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Lattices with lots of congruence energy[1]

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Dedicated to Professors Ágnes Szendrei and Mária B. Szendrei

Abstract. In 1978, motivated by E. Hückel's work in quantum chemistry, I. Gutman introduced the concept of the *energy* of a finite simple graph G as the sum of the absolute values of the eigenvalues of the adjacency matrix of G. At the time of writing, the MathSciNet search for "Title=(graph energy) AND Review Text=(eigenvalue)" returns 365 publications, most of which going after Gutman's definition. A congruence α of a finite algebra A turns A into a simple graph: we connect $x \neq y \in A$ by an edge iff $(x, y) \in \alpha$; we let $\mathrm{En}(\alpha)$ be the energy of this graph. We introduce the *congruence energy* $\mathrm{CE}(A)$ of A by $\mathrm{CE}(A) := \sum\{\mathrm{En}(\alpha) : \alpha \in \mathrm{Con}(A)\}$. Let $\mathrm{LAT}(n)$ and $\mathrm{CDA}(n)$ stand for the class of n-element lattices and that of n-element congruence distributive algebras of any type. For a class $\mathcal{X}$, let $\mathrm{CE}(\mathcal{X}) := \{\mathrm{CE}(A) : A \in \mathcal{X}\}$. We prove the following. (1) For $\alpha \in A$, $\mathrm{En}(\alpha)/2$ is the height of α in the *equivalence* lattice of A. (2) The largest number and the second largest number in $\mathrm{CE}(\mathrm{LAT}(n))$ are $(n-1)\cdot 2^{n-1}$ and, for $n \geq 4$, $(n-1)\cdot 2^{n-2} + 2^{n-3}$; these numbers are only witnessed by chains and lattices with exactly one two-element antichain, respectively. (3) The largest number in $\mathrm{CE}(\mathrm{CDA}(n))$ is also $(n-1)\cdot 2^{n-1}$, and if $\mathrm{CE}(A) = (n-1)\cdot 2^{n-1}$ for an $A \in \mathrm{CDA}(n)$, then $\mathrm{Con}(A)$ is a boolean lattice with size $|\mathrm{Con}(A)| = 2^{n-1}$.

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1. Targeted readership

Most mathematicians are expected to read the *results* of this paper easily. These result might motivate analogous investigation of some algebraic structures not mentioned here. To follow the *proofs*, a little familiarity with lattice theory is assumed.

2. Outline

Sections 3 and 5 give some history and motivations. Section 4 introduces the key concepts. Section 6 translates these concepts from linear algebra to lattice theory. Section 7 states the main result of the paper, Theorem 7.1. Section 8, which comprises the majority of the paper, proves the main result.

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3. Motivations coming from quantum chemistry and graph theory

The research aiming at the energy of a graph goes back to Hückel [12], which is a quantum chemical paper published more then nine decades ago. It would be difficult to present a short survey of how the research of the energy of an unsaturated conjugated hydrocarbon molecule lead to a concept on the border line between graph theory and linear algebra. The reader can find such a survey in the introductory part of Majstorović, Klobučar, and Gutman [13]. What is important for us is that Gutman's pioneering paper [11] introduced the concept of the energy of a graph in 1978, and this concept has been studied in quite many publications since then.

The concept of energy can be extended to mathematical structures that are accompanied by graphs. This is exemplified by Pawar and Bhamre [15]. As opposed to Gutman [13], Pawar and Bhamre [15] use non-simple graphs. Here we stick to simple graphs but we need a family of them.

4. The concept we introduce

A *simple graph* is an undirected graph without loop edges and multiple edges. Let $v_1, \dots, v_n$ be a repetition-free list of all vertices of a finite simple graph G. (The order of the vertices in this list will turn out to be unimportant in (4.1) later.) The *adjacency matrix* of G is the n-by-n matrix $B = (a_{ij})_{n\times n}$ with entries 0 and 1 according to the rule that $a_{ij} = 1$ iff v_i and v_j are connected by an edge. The n-by-n unit matrix is denoted by I_n; every diagonal entry of I_n is 1 while any other entry of I_n is 0. Since B is a symmetric matrix, its characteristic polynomial is known to be the product of linear factors over $\mathbb{R}$, that is, $\det(xI_n - B) = \prod_{j:=1}^{n}(x - x_j)$ with real numbers (called eigenvalues) $x_1, \dots, x_n$. According to Gutman [11], the energy of the graph G in question is defined to be $\mathrm{En}(G) := \sum_{j=1}^{n} |x_j|$. Note that

$$\text{(4.1)}\qquad \begin{cases} \text{if we change the order of elements in the list } v_1, \dots, v_n, \\ \text{then } B \text{ turns into another matrix } B'\text{; however, } B \text{ and } B' \\ \text{are similar matrices with the same characteristic poly-} \\ \text{nomial, whereby } \mathrm{En}(G) \text{ is well defined.} \end{cases}$$

Next, we start from a finite algebra $A = (A, F)$. A congruence α of A, in notation, $\alpha \in \mathrm{Con}(A)$, determines a graph $G_{A,\alpha}$ in quite a natural way: the vertices are the elements of A while $a, b \in A$ are connected by an edge of G iff $a \neq b$ but $(a, b) \in \alpha$. We define *the energy* $\mathrm{En}(\alpha)$ *of the congruence* α by letting $\mathrm{En}(\alpha) := \mathrm{En}(G_{A,\alpha})$.

