prec \dots \prec c_{m_{\rm l}}$, $d_0 \prec \dots \prec d_{m_{\rm r}}$, $c_0' \prec \dots \prec c_{m_{\rm l}'}'$, and $d_0' \prec \dots \prec d_{m_{\rm r}'}'$. Since $\uparrow c_{m_{\rm l}}$ and $\uparrow d_{m_{\rm r}}$ are chains by (4.11),

$$\begin{aligned} \mathrm{C}_{\rm l}(D) &= {\downarrow} c_{m_{\rm l}} \cup {\uparrow} c_{m_{\rm l}}, \quad & \mathrm{C}_{\rm r}(D) &= {\downarrow} d_{m_{\rm r}} \cup {\uparrow} d_{m_{\rm r}}, \\ \mathrm{C}_{\rm l}(D') &= {\downarrow} c_{m_{\rm l}'}' \cup {\uparrow} c_{m_{\rm l}'}', \quad & \mathrm{C}_{\rm r}(D') &= {\downarrow} d_{m_{\rm r}'}' \cup {\uparrow} d_{m_{\rm r}'}'. \end{aligned} \tag{5.4}$$

We know from G. Czédli and E. T. Schmidt [15, (2.14)] that, with the exceptions of $c_{m_{\rm l}}$, $d_{m_{\rm r}}$, $c_{m_{\rm l}'}'$ and $d_{m_{\rm r}'}'$, the elements given in (5.3) are meet-reducible. Thus, each of D and D' has exactly two doubly irreducible elements, and they are $c_{m_{\rm l}}, d_{m_{\rm r}} \in D$ and $c_{m_{\rm l}'}', d_{m_{\rm r}'}' \in D'$, respectively. Hence, $\{\varphi(c_{m_{\rm l}}), \varphi(d_{m_{\rm r}})\} = \{c_{m_{\rm l}'}', d_{m_{\rm r}'}'\}$. Thus, Lemma 5.2 follows from (5.4). □

5.2. Diagrams with normal slopes on their boundaries. Although the title of this subsection does not define the class $\mathcal{C}_1$ of diagrams, it reveals a property, to be defined soon, of diagrams in $\mathcal{C}_1$. In the rest of the section, we often consider the plane as $\mathbb{C}$, the field of complex numbers. However, a comment is useful at this point. When dealing with diagrams, they are on the blackboard or in a page of an article or a book. In all these cases, the direction "up" is fixed, but $0 \in \mathbb{C}$ and (to the right of 0) $1 \in \mathbb{C}$ are not necessarily. In other words, the position of the origin and the unit distance is our choice. Let

$$\begin{aligned}\boldsymbol{\epsilon} &= \cos(\pi/4) + i\,\sin(\pi/4) = \sqrt{2}/2 + i\,\sqrt{2}/2,\\ \boldsymbol{\epsilon}^3 &= \cos(3\pi/4) + i\,\sin(3\pi/4) = -\sqrt{2}/2 + i\,\sqrt{2}/2.\end{aligned}$$

We use these 8th roots of 1 to coordinatize the location of vertices of diagrams in $\mathcal{C}_1$, which we want to define.

For finite sequences $\vec{x} = \langle x_1, \dots, x_j\rangle$ and $\vec{y} = \langle y_1, \dots, y_k\rangle$, we can *glue* these two sequences to obtain a new sequence $\vec{x} +^{\text{gl}} \vec{y} := \langle x_1, \dots, x_j, y_1, \dots, y_k\rangle$; we can also glue more than two sequences. For $D \in \mathcal{C}_0$, let

$$m_{\text{l}}(D) = \max\{\text{ljc}_{D'}(x) : x \in D'\} \text{ and } m_{\text{r}}(D) = \max\{\text{rjc}_{D'}(x) : x \in D'\},$$

where D' is the full slimming subdiagram of D. (That is, D determines a unique full slimming sublattice L' of the lattice L defined by D, and D' consists of the vertices that represent the elements of L'.) Let $\mathsf{C_n}$ denote the *chain of length* n; it consists of $n+1$ elements. The superscripts ft and gh below come from "left" and "right", respectively.

Definition 5.3.

(A) A planar diagram D of a *glued sum indecomposable* finite planar semimodular lattice L belongs to $\mathcal{C}_1$ if there exist a complex number $\boldsymbol{\delta} \in \mathbb{C}$ and sequences

$$\vec{r}^{\,\text{ft}} = \langle r_1^{\text{ft}}, \dots, r_{m_{\text{l}}(D)}^{\text{ft}}\rangle \quad \text{and} \quad \vec{r}^{\,\text{gh}} = \langle r_1^{\text{gh}}, \dots, r_{m_{\text{r}}(D)}^{\text{gh}}\rangle \tag{5.5}$$

of positive real numbers such that the following conditions hold.

(i) L is a planar semimodular lattice. The full slimming subdiagram of D and the corresponding sublattice of L are denoted by D' and L', respectively.

(ii) For every $x \in L'$, the corresponding vertex of D' is

$$\boldsymbol{\delta} + \boldsymbol{\epsilon}^3 \cdot \sum_{j=1}^{\text{ljc}_{D'}(x)} r_j^{\text{ft}} + \boldsymbol{\epsilon} \cdot \sum_{j=1}^{\text{rjc}_{D'}(x)} r_j^{\text{gh}} \in \mathbb{C}. \tag{5.6}$$

(iii) We know that for each "eye" $x \in L \setminus L'$, there exists a unique 4-cell U of D' whose interior contains x; the condition is that the eyes in the interior of U should belong to the (not drawn) line segment connecting the left corner and the right corner of U and, furthermore, these eyes should divide this line segment into equal-sized parts.

In this case, we say that D is *determined by* $\langle \boldsymbol{\delta}, \vec{r}^{\,\mathrm{ft}}, \vec{r}^{\,\mathrm{gh}} \rangle$. We also say that $\langle \boldsymbol{\delta}, \vec{r}^{\,\mathrm{ft}}, \vec{r}^{\,\mathrm{gh}} \rangle$ is the *complex coordinate triplet* of $D \in \mathcal{C}_1$.

(B) For a *chain* $C = \{0 = c_0 \prec c_1 \prec \cdots \prec c_n = 1\}$ of length $n \in \mathbb{N}_0$, a planar diagram D of C belongs to $\mathcal{C}_1$ if there exists a $\boldsymbol{\delta} \in \mathbb{C}$ such that one of the following three possibilities holds.

  (i) There is a sequence $\vec{r}^{\,\mathrm{ft}} = \langle r_1^{\mathrm{ft}}, \dots, r_n^{\mathrm{ft}} \rangle$ of positive real numbers such that, for $j \in \{0, \dots, n\}$, the vertex representing c_j is $\boldsymbol{\delta} + \boldsymbol{\epsilon}^3 \cdot (r_1^{\mathrm{ft}} + \cdots + r_j^{\mathrm{ft}})$. In this case we let $m_{\mathrm{l}}(D) := n$, $m_{\mathrm{r}}(D) := 0$, and let $\vec{r}^{\,\mathrm{gh}}$ be the empty sequence.

  (ii) There is a sequence $\vec{r}^{\,\mathrm{gh}} = \langle r_1^{\mathrm{gh}}, \dots, r_n^{\mathrm{gh}} \rangle$ of positive real numbers such that, for $j \in \{0, \dots, n\}$, the vertex representing c_j is $\boldsymbol{\delta} + \boldsymbol{\epsilon} \cdot (r_1^{\mathrm{gh}} + \cdots + r_j^{\mathrm{gh}})$. In this case we let $m_{\mathrm{l}}(D) := 0$, $m_{\mathrm{r}}(D) := n$, and let $\vec{r}^{\,\mathrm{ft}}$ be the empty sequence.

  (iii) There are positive integers j and k with $n = j + k$, sequences $\vec{r}^{\,\mathrm{ft}} = \langle r_1^{\mathrm{ft}}, \dots, r_j^{\mathrm{ft}} \rangle$ and $\vec{r}^{\,\mathrm{gh}} = \langle r_1^{\mathrm{gh}}, \dots, r_k^{\mathrm{gh}} \rangle$ of positive real numbers such that D is a cover-preserving $\{0, 1\}$-subdiagram of the diagram $E \in \mathcal{C}_1$ of $\mathsf{C}_\mathsf{j} \times \mathsf{C}_\mathsf{k}$ determined by $\langle \boldsymbol{\delta}, \vec{r}^{\,\mathrm{ft}}, \vec{r}^{\,\mathrm{gh}} \rangle$. Then $\langle \boldsymbol{\delta}, \vec{r}^{\,\mathrm{ft}}, \vec{r}^{\,\mathrm{gh}} \rangle$ is said to be the *complex coordinate triplet* of D. However, this vector does not determine D, which can be any of the "zigzags" from 0_E up to 1_E.

(C) If the canonical glued sum decomposition $D_1 +^{\mathrm{gl}} \dots +^{\mathrm{gl}} D_t$ of $D \in \mathcal{C}_0$, see (5.1), consists of $t \geq 2$ components, then we say that D belongs to $\mathcal{C}_1$ if so do its components, $D_1, \dots, D_t$. With the self-explanatory notation, the complex coordinate triplet of D is $\langle \boldsymbol{\delta}, \vec{r}^{\,\mathrm{ft}}, \vec{r}^{\,\mathrm{gh}} \rangle$ defined as

$$\langle \boldsymbol{\delta}^{(1)}, \vec{r}^{\,\mathrm{ft}(1)} +^{\mathrm{gl}} \dots +^{\mathrm{gl}} \vec{r}^{\,\mathrm{ft}(t)}, \vec{r}^{\,\mathrm{gh}(1)} +^{\mathrm{gl}} \dots +^{\mathrm{gl}} \vec{r}^{\,\mathrm{gh}(t)} \rangle. \tag{5.7}$$

We define $m_{\mathrm{l}}(D)$ and $m_{\mathrm{r}}(D)$ as the number of components of $\vec{r}^{\,\mathrm{ft}}$ and that of $\vec{r}^{\,\mathrm{gh}}$, respectively.

(D) We say that $\langle \boldsymbol{\delta}, \vec{r}^{\,\mathrm{ft}}, \vec{r}^{\,\mathrm{gh}} \rangle$ is a *triplet compatible with* L, if $\vec{r}^{\,\mathrm{ft}}$ and $\vec{r}^{\,\mathrm{gh}}$ are finite sequences of positive real numbers, $\boldsymbol{\delta} \in \mathbb{C}$, and there exists a planar diagram D of L (that is, D is in $\mathcal{C}_0$ but not necessarily in $\mathcal{C}_1$) such that one of the following three possibilities holds.

  (i) L is a nontrivial chain and the length of L is the sum of the length (= number of components) of $\vec{r}^{\,\mathrm{ft}}$ and that of $\vec{r}^{\,\mathrm{gh}}$. Here we allow that $\vec{r}^{\,\mathrm{ft}}$ or $\vec{r}^{\,\mathrm{gh}}$ is the empty sequence with length 0.

  (ii) D is glued sum indecomposable, $\vec{r}^{\,\mathrm{ft}}$ is of length $m_{\mathrm{l}}(D)$, and $\vec{r}^{\,\mathrm{gh}}$ is of length $m_{\mathrm{r}}(D)$.

  (iii) In the canonical decomposition $D = D_1 +^{\mathrm{gl}} \dots +^{\mathrm{gl}} D_t$, see (5.1), $t \geq 2$, $\vec{r}^{\,\mathrm{ft}}$ is of length $m_{\mathrm{l}}(D_1) + \cdots + m_{\mathrm{l}}(D_t)$, and $\vec{r}^{\,\mathrm{gh}}$ is of length $m_{\mathrm{r}}(D_1) + \cdots + m_{\mathrm{r}}(D_t)$.

(E) For $D \in \mathcal{C}_1$, we say that D is *collinear* if $0 \in \{m_{\mathrm{l}}(D), m_{\mathrm{r}}(D)\}$. Otherwise, D is *non-collinear*.

For example, D_{t-1} and D_t in Figure 6, which happen to be slim rectangular diagrams, belong to $\mathcal{C}_1$ but not to $\mathcal{C}_2$, to be defined soon. In these diagrams, $\vec{r}^{\,\mathrm{ft}}$ and $\vec{r}^{\,\mathrm{gh}}$ are indicated. No matter if the pentagon-shaped grey-filled elements are considered or not, the diagram of L in Figure 2 is also in $\mathcal{C}_1$; this lattice is neither slim, nor rectangular, $\vec{r}^{\,\mathrm{ft}} = \langle 1,1,1\rangle$ and $\vec{r}^{\,\mathrm{gh}} = \langle 1,2,1,1\rangle$. There are also many earlier examples, including G. Czédli [2, Figure 7], [3, M in Figure 3] , [5, D in Figures 2, 3], which belong to $\mathcal{C}_1 \setminus \mathcal{C}_2$. The examples in $\mathcal{C}_2$, to be mentioned later, are also in $\mathcal{C}_1$. Our examples are non-collinear, since only nontrivial chains have collinear diagrams in $\mathcal{C}_1$. However, the chain C_n with $n \geq 2$ also has non-collinear diagrams in $\mathcal{C}_1$.

Remark 5.4. One may ask why we need (B) and (C) of Definition 5.3 and why we do not apply (A) and (5.6) without assuming glued sum indecomposability. For the answer, see Remark 6.6.

Assume that $D \in \mathcal{C}_1$ is a diagram of a glued sum indecomposable planar semimodular lattice L with complex coordinate triplet $\langle \boldsymbol{\delta}, \vec{r}^{\,\mathrm{ft}}, \vec{r}^{\,\mathrm{gh}}\rangle$ and that the full slimming subdiagram of D is D'. If we change $\langle \boldsymbol{\delta}, \vec{r}^{\,\mathrm{ft}}, \vec{r}^{\,\mathrm{gh}}\rangle$ to some $\langle \boldsymbol{\delta}^*, \vec{r}^{\,*\mathrm{ft}}, \vec{r}^{\,*\mathrm{gh}}\rangle$ where

$$\vec{r}^{\,*\mathrm{ft}} = \langle r_1^{*\mathrm{ft}}, \dots, r_{m_{\mathrm{l}}(D)}^{*\mathrm{ft}}\rangle \quad \text{and} \quad \vec{r}^{\,*\mathrm{gh}} = \langle r_1^{*\mathrm{gh}}, \dots, r_{m_{\mathrm{r}}(D)}^{*\mathrm{gh}}\rangle, \tag{5.8}$$

then (5.6), in which $\mathrm{ljc}_{D'}(x)$ and $\mathrm{rjc}_{D'}(x)$ are still understood in the full slimming of the *original* diagram, defines another diagram D^* of L. We say that D^* is obtained from D by *rescaling*. We can rescale a diagram $D \in \mathcal{C}_1$ of a chain similarly, keeping $m_{\mathrm{l}}(D)$ and $m_{\mathrm{r}}(D)$ unchanged. Finally, if $D \in \mathcal{C}_1$ and we rescale its components in the canonical decomposition (5.1), then we obtain another diagram of the same lattice, and we say that it is obtained from D by *piecewise rescaling*. Also, we can reflect some of the D_j in (5.1) over a vertical axis. (Of course, we may have to move several D_j's to the left or to the right in order not to "tear" the glued sum.) We say that the new diagram is obtained by *component-flipping*. Finally, *parallel shifting* means that we change $\boldsymbol{\delta}$ in (5.6). Obviously, $\mathcal{C}_1$ is closed with respect to component-flipping. Since the compatibility of a triplet does not depend on the magnitudes of its real number components, if $\langle \boldsymbol{\delta}, \vec{r}^{\,\mathrm{ft}}, \vec{r}^{\,\mathrm{gh}}\rangle$ is the complex coordinate triplet of $D \in \mathcal{C}_1$, then (5.8) gives a triplet compatible with L. Hence, Theorem 5.5(i) below implies that $\mathcal{C}_1$ is also closed with respect to piecewise rescaling; this is not obvious, because we have to shows that rescaling does not ruin planarity.

Theorem 5.5. *For a planar semimodular lattice L, the following hold.*

(i) *If $\langle \boldsymbol{\delta}, \vec{r}^{\,\mathrm{ft}}, \vec{r}^{\,\mathrm{gh}}\rangle$ is a triplet compatible with L, then this triplet* (*uniquely*) *determines a diagram $D \in \mathcal{C}_1$ of L.*

(ii) *In particular, L has a diagram in $\mathcal{C}_1$.*

(iii) *The diagram of L in $\mathcal{C}_1$ is unique up to component-flipping, parallel shifting, and piecewise rescaling.*

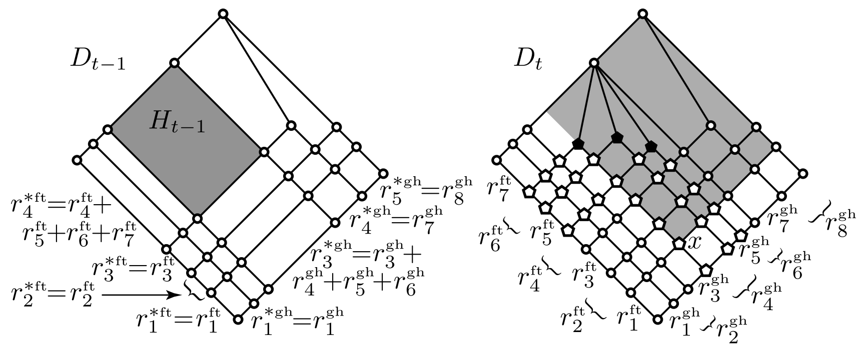

FIGURE 6. D_t is a 3-fold multifork extension of D_{t-1} at H_{t-1}

Before proving this theorem, it is necessary to recall a construction from G. Czédli [5]. Let D be a planar diagram of a slim semimodular lattice L. A 4-cell H of D is *distributive* if the ideal $\downarrow 1_H$ is a distributive lattice. To obtain a *multifork extension* D' of D at the 4-cell H, we have to perform two steps. As the first step, we insert k new lower covers of 1_H into the interior of H. For $\langle D, D', H, k\rangle = \langle D_t, D_{t-1}, H_{t-1}, 3\rangle$, the situation is exemplified in Figure 6, where $H = H_{t-1}$ is the grey 4-cell on the left and the new lower covers of 1_H are the black-filled pentagon-shaped elements on the right. (Except for $D = D_{t-1}$, $D' = D_t$, and $H = H_{t-1}$, the reader is advised to disregard the labels in the figure at present.) In the second step, we proceed downwards by inserting new elements (the empty-filled pentagon-shaped ones in the figure) into the 4-cells of the two trajectories through H, and we obtain D' in this way. We say that D' and L' are obtained by a (k-fold) *multifork extension* at the 4-cell H from D and from L, respectively. The maximal elements in $L' \setminus L$ or, equivalently, the new meet-irreducible elements, are called the *source elements* of the fork extension. (They are the black-filled pentagon-shaped elements in the figure.) For more details, the reader might want but need not resort to [5, Definition 3.1]. Note that this construction also makes sense for slim semimodular lattices without rectangularity.

The importance of this construction is given by the following lemma. Remember that a *grid* is the direct product of two finite chains.

Lemma 5.6 (G. Czédli [5, Theorem 3.7])**.** *If $D \in \mathcal{C}_0$ is a slim rectangular diagram, then there exist a $t \in \mathbb{N}_0 = \{0, 1, 2, \dots\}$,*

$$\begin{aligned}&\text{a sequence of diagrams } D_0 \subseteq D_1 \subseteq \cdots \subseteq D_t = D,\\ &\text{and distributive 4-cells } H_j \text{ of } D_j \text{ for } j = 0, 1, \dots, t-1\end{aligned} \tag{5.9}$$

such that $D_0, \dots, D_{t-1} \in \mathcal{C}_0$, D_0 is a grid, and that D_{j+1} is obtained from D_j by a multifork extension at H_j, for $j = 0, 1, \dots, t-1$.

The sequence in (5.9) is not unique, since the order of multifork extensions is not unique in general. However, t is uniquely determined, because it is clearly the number of elements with more than two lower covers. Now, we tailor Lemma 5.6 to our needs as follows.

Lemma 5.7. *Let L be a slim rectangular lattice, and let t be the number of its elements with more than two lower covers. If $\langle \boldsymbol{\delta}, \vec{r}^{\,\mathrm{ft}}, \vec{r}^{\,\mathrm{gh}}\rangle$ is a triplet compatible with L, then it is the complex coordinate triplet of a unique diagram D of L in $\mathcal{C}_1$ and, furthermore, there exist*

$$\begin{aligned}&\text{a sequence of diagrams } D_0 \subseteq D_1 \subseteq \cdots \subseteq D_t = D,\\ &\text{and distributive 4-cells } H_j \text{ of } D_j \text{ for } j = 0, 1, \ldots, t-1\end{aligned} \tag{5.10}$$

such that $D_0, \ldots, D_{t-1} \in \mathcal{C}_1$, D_0 is a grid, and that D_{j+1} is obtained from D_j by a multifork extension at H_j, for $j = 0, 1, \ldots, t-1$.

Again, the sequence in (5.10) is not unique in general. However, unless otherwise stated, we *fix* such a sequence and call it *the multifork construction sequence* of D. Before proving Lemma 5.7, we need an auxiliary statement.

Lemma 5.8.

(i) *Let x be an element of a slim rectangular lattice L. If $\downarrow x$ is distributive, then it is a grid (= direct product of two chains) or a chain.*
(ii) *A distributive rectangular lattice is a grid.*

Proof. In order to prove part (i), assume that $\downarrow x$ is not a chain. Since $\mathrm{Ji}(\downarrow x) \subseteq \mathrm{Ji}(L)$, $\mathrm{Ji}(\downarrow x)$ satisfies the condition given in (2.1). Hence, there is a grid G such that the ordered sets $\mathrm{Ji}(G)$ and $\mathrm{Ji}(\downarrow x)$ are isomorphic. By the classical structure theory of finite distributive lattices, see G. Grätzer [17, Corollary 108], $\downarrow x \cong G$, as required. This proves part (i). Part (ii) follows from part (i), applied to $x = 1$, and (4.13). □

Proof of Lemma 5.7. We prove the lemma by induction on t. If $t = 0$, then L is a grid by Lemma 5.8(ii) and the statement is trivial. Assume that $t > 0$ and the lemma holds for $t-1$. By Lemma 5.6, there exist a slim rectangular lattice L' with exactly $t-1$ of its elements having more than two lower covers, a fixed diagram $D'_0 \in \mathcal{C}_0$ of L', a distributive covering square (equivalently, a distributive 4-cell in D'_0) H_{t-1} of L', and $k \in \mathbb{N} = \{1, 2, \ldots\}$ such that L is obtained from L' by a k-fold multifork extension at H_{t-1}. With respect to D'_0, let $i = \mathrm{ljc}_{L'}(1_{H_{t-1}})$ and $j = \mathrm{rjc}_{L'}(1_{H_{t-1}})$. Define

$$\begin{aligned}\vec{r}^{\,*\mathrm{ft}} &= \langle r^{\mathrm{ft}}_1, \ldots, r^{\mathrm{ft}}_{i-1}, r^{\mathrm{ft}}_i + \cdots + r^{\mathrm{ft}}_{i+k}, r^{\mathrm{ft}}_{i+k+1}, \ldots, r^{\mathrm{ft}}_{k+m_{\mathrm{l}}(D'_0)}\rangle \text{ and}\\ \vec{r}^{\,*\mathrm{gh}} &= \langle r^{\mathrm{gh}}_1, \ldots, r^{\mathrm{gh}}_{j-1}, r^{\mathrm{gh}}_j + \cdots + r^{\mathrm{gh}}_{j+k}, r^{\mathrm{gh}}_{j+k+1}, \ldots, r^{\mathrm{gh}}_{k+m_{\mathrm{r}}(D'_0)}\rangle.\end{aligned}$$

Since D'_0 witnesses that $\langle \boldsymbol{\delta}, \vec{r}^{\,*\mathrm{ft}}, \vec{r}^{\,*\mathrm{gh}}\rangle$ is a triplet compatible with L', the induction hypothesis applies to this triplet and L'. Therefore, there exists a diagram $D' \in \mathcal{C}_1$ of L' whose complex coordinate triplet is $\langle \boldsymbol{\delta}, \vec{r}^{\,*\mathrm{ft}}, \vec{r}^{\,*\mathrm{gh}}\rangle$ such that (5.10) holds with $t-1$, D', and L' instead of t, D, and L; see Figure 6 for an illustration with $\langle i, j, k\rangle = \langle 4, 3, 3\rangle$. (In the figure, H_{t-1} is the grey covering square on the left; disregard the grey area on the right.) The ideal $\downarrow 1_{H_{t-1}}$ in D' is a distributive lattice, so it is a grid. Hence, clearly, if we insert a k-multifork at H_{t-1} according to $\langle r_i^{\mathrm{ft}}, \dots, r_{i+k}^{\mathrm{ft}} \rangle$ and $\langle r_j^{\mathrm{gh}}, \dots, r_{j+k}^{\mathrm{gh}} \rangle$ as in the figure, then we obtain a *planar* diagram D, which belongs to $\mathcal{C}_1$. The definition of $\vec{r}^{*\mathrm{ft}}$ and $\vec{r}^{*\mathrm{gh}}$ imply that $\langle \boldsymbol{\delta}, \vec{r}^{\mathrm{ft}}, \vec{r}^{\mathrm{gh}} \rangle$ is the complex coordinate triplet of D. This completes the induction step and proves the lemma. □

Proof of Theorem 5.5. Part (iii) follows from Proposition 5.1. Part (ii) is an obvious consequence of part (i), so we only focus on part part (i).

It is straightforward to see that if part (i) holds for all the L_i in the canonical glued sum decomposition (5.1) of L, then it also holds for L. Part (i) is evident if L_i is a chain. Part (i) follows from Lemma 5.7 if L_i is a slim rectangular lattice. So, it suffices to show the validity of part (i) if L_i is a glued sum indecomposable planar semimodular lattice. To ease the notation, we write L rather than L_i. Actually, since the application of Definition 5.3(Aiii) cannot destroy planarity, we can assume that L is a *slim* semimodular lattice. Let $\langle \boldsymbol{\delta}, \vec{r}^{\mathrm{ft}}, \vec{r}^{\mathrm{gh}} \rangle$ be a triplet compatible with L. Theorem 2.2 allows us to consider the normal rectangular extension L' of L. Since this is only the question of the diagram-dependent values m_{l} and m_{r}, it follows from Lemma 4.6 and Proposition 5.1 that $\langle \boldsymbol{\delta}, \vec{r}^{\mathrm{ft}}, \vec{r}^{\mathrm{gh}} \rangle$ is compatible with L'. Thus, Lemma 5.7 gives us a diagram $D' \in \mathcal{C}_1$ of L' such that $\langle \boldsymbol{\delta}, \vec{r}^{\mathrm{ft}}, \vec{r}^{\mathrm{gh}} \rangle$ is the complex coordinate triplet of D'. We conclude from Lemma 4.6 that the elements of L in D' are exactly in the appropriate places that (5.6) demands for L. These elements form a subdiagram D. By Lemma 4.1, D is a region of D'. As a region of a planar diagram, D is also planar. It is clear, again by Lemma 4.6, that $\langle \boldsymbol{\delta}, \vec{r}^{\mathrm{ft}}, \vec{r}^{\mathrm{gh}} \rangle$ is the complex coordinate triplet of D. In particular, $D \in \mathcal{C}_1$. □

Although $\mathcal{C}_2$ is not yet defined, the diagrams in $\mathcal{C}_j$, $j \in \{1, 2\}$, of a *rectangular* lattice are particularly easy to draw. Hence, we formulate the following remark, which follows from Lemma 4.6. Note, however, that (5.6) allows us to draw a diagram directly, without drawing its normal rectangular extension.

Remark 5.9. For $j \in \{1, 2\}$, a diagram $D \in \mathcal{C}_j$ of a planar semimodular lattice L with more than two elements can be constructed as follows.

(i) Take a normal rectangular extension R of L.
(ii) Find a diagram $E \in \mathcal{C}_j$ of R.
(iii) Remove the vertices corresponding to $R \setminus L$ and the edges not in L.

As a counterpart of this remark, we formulate the following statement here, even if $\mathcal{C}_2$ is not yet defined. (We need this statement before introducing $\mathcal{C}_2$, and its validity for $\mathcal{C}_1$ will trivially imply that it holds for $\mathcal{C}_2$.) We say that E is a *normal rectangular extension diagram* of a planar semimodular diagram D if E is a planar diagram of a normal rectangular extension of the lattice determined by D and we can obtain D from E by omitting some vertices and edges. The equation $E_1 = E_2$ below is understood in the sense that the two diagrams consist of the same complex numbers as vertices and the same edges. Note that a glued sum indecomposable lattice cannot have a collinear diagram; see Definition 5.3(E).