Note that $\mathrm{En}(\alpha)$ is meaningful for any equivalence relation α of A, in notation, $\alpha \in \mathrm{Equ}(A)$ since $\mathrm{Equ}(A) = \mathrm{Con}(A, \emptyset)$ (the case of no operation). In fact, $\mathrm{En}(\alpha)$ is meaningful for any symmetric relation α of A but in this paper we restrict ourselves to congruence relations. To explain the definition of $\mathrm{En}(\alpha)$ more directly and for the sake of later reference, assume that $A = \{a_1, \dots, a_n\}$ is an n-element algebra and $\alpha \in \mathrm{Con}(A)$. Let $\Delta_A := \{(x, x) : x \in A\}$ denote the smallest congruence of A. Up to matrix similarity, the *adjacency matrix* of α is

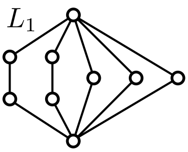
$$M(\alpha) = (m_{ij})_{n\times n} \quad\text{where}\quad m_{ij} = \begin{cases} 1, & \text{if } (a_i, a_j) \in \alpha \setminus \Delta_A, \\ 0, & \text{otherwise.} \end{cases} \tag{4.2}$$

Then the characteristic polynomial of $M(\alpha)$ is $\chi_{M(\alpha)} = \prod_{j:=1}^{n}(x - x_j)$. Keeping (4.1) in mind, we define the

$$\boxed{\textit{energy } \mathrm{En}(\alpha) \text{ of } \alpha \text{ by } \mathrm{En}(\alpha) := \sum_{j=1}^{n} |x_j|.}$$

If we take all congruences α of A and form the sum of their $\mathrm{En}(\alpha)$'s, then we obtain the *congruence energy* $\mathrm{CE}(A)$ of our algebra. So the key definition in the paper is the following: for a finite algebra A,

$$\boxed{\text{the } \textit{congruence energy } \mathrm{CE}(A) \text{ of } A \text{ is } \mathrm{CE}(A) := \sum_{\alpha\in\mathrm{Con}(A)} \mathrm{En}(\alpha).} \tag{4.3}$$



$|L_1| = |L_2|$ and $|\mathrm{Con}(L_1)| = 5 > 4 = |\mathrm{Con}(L_2)|$,
but $\mathrm{CE}(L_1) = 24 < 32 = \mathrm{CE}(L_2)$

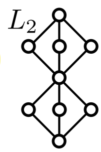


Figure 1: An example with two lattices

5. Motivations coming from algebra

A straightforward way to measure the complexity of the collection of congruences of a finite algebra A is to take $|\mathrm{Con}(A)|$; $\mathrm{CE}(A)$ offers another way. Figure 1 shows that none of the inequalities $\mathrm{CE}(A_1) < \mathrm{CE}(A_2)$ and $|\mathrm{Con}(A_1)| < |\mathrm{Con}(A_2)|$ implies the other one. Among the n-element algebras A, those with *minimal* $|\mathrm{Con}(A)|$ could be the involved *the building stones* of other algebras, like finite simple groups. On the other hand, n-element algebras A with *maximal* or close to maximal $|\mathrm{Con}(A)|$ are often *nice buildings* with well-understood structures and nice properties; see, for example, Czédli [5, 6, 7] and Mureşan and Kulin [14]. Note that the same holds for n-element semilattices or lattices with maximal or close to maximal numbers of subalgebras or weak congruences (in the sense of v Šešelja, B. and Tepavčević [16]); see Ahmed and Horváth [1, 2] and Ahmed, Horváth, and Németh [3]. In addition to Section 3, these ideas also motivate the present paper.

6. Two easy remarks

For a finite algebra A and $\alpha \in \mathrm{Con}(A)$, the quotient algebra A/α consists of the α-blocks, whereby $|A/\alpha|$ is the *number of the blocks of* α. We will denote by $\mathrm{Equ}(A) = (\mathrm{Equ}(A), \subseteq)$ the *equivalence lattice* of A; note that $\mathrm{Con}(A)$ is a sublattice of $\mathrm{Equ}(A)$ containing the least equivalence $\Delta_A = \{(x,x) : x \in A\}$ and the largest equivalence $\nabla_A = A \times A$. The covering relation understood in $\mathrm{Equ}(A)$ is denoted by $\prec_e$. For $\alpha \in \mathrm{Con}(A)$, the *height* of α in $\mathrm{Equ}(A)$ will be denoted by $h_{\mathrm{eq}}(\alpha)$. In particular, $h_{\mathrm{eq}}(\Delta_A) = 0$ and $h_{\mathrm{eq}}(\nabla_A) = |A| - 1$. Using the semimodularity of $\mathrm{Equ}(A)$, see, for example, Grätzer [9, Theorem 404], we obtain trivially that

$$\text{for } \alpha < \beta \text{ in } \mathrm{Equ}(A),\ \ \alpha \prec_e \beta \text{ if and only if } |A/\beta| = |A/\alpha| - 1. \tag{6.1}$$

It follows from (6.1) that

$$\text{if } |A| = n \text{ and } \alpha \in \mathrm{Equ}(A), \text{ then } h_{\mathrm{eq}}(\alpha) + |A/\alpha| = n. \tag{6.2}$$

Remark 6.1. *For an n-element finite algebra A and $\Theta \in \mathrm{Con}(A)$,*

$$\mathrm{En}(\Theta) = 2 \cdot (n - |A/\Theta|) = 2 \cdot h_{\mathrm{eq}}(\Theta) \qquad \textit{and} \tag{6.3}$$

$$\mathrm{CE}(A) = 2n \cdot |\mathrm{Con}(A)| - 2 \cdot \sum_{\Theta \in \mathrm{Con}(A)} |A/\Theta| = 2 \cdot \sum_{\alpha \in \mathrm{Con}(A)} h_{\mathrm{eq}}(\Theta). \tag{6.4}$$

Proof. Consider the following k-by-k matrices:

$$M_k := \begin{pmatrix} 0 & 1 & 1 & \dots & 1 \\ 1 & 0 & 1 & \dots & 1 \\ 1 & 1 & 0 & \dots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 1 & \dots & 0 \end{pmatrix}, \quad Q_k := \begin{pmatrix} -1 & k-1 & -1 & \dots & -1 \\ -1 & -1 & k-1 & \dots & -1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -1 & -1 & -1 & \dots & k-1 \\ 1 & 1 & 1 & \dots & 1 \end{pmatrix},$$

$$P_k := \begin{pmatrix} -1 & -1 & \dots & -1 & 1 \\ 1 & 0 & \dots & 0 & 1 \\ 0 & 1 & \dots & 0 & 1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & 1 \end{pmatrix}, \quad H_k := \begin{pmatrix} -1 & 0 & \dots & 0 & 0 \\ 0 & -1 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & -1 & 0 \\ 0 & 0 & \dots & 0 & k-1 \end{pmatrix}.$$