Proposition 5.10. *If $j \in \{0,1,2\}$, $D \in \mathcal{C}_j$, and D has at least three vertices, then the following assertions hold.*

(i) *If $j = 0$, then D has a normal rectangular extension diagram in $\mathcal{C}_0$.*

(ii) *If $j \in \{1,2\}$ and D is non-collinear, then D has a normal rectangular extension diagram in $\mathcal{C}_j$.*

(iii) *Assume, in addition, that D is glued sum indecomposable. Let $E_1 \in \mathcal{C}_j$ and $E_2 \in \mathcal{C}_j$ be normal rectangular extension diagrams of D. If j is in $\{1,2\}$, then $E_1 = E_2$. If $j = 0$, then E_1 is similar to E_2.*

Besides that $\mathcal{C}_3$ has not been defined yet, Remark 5.15 will explain why $j = 3$ is not allowed above.

Proof of Proposition 5.10(iii)*.* We can assume that D is slim; then its normal rectangular extension is also slim by Theorem 2.2(ii). The reason is that if D is not slim, then we can work with its full slimming subdiagram D', and we can put the eyes back in the normal rectangular extension later. For *lattices*, the ambiguity of the full slimming can cause some difficulties, see Lemma 4.9. However, for *diagrams*, the full slimming is uniquely determined and cannot cause any problem; see also Definition 5.3(Aiii).

For $j \in \{1,2\}$, part (iii) follows from Lemma 4.6, (4.38), and (5.6).

Next, we assume that $j = 0$. So let $D \in \mathcal{C}_0$ and let $E_1, E_2 \in \mathcal{C}_0$ be normal rectangular extension diagrams of D. Let L be the lattice determined by D. By (4.38) and Remark 4.8, $\mathrm{ACP}_{E_1}(L) = \mathrm{ACP}(L) = \mathrm{ACP}_{E_2}(L)$. For $k \in \{1,2\}$, take the coordinatization map $\delta_k : E_k \to \mathrm{ACP}_{E_k}(L)$, given in the last sentence of Lemma 4.7. Since δ_1 and δ_2 are lattice isomorphisms by Lemma 4.7, so is $\eta := \delta_2^{-1} \circ \delta_1 : E_1 \to E_2$. Clearly, η preserves the join-coordinate pairs. Hence, it follows from (4.15) that η is a similarity map. □

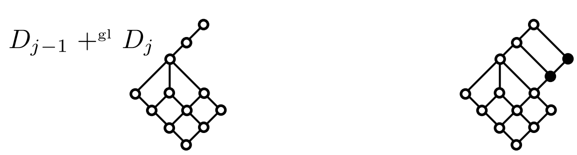

FIGURE 7. Getting rid of a collinear chain D_j

Outline for Proposition 5.10(i)-(ii)*.* As opposed to part (iii), we will not use parts (i) and (ii) in the paper. Hence, and also because of space considerations, we only give the main ideas. Consider the canonical glued sum decomposition $D = D_1 +^{\mathrm{gl}} \dots +^{\mathrm{gl}} D_t$; see (5.1). In the simplest case, we can take a normal rectangular extension E_j of D_j for every j; either by following the argument in the proof of Proposition 5.10(iii) for $j = 0$, see also (4.31), or trivially for chain components. Then Figure 4 indicates how to continue by successively replacing the glued sum of two consecutive rectangular diagrams by their normal rectangular extension. However, there are less simple cases, where some

D_j are collinear or $|D_j| = 2$. Then we can exploit the fact that $D_j \neq D$, and so at least one of D_{j-1} and D_{j+1} exists and it is glued sum indecomposable. If, say, D_{j-1} is glued sum indecomposable, then $D_{j-1} +^{\mathrm{gl}} D_j$, see on the left of Figure 7, can be replaced by the diagram on the right of the same figure.

The straightforward but tedious details proving that our method yields a normal rectangular extension diagram of D are omitted. □

5.3. Equidistant diagrams with normal slopes on their boundaries. We define a subclass $\mathcal{C}_2$ of $\mathcal{C}_1$ as follows

Definition 5.11. A diagram $D \in \mathcal{C}_1$ belongs to $\mathcal{C}_2$ if its complex coordinate triplet is of the form

$$\langle \boldsymbol{\delta}, \vec{r}^{\,\mathrm{ft}}, \vec{r}^{\,\mathrm{gh}} \rangle = \langle \boldsymbol{\delta}, \langle r, \dots, r\rangle, \langle r, \dots, r\rangle\rangle \tag{5.11}$$

for a positive constant $r \in \mathbb{R}$. "*Rescaling*" in $\mathcal{C}_2$ means to change r.

From Theorem 5.5, we clearly obtain the following statement.

Corollary 5.12. *Every planar semimodular lattice has a diagram in $\mathcal{C}_2$, which is unique up to rescaling in $\mathcal{C}_2$, parallel shifting, and component-flipping.*

The diagrams in Figures 3, 4, and $\widehat{R}$ in Figure 2, and, for example, the diagrams in G. Czédli [2, Figures 1, 2, 3, 4, 5], [3, Figures 2, 4, 5], and [5, Figures 1, 8, 9] belong to $\mathcal{C}_2$. Furthermore, the fact that the diagrams in G. Czédli [7, Figure 5] belong to $\mathcal{C}_2$ is more than an esthetic issue; it is an integral part of the proof of [7, Lemma 3.9]. Generally, for a planar semimodular lattice, we use a diagram outside $\mathcal{C}_2$ only in the following two cases: a diagram is extended or a subdiagram is taken, or if there are many eyes in the interior of a covering square. (In the first but not the second case, $\mathcal{C}_1$ is recommended.)

5.4. Uniqueness without compromise. The "up" direction in our plane (blackboard, page of an article, etc.) is usually fixed. Hence, for a diagram $D \in \mathcal{C}_2$, the parameters $\boldsymbol{\delta}$ and r in (5.11) does not effect the geometric shape and the orientation of D. So, we can choose $\langle \boldsymbol{\delta}, r\rangle = \langle 0, 1\rangle$. As we will see soon, this means that we choose the complex plain $\mathbb{C}$ so that 0_D is placed at $0 \in \mathbb{C}$ and the leftmost atom of D is placed at $\boldsymbol{\epsilon}^3$. However, reflecting some of the D_j in the canonical decomposition (5.1) across a vertical axis may effect the geometric shape of D, and we want to get rid of this possibility. To achieve this goal, we need some preparation.

Let D be a planar diagram of a slim semimodular lattice. Recall from G. Czédli and E. T. Schmidt [16] that the *Jordan–Hölder permutation* π_D, which was associated with D first by H. Abels [1] and R. P. Stanley [25], can be defined as follows. Let

$$\mathrm{C_l}(D) = \{0 = e_0 \prec e_1 \prec \cdots \prec e_n = 1\} \text{ and}$$
$$\mathrm{C_r}(D) = \{0 = f_0 \prec f_1 \prec \cdots \prec f_n = 1\},$$

and let S_n denote the symmetric group consisting of all $\{1,\dots,n\}\to\{1,\dots,n\}$ permutations. We define $\pi_D\in S_n$ by the rule

$$\pi_D(i)=j \iff [e_{i-1},e_i] \text{ and } [f_{j-1},f_j] \text{ belong to the same trajectory.}$$

Obviously, for slim semimodular lattices diagrams D_1 and D_2,

$$\text{if } D_1 \text{ is similar to } D_2, \text{ then } \pi_{D_1}=\pi_{D_2}. \tag{5.12}$$

For $\sigma,\tau\in S_n$, σ *lexicographically precedes* τ, in notation $\sigma\le_{\text{lex}}\tau$, if

$$\langle\sigma(1),\dots,\sigma(n)\rangle\le\langle\tau(1),\dots,\tau(n)\rangle \tag{5.13}$$

in the lexicographic order. Although (5.13) is meaningful for all slim semimodular diagrams, Section 4 does not work for chains. For example, the diagrams in $\mathcal{C}_2$ of a chain cannot be distinguished by means of join-coordinates. Hence, chain components in the canonical decomposition (5.1) would lead to difficulties. Therefore, we assume glued sum indecomposability here. So let $D'_j\in\mathcal{C}_0$ be the full slimming diagram of $D_j\in\mathcal{C}_0$ for $j\in\{1,2\}$ such that D'_1 is similar to D'_2 and, in addition, let the D_j be glued sum indecomposable. Note that if $\text{height}(x)=\text{height}(y)$ and $x\ne y$, then $x\parallel y$ and, by Lemma 3.2(ii), either $x\;\lambda\;y$, or $y\;\lambda\;x$. Hence, we can consider the unique list $\langle x_1^{(j)},x_2^{(j)},\dots,x_k^{(j)}\rangle$ of elements of D'_j such that, for all $1\le s<t\le k$, either $\text{height}(x_s^{(j)})<\text{height}(x_t^{(j)})$, or $\text{height}(x_s^{(j)})=\text{height}(x_t^{(j)})$ and $x_s^{(j)}\;\lambda\;x_t^{(j)}$. Clearly, this list is repetition-free. Denoting the similarity map $D'_1\to D'_2$ by φ, note that

$$\varphi \text{ preserves the list, that is, } \varphi(x_s^{(1)})=x_s^{(2)} \text{ for } \forall s\in\{1,\dots,k\}. \tag{5.14}$$

We say that $D_1\sqsubseteq_{\text{lex}}D_2$ if the k-tuple $\langle f^{\text{nc}}_{D'_1\subseteq D_1}(x_1^{(1)}),\dots,f^{\text{nc}}_{D'_1\subseteq D_1}(x_k^{(1)})\rangle$ equals or lexicographically precedes $\langle f^{\text{nc}}_{D'_2\subseteq D_2}(x_1^{(2)}),\dots,f^{\text{nc}}_{D'_2\subseteq D_2}(x_k^{(2)})\rangle$. Let us emphasize that $D_1\sqsubseteq_{\text{lex}}D_2$ only makes sense if the full slimming sublattice of D_1 is similar to that of D_2. The upper integer part of a real number x is denoted by $\lceil x\rceil$; for example, $\lceil\sqrt{3}\rceil=2=\lceil 2\rceil$. Now we are in the position to define a class $\mathcal{C}_3\subset\mathcal{C}_2$ of diagrams as follows.

Definition 5.13. Let $D\in\mathcal{C}_2$ be a diagram, and let L denote the planar semimodular lattice it determines. Let D' and L' denote the full slimming subdiagram of D and the corresponding full slimming sublattice of L, respectively. Then D belongs to $\mathcal{C}_3$ if one of the conditions (A), (B), and (C) below holds.

(A) D is glued sum indecomposable and the following three conditions hold.
  (i) The complex coordinate triplet of D is $\langle 0,\langle 1,\dots,1\rangle,\langle 1\dots,1\rangle\rangle$.
  (ii) For every diagram $E'\in\mathcal{C}_0$ of L', $\pi_{D'}\le_{\text{lex}}\pi_{E'}$.
  (iii) For every diagram $E\in\mathcal{C}_0$ of L, if the full slimming of E is similar to D, then $E\sqsubseteq_{\text{lex}}D$.

(B) D is a chain $D=\{0=d_0\prec\dots\prec d_n=1\}$ and, for $j\in\{0,\dots,n\}$,

$$d_j=\begin{cases} j\boldsymbol{\epsilon}^3, & \text{if } j\le\lceil n/2\rceil,\\ \lceil n/2\rceil\boldsymbol{\epsilon}^3+(j-\lceil n/2\rceil)\boldsymbol{\epsilon}, & \text{if } j>\lceil n/2\rceil. \end{cases}$$

(C) The canonical glued sum decomposition (5.1) consists of more than one components, that is, $t > 1$, and, for every $j \in \{1, \dots, t\}$, an appropriate parallel shift (that is, changing the first component of the complex coordinate triplet) turns D_j into a diagram in $\mathcal{C}_3$.

For example, the diagrams in Figures 3, 10 and L_2, R_2 in Figure 4 are in $\mathcal{C}_3$; see also Figure 8. Observe that in (Aii) and (Aiii) of Definition 5.13, E' and E range in $\mathcal{C}_0$ rather than only in $\mathcal{C}_2$. Of course, there could be other definitions to make the following proposition valid. Our vague idea is that "at low level", we want more elements on the left than on the right.

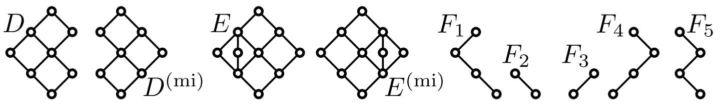

FIGURE 8. $D, E, F_1, F_2 \in \mathcal{C}_3$ but $D^{(\mathrm{mi})}, E^{(\mathrm{mi})}, F_3, F_4, F_5 \notin \mathcal{C}_3$

Proposition 5.14. *Every planar semimodular lattice L has a* unique *diagram D in $\mathcal{C}_3$. The uniqueness means that if $D^*, D^\natural \in \mathcal{C}_3$ are diagrams of L, then their vertex sets are exactly the same subsets of $\mathbb{C}$, and their edge sets are also the same sets of straight line segments in the complex plane.*

Proof. By G. Czédli and E. T. Schmidt [16, Lemma 4.9], the converse implication in (5.12) also holds; alternatively, see G. Czédli and G. Grätzer [10, Theorem 3-9.6]. Hence, the uniqueness part or the proposition follows.

In order to verify the existence part, let L' be a full slimming sublattice of L. We obtain from Corollary 5.12 that L' has a diagram $D' \in \mathcal{C}_2$. We can assume that L and, consequently, L' are glued sum indecomposable. After rescaling in $\mathcal{C}_2$ and parallel shifting if necessary, we can assume that

$$\text{the complex coordinate triplet of } D' \text{ is } \langle 0, \langle 1, \dots, 1\rangle, \langle 1, \dots, 1\rangle\rangle. \tag{5.15}$$

Of course, the same holds for $D'^{(\mathrm{mi})}$, obtained from D' by reflecting it over the "imaginary" axis $\{ri : r \in \mathbb{R}\}$. It follows from Proposition 5.1 that every diagram $E' \in \mathcal{C}_0$ of L' is similar to D' or $D'^{(\mathrm{mi})}$. Hence, by (5.12), all the permutations we have to consider belong to $\{\pi_{D'}, \pi_{D'^{(\mathrm{mi})}}\}$. This and (5.13) give that D' or $D'^{(\mathrm{mi})}$ belongs to $\mathcal{C}_3$, depending on $\pi_{D'} \leq_{\mathrm{lex}} \pi_{D'^{(\mathrm{mi})}}$ or $\pi_{D'^{(\mathrm{mi})}} \leq_{\mathrm{lex}} \pi_{D'}$, because both represent L' and satisfy (5.15). Let, say, $D' \in \mathcal{C}_3$.