Note that each of P_k, Q_k, and H_k contains a $(k-1)$-by-$(k-1)$ submatrix in which the diagonal elements are all equal and so do the non-diagonal elements; these submatrices are the bottom left $(k-1)$-by-$(k-1)$ submatrix of P_k, the top right one of Q_k, and the top left one of H_k. An easy computation shows that $P_kQ_k = kI_k$, implying that $P_k^{-1} = k^{-1}Q_k$. Another computation yields that $P_kH_kQ_k = kM_k$, whereby $M_k = P_kH_k(k^{-1}Q_k) = P_kH_kP_k^{-1}$. This shows that M_k and H_k are similar matrices with the same characteristic polynomial and eigenvalues. Hence, the sum of the absolute values of the eigenvalues of M_k is $2(k-1)$. Let $U_1, \dots, U_t$ be the Θ-blocks where $t = |A/\Theta|$. For $i = 1, \dots, t$, let $k_i := |U_i|$. List the elements of A so that first we list the elements of U_1, then the elements of U_2, and so on. Then $M(\Theta)$ is the matrix we obtain by placing $M_{k_1}, \dots, M_{k_t}$ along the diagonal and putting zeros everywhere else. Then the system of the eigenvalues of $M(\Theta)$ is the union of the systems of the eigenvalues of the M_{k_i}, $i = 1, \dots, t$. Thus, using that $k_1 + \dots + k_t = |A| = n$, $\mathrm{En}(\Theta) = 2(k_1 - 1) + \dots + 2(k_t - 1) = 2n - 2t = 2(n - |A/\Theta|)$, as required. The rest of Remark 6.1 is now trivial. □

For a positive integer n, let $[n] := \{1, 2, \dots, n\}$, and let $k \in [1]$. Recall that $B(n) := |\mathrm{Equ}([n])|$ is the n-th *Bell number*, $S_2(n,k) := |\{\alpha \in \mathrm{Equ}([n]) : |[n]/\alpha| = k\}|$ is a *Stirling number of the second kind*, and $B_2(n) := \sum_{i=1}^{n} i \cdot S_2(n,i)$ is the n-th 2-*Bell number*. They are frequently studied numbers; see the sequences A000110, A008277, and A005493 and A138378 in Sloan's OEIS [17].

Remark 6.2. *For a positive integer n and an n-element algebra A, we have that*

$$\mathrm{CE}(A) \le 2nB(n) - 2B_2(n). \tag{6.5}$$

In (6.5)*, equality holds if and only if* $\mathrm{Con}(A) = \mathrm{Equ}(A)$.

The straightforward details of the proof are omitted. Let (6.5)(n) stand for $2nB(n) - 2B_2(n)$; the first ten values of (6.5)(n) are given in the following table.

n	1	2	3	4	5
(6.5)(n)	0	2	10	46	218
n	6	7	8	9	10
(6.5)(n)	1 088	5 752	32 226	190 990	1 194 310

7. The main result

An algebra A is *congruence distributive* if the lattice $\mathrm{Con}(A) = (\mathrm{Con}(A), \subseteq)$ is distributive. Lattices are congruence distributive. *Chains* are lattices in which any two elements x and y are comparable, in notation, $x \nparallel y$. The n-element chain is denoted by C_n, and let B_4 be the 4-element boolean lattice. The *glued sum* $U +_{\mathrm{glu}} V$ of disjoint finite lattices U and V is $(U \cup (V \setminus \{0_V\}), \le)$ where $x \le y$ iff $x \le_U y$, $x \le_V y$, or $(x, y) \in U \times V$. Note that the $U +_{\mathrm{glu}} V$ is a particular case of Hall–Dilworth gluing. In order to formulate the main result of the paper, we define

$$g_{\mathrm{mx}}(n) := (n-1) \cdot 2^{n-1} \quad \text{and} \quad g_{\mathrm{sb}}(n) := (n-1) \cdot 2^{n-2} + 2^{n-3}. \tag{7.1}$$

The acronyms in the subscripts come from "MaXimal" and "SuBmaximal'.

Theorem 7.1. *For any positive integer n, the following three assertions hold.*

(a) *Let A be an n-element congruence distributive algebra. Then we have that $\mathrm{CE}(A) \le g_{\mathrm{mx}}(n)$. Furthermore, if $\mathrm{CE}(A) = g_{\mathrm{mx}}(n)$, then $\mathrm{Con}(A)$ is a boolean lattice and $|\mathrm{Con}(A)| = 2^{n-1}$.*

(b) *Let L be an n-element lattice. Then $\mathrm{CE}(L) \le g_{\mathrm{mx}}(n)$. Furthermore, $\mathrm{CE}(L) = g_{\mathrm{mx}}(n)$ if and only if L is the n-element chain.*

(c) *Let L be an n-element lattice such that $\mathrm{CE}(L) < g_{\mathrm{mx}}(n)$. Then $n \ge 4$ and $\mathrm{CE}(L) \le g_{\mathrm{sb}}(n)$. Furthermore, $\mathrm{CE}(L) = g_{\mathrm{sb}}(n)$ if and only if there is exactly one 2-element antichain in L. Equivalently, $\mathrm{CE}(L) = g_{\mathrm{sb}}(n)$ if and only if there are finite chains C' and C'' such that $L = C' +_{\mathrm{glu}} B_4 +_{\mathrm{glu}} C''$.*

8. Proving Theorem 7.1

To prove the theorem, we need several preparatory statements. For elements x and y of a lattice L, the least congruence $\Theta \in \mathrm{Con}(L)$ containing (x, y) is denoted by $\mathrm{con}(x, y)$. Similarly, $\mathrm{equ}(x, y)$ stands for the least equivalence relation containing the pair (x, y). An element $a \in L$ is an *atom* if $0 \prec a$. The set of atoms of a lattice L will be denoted by $\mathrm{At}(L)$. Lemma 8.1 below is a counterpart of Theorem 7.1.