Since D' has finitely many 4-cells and the positions of the eyes in a given 4-cell are determined by Definition 5.3(Aiii), we conclude that there are only finitely many antislimmings $D_1, \dots, D_k$ of D' in $\mathcal{C}_2$ that define L. By changing the subscripts is necessary, we can assume that $D_j \sqsubseteq_{\mathrm{lex}} D_k$ holds for all $j \in \{1, \dots, k\}$. We assert that $D_k \in \mathcal{C}_3$. In order to prove this, consider an arbitrary diagram $E \in \mathcal{C}_0$ of L such that its full slimming subdiagram E' is similar to D'. We have to show that $E \sqsubseteq_{\mathrm{lex}} D_k$. Let $\varphi\colon D' \to E'$ be similarity map, and define a map $g\colon D' \to \mathbb{N}_0$ as $g = f^{\mathrm{nc}}_{E' \subseteq E} \circ \varphi$; see (4.43).

For each 4-cell H of D', let us add $g(0_H)$ eyes into the interior of H, keeping Definition 5.3(Aiii) in mind. In this way, we obtain a diagram $D \in \mathcal{C}_3$, which is an antislimming of D'. Since $g = f^{\mathrm{nc}}_{D' \subseteq D}$ obviously holds, the similarity map φ is an f^{nc}-preserving isomorphism. Applying (4.44) to the lattices our diagrams determine, it follows that E and D define isomorphic lattices. Hence, $D \in \mathcal{C}_3$ defines L, and we obtain that $D = D_j$ for some $j \in \{1, \dots, k\}$. Since φ is f^{nc}-preserving and it preserves the list of (5.14), $E \sqsubseteq_{\mathrm{lex}} D_k \iff D_j \sqsubseteq_{\mathrm{lex}} D_k$. Therefore, by the choice of D_k, $E \sqsubseteq_{\mathrm{lex}} D_k$, as required. □

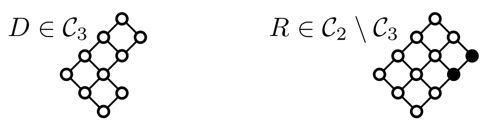

FIGURE 9. In $\mathcal{C}_3$, D has no normal rectangular extension diagram

Consider D and R in Figure 9. By Proposition 5.10(iii), R is the only normal rectangular extension of D in $\mathcal{C}_2$. Hence, we obtain the following remark.

Remark 5.15. Part (ii) of Proposition 5.10 fails for $j = 3$.

6. A toolkit for diagrams in $\mathcal{C}_1$

For $x = x_1 + x_2 i$ and $y = y_1 + y_2 i$ in $\mathbb{C}$, where $x_1, x_2, y_1, y_2 \in \mathbb{R}$, we say that x is *geometrically below* y if $x_2 \le y_2$. In addition to Theorem 5.5(iii), the following statement also indicates well the advantage of $\mathcal{C}_1$ over $\mathcal{C}_0$; note that this statement would fail without assuming slimness.

Corollary 6.1. *Let $D \in \mathcal{C}_1$ be a slim semimodular diagram. For distinct $x, y \in D$, we have $x < y$ iff x is geometrically below y and the slope of the line through x and y is in the interval $[\pi/4, 3\pi/4]$ (that is, between $45°$ and $135°$).*

Proof. First, we deal with the case where D is glued sum indecomposable. Let $x \ne y \in D$, and denote the line through x and y by ℓ. Assume that $x < y$. Since ljc_D and rjc_D are monotone, we obtain from (5.6) that $y - x = r_1 \boldsymbol{\epsilon}^3 + r_2 \boldsymbol{\epsilon} \in \mathbb{C}$ with nonnegative $r_1, r_2 \in \mathbb{R}$. This implies that the slope of ℓ is in $[\pi/4, 3\pi/4]$ and x is geometrically below y. Conversely, assume that the slope of ℓ is in $[\pi/4, 3\pi/4]$ and x is geometrically below y. Again, we can write the complex number $y - x$ in the form $y - x = t_1 \boldsymbol{\epsilon}^3 + t_2 \boldsymbol{\epsilon} \in \mathbb{C}$ with $t_1, t_2 \in \mathbb{R}$. Since x is geometrically below y, the assumption on the slope of ℓ implies that t_1 and t_2 are nonnegative. Thus, we can extract from (5.6) that $\mathrm{ljc}_D(x) \le \mathrm{ljc}_D(y)$ and $\mathrm{rjc}_D(x) \le \mathrm{rjc}_D(y)$. Hence, $x \le y$ by (4.16). So, Corollary 6.1 holds for the glued sum indecomposable case, which easily implies its validity for the general case. □

In view of Remark 5.9 and the simplicity of the constructive step described in Definition 5.3(Aiii), we will mainly focus on *slim rectangular* diagrams. Let $D \in \mathcal{C}_1$, and let $[u,v]$ or, in other words, $u \prec v$ be an edge of the diagram D. If the angle this edge makes with a horizontal line is $\pi/4$ (45°) or $3\pi/4$ (135°), then we say that the edge is of *normal slope*. If this angle is strictly between $\pi/4$ and $3\pi/4$, then the edge is *precipitous* or, in other words, it is of *high slope*. The following observation shows that edges of "low slopes" do not occur. The *boundary* and the *interior* of a diagram D are $\mathrm{Bnd}(D) := \mathrm{C}_\mathrm{l}(D) \cup \mathrm{C}_\mathrm{r}(D)$ and $D \setminus \mathrm{Bnd}(D)$, respectively. Remember that $\mathrm{Mi}(D)$, the set of *meet-irreducible elements*, is $\{x \in D : x$ has exactly one cover$\}$.

Observation 6.2. *Let $D \in \mathcal{C}_1$ be a slim rectangular lattice diagram. If $u \prec v$ in D, then exactly one of the following two possibilities holds:*

(i) *the edge $[u,v]$ is of normal slope and $u \in \mathrm{Bnd}(D) \cup (D \setminus \mathrm{Mi}(D))$;*
(ii) *the edge $[u,v]$ is precipitous, $u \in \mathrm{Mi}(D)$, u is in $D \setminus \mathrm{Bnd}(D)$, the interior of D, and v has at least three lower covers.*

Proof of Observation 6.2. Take a multifork construction sequence (5.10) . Obviously, the statement holds for D_0. If it holds for D_j, then it is easy to see that it also holds for D_{j+1}. □

The following observation follows by a trivial induction based on Lemma 5.7. The case $y \in \mathrm{Ji}(D)$, equivalently, $y \in C_{\mathrm{ll}}(D) \cup C_{\mathrm{lr}}(D)$, is not considered in it.

Observation 6.3. *If $D \in \mathcal{C}_1$ is a slim rectangular lattice diagram, $x \prec y \in D$, and $y \notin \mathrm{Ji}(D)$, then the following three conditions are equivalent.*

(i) *The edge $[x,y]$ is of slope $\pi/4$ (respectively, $3\pi/4$).*
(ii) *x is the leftmost (respectively, rightmost) lower cover of y.*

Let u be a trajectory of a slim semimodular lattice diagram such that its edges, from left to right, are listed as $[x_0,y_0]$, $[x_1,y_1]$, …, $[x_k,y_k]$. For $a \not\parallel b$, let $[a,b]^*$ denote $[a,b]$ if $a \leq b$, and let it denote $[b,a]$ if $b \leq a$. That is, $[a,b]^* = [a \wedge b, a \vee b]$. The *lower border* of u is the set $\{[x_{j-1},x_j]^* : 1 \leq j \leq k\}$ of edges. Similarly, the *upper border* of u is $\{[y_{j-1},y_j]^* : 1 \leq j \leq k\}$.

Corollary 6.4. *Let $D \in \mathcal{C}_1$ be a diagram of a slim semimodular lattice L. If T is trajectory of D, then every edge of its lower border is of normal slope.*

Proof. Clearly, we can assume that D is glued sum indecomposable. Let j be in $\{1,\dots,k\}$. First, assume that, in addition, L is rectangular. By (2.2), so is D. We assume that $y_{j-1} < y_j$, because otherwise we can work in $D^{(\mathrm{mi})}$. Thus, T goes upwards at $[x_{j-1},y_{j-1}]$. Hence, $x_{j-1} < x_j$, $[x_{j-1},x_j]^* = [x_{j-1},x_j]$, and $x_{j-1} = x_j \wedge y_{j-1} \notin \mathrm{Mi}(L)$. Therefore, Observation 6.2 yields that $[x_{j-1},x_j]^*$ is of normal slope. Second, we do not assume that D is rectangular. Then, by Lemma 4.1 and Proposition 5.10, D is a region of a unique slim rectangular diagram $E \in \mathcal{C}_1$, and T is a section from $\mathrm{C}_\mathrm{l}(D)$ to $\mathrm{C}_\mathrm{r}(D)$ of a trajectory T' of E. Since $[x_{j-1},x_j]^*$ is on the lower border of T', it is of normal slope in E. By Remark 5.9, it is of the same slope in D. □

For a 4-cell H, we say that H is a 4-*cell with normal slopes* if each of the four sides of H is of normal slope.

Corollary 6.5. *If H is a distributive 4-cell of a diagram $D \in \mathcal{C}_1$, then H is of normal slopes and, moreover, every edge in $\downarrow 1_H$ is of normal slope.*

Proof. It is a folklore result, see the Introduction in G. Grätzer and E. Knapp [19] or see G. Czédli and E. T. Schmidt [14, Lemmas 2 and 16], that

no element of a planar distributive lattice covers more than two elements. (6.1)

Hence, the corollary follows from Observation 6.2. □

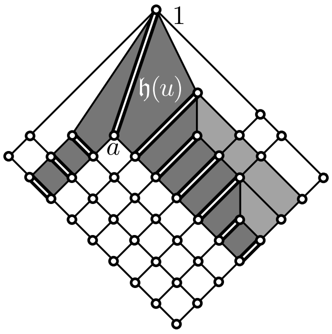

FIGURE 10. Territories: $\mathrm{Terr}(u)$ and $\mathrm{Terr}_{\mathrm{orig}}(u)$

Remark 6.6. In order to answer the question in Remark 5.4, let L be a non-chain slim semimodular lattice consisting of at least 3 elements such that at least one of its glued sum components is a chain; see (5.1). As opposed to Proposition 5.10(ii), if we applied (5.6) to obtain a diagram D of L, then D would not have a normal rectangular extension diagram in $\mathcal{C}_1$.

Proof of Remark 6.6. Suppose, for a contradiction, that D has a rectangular extension diagram $E \in \mathcal{C}_1$. Pick a and b in a chain component of D such that $a \prec b$. By (5.6), $[a, b]$ is a vertical edge. Since a is the only lower cover of b in D but Observation 6.2 yields that b has at least three lower covers in E, Definition 2.1(iii) is violated. This contradiction proves Remark 6.6. □

For a slim $D \in \mathcal{C}_0$, the set of trajectories of D is denoted by $\mathrm{Traj}(D)$.

Definition 6.7. Let $D \in \mathcal{C}_1$ be a slim rectangular diagram, and let u be a trajectory of D.

(i) The *top edge* of a trajectory u, denoted by $\mathfrak{h}(u)$, belongs to u and is defined by the property that $y \leq 1_{\mathfrak{h}(u)}$ for all $[x, y] \in u$; see (3.4).
(ii) For a region or a 4-cell A, the territory of A is denoted by $\mathrm{Terr}(A)$. It is a closed polygon in the plane.

(iii) Similarly, the *territory* of u, denoted by $\mathrm{Terr}(u)$, is the closed polygon of the plane covered by the squares of u. An example is given in Figure 10, where u is the hat-trajectory through $\mathfrak{h}(u) = [a, 1]$ and it consists of the double (thick) edges; $\mathrm{Terr}(u)$ is the dark grey area.

(iv) With reference to a *fixed* multifork construction sequence (5.10), for each $x \in D$, there is a smallest j such that $x \in D_j$. We denote this smallest j by $\mathrm{yb}(x)$; the acronym comes from "year of birth". For an interval $\mathfrak{g}$, $\mathrm{yb}(\mathfrak{g}) = \max\{\mathrm{yb}(0_\mathfrak{g}), \mathrm{yb}(1_\mathfrak{g})\}$ is the smallest j such that $\mathfrak{g}$ is an edge of D_j. For $x, y \in D$, x is *younger* than y if $\mathrm{yb}(x) > \mathrm{yb}(y)$, and similar terminology applies for intervals. Note that an interval $\mathfrak{g}$ can contain elements younger than $\mathfrak{g}$ itself.

(v) For $u \in \mathrm{Traj}(D)$, we define $\mathrm{yb}(u)$ as $\mathrm{yb}(\mathfrak{h}(u))$. The trajectory of $D_{\mathrm{yb}(u)}$ that contains $\mathfrak{h}(u)$ is denoted by $\mathrm{btr}(u)$; now the acronym comes from "birth trajectory". Clearly, u is a straight trajectory iff $\mathrm{yb}(u) = 0$. Also, u is a hat-trajectory iff $\mathrm{yb}(u) > 0$.

(vi) For $\mathrm{yb}(u) \le j \le t$, the trajectory of D_j through the edge $\mathfrak{h}(u)$ is denoted by $\mathrm{anc}(u, j)$, and it is called an *ancestor* of u. (Observation 6.8(iv) will show that $\mathrm{anc}(u, j)$ exists.) In particular, we have that $\mathrm{anc}(u, \mathrm{yb}(u)) = \mathrm{btr}(u)$.

(vii) The *original territory* of u, denoted by $\mathrm{Terr}_{\mathrm{orig}}(u)$, is the territory of $\mathrm{btr}(u)$ in $D_{\mathrm{yb}(u)}$. For example, in Figure 10, $\mathrm{Terr}_{\mathrm{orig}}(u)$ is the grey area (dark grey and light grey together). The *original upper border* of u is the upper border of $\mathrm{btr}(u)$ in $D_{\mathrm{yb}(u)}$; it is a broken line consisting of several (possibly, one) straight line segments in the plane. Similarly, the *original lower border* of u is the lower border of $\mathrm{btr}(u)$ in $D_{\mathrm{yb}(u)}$.

(viii) The *halo square* of u is the 4-cell $H_{\mathrm{yb}(u)-1}$ of $D_{\mathrm{yb}(u)-1}$ into which the multifork giving birth to u is inserted. A straight trajectory has no halo square.