Lemma 8.1 (Czédli [5])**.** *Let n be a positive integer. For an n-element lattice L and an n-element congruence distributive algebra A, the following hold.*

(a) $|\mathrm{Con}(A)| \le 2^{n-1}$. *Also, if* $|\mathrm{Con}(A)| = 2^{n-1}$, *then* $\mathrm{Con}(A)$ *is a boolean lattice.*

(b) $|\mathrm{Con}(L)| \le 2^{n-1}$. *Furthermore,* $|\mathrm{Con}(L)| = 2^{n-1}$ *if and only if L is a chain.*

(c) *If* $|\mathrm{Con}(L)| < 2^{n-1}$, *then* $|\mathrm{Con}(L)| \le 2^{n-2}$. *Also,* $|\mathrm{Con}(L)| = 2^{n-2}$ *if and only if there are finite chains C' and C'' such that $L = C' +_{\mathrm{glu}} B_4 +_{\mathrm{glu}} C''$.*

Prior to Czédli [5], part (b) of this lemma was proved by Freese [8]. Note in advance that the proof of Theorem 7.1 will use lots of ideas from the proof of Lemma 8.1; sometimes, this will happen implicitly. However, Figure 1 indicates that new ideas and computations are also needed and some earlier arguments need modifying, and so Theorem 7.1 needs a more involved proof than Lemma 8.1. To make the paper easy to read, we are going to present a self-contained proof of Theorem 7.1 rather than explaining how to interpret or modify fragments of [5] scattered in the rest of the paper.

Figure 2: B_4, M_3, and N_5

Up congruence perspectivity and *down congruence perspectivity* will be denoted by $\twoheadrightarrow_{\mathrm{up}}$ and $\twoheadrightarrow_{\mathrm{dn}}$, respectively. That is, for intervals $[a, b]$ and $[c, d]$ of a lattice L, $[a, b] \twoheadrightarrow_{\mathrm{up}} [c, d]$ means that $b \vee c = d$ and $a \le c$ while $[a, b] \twoheadrightarrow_{\mathrm{dn}} [c, d]$ stands for the conjunction of $a \wedge d = c$ and $b \ge d$. Congruence perspectivity and *congruence projectivity* are denoted by $\twoheadrightarrow$ and $\twoheadrightarrow^*$, respectively; $[a, b] \twoheadrightarrow [c, d]$ means that $[a, b] \twoheadrightarrow_{\mathrm{up}} [c, d]$ or $[a, b] \twoheadrightarrow_{\mathrm{dn}} [c, d]$ while $\twoheadrightarrow^*$ is the transitive and reflexive closure of $\twoheadrightarrow$. An interval $[a, b]$ is *prime* if $a \prec b$. The least element and the largest element of an interval I are denoted by 0_I and 1_I, respectively. Except for its part (1), the following lemma belongs to the folklore.

Lemma 8.2. *Let L be a* finite *lattice. Then the following assertions hold.*

(1) *By Grätzer [10], an $\alpha \in \mathrm{Equ}(L)$ is a congruence of L if and only if the α-blocks are intervals and for any $x, y, z \in L$ the following implication and its dual hold:*

(8.1) $$\text{if } x \prec y,\ x \prec z, \text{ and } (x, y) \in \alpha, \text{ then } (z, y \vee z) \in \alpha.$$

(2) $\mathrm{J}(\mathrm{Con}(L)) = \{\mathrm{con}(a,b) : a \prec b\}$. *Consequently, a congruence is determined by the prime intervals it collapses.*

(3) *For prime intervals* $[a,b]$ *and* $[c,d]$ *of* L,

$$(c,d) \in \mathrm{con}(a,b) \iff \mathrm{con}(a,b) \geq \mathrm{con}(c,d) \iff [a,b] \twoheadrightarrow^{*} [c,d]. \tag{8.2}$$

(4) *Let* $\Theta \in \mathrm{Con}(L)$ *and assume that* X, Y, U, V, S, T *are* Θ*-blocks. Then*

$$X \vee Y = U \iff 0_X \vee 0_Y = 0_U, \quad X \wedge Y = V \iff 1_X \wedge 1_Y = 1_V, \tag{8.3}$$

$$\text{whereby} \quad S \leq T \iff 0_S \leq 0_T \iff 1_S \leq 1_T, \tag{8.4}$$

$$\text{and so} \quad S = T \iff 0_S = 0_T \iff 1_S = 1_T. \tag{8.5}$$

For an element a of a lattice L, we use the notation $\uparrow a = \uparrow_L a =: \{x \in L : x \geq a\}$ and $\downarrow a = \downarrow_L a := \{x \in L : x \leq a\}$. To formulate an easy and mostly folkloric lemma, we need the following map (AKA function):

$$f_a \colon L \setminus \uparrow a \to \uparrow a, \text{ defined by } x \mapsto a \vee x. \tag{8.6}$$

Lemma 8.3. *Let* L *be a* finite *distributive lattice.*

(i) *If* $a \in \mathrm{At}(L)$, *then* f_a *defined in* (8.6) *is a lattice embedding.*

(ii) *If* $a \in \mathrm{At}(L)$ *has a complement, then* f_a *is an isomorphism.*

(iii) L *is a boolean if and only if* f_a *is an isomorphism for (equivalently, if* f_a *is bijective) for each atom* a *of* L.

Having no direct reference at hand, we present an easy proof.

Proof of Lemma 8.3. Clearly, f_a is a lattice homomorphism by distributivity. Assume that $b_1, b_2 \in L \setminus \uparrow a$ such that $f_a(b_1) = f_a(b_2)$. For $i \in \{1,2\}$, we have that $a \wedge b_i = 0$ since $b_i \not\geq a \succ 0$. Hence, b_i is a complement of a in the interval $[0, a \vee b_1] = [0, a \vee b_2]$. But this interval is a distributive lattice, whereby the uniqueness of complements in distributive lattices imply that $b_1 = b_2$. That is, f_a is injective, proving part (i).

Next, assume that $a \in \mathrm{At}(L)$ with a complement a'. Let $c \in \uparrow a$. Then $f_a(c \wedge a') = (c \wedge a') \vee a = (c \vee a) \wedge (a' \vee a) = c \wedge 1 = c$. If we had that $c \wedge a' \in \uparrow a$, then $a \leq (c \wedge a') \wedge a = c \wedge (a' \wedge a) = c \wedge 0 = 0$ would be a contradiction. Hence, $c \wedge a' \in L \setminus \uparrow a$. Thus, f_a is surjective and so it is an isomorphism, proving part (ii).

The "only if" part of (iii) follows from part (ii). To prove the "if" part, assume that L is a finite distributive lattice such that f_a is bijective for every $a \in \mathrm{At}(L)$. Let $\mathrm{J}(L)$ denote the poset of (nonzero) join-irreducible elements of L, and let $\mathcal{S}_{\mathrm{dn}}(\mathrm{J}(L)) = (\mathcal{S}_{\mathrm{dn}}(\mathrm{J}(L)), \cup, \cap)$ be the lattice of its down-sets. By the well-known structure theorem of finite distributive lattices, see, e.g., Grätzer [9, Theorem 107],

$$L \cong \mathcal{S}_{\mathrm{dn}}(\mathrm{J}(L)). \tag{8.7}$$

We claim that $\mathrm{J}(L)$ is an antichain. Supposing the contrary, let $a, b \in \mathrm{J}(L)$ such that $a < b$. We can assume that $a \in \mathrm{At}(L)$ since otherwise we can replace it by an atom of $\downarrow a$. The join-irreducibility of b implies that $b \neq f_a(x)$ for any $x \in L \setminus \uparrow a$, contradicting the bijectivity of f_a. Hence, $\mathrm{J}(L)$ is an antichain and $\mathcal{S}_{\mathrm{dn}}(\mathrm{J}(L))$ is the (boolean) powerset lattice. By (8.7), L is boolean, completing the proof. $\square$

Proof of Theorem 7.1. As a convention for the whole proof, α always denotes an *atom* of the congruence lattice of our n-element algebra or lattice. We prove our statements by induction on n. For $n \in \{1,2\}$, $\mathrm{Con}(A) = \mathrm{Equ}(A)$ and the statement is clear. So let $n \geq 3$, and assume that all the three parts of the theorem hold for all algebras and lattices that have fewer than n elements. Let $A = (A, F)$ be an n-element congruence distributive algebra. For an atom $\alpha \in \mathrm{At}(\mathrm{Con}(A))$, we define

$$\begin{cases} \mathrm{CA}(A,\alpha) := \{\Theta \in \mathrm{Con}(A) : \alpha \leq \Theta\} \text{ and} \\ \mathrm{CB}(A,\alpha) := \mathrm{Con}(A) \setminus \mathrm{CA}(A,\alpha). \end{cases} \tag{8.8}$$

According to (8.6), f_α is the map $\mathrm{CB}(A,\alpha) \to \mathrm{CA}(A,\alpha)$ defined by $\beta \mapsto \alpha \vee \beta$.

For $\beta \in \mathrm{CA}(A,\alpha)$, we define $\beta/\alpha \in \mathrm{Con}(A/\alpha) \subseteq \mathrm{Equ}(A/\alpha)$ in the usual way: $\beta/\alpha := \{(x/\alpha, y/\alpha) : (x,y) \in \beta\}$. By the Correspondence Theorem, see, for example, Burris and Sankappanavar [4, Thm. 6.20], $\mathrm{CA}(A,\alpha) \cong \mathrm{Con}(A/\alpha)$. Applying the Second Isomorphism Theorem, see, e.g., Burris and Sankappanavar [4, Theorem 6.15], to the algebra $(A, \emptyset)$ with no operation, we obtain that $|(A/\alpha)/\beta/\alpha| = |A/\beta|$. Hence, it follows from (6.2) that $h_{\mathrm{eq}}(\beta) = n - |A/\beta| = n - |A/\alpha| + |A/\alpha| - |(A/\alpha)/\beta/\alpha| = h_{\mathrm{eq}}(\alpha) + |A/\alpha| - |A/\alpha/\beta/\alpha| = h_{\mathrm{eq}}(\alpha) + h_{\mathrm{eq}}(\beta/\alpha)$. That is,

$$\text{for every } \beta \in \mathrm{CA}(A,\alpha), \quad h_{\mathrm{eq}}(\beta) = h_{\mathrm{eq}}(\beta/\alpha) + h_{\mathrm{eq}}(\alpha). \tag{8.9}$$

Since the map $\mathrm{CA}(A,\alpha) \to \mathrm{Con}(A/\alpha)$, defined by $\beta \to \beta/\alpha$ is a lattice isomorphism by the Correspondence Theorem, (6.3), (6.4), and (8.9) imply that

$$|\mathrm{CA}(A,\alpha)| = |\mathrm{Con}(A/\alpha)|, \tag{8.10}$$

$$\mathrm{En}(\beta) = \mathrm{En}(\beta/\alpha) + \mathrm{En}(\alpha) \text{ for every } \beta \in \mathrm{CA}(A,\alpha), \text{ and} \tag{8.11}$$

$$\sum_{\beta \in \mathrm{CA}(A,\alpha)} \mathrm{En}(\beta) = \mathrm{CE}(A/\alpha) + \mathrm{En}(\alpha) \cdot |\mathrm{Con}(A/\alpha)|. \tag{8.12}$$

Next, let $\gamma \in \mathrm{CB}(A,\alpha)$. Then $\gamma < \alpha \vee \gamma = f_\alpha(\gamma)$ gives that $h_{\mathrm{eq}}(\gamma) < h_{\mathrm{eq}}(f_\alpha(\gamma))$. Hence, using (6.4) and the fact that the function h_{eq} takes integer values,

$$\begin{cases} \text{for } \gamma \in \mathrm{CB}(A,\alpha), \quad h_{\mathrm{eq}}(\gamma) \leq h_{\mathrm{eq}}(f_\alpha(\gamma)) - 1 \text{ and} \\ \mathrm{En}(\gamma) \leq \mathrm{En}(f_\alpha(\gamma)) - 2. \end{cases} \tag{8.13}$$

At $\leq'$ and $\leq^*$ below, we use (8.13) and the injectivity of f_α (see Lemma 8.3), while we use (8.12) and $|\mathrm{CA}(A,\alpha)| = |\mathrm{Con}(A/\alpha)$ at $=^\dagger$.