By a straight line segment *compatible with a diagram* or, if the diagram is understood, a *compatible straight line segment* we mean a straight line segment composed from consecutive edges $[x_0, x_1]$, $[x_1, x_2]$, …, $[x_{k-1}, x_k]$ of the same slope. In particular, every edge is a compatible straight line segment. When we pass from D_j to D_{j+1} in (5.10), then every edge of D_j either remains an edge of D_{j+1}, or it is divided into several new edges by new vertices. A 4-cell is formed from two top edges and two bottom edges. Observe that, by (6.1), the halo square H_j will not remain distributive in D_{j+1}. Hence, the top edges of H_j do not belong to the trajectory through a top edge of H_k for $k > j$. However, Corollary 6.5 applies to H_j when we consider it in D_j. To summarize the present paragraph, we conclude the following statement; its part (iv) follows from part (iii).

Observation 6.8. *If $D \in \mathcal{C}_1$ is a slim rectangular diagram and $u \in \mathrm{Traj}(D)$, then the following hold.*

(i) *If* $0 \le j < k \le t$*, then every compatible straight line segment of* D_j *is also a compatible straight line segment of* D_k *and, in particular, of* D*. Note that a straight light segment of* D_j *can consist of more edges in* D_k *than in* D_j*.*

(ii) *The sides of the planar polygon* $\mathrm{Terr}_{\mathrm{orig}}(u)$ *are compatible straight line segments of* D*. In particular, the upper border and the lower border of* $\mathrm{Terr}_{\mathrm{orig}}(u)$ *consist of compatible straight line segments of* D*.*

(iii) *With reference to* (5.10)*, let* $j < k \le t$*. The upper edges of the halo square* H_j *are of normal slopes, and they are also edges of* D_k *and, in particular, of* D*. Furthermore, denoting* 1_{H_j} *by* 1_j*, the edges of the form* $[x, 1_j]$ *are the same in* D_k *and, in particular, in* D *as in* D_{j+1}*. That is,* $\{x \in D_{j+1} : x \prec 1_j\} = \{x \in D_k : x \prec 1_j\} = \{x \in D : x \prec 1_j\}$*.*

(iv) *For* $\mathrm{yb}(u) \le j \le t$*,* $\mathrm{anc}(u, j)$ *exists and* $\mathfrak{h}(\mathrm{anc}(u, j)) = \mathfrak{h}(u)$*. In particular,* $\mathfrak{h}(\mathrm{btr}(u)) = \mathfrak{h}(u)$*.*

As a straightforward consequence of Corollary 6.4, we have

Remark 6.9. If $u \in \mathrm{Traj}(D)$ for a slim rectangular $D \in \mathcal{C}_1$, then the lower border B of u and the original lower border of u are the same (straight or broken) lines in the plane and they consists of compatible straight line segments. Furthermore, for all $j \in \{\mathrm{yb}(u), \dots, t\}$, the lower border of $\mathrm{anc}(u, j)$ is also B.

Proof. A trivial induction based on Lemma 5.7. □

As an illustration for the following lemma, see Figure 10.

Lemma 6.10. *Let* $D \in \mathcal{C}_1$ *be a slim rectangular diagram, and let* u *be a trajectory of* D*. If* u *is a straight trajectory, then its original territory, denoted by* $\mathrm{Terr}_{\mathrm{orig}}(u)$*, is a rectangle whose sides are compatible straight line segments with normal slopes. If* u *is a hat-trajectory, then the polygon* $\mathrm{Terr}_{\mathrm{orig}}(u)$ *is bordered by one or two precipitous edges belonging to its upper border and containing* $1_{\mathfrak{h}(u)}$ *as an endpoint, and compatible straight line segments of normal slopes.*

Proof. Clearly, all edges of D_0 in (5.10), are of normal slope. Hence, the first part of the lemma follows, because $\mathrm{yb}(u) = 0$, provided u is a straight trajectory. Next, assume that u is a hat-trajectory, that is, $\mathrm{yb}(u) > 0$. Since the halo square $H_{\mathrm{yb}(u)-1}$ of u is a distributive 4-cell of $D_{\mathrm{yb}(u)-1}$, (6.1) implies that no element of the ideal $\downarrow 1_{\mathfrak{h}(u)}$ can have more than 2 lower covers in $D_{\mathrm{yb}(u)-1}$. Hence, the rest of the lemma follows from Observation 6.2. □

As a useful supplement to Observation 6.2, we formulate the following.

Observation 6.11. *If* $D \in \mathcal{C}_1$ *is a slim rectangular lattice diagram,* $x, y \in D$*, and* $x \prec y$*, then the following three conditions are equivalent.*

(i) *The edge* $[x, y]$ *is precipitous.*

(ii) y *has at least three lower covers and* x *is neither the leftmost, nor the rightmost of them.*

(iii) *The trajectory* u *containing* $[x, y]$ *is a hat-trajectory and* $[x, y]$ *is* $\mathfrak{h}(u)$*.*

Proof. A trivial induction based on Lemma 5.7. □

7. Another version of the Trajectory Coloring Theorem

The set of prime intervals of a finite lattice M is denoted by $\mathrm{PrInt}(M)$. (An interval $[x, y]$ is *prime* if $x \prec y$.) For a quasiordering (reflexive and transitive relation) ν, $x \le_\nu y$ stands for $\langle x, y\rangle \in \nu$.

Definition 7.1 (G. Czédli [3, page 317])**.** A *quasi-colored lattice* is a finite lattice M with a surjective map γ, called *quasi-coloring*, from $\mathrm{PrInt}(M)$ onto a quasiordered set $\langle A; \nu\rangle$ such that γ satisfies the following two properties:

(C1) if $\gamma(\mathfrak{p}) \ge_\nu \gamma(\mathfrak{q})$, then $\mathrm{con}(\mathfrak{p}) \ge \mathrm{con}(\mathfrak{q})$,
(C2) if $\mathrm{con}(\mathfrak{p}) \ge \mathrm{con}(\mathfrak{q})$, then $\gamma(\mathfrak{p}) \ge_\nu \gamma(\mathfrak{q})$.

If, in addition, $\langle A; \nu\rangle$ is an ordered set, then γ is called a *coloring*; this concept is due to G. Grätzer and E. Knapp [19].

For $u \in \mathrm{Traj}(D)$, the top edge $\mathfrak{h}(u)$ was defined in Definition 6.7(i).

Definition 7.2 (G. Czédli [5, Definitions 4.3 and 7.1])**.** Let D be a slim rectangular diagram.

(i) On the set $\mathrm{Traj}(D)$ of all trajectories of D, we define a relation σ as follows. For $u, v \in \mathrm{Traj}(D)$, we let $\langle u, v\rangle \in \sigma$ iff u is a hat-trajectory, $1_{\mathfrak{h}(u)} \le 1_{\mathfrak{h}(v)}$, but $0_{\mathfrak{h}(u)} \not\le 0_{\mathfrak{h}(v)}$.
(ii) For $u, v \in \mathrm{Traj}(D)$, we let $\langle u, v\rangle \in \Theta$ iff $u = v$, or both u and v are hat trajectories such that $1_{\mathfrak{h}(u)} = 1_{\mathfrak{h}(v)}$. The quotient set $\mathrm{Traj}(D)/\Theta$ of $\mathrm{Traj}(D)$ by the equivalence Θ is denoted $\widehat{\mathrm{Traj}}(D)$. Its elements are denoted by u/Θ, where $u \in \mathrm{Traj}(D)$.
(iii) On the set $\widehat{\mathrm{Traj}}(D)$, we define a relation $\widehat{\sigma}$ as follows. For u/Θ and v/Θ in $\widehat{\mathrm{Traj}}(D)$, we let $\langle u/\Theta, v/\Theta\rangle \in \widehat{\sigma}$ iff $u/\Theta \ne v/\Theta$ and there exist $u', v' \in \mathrm{Traj}(D)$ such that $\langle u, u'\rangle, \langle v, v'\rangle \in \Theta$ and $\langle u', v'\rangle \in \sigma$.
(iv) We let $\widehat{\tau} = \mathrm{quor}(\widehat{\sigma})$, the reflexive transitive closure of $\widehat{\sigma}$ on $\widehat{\mathrm{Traj}}(D)$.
(v) The *trajectory coloring* of D is the coloring $\widehat{\xi}$ from $\mathrm{PrInt}(D)$ onto the ordered set $\langle \widehat{\mathrm{Traj}}(D); \widehat{\tau}\rangle$, defined by the rule that $\widehat{\xi}(\mathfrak{p})$ is the Θ-block of the unique trajectory containing $\mathfrak{p}$.

We recall the following result, which carries a lot of information on the congruence lattice of a slim rectangular lattice. (By [5, Remark 8.5], the case of slim semimodular lattices reduces to the slim rectangular case.) Note that the original version of the proposition below assumes slightly less, $D \in \mathcal{C}_0$.

Proposition 7.3 (G. Czédli [5, Theorem 7.3(i)])**.** *If L is a slim rectangular lattice with a diagram $D \in \mathcal{C}_1$, then $\langle \widehat{\mathrm{Traj}}(D); \widehat{\tau}\rangle$ is an ordered set and it is isomorphic to $\langle \mathrm{Ji}(\mathrm{Con}\, L); \le\rangle$. Furthermore, $\widehat{\xi}$ in Definition 7.2*(v) *is a coloring.*

The fact that the key relation $\widehat{\tau}$ is defined as a transitive (and reflexive) closure is probably inevitable. However, the complicated definition of $\widehat{\sigma}$, whose reflexive transitive closure is taken, makes Proposition 7.3 a bit difficult to use. Hence, we introduce the following concept. For $u, v \in \mathrm{Traj}(D)$, we say that u is a *descendant* of v, in notation $u <_{\mathrm{desc}} v$, if $\mathrm{yb}(u) > \mathrm{yb}(v)$ and the halo square of u, as a geometric quadrangle, is within the original territory $\mathrm{Terr}_{\mathrm{orig}}(v)$ of v. Note that "descendant" is an irreflexive relation. Note also that, as opposed to "in" for containment, in geometric sense we always use the preposition "within". That is, "A is within B" means that A and B are geometric polygons (closed subsets of the complex plane that contain their inner points) such that A is a subset of B. For a point x, if $x \in B$, then we also say that x is within B to express that B is a polygon.

We are now in the position to formulate the main achievement of the present section. Since it looks quite technical in itself, let us emphasize that the following theorem is to be used together with Proposition 7.3, where $\widehat{\tau}$ is the transitive reflexive closure of $\widehat{\sigma}$, described pictorially in the theorem below.

Theorem 7.4. *For a slim rectangular diagram $D \in \mathcal{C}_1$ and $u, v \in \mathrm{Traj}(D)$, $\langle u/\Theta, v/\Theta\rangle \in \widehat{\sigma}$ iff there are $u' \in u/\Theta$ and $v' \in v/\Theta$ such that $u' <_{\mathrm{desc}} v'$.*

Proof. First of all, note that for any $w \in \mathrm{Traj}(D)$ and $w' \in w/\Theta$, we have that $\mathrm{yb}(w') = \mathrm{yb}(w)$. This allows us to define $\mathrm{yb}(w/\Theta)$ as $\mathrm{yb}(w)$.

In order to prove the "if" part, assume that $u' <_{\mathrm{desc}} v'$. Since $\langle u/\Theta, v/\Theta\rangle = \langle u'/\Theta, v'/\Theta\rangle$, what we have to show is that $\langle u'/\Theta, v'/\Theta\rangle \in \widehat{\sigma}$. Actually, to ease the notation, we can assume that $u <_{\mathrm{desc}} v$, and we want to show that

$$\langle u/\Theta, v/\Theta\rangle \in \widehat{\sigma}. \tag{7.1}$$

We know from $u <_{\mathrm{desc}} v$ that the halo square of u is within $\mathrm{Terr}_{\mathrm{orig}}(v)$. Clearly, u is a hat-trajectory. Since $0_{\mathfrak{h}(u)}$ is within the interior of this square, it is geometrically (strictly) above the original lower border of v. By Remark 6.9, $0_{\mathfrak{h}(u)}$ is geometrically above the lower border of v. Hence, Corollaries 6.1 and 6.4 imply that $0_{\mathfrak{h}(u)} \nleq 0_{\mathfrak{h}(v)}$. On the other hand, the position of the halo square of u yields that $1_{\mathfrak{h}(u)}$ is within the original territory of v. Hence, Corollary 6.1 and Lemma 6.10 imply that $1_{\mathfrak{h}(u)} \leq 1_{\mathfrak{h}(v)}$. Thus, we conclude that $\langle u, v\rangle \in \sigma$, which implies (7.1) and the "if" part of Theorem 7.4.

In order to prove the "only if" part, assume that $\langle u/\Theta, v/\Theta\rangle \in \widehat{\sigma}$. Hence, there are $u' \in u/\Theta$ and $v^* \in v/\Theta$ such that $\langle u', v^*\rangle \in \sigma$. This means that u' is a hat-trajectory, $0_{\mathfrak{h}(u')} \nleq 0_{\mathfrak{h}(v^*)}$, and $1_{\mathfrak{h}(u')} \leq 1_{\mathfrak{h}(v^*)} = 1_{\mathfrak{h}(v)}$. Our purpose is to find a $v' \in v/\Theta$ such that $u' <_{\mathrm{desc}} v'$. We claim that u' is "younger" than v^*, that is,

$$i := \mathrm{yb}(u) = \mathrm{yb}(u') > \mathrm{yb}(v^*) = \mathrm{yb}(v) =: j. \tag{7.2}$$

The equalities are clear by the first sentence of the proof. To show the inequality in (7.2), there are two cases to consider. First, assume that $1_{\mathfrak{h}(u')} = 1_{\mathfrak{h}(v^*)}$. Since $\langle u', v^*\rangle \notin \Theta$ by the definition of $\widehat{\sigma}$ and u' is a hat-trajectory, we obtain that v^* is a straight trajectory. Thus, we conclude that $i = \mathrm{yb}(u') >$

$0 = \mathrm{yb}(v^*) = j$. Second, assume that $1_{\mathfrak{h}(u')} < 1_{\mathfrak{h}(v^*)}$. Clearly, $i = \mathrm{yb}(u') \neq \mathrm{yb}(v^*) = j$. Suppose, for a contradiction, that $i < j$. Then v is a hat-trajectory and $1_{\mathfrak{h}(u')} < 1_{\mathfrak{h}(v^*)} = 1_{\mathfrak{h}(v)}$. Since u' is a hat-trajectory, $1_{\mathfrak{h}(u')}$ has at least three lower covers in D_i , and the same is true in D_{j-1} by Observation 6.8(iii). But this contradicts (6.1), because $1_{\mathfrak{h}(v)}$ is the top of the halo square H_{j-1}, which is distributive in D_{j-1}. We have proved (7.2).