$$(8.14)\quad \begin{cases} \displaystyle\sum_{\gamma\in\mathrm{CB}(A,\alpha)} \mathrm{En}(\gamma) \leq' \sum_{\gamma\in\mathrm{CB}(A,\alpha)} \bigl(\mathrm{En}(f_\alpha(\gamma)) - 2\bigr) \\ \displaystyle\qquad\qquad\qquad \leq^* \sum_{\beta\in\mathrm{CA}(A,\alpha)} \bigl(\mathrm{En}(\beta) - 2\bigr) \\ \displaystyle\qquad\qquad\qquad = -2|\mathrm{CA}(A,\alpha)| + \sum_{\beta\in\mathrm{CA}(A,\alpha)} \mathrm{En}(\beta) \\ \qquad\qquad\qquad =^\dagger \mathrm{CE}(A/\alpha) + \bigl(\mathrm{En}(\alpha) - 2\bigr)\cdot|\mathrm{Con}(A/\alpha)|. \end{cases}$$

It follows from (8.12) and (8.14) that

$$(8.15)\qquad \mathrm{CE}(A) \leq 2\cdot\mathrm{CE}(A/\alpha) + \bigl(2\cdot\mathrm{En}(\alpha) - 2\bigr)\cdot|\mathrm{Con}(A/\alpha)|.$$

Next, we claim that

$$(8.16)\quad \begin{cases} \text{if the inequality in (8.15) happens to be an equality, then } f_\alpha \text{ is} \\ \text{bijective and } h_{\mathrm{eq}}(\gamma) = h_{\mathrm{eq}}(f_\alpha(\gamma)) - 1 \text{ holds for every } \gamma\in\mathrm{CB}(A,\alpha); \\ \text{in particular, it holds for } \gamma = \Delta_A \text{ and so } h_{\mathrm{eq}}(\alpha) = 1. \end{cases}$$

To see this, note that $\alpha = f_\alpha(\Delta_A)$ is an f_α-image and $\mathrm{En}(\beta) - 2 = 2(h_{\mathrm{eq}}(\beta) - 1) > 0$ for every $\beta \in \mathrm{CA}(A,\alpha)\setminus\{\alpha\}$. Hence if f_α was not surjective, then $\leq^*$ above would be a strict inequality and so would (8.15). This yields that f_α is surjective, whereby it is bijective by Lemma 8.3(i). We know from (6.4) that the two inequalities occurring in (8.13) are equivalent and so are the corresponding strict inequalities. So if $h_{\mathrm{eq}}(\gamma) = h_{\mathrm{eq}}(f_\alpha(\gamma)) - 1$ failed for some $\gamma \in \mathrm{CB}(A,\alpha)$, then (8.13) would give that $\mathrm{En}(\gamma) < \mathrm{En}(f_\alpha(\gamma)) - 2$, whence $\leq'$ and (8.15) would be strict inequalities, contradicting our assumption. Thus, we have verified (8.16). Next, we claim that

$$(8.17)\quad \begin{cases} \text{if } \mathrm{Con}(A) \text{ is distributive, } \alpha\in\mathrm{At}(\mathrm{Con}(A)) \text{ has} \\ \text{a complement in } \mathrm{Con}(A)\text{, and } h_{\mathrm{eq}}(\alpha) = 1\text{, then} \\ \mathrm{CE}(A) = 2\cdot\mathrm{CE}(A/\alpha) + 2\cdot|\mathrm{Con}(A/\alpha)|. \end{cases}$$

To show (8.17), note that α is an atom of $\mathrm{Equ}(A)$ since $h_{\mathrm{eq}}(\alpha) = 1$. Hence, by the semimodularity of $\mathrm{Equ}(A)$, $f_\alpha(\gamma) = \alpha \vee_{\mathrm{Con}(A)} \gamma = \alpha \vee_{\mathrm{Equ}(A)} \gamma$ covers γ in $\mathrm{Equ}(A)$. So $h_{\mathrm{eq}}(\gamma) = h_{\mathrm{eq}}(f_\alpha(\gamma)) - 1$ and $\mathrm{En}(\gamma) = \mathrm{En}(f_\alpha(\gamma)) - 2$ for all $\gamma \in \mathrm{CB}(A,\alpha)$. Hence, $\leq'$ in (8.14) is an equality. So is $\leq^*$ in (8.14) since f_α is bijective by Lemma 8.3(ii). Thus, both (8.14) and (8.15) are equalities, implying the validity of (8.17).

Next, we define an integer-valued function with domain $\{4,5,6,7,\dots\}$ as follows.

$$(8.18)\quad \begin{cases} \text{With the initial value } g_{\mathrm{pn}}(4) := 17/2,\ g_{\mathrm{pn}}(k) \text{ for } k\geq 5 \text{ is given} \\ \text{by the recursive formula } g_{\mathrm{pn}}(k) := 2g_{\mathrm{pn}}(k-1) + 5\cdot 2^{k-5}. \end{cases}$$

The "pentagon" lattice N_5 is drawn in Figure 2. The subscript of g_{pn} comes from "PeNtagon"; this is motivated by the following claim, in which k denotes an integer.

$$\text{(8.19)}\qquad \begin{cases} \text{If a } k\text{-element lattice } K \text{ is of the form} \\ K = C' +_{\text{glu}} N_5 +_{\text{glu}} C'' \text{ with chains} \\ C' \text{ and } C'', \text{ then } \mathrm{CE}(K) = g_{\text{pn}}(k) \text{ and} \\ |\mathrm{Con}(K)| = 5\cdot 2^{k-5}. \end{cases}$$

We prove this by induction on k. If $k = 5$, then $K \cong N_5$, whereby Lemma 8.2 yields that $|\mathrm{Con}(N_5)| = 5 = 5\cdot 2^{5-5}$ and $\mathrm{CE}(N_5) = 22 = g_{\text{pn}}(5)$. Hence, (8.19) holds for $k = 5$. So assume that $k > 5$ and (8.19) holds for $k-1$. Since $|C'| > 1$ or $|C''| > 1$, duality allows us to assume that $|C'| > 1$. Then K has a unique atom b. By parts (2) and (3) of Lemma 8.2, $\beta := \mathrm{equ}(0, b) \in \mathrm{At}(\mathrm{Con}(K))$, and $[b, 1]$ is the only non-singleton block of $\gamma := \mathrm{con}(b, 1)$. For $K^\dagger := K/\beta$, (6.2) gives that $|K^\dagger| = |K| - h_{\text{eq}}(\beta) = k-1$ and, in addition, $K^\dagger = C'_\dagger +_{\text{glu}} N_5^\dagger +_{\text{glu}} C''_\dagger$ where $C'_\dagger$ and $C''_\dagger$ are chains. Since γ is a complement of β and $h_{\text{eq}}(\beta) = 1$, (8.17) and the induction hypothesis imply that $\mathrm{CE}(K) = 2\cdot \mathrm{CE}(K^\dagger) + 2|\mathrm{Con}(K^\dagger)| = 2g_{\text{pn}}(k-1) + 2\cdot 5\cdot 2^{k-1-5} = 2g_{\text{pn}}(k-1) + 5\cdot 2^{k-5} = g_{\text{pn}}(k)$, as required. Furthermore, since f_β is bijective by Lemma 8.3(ii) and $|{\uparrow_{\mathrm{Con}(K)}}\,\beta| = |\mathrm{Con}(K^\dagger)| = 5\cdot 2^{k-6}$ by the Correspondence Theorem and the induction hypothesis, we have that $|\mathrm{Con}(K)| = 2\cdot |\mathrm{Con}(K)^\dagger| = 5\cdot 2^{k-5}$. This completes the induction step and the proof of (8.19).