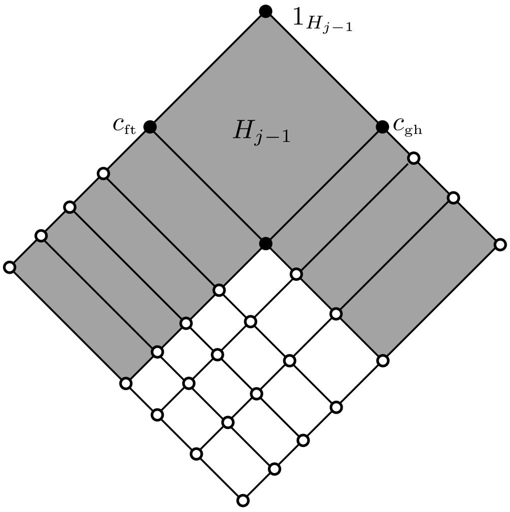

FIGURE 11. $\downarrow 1_{H_{j-1}}$ in D_{j-1} and the territory S

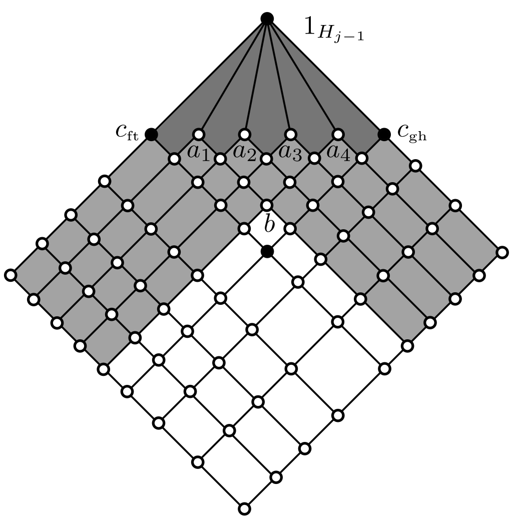

FIGURE 12. $\downarrow 1_{H_{j-1}}$ in D_j and the territory S'

Next, there are two cases to consider depending on $j > 0$ or $j = 0$.

First, we assume that $j > 0$. Then v and v^* are hat-trajectories. Note that $1_{\mathfrak{h}(v)}$ belongs to D_{j-1} and equals $1_{H_{j-1}}$. The left and right corners of H_{j-1} are denoted by $c_{\rm ft}$ and $c_{\rm gh}$, respectively. Since the halo square H_{j-1} is distributive in D_{j-1}, the ideal $\downarrow 1_{H_{j-1}}$ of D_{j-1} is a grid by Lemma 5.8(i). This ideal is illustrated in Figure 11. Corollary 6.5 yields that the edges of this ideal are of normal slopes. We denote by S the planar territory that consists of the 4-cells (in D_{j-1}) of the trajectory through $[c_{\rm gh}, 1_{H_{j-1}}]$ that are before (to the left of) $[c_{\rm gh}, 1_{H_{j-1}}]$ and also of the 4-cells of the trajectory through $[c_{\rm ft}, 1_{H_{j-1}}]$ that are after (to the right of) $[c_{\rm ft}, 1_{H_{j-1}}]$. Note that S is usually concave and that H_{j-1} is within S. In Figure 11, S is the grey-colored polygon. Since the edges of the grid $\downarrow 1_{H_{j-1}}$ are of normal slopes, S in D_{j-1} is bordered by compatible straight lines of normal slopes. Hence, by Observation 6.8(i), S in D_j, and also in D_{i-1}, is bordered by compatible straight line segments of normal slopes. Listed from left to right, let $a_1, \dots, a_k$ be the new lower covers of $1_{\mathfrak{h}(v)} = 1_{H_{j-1}}$ in D_j; for $k = 4$, see Figure 12. By Observation 6.8(iii), $c_{\rm ft}, a_1, \dots, a_k, c_{\rm gh}$ is the full list, again from left to right, of all lower covers of $1_{\mathfrak{h}(v)}$ in D. With the notation $b = a_1 \wedge a_k = a_1 \wedge \cdots \wedge a_k$, the ideal $\downarrow b$ of D_j determines a territory I. Since every element of $\downarrow 1_{H_{j-1}}$ in D_{j-1} has at most two lower covers by (6.1), it follows from the multifork construction that every element of $\downarrow b$ in D_j has at most two lower covers. Therefore, Observation 6.2 or 6.11 and Observation 6.8(i) yield that the territory I is bordered by edges of normal slopes in D_j and by compatible straight line segments of normal slopes in D_{i-1}. Consequently, the territory $S' = S \setminus \mathrm{Interior}(I)$ is again bordered by edges of D_j and by compatible straight line segments of D_{i-1} with normal slopes. In Figure 12, S' is the grey (dark and light grey together) territory. As earlier, a 4-cell is 4-*cell with normal slopes* if all of its edges are of normal slopes. In D_j, S' is a union of 4-cells. Namely, it is the union of 4-cells that belong to the new trajectories that the latest (the j-th) fork extension yielded. Among these 4-cells, those containing $1_{\mathfrak{h}(v)}$ are not with normal slopes. They will be called the *dark-grey cells*, and they are depicted in Figure 12 accordingly. We know from G. Czédli and E. T. Schmidt [14, Lemma 13], see also G. Czédli and G. Grätzer [10, Ex. 3.41], that two neighboring lower covers of an element in a slim semimodular diagram always generate a cover-preserving square, that is, a 4-cell. Hence, we obtain from Observation 6.8 (iii) that

$$\text{the dark-grey 4-cells are also 4-cells in } D \text{ and in } D_i. \tag{7.3}$$

It follows from the multifork construction that $1_{H_{j-1}}$ is the only element of $\downarrow 1_{H_{j-1}}$ that has more than two lower covers. Hence, by Observation 6.2, the rest of the 4-cells of D_j within S' are of normal slopes; they are called *light-grey* 4-*cells*, and so they are depicted in Figure 12. Although the light-grey 4-cells are not necessarily 4-cells of D_{i-1}, we know from Observation 6.8(i) that they are bordered by compatible straight line segments of D_{i-1}. Finally, S includes some additional 4-cells that are not in S'; they are of normal slopes by Observation 6.2, so they are also bordered by compatible straight line segments of D_{i-1}, and they are uncolored in the figure. Clearly, the *compatible* straight line segments of D_{i-1}

$$\text{cannot cut the 4-cell } H_{i-1} \text{ into two halves of positive area.} \tag{7.4}$$

Furthermore, since H_{i-1} has at least one interior element in D_i, (7.3) gives that H_{i-1} cannot be within a dark-grey 4-cell. Hence, we conclude from (7.4) that there is a unique light-grey or uncolored 4-cell C of D_j within S such that H_{i-1} is within the territory determined by C. (Possibly but not necessarily, $H_{i-1} = C$.) By definitions, $\mathfrak{h}(v^*)$ and $\mathfrak{h}(v)$ are in the set $\{[a_1, 1_{\mathfrak{h}(v)}], \dots, [a_k, 1_{\mathfrak{h}(v)}]\}$ of edges. If $0_{\mathfrak{h}(u')} \le b = a_1 \wedge \cdots \wedge a_k$, then $0_{\mathfrak{h}(u')} \le a_m$ for all $m \in \{1, \dots, k\}$, which contradicts $0_{\mathfrak{h}(u')} \not\le 0_{\mathfrak{h}(v^*)}$. Thus, $0_{\mathfrak{h}(u')} \not\le b$. Combining this with $0_{\mathfrak{h}(u')} \le 1_C$, it follows that C cannot be an uncolored 4-cell. Hence, C is a light-grey colored 4-cell in D_j. Therefore, there is a (unique) $m \in \{1, \dots, k\}$ such that C is a 4-cell of the hat-trajectory w_m of D_j with top edge $[a_m, 1_{\mathfrak{h}(v)}]$. Since $a_m \prec_D 1_{\mathfrak{h}(v)}$ by Observation 6.8(iii), we can also consider the trajectory $v' \in \mathrm{Traj}(D)$ that contains $[a_m, 1_{\mathfrak{h}(v)}]$; actually, $[a_m, 1_{\mathfrak{h}(v)}]$ is the top edge of v' by Observation 6.11. Clearly, $w_m = \mathrm{btr}(v')$, C is within $\mathrm{Terr}_{\mathrm{orig}}(v') = \mathrm{Terr}(w_m)$, and $v' \in v/\Theta$. But H_{i-1} is within C, so H_{i-1} is also within the original territory $\mathrm{Terr}_{\mathrm{orig}}(v')$ of v'. Consequently, $u' <_{\mathrm{desc}} v'$.

Second, we assume that $j = 0$. Then v is a straight trajectory, v/Θ is a singleton, and $u' <_{\mathrm{desc}} v'$ follows in a similar but in a much easier way; the details are omitted. This completes the proof of Theorem 7.4 □

8. G. Grätzer's Swing Lemma

For a slim rectangular lattice diagram $D \in \mathcal{C}_0$ and prime intervals $\mathfrak{p}$ and $\mathfrak{q}$ of D, we say that $\mathfrak{p}$ *swings* to $\mathfrak{q}$, in notation, $\mathfrak{p} \curvearrowright \mathfrak{q}$, if $1_{\mathfrak{p}} = 1_{\mathfrak{q}}$, $1_{\mathfrak{p}}$ has at least three lower covers, and $0_{\mathfrak{q}}$ is neither the leftmost, nor the rightmost lower cover of $1_{\mathfrak{p}}$. If D is in $\mathcal{C}_1$, not only in $\mathcal{C}_0$, then Observation 6.11 implies that

$$\mathfrak{p} \curvearrowright \mathfrak{q} \quad \text{iff} \quad 1_{\mathfrak{p}} = 1_{\mathfrak{q}} \text{ and } \mathfrak{q} \text{ is a precipitous edge.} \tag{8.1}$$

As usual, $\mathfrak{p}$ is *up-perspective* to $\mathfrak{q}$, in notation, $\mathfrak{p} \overset{\mathrm{up}}{\sim} \mathfrak{q}$, if $1_{\mathfrak{p}} \vee 0_{\mathfrak{q}} = 1_{\mathfrak{q}}$ and $1_{\mathfrak{p}} \wedge 0_{\mathfrak{q}} = 0_{\mathfrak{p}}$. Down-perspectivity is just the converse relation defined by $\mathfrak{p} \overset{\mathrm{dn}}{\sim} \mathfrak{q} \iff \mathfrak{q} \overset{\mathrm{up}}{\sim} \mathfrak{p}$. Although here we only formulate the Swing Lemma for slim *rectangular* lattices, the original version in G. Grätzer [18] is the same statement for slim *semimodular* lattices. Speaking of diagrams rather than lattices is not an essential change.

Lemma 8.1 (Swing Lemma in G. Grätzer [18])**.** *Let $D \in \mathcal{C}_0$ be a slim rectangular diagram, and let $\mathfrak{p}$ and $\mathfrak{q}$ be edges (that is, prime intervals) of D. Then the following two conditions are equivalent.*

(i) $\mathrm{con}(\mathfrak{p}) \ge \mathrm{con}(\mathfrak{q})$ *in the lattice of all congruences of D.*

(ii) *There exist an $n \in \mathbb{N}_0$ and edges $\mathfrak{r} = \mathfrak{r}_0, \mathfrak{r}_1, \dots, \mathfrak{r}_n = \mathfrak{q}$ in D such that $\mathfrak{p} \overset{\mathrm{up}}{\sim} \mathfrak{r}$, $\mathfrak{r}_{i-1} \overset{\mathrm{dn}}{\sim} \mathfrak{r}_i$ for $i \in \{1, \dots, n\}$, i odd, and $\mathfrak{r}_{i-1} \curvearrowright \mathfrak{r}_i$ for $i \in \{1, \dots, n\}$, i even.*

Proof. Our argument relies, among other ingredients, on the multifork construction sequence of D, and the notation in (5.10) will be in effect.

Let L be the lattice determined by D. By Theorem 5.5(ii), L also has a diagram D' in $\mathcal{C}_1$. Obviously, L is glued sum indecomposable. Thus, Proposition 5.1 implies that every planar diagram of L is similar to D or $D^{(\mathrm{mi})}$. Hence, D' is similar to D or to $D^{(\mathrm{mi})}$. Since the statement of the lemma is obviously invariant under left-right similarity, we can assume that $D = D' \in \mathcal{C}_1$.

Assume (ii). We claim that for prime intervals $\mathfrak{r}'$ and $\mathfrak{r}''$ of D,

$$\text{if } \mathfrak{r}' \hookrightarrow \mathfrak{r}'', \text{ then } \operatorname{con}(\mathfrak{r}') \supseteq \operatorname{con}(\mathfrak{r}''). \tag{8.2}$$

Assume that $\mathfrak{r}' \hookrightarrow \mathfrak{r}''$. It follows from the definition of $\hookrightarrow$ and Observation 6.11(iii) that $0_{\mathfrak{r}''}$ is a source element in $D_{\mathrm{yb}(\mathfrak{r}'')} \setminus D_{\mathrm{yb}(\mathfrak{r}'')-1}$, and either the same holds for $0_{\mathfrak{r}'}$, or $0_{\mathfrak{r}'}$ is a corner of the halo square $H_{\mathrm{yb}(\mathfrak{r}'')-1}$. In both cases, since $1_{\mathfrak{r}'} = 1_{\mathfrak{r}''}$, $\operatorname{con}(\mathfrak{r}') \supseteq \operatorname{con}(\mathfrak{r}'')$ follows in a straightforward way. This proves (8.2). On the other hand, if $\mathfrak{r}' \overset{\mathrm{dn}}{\sim} \mathfrak{r}''$, then $\operatorname{con}(\mathfrak{r}') = \operatorname{con}(\mathfrak{r}'')$. Combining this with (8.2), we obtain (i). Thus, (ii) implies (i).