If $\alpha \in \mathrm{At}(\mathrm{Con}(A))$ is fixed and so no ambiguity threatens, we let

$$\text{(8.20)}\qquad m := h_{\text{eq}}(\alpha) = \mathrm{En}(\alpha)/2;\ \text{note that } |A/\alpha| = n - m.$$

Equalities obtained by straightforward computations will be denoted by $=$ signs.

$$\text{(8.21)}\qquad \text{Let}\quad w(x) := g_{\text{mx}}(n) - \Big(2\cdot g_{\text{mx}}(n-x) + (2\cdot 2x - 2)\cdot 2^{n-x-1}\Big)$$

$$\text{(8.22)}\qquad = 2^{n-x}\cdot \Big((n-1)\cdot 2^{x-1} - n - x + 2\Big).$$

Keeping $n \geq 3$ in mind, we claim that this auxiliary function has the property that

$$\text{(8.23)}\qquad \text{for } 1 \leq x \leq n-2,\ \ w(x) \geq 0 \text{ and } w(x) = 0 \iff x = 1.$$

Let $w_2(x)$ denote the second factor of (8.22). It suffices to show that (8.23) holds for $w_2(x)$ instead of $w(x)$. We denote $\frac{d}{dx} w_2(x)$ by $w_2'(x)$. Since $w_2(1) = 0$ and $w_2'(x) = \big((n-1)\cdot 2^{x-1} - n - x + 2\big)' = (n-1)\cdot 2^{x-1}\cdot \ln 2 - 1 \geq 2\cdot \ln 2\cdot 1 - 1 = \ln 4 - 1 > 0$ implies that $w_2(x)$ is strictly increasing in the interval $[1, \infty)$, we conclude (8.23).

If $m = h_{\text{eq}}(\alpha) = n-1$, then A is a simple algebra and part (a) as well as parts (b) and (c) of the theorem are trivial. Hence, we can always assume that $m \leq n-2$. By the induction hypothesis, (8.20), and Lemma 8.1,

$$\text{(8.24)}\qquad \mathrm{CE}(A/\alpha) \leq g_{\text{mx}}(n-m) \quad \text{and} \quad |\mathrm{Con}(A/\alpha)| \leq 2^{n-m-1}.$$

Hence, letting $m = h_{\mathrm{eq}}(\alpha) = \mathrm{En}(\alpha)/2$ play the role of x, we have that

$$\mathrm{CE}(A) \overset{(8.15),(8.24)}{\leq} 2\cdot g_{\mathrm{mx}}(n-m)+(2\cdot 2m-2)\cdot 2^{n-m-1} \overset{(8.21),(8.23)}{\leq} g_{\mathrm{mx}}(n), \tag{8.25}$$

proving that $\mathrm{CE}(A) \leq g_{\mathrm{mx}}(n)$, as required. Next, assume that $\mathrm{CE}(A) = g_{\mathrm{mx}}(n)$. Then both inequalities in (8.25) are equalities, whereby the same holds for the inequalities in (8.15) and (8.24), and $h_{\mathrm{eq}}(\alpha) = m = x = 1$ by (8.23). Note that (8.16) also gives that $h_{\mathrm{eq}}(\alpha) = 1$ and, furthermore, it gives that f_α is bijective. Since it is irrelevant how the atom $\alpha \in \mathrm{Con}(A)$ was fixed,

$$\text{for every } \alpha \in \mathrm{At}(\mathrm{Con}(A)),\ f_\alpha \text{ is bijective and } h_{\mathrm{eq}}(\alpha) = 1. \tag{8.26}$$

Thus, Lemma 8.3(iii) implies that $\mathrm{Con}(A)$ is a boolean lattice. To show that this boolean lattice is of size 2^{n-1}, we consider α fixed again. We have already mentioned that the inequalities in (8.24) are equalities, whence (8.10), (8.24), and the equality in (8.26) give that $\mathrm{CA}(A, \alpha) = 2^{n-h_{\mathrm{eq}}(\alpha)-1} = 2^{n-2}$. Thus, using that f_α is bijective, we obtain that $\mathrm{Con}(A) = 2 \cdot \mathrm{CA}(A, \alpha) = 2^{n-1}$. Therefore, $\mathrm{Con}(A)$ is the 2^{n-1}-element boolean lattice, and we have proved part (a) of the theorem.

Next, we turn our attention to part (b). The inequality in it follows from part (a) since lattices are congruence distributive. Let $L := C_n$, the n-element chain, and let u be the unique atom of L. It follows easily from Lemma 8.2 that $\alpha := \mathrm{equ}(0, u)$ is an atom of $\mathrm{Con}(L)$. Hence, the chain $L' := L/\alpha$ is of size $|L'| = n - h_{\mathrm{eq}}(\alpha) = n - 1$ by (6.2). By Lemma 8.1(b), $|\mathrm{Con}(L')| = 2^{n-2}$. Since $\mathrm{Con}(L)$ is boolean by Lemma 8.1 and $h_{\mathrm{eq}}(\alpha) = 1$, (8.17) gives that $\mathrm{CE}(L) = 2\mathrm{CE}(L')+2|\mathrm{Con}(L')|$. Using these facts and the induction hypothesis, we obtain that $\mathrm{CE}(L)$ is

$$2g_{\mathrm{mx}}(n-1) + 2 \cdot 2^{n-2} = 2\big((n-2) \cdot 2^{n-2} + 2^{n-2}\big) = g_{\mathrm{mx}}(n), \tag{8.27}$$

proving the “if part” of part (b).