Before proving the converse implication, some preparations are necessary. For edges $\mathfrak{e}$ and $\mathfrak{e}'$ of D, we say that $\mathfrak{e}'$ is $\hookrightarrow\overset{\mathrm{dn}}{\sim}$-*accessible* from $\mathfrak{e}$ if there exists a finite sequence $\mathfrak{r}_0 = \mathfrak{e}, \mathfrak{r}_1, \dots, \mathfrak{r}_n = \mathfrak{e}'$ of edges such that, for every $i \in \{1, \dots, n\}$, either $\mathfrak{r}_{i-1} \hookrightarrow \mathfrak{r}_i$, or $\mathfrak{r}_{i-1} \overset{\mathrm{dn}}{\sim} \mathfrak{r}_i$. For $j \in \{0, \dots, t\}$, D_j determines a sublattice in D. We claim that for common edges $\mathfrak{e}$ and $\mathfrak{e}'$ of D_j and D,

$$\begin{array}{l}\text{if } \mathfrak{e}' \text{ is } \hookrightarrow\overset{\mathrm{dn}}{\sim}\text{-accessible from } \mathfrak{e} \text{ in } D_j, \text{ then}\\ \text{it is also } \hookrightarrow\overset{\mathrm{dn}}{\sim}\text{-accessible from } \mathfrak{e} \text{ in } D.\end{array} \tag{8.3}$$

In order to prove this, we have to show that $\mathfrak{r}_n = \mathfrak{e}', \mathfrak{r}_{n-1}, \dots$ are also edges in D. If $\mathfrak{r}_i$ is an edge, that is, a prime interval of D, $i > 0$, and $\mathfrak{r}_{i-1} \overset{\mathrm{dn}}{\sim} \mathfrak{r}_i$, then $\mathfrak{r}_{i-1}$ is also a prime interval in D by semimodularity. If $\mathfrak{r}_i$ is a prime interval of D, $i > 0$, and $\mathfrak{r}_{i-1} \hookrightarrow \mathfrak{r}_i$, then $\mathfrak{r}_{i-1}$ is also a prime interval in D by Observation 6.8(iii). This completes the induction proving (8.3).

For a trajectory $w \in \operatorname{Traj}(D)$, the original territory $\operatorname{Terr}_{\mathrm{orig}}(w)$ of w can be divided into two parts; note that one of these parts is empty iff w is a straight trajectory. The union of the 4-cells (as quadrangles in the plane) of $\operatorname{btr}(w)$ before $\mathfrak{h}(\operatorname{btr}(w))$ (if we walk from left to right along $\operatorname{btr}(w)$) is the "*before the top edge*" part, and this polygon is denoted by $B(w)$. Similarly, the union of the 4-cells of $\operatorname{btr}(w)$ after $\mathfrak{h}(\operatorname{btr}(w))$ is the "*after the top edge*" part, and it is denoted by $A(w)$. Note that

$$\operatorname{Terr}_{\mathrm{orig}}(w) = B(w) \cup A(w). \tag{8.4}$$

If w is a hat-trajectory, then both $B(w)$ and $A(w)$ are polygons of positive area and

$$\begin{array}{l}\text{each of } B(w) \text{ and } A(w) \text{ has one or two precipitous}\\ \text{sides, which contain (that is, end at) } 1_{\mathfrak{h}(w)}, \text{ and the}\\ \text{rest of the sides are of normal slopes;}\end{array} \tag{8.5}$$

this follows from Observation 6.11, the construction of the multifork construction sequence (5.10), and Corollary 6.5. If w is a straight trajectory, then

Corollary 6.5 yields that one of $B(w)$ and $A(w)$ is a rectangle whose sides are of normal slope while the other one is the edge $\mathfrak{h}(\mathrm{btr}(w))$, that is, a degenerate rectangle. No matter if w is a hat-trajectory or a straight one, an edge $\mathfrak{g}$ of D is said to be *quasi-parallel* to $\mathfrak{h}(w)$, in notation, $\mathfrak{g} \parallel_{\mathrm{quasi}} \mathfrak{h}(w)$, if $\mathfrak{g}$ is within (that is, both $0_{\mathfrak{g}}$ and $1_{\mathfrak{g}}$ are within) the original territory $\mathrm{Terr}_{\mathrm{orig}}(w)$ of w, and either $\mathfrak{g}$ is in $B(w)$ and it is of slope $3\pi/4$ (that is, $135°$), or $\mathfrak{g}$ is in $A(w)$ and it is of slope $\pi/4$. If $\mathfrak{g} \parallel_{\mathrm{quasi}} \mathfrak{h}(w)$, then $\mathfrak{g}$ is of normal slope by definition. Observe that $\parallel_{\mathrm{quasi}}$ is not a symmetric relation. Let us emphasize that, by definition, $\mathfrak{g} \parallel_{\mathrm{quasi}} \mathfrak{h}(w)$ implies that $\mathfrak{g}$ is within $\mathrm{Terr}_{\mathrm{orig}}(w)$. Note that if $\mathfrak{g}$ is within $\mathrm{Terr}_{\mathrm{orig}}(w)$, then it is within $B(w)$ or within $A(w)$, but it is not necessarily quasi-parallel to $\mathfrak{h}(w)$. We say that an edge $\mathfrak{f}$ is, say, on the lower border of $\mathrm{Terr}_{\mathrm{orig}}(w)$ if both $0_{\mathfrak{f}}$ and $1_{\mathfrak{f}}$ are on this lower border. We conclude from Lemma 6.10 that

$$\text{if } \mathfrak{g} \parallel_{\mathrm{quasi}} \mathfrak{h}(w), \text{ then } \mathfrak{g} \text{ is neither on the lower border, nor on the upper border of } \mathrm{Terr}_{\mathrm{orig}}(w). \tag{8.6}$$

We claim that, for every edge $\mathfrak{g}$ of D and every $w \in \mathrm{Traj}(D)$,

$$\text{if } \mathfrak{g} \parallel_{\mathrm{quasi}} \mathfrak{h}(w), \text{ then } \mathfrak{g} \text{ is } \hookrightarrow\!\overset{\mathrm{dn}}{\sim}\text{-accessible from } \mathfrak{h}(w). \tag{8.7}$$

We prove this by induction on $\mathrm{yb}(\mathfrak{g})$. Assume that $\mathfrak{g} \parallel_{\mathrm{quasi}} \mathfrak{h}(w)$. We can also assume that $\mathfrak{g} \neq \mathfrak{h}(w)$, since otherwise (8.7) trivially holds. By definitions, $\mathfrak{g} \in \mathrm{Terr}_{\mathrm{orig}}(w) = B(w) \cup A(w)$. By left-right symmetry, we assume that $\mathfrak{g} \in B(w)$. Since $\mathfrak{g}$ is within $B(w)$ and $\mathfrak{g} \neq \mathfrak{h}(w)$, $B(w)$ is of positive area.

It follows from (8.6), the description of the multifork extension, and that of the multifork construction sequence (5.10) that $\mathrm{yb}(w) \leq \mathrm{yb}(\mathfrak{g})$. Remember that t denotes the length of the sequence (5.10). Combining Observation 6.8(iv) and (8.3), it follows that we can assume that $\mathrm{yb}(\mathfrak{g}) = t$. (Less formally speaking with more details, if $\mathfrak{g}$ came to existence earlier but not before w, then first we could show (8.7) in $D_{\mathrm{yb}(\mathfrak{g})}$ for $\mathfrak{g}$ and the ancestor $\mathrm{anc}(w, \mathrm{yb}(\mathfrak{g}))$ of w the same way we are going to show (8.7) in D, and then we could apply (8.3).)

That is, $\mathfrak{g}$ came to existence only in the last step of the multifork construction sequence, and the induction hypothesis is that for every edge $\mathfrak{g}'$ of D, if $\mathfrak{g}' \parallel_{\mathrm{quasi}} \mathfrak{h}(w)$ and $\mathfrak{g}'$ is an edge of D_{t-1}, then $\mathfrak{g}'$ is $\hookrightarrow\!\overset{\mathrm{dn}}{\sim}$-accessible from $\mathfrak{h}(w)$ in D. We can also assume that $\mathrm{yb}(w) < t$, because otherwise $\mathfrak{g} \parallel_{\mathrm{quasi}} \mathfrak{h}(w)$ gives $\mathfrak{g} \in w$ and (8.7) follows from $\mathfrak{h}(w) \overset{\mathrm{dn}}{\sim} \mathfrak{g}$. It follows from the description of multifork extensions and (5.10) that there is a hat-trajectory z of D that is "responsible" for the fact that $\mathfrak{g}$ came to existence. Since there are two essentially different ways of the above-mentioned responsibility, we have to distinguish two cases.

Case 1. We assume that $\mathfrak{g} \in z$. Let

$$U = \{\mathfrak{g}' \in z : \mathfrak{g}' \parallel_{\mathrm{quasi}} \mathfrak{h}(w) \text{ and } \mathfrak{g}' \text{ is on the right of } \mathfrak{g}\}.$$

Being "on the right" above means that when we walk along z, then $\mathfrak{g}'$ comes later than $\mathfrak{g}$ or $\mathfrak{g}' = \mathfrak{g}$. Note that $\mathfrak{g} \in U$. For an illustration, see Figure 13, where $B(w)$ is the (light and dark) grey area and $U = \{\mathfrak{g}_0, \dots, \mathfrak{g}_4\}$. If w is a hat-trajectory, then, in accordance with (8.5), we denote the vertices of the polygon $B(w)$ by a, $b = 0_{\mathfrak{h}(w)}$, $c = 1_{\mathfrak{h}(w)}$, d, and e; anticlock-wise, starting from the bottom a. Except possibly for the edge $[d, c]$, which could be of slope $\pi/4$ (and then d is not a vertex of the polygon), the slopes of the sides of $B(w)$ are faithfully depicted in Figure 13. In particular, $\mathfrak{h}(w) = [b, c]$ is precipitous, if w is a hat-trajectory. On the other hand, if w is a straight trajectory, then $\mathfrak{h}(w)$ is on the upper right boundary of D and D_0 (because otherwise $B(w)$ would not be of positive area), the edges $[b, c]$ and $[d, c]$ are of slopes $3\pi/4$ and $\pi/4$, respectively, while the slopes of the other sides of the polygon $B(w)$ are faithfully depicted. (Note that d is not a vertex of the polygon in this case.) We claim that, for every edge $\mathfrak{g}'$ of D,

$$\text{if } \mathfrak{g}' \in U, \quad \text{then } \mathfrak{g}' \text{ is not on } \mathrm{C_r}(D). \tag{8.8}$$

In order to prove this, assume that $\mathfrak{g}' \in U$. Since $\mathrm{yb}(\mathfrak{g}') = t > \mathrm{yb}(w)$, $\mathfrak{g}' \neq \mathfrak{h}(w)$. We know from $\mathfrak{g}' \parallel_{\text{quasi}} \mathfrak{h}(w)$ that $\mathfrak{g}'$ is of slope $3\pi/4$. Observe that that $1_{\mathfrak{g}'} \neq 1_{\mathfrak{h}(w)}$, because otherwise either $\mathfrak{h}(w)$ is precipitous and $0_{\mathfrak{g}'}$ is not within $B(w)$, or the edge $\mathfrak{h}(w)$ is of slope $3\pi/4$ and $\mathfrak{g}' = \mathfrak{h}(w)$. Being within $B(w)$, $1_{\mathfrak{g}'}$ cannot be strictly greater than $1_{\mathfrak{h}(w)}$. Hence, using that $1_{\mathfrak{h}(w)}$ is the only cover of $0_{\mathfrak{h}(w)}$ in D, we obtain that $1_{\mathfrak{g}'} \not\geq 0_{\mathfrak{h}(w)}$. We also obtain that $1_{\mathfrak{g}'} \not\leq 0_{\mathfrak{h}(w)}$, because otherwise Corollary 6.1 yields that $1_{\mathfrak{g}'}$, which is within $B(w)$, is on the lower right border (from a to b) of $B(w)$, but then $0_{\mathfrak{g}'}$ cannot be within $B(w)$ since $\mathfrak{g}'$ is of slope $3\pi/4$. So, $1_{\mathfrak{g}'} \parallel 0_{\mathfrak{h}(w)}$. Since the lower right border of $B(w)$, from a to b, is a compatible straight line by Observation 6.8(i)–(ii), it is also a chain in D. Extend this chain to a maximal chain $\tilde{C}$ of D such that $c = 1_{\mathfrak{h}(w)} \in \tilde{C}$. Being within $B(w)$, $1_{\mathfrak{g}'}$ is on the left of $\tilde{C}$. Since $1_{\mathfrak{g}'} \parallel 0_{\mathfrak{h}(w)} = b \in \tilde{C}$, $1_{\mathfrak{g}'}$ is strictly on the left of $\tilde{C}$. Using Lemma 3.2, it follows that $1_{\mathfrak{g}'} \lambda\, 0_{\mathfrak{h}(w)} = b$ and $1_{\mathfrak{g}'} \notin \mathrm{C_r}(D)$. This proves (8.8).

Trajectories go from left to right. We claim that, for every every $\mathfrak{g}' \in U$,

$$z \text{ does not terminate at } \mathfrak{g}' \text{ and goes upwards at } \mathfrak{g}'. \tag{8.9}$$

The first part follows from (8.8). Suppose, for a contradiction, that z goes downwards at $\mathfrak{g}' \in U$ or z makes a turn to the lower right at $\mathfrak{g}' \in U$. This means that $\mathfrak{g}'$ is the upper left edge of a 4-cell. The slope of the upper right edge of this 4-cell is greater than that of $\mathfrak{g}'$, which is $3\pi/4$ since $\mathfrak{g}' \parallel_{\text{quasi}} \mathfrak{h}(w)$ and $\mathfrak{g}'$ is within $B(w)$. Hence, D has an edge with slope greater than $3\pi/4$. This is a contradiction, because every edge is either precipitous or is of normal slope by Observation 6.2 . Thus, we conclude (8.9).

Listing from left to right, let $\mathfrak{g} = \mathfrak{g}_0, \dots, \mathfrak{g}_k$ be the edges of U, let $\mathfrak{g}_{k+1}$ be the next edge of z, and let C_k be the 4-cell of z formed by $\mathfrak{g}_k$ and $\mathfrak{g}_{k+1}$. In Figure 13, $k = 4$ and $\mathfrak{g}_0, \dots, \mathfrak{g}_{k+1}$ are the thick edges. (8.9) yields that $\mathfrak{g}_{k+1}$ exists. Since $\mathfrak{g}_k$ belongs to U, it is of slope $3\pi/4$. Hence, except possibly for the

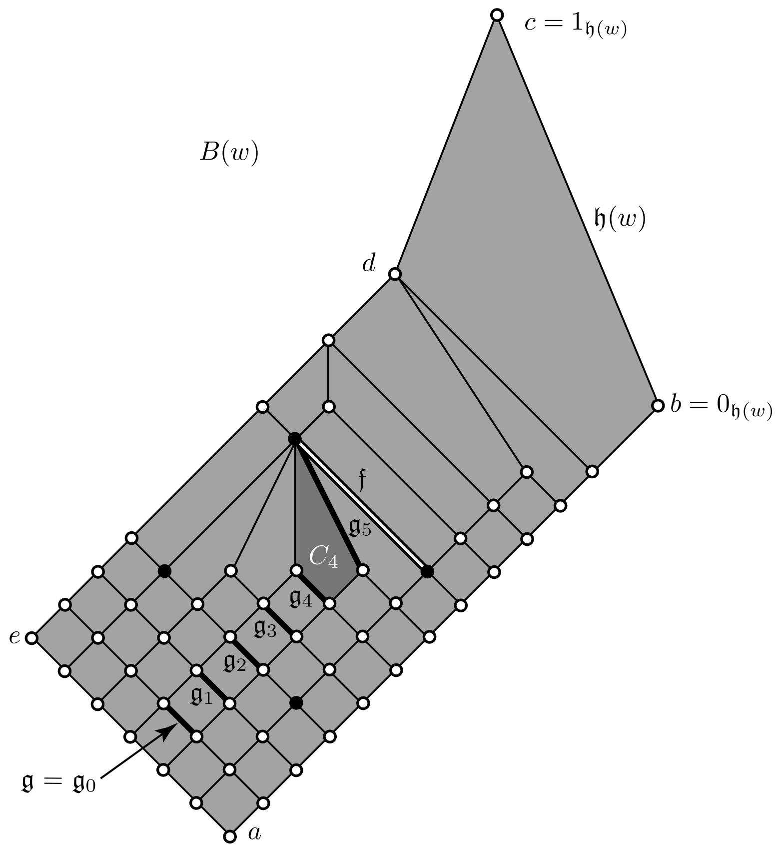

FIGURE 13. $B(w)$ and $\mathfrak{g} \in z$

side from a to e, $\mathfrak{g}_k$ does not lie on the sides of the polygon $B(w)$. Therefore, since $\mathfrak{g}_k$ is within $B(w)$, $B(w) \cap \operatorname{Terr}(C_k)$ is of positive area. However, the sides of $B(w)$, which are compatible straight line segments, cannot divide $\operatorname{Terr}(C_k)$, formed by edges of D, into two parts of positive area. Hence, $\operatorname{Terr}(C_k)$ is within $B(w)$, that is, $\operatorname{Terr}(C_k) \subseteq B(w)$. In particular, $\mathfrak{g}_{k+1}$ is within $B(w)$. So, the definition of U implies that $\mathfrak{g}_{k+1} \nparallel_{\text{quasi}} \mathfrak{h}(w)$. This and Observation 6.2 imply that either $\mathfrak{g}_{k+1}$ is precipitous, or it is of slope $\pi/4$. Applying Corollary 6.4 to z, we obtain that the slope of the edge $[0_{\mathfrak{g}_k}, 0_{\mathfrak{g}_{k+1}}]$, which is distinct from that of $\mathfrak{g}_k$, is $\pi/4$. It follows that $\mathfrak{g}_{k+1}$ cannot be of slope $\pi/4$, because otherwise the slope of the edge $[1_{\mathfrak{g}_k}, 1_{\mathfrak{g}_{k+1}}]$ is less than $\pi/4$, contradicting Observation 6.2. Therefore, $\mathfrak{g}_{k+1}$ is precipitous, and Observation 6.11 implies that $\mathfrak{g}_{k+1} = \mathfrak{h}(z)$.

Consider the halo square H_{t-1} in D_{t-1}. Its four elements in $D = D_t$ are the black-filled elements in the figure. Since the upper edges of H_{t-1} are of normal slopes and $1_{C_k} = 1_{\mathfrak{g}_{k+1}} = 1_{\mathfrak{h}(z)} = 1_{H_{t-1}}$, Observation 6.2 implies that $\operatorname{Terr}(H_{t-1}) \cap \operatorname{Terr}(C_k)$ is of positive area. But $\operatorname{Terr}(C_k) \subseteq B(w)$, so a part of $\operatorname{Terr}(H_{t-1})$ with positive area is also within $B(w)$. This is also true in D_{t-1}. In

D_{t-1}, where H_{t-1} is a 4-cell, the sides of $B(w)$, which are compatible straight line segments, cannot divide $\mathrm{Terr}(H_{t-1})$ into two parts of positive area. Hence, $\mathrm{Terr}(H_{t-1}) \subseteq B(w)$. In particular, both upper edges of H_{t-1} are within $B(w)$. The halo square H_{t-1} is distributive in D_{t-1}. Hence, Corollary 6.5 gives that its upper edges are of normal slopes. Hence, exactly one of these upper edges, which we denote by $\mathfrak{f}$, is quasi-parallel to $\mathfrak{h}(w)$. In the figure, $\mathfrak{f}$ is drawn with double lines. Since $\mathrm{yb}(\mathfrak{f}) \leq t-1$, $\mathfrak{f}$ is $\hookrightarrow\overset{\mathrm{dn}}{\sim}$-accessible from $\mathfrak{h}(w)$ by the induction hypothesis. On the other hand, $\mathfrak{f} \hookrightarrow \mathfrak{g}_{k+1} \overset{\mathrm{dn}}{\sim} \mathfrak{g}$. Thus, transitivity yields that $\mathfrak{g}$ is $\hookrightarrow\overset{\mathrm{dn}}{\sim}$-accessible from $\mathfrak{h}(w)$, as required.

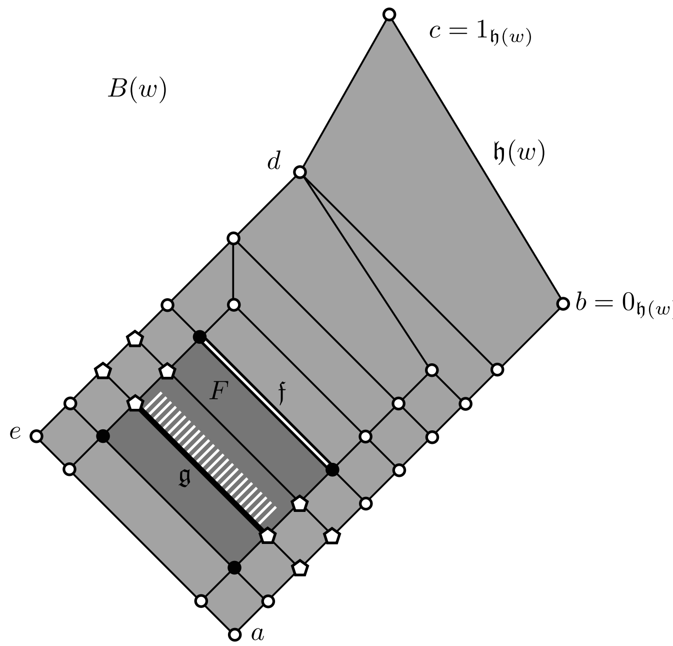

FIGURE 14. $B(w)$ and $\mathfrak{g} \notin z$

Case 2. We assume that $\mathfrak{g} \notin z$. It follows from the description of a multifork extension that there is a 4-cell F of D_{t-1} that is divided into new cells in $D = D_t$, and $\mathfrak{g}$ is one of the new edges that divide F into parts; see Figure 14, where $B(w)$ is the grey area as before, and $\mathrm{Terr}(F)$ in D is dark grey. The slopes of the sides of $B(w)$ in Figure 14 are depicted with the same accuracy as in case of Figure 13. Corollary 6.5, applied to D_{t-1} and the halo square H_{t-1} whose top is $1_{\mathfrak{h}(z)}$, implies that F is of normal slope. Since $\mathfrak{g} \parallel_{\mathrm{quasi}} \mathfrak{h}(w)$, $\mathfrak{g}$ is of slope $3\pi/4$. Using that $\mathfrak{g}$ is within $B(w)$, both $\mathfrak{g}$ and $\mathfrak{h}(w)$ are edges of D_{t-1}, and $\mathfrak{g} \neq \mathfrak{h}(w)$, we conclude that there is a narrow rectangular zone $S \subseteq B(w)$ of positive area and of normal slopes such that S is on the right of and adjacent to $\mathfrak{g}$. In the figure, S is indicated by the striped area. Also, choosing it narrow enough, S is within $\mathrm{Terr}(F)$. Since $S \subseteq \mathrm{Terr}(F) \cap B(w)$ holds not only in D but also in D_{t-1}, where F is a 4-cell, we conclude that $\mathrm{Terr}(F) \subseteq B(w)$ as in Case 1. Hence, one of the upper edges of F, which we denote by $\mathfrak{f}$, is quasi-parallel to $\mathfrak{h}(w)$. Using $\mathrm{yb}(f) \le t-1$ and the induction hypothesis, we obtain that $\mathfrak{f}$ is $\hookrightarrow\overset{\mathrm{dn}}{\sim}$-accessible from $\mathfrak{h}(w)$. So is $\mathfrak{g}$, since $\mathfrak{f} \overset{\mathrm{dn}}{\sim} \mathfrak{g}$. This completes the induction, and the proof of (8.7)

Now, we are in the position to prove the converse implication of Lemma 8.1. Assume that (i) holds, that is, $\mathrm{con}(\mathfrak{p}) \ge \mathrm{con}(\mathfrak{q})$. Denote by u and v the trajectories of D that contain $\mathfrak{p}$ and $\mathfrak{q}$, respectively. We claim that

$$\mathfrak{h}(v) \text{ is } \hookrightarrow\overset{\mathrm{dn}}{\sim}\text{-accessible from } \mathfrak{h}(u). \tag{8.10}$$

Since $\widehat{\xi}$ from Definition 7.2(v) is a coloring by Proposition 7.3, (C2) yields that

$$\langle v/\Theta, u/\Theta\rangle = \langle \widehat{\xi}(\mathfrak{q}), \widehat{\xi}(\mathfrak{p})\rangle \in \widehat{\tau} = \mathrm{quor}(\widehat{\sigma}).$$

Thus, there exist an $n \in \mathbb{N}_0 = \{0, 1, 2, \dots\}$ and a sequence $w_0 = v$, w_1, $\dots$, $w_n = u$ of trajectories of D such that $\langle w_{j-1}/\Theta, w_j/\Theta\rangle \in \widehat{\sigma}$ for $j \in \{1, \dots, n\}$. By Theorem 7.4, there are $w_0', w_1', w_1'', w_2', w_2'', \dots, w_{n-1}', w_{n-1}'', w_n'' \in \mathrm{Traj}(D)$ such that $w_j', w_j'' \in w_j/\Theta$ for $j \in \{1, \dots, n-1\}$, $w_0' \in w_0/\Theta$, $w_n'' \in w_n/\Theta$, and $w_{j-1}' <_{\mathrm{desc}} w_j''$ for $j \in \{1, \dots, n\}$. Let F_{j-1} denote the halo square of w_{j-1}' when this trajectory is born. By the definition of $<_{\mathrm{desc}}$, F_{j-1} is within $\mathrm{Terr}_{\mathrm{orig}}(w_j'')$. So are its upper edges, which are edges of D with normal slopes by Observation 6.8(iii). Hence one of these two upper edges, which we denote by $\mathfrak{f}_{j-1}$, is quasi-parallel to $\mathfrak{h}(w_j'')$. Applying (8.7), we obtain that $\mathfrak{f}_{j-1}$ is $\hookrightarrow\overset{\mathrm{dn}}{\sim}$-accessible from $\mathfrak{h}(w_j'')$. Since $\mathfrak{f}_{j-1} \hookrightarrow \mathfrak{h}(w_{j-1}')$, transitivity yields that

$$\text{for } j \in \{1, \dots, n\},\ \mathfrak{h}(w_{j-1}') \text{ is } \hookrightarrow\overset{\mathrm{dn}}{\sim}\text{-accessible from } \mathfrak{h}(w_j''). \tag{8.11}$$

The top edges of any two trajectories in the same Θ-block are $\hookrightarrow\overset{\mathrm{dn}}{\sim}$-accessible from each other; either because they are equal, or by using a $\hookrightarrow$ step. Thus,

$\mathfrak{h}(w_n'')$ and $\mathfrak{h}(w_0)$ are $\hookrightarrow\overset{\mathrm{dn}}{\sim}$-accessible from $\mathfrak{h}(w_n)$ and $\mathfrak{h}(w_0')$, respectively, and, for $j \in \{1, \dots, n-1\}$, $\mathfrak{h}(w_j'')$ is $\hookrightarrow\overset{\mathrm{dn}}{\sim}$-accessible from $\mathfrak{h}(w_j')$. (8.12)

Using transitivity, (8.11), and (8.12), we conclude (8.10). Finally, let $\mathfrak{r} = \mathfrak{h}(u)$. Since $\mathfrak{p} \in u$, we have that $\mathfrak{p} \overset{\mathrm{up}}{\sim} \mathfrak{r}$. Similarly, $\mathfrak{h}(v) \overset{\mathrm{dn}}{\sim} \mathfrak{q}$. Combining these facts with (8.10), we obtain that $\mathfrak{q}$ is $\hookrightarrow\overset{\mathrm{dn}}{\sim}$-accessible from $\mathfrak{r}$. Hence, there exists a finite sequence $\mathfrak{r} = \mathfrak{r}_0, \mathfrak{r}_1, \dots, \mathfrak{r}_k = \mathfrak{q}$ of edges such that, for each $j \in \{1, \dots, k\}$, $\mathfrak{r}_{j-1} \overset{\mathrm{dn}}{\sim} \mathfrak{r}_j$ or $\mathfrak{r}_{j-1} \hookrightarrow \mathfrak{r}_j$. However, we still have to show that the relations $\overset{\mathrm{dn}}{\sim}$ and $\hookrightarrow$ alternate and that $\overset{\mathrm{dn}}{\sim}$ is applied first, that is, $\mathfrak{r}_0 \overset{\mathrm{dn}}{\sim} \mathfrak{r}_1$.

To do so, first we prefix $\mathfrak{r}_0 \overset{\mathrm{dn}}{\sim} \mathfrak{r}_0$ to the sequence, if necessary. Next, we get rid of the unnecessary repetitions. Namely, whenever we see the pattern $\mathfrak{b} \hookrightarrow \mathfrak{b}' \hookrightarrow \mathfrak{b}''$ in the sequence, we correct it either to $\mathfrak{b} \hookrightarrow \mathfrak{b}''$ or to $\mathfrak{b}$, depending on $\mathfrak{b} \neq \mathfrak{b}''$ or $\mathfrak{b} = \mathfrak{b}''$. Knowing that $\overset{\mathrm{dn}}{\sim}$ is transitive, we correct every pattern $\mathfrak{b} \overset{\mathrm{dn}}{\sim} \mathfrak{b}' \overset{\mathrm{dn}}{\sim} \mathfrak{b}''$ to $\mathfrak{b} \overset{\mathrm{dn}}{\sim} \mathfrak{b}''$. Finally, there is no pattern to correct, and part (ii) of Lemma 8.1 holds. This completes the proof of Lemma 8.1. □

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