Next, for later reference, we prove that

$$\begin{cases} \text{if } \delta \in \mathrm{Con}(L) \text{ such that } h_{\mathrm{eq}}(\delta) = 1 \text{ and} \\ L/\delta \text{ is a chain, then } L \text{ is also a chain.} \end{cases} \tag{8.28}$$

To prove (8.28), observe that L/δ consists of a unique 2-element δ-block $B = \{0_B, 1_B\}$, and the rest of the δ-blocks are singletons. Let $H := \{h\}$ be a singleton δ-block. Since L/δ is a chain, B and H are comparable; duality allows us to assume that $B < H$ holds in L/δ. It follows from (8.4) that $0_B < 1_B \leq 1_H = h$. Hence, h is comparable with the elements of B, and it is trivially comparable with every element that forms a singleton block. Therefore, L is a chain, proving (8.28).

To prove the “only if” part of part (b), assume that L is an n-element lattice and $\mathrm{CE}(L) = g_{\mathrm{mx}}(n)$. By part (a) of the theorem, $\mathrm{Con}(L)$ is the 2^{n-1}-element boolean lattice; let $\alpha_1, \dots, \alpha_{n-1}$ be its atoms. They are independent in the semimodular lattice $\mathrm{Equ}(L)$, whereby it is known, e.g. from Theorem 380 of

Grätzer [9], that $h_{\mathrm{eq}}(\alpha_1)+\cdots+h_{\mathrm{eq}}(\alpha_{n-1}) = h_{\mathrm{eq}}(\alpha_1\vee\cdots\vee\alpha_{n-1}) = h_{\mathrm{eq}}(\nabla_L) = n-1$. Hence each of the positive integers $h_{\mathrm{eq}}(\alpha_1), \dots, h_{\mathrm{eq}}(\alpha_{n-1})$ equals 1. In particular, letting $\alpha := \alpha_1$, $h_{\mathrm{eq}}(\alpha) = 1$. Thus $L' := L/\alpha$ is an $(n-1)$-element lattice by, say, (6.2). By (8.17),

$$2\cdot \mathrm{CE}(L') + 2\cdot |\mathrm{Con}(L')| = \mathrm{CE}(L) = g_{\mathrm{mx}}(n). \tag{8.29}$$

However, $\mathrm{CE}(L') \le g_{\mathrm{mx}}(n-1)$ by part (a) of the theorem and $|\mathrm{Con}(L')| \le 2^{n-2}$ by Lemma 8.1(b). Hence, comparing (8.27) and (8.29), we obtain that $\mathrm{CE}(L') = g_{\mathrm{mx}}(n-1)$. Thus, the induction hypothesis implies that L' is a chain. By (8.28), so is L, proving part (b) of the theorem.

Next, note that $g_{\mathrm{sb}}(k)$ is not an integer for an integer $k<3$. We claim that

$$\text{for } k\ge 3,\quad g_{\mathrm{sb}}(k) < g_{\mathrm{mx}}(k), \tag{8.30}$$
$$\text{for } k\ge 4,\quad g_{\mathrm{sb}}(k) = 2g_{\mathrm{sb}}(k-1) + 2^{k-2},\ \text{ and} \tag{8.31}$$
$$\text{for } k\ge 5,\quad g_{\mathrm{pn}}(k) < g_{\mathrm{sb}}(k). \tag{8.32}$$

Indeed, (8.30) follows trivially from (7.1) while a trivial induction based on (8.18), (8.31), and $22 = g_{\mathrm{pn}}(5) = 22 < 36 = g_{\mathrm{sb}}(5)$ and $5\cdot 2^{k-5} < 2^{k-2}$ yields (8.32).

Next, we prove part (c) of the theorem by induction on n. If L is an n-element lattice such that $\mathrm{CE}(L) < g_{\mathrm{mx}}(L)$, then part (b) of the theorem implies that L is not a chain, whereby $n\ge 4$. So the base of the induction is $n=4$. For $n=4$, if $\mathrm{CE}(L) < g_{\mathrm{mx}}(n)$, then $L = B_4$, the only 4-element non-chain, and $\mathrm{CE}(L) = 14 = g_{\mathrm{sb}}(4)$, whereby part (c) of the theorem clearly holds for $n=4$. Thus, from now on, we assume that $n\ge 5$ and L is an n-element lattice such that $\mathrm{CE}(L) < g_{\mathrm{mx}}(n)$ and part (c) of the theorem holds for all lattices consisting of fewer than n elements. By part (b), L is not a chain. There are two cases.

Case 1. We assume that there is an $\alpha\in \mathrm{At}(\mathrm{Con}(L))$ such that $L' := L/\alpha$ is not a chain. For such an atom α and $m := h_{\mathrm{eq}}(\alpha) = \mathrm{En}(\alpha)/2$, (8.20) gives that $|L'| = n-m$. Hence $|\mathrm{Con}(L')| \le 2^{n-m-2}$ by Lemma 8.1. By the induction hypothesis, $\mathrm{CE}(L') \le g_{\mathrm{sb}}(n-m)$. Thus, (8.15) yields that

$$\mathrm{CE}(L) \le 2g_{\mathrm{sb}}(n-m) + (4m-2)\cdot 2^{n-m-2}. \tag{8.33}$$

This motivates us to consider the auxiliary function

$$u_n(x) := g_{\mathrm{sb}}(n) - \bigl(2g_{\mathrm{sb}}(n-x) + (4x-2)\cdot 2^{n-x-2}\bigr), \tag{8.34}$$

where $x\in\mathbb{R}$ is a real variable. With the usual notation $u_n'(x) := \frac{d}{dx}u_n(x)$,

$$u_n(x) = (2n-1)\cdot 2^{n-3} - (4n+4x-6)\cdot 2^{n-x-3}, \tag{8.35}$$
$$u_n(1) = 0,\ \text{ and} \tag{8.36}$$
$$u_n'(x) = \bigl((2(n+x)-3)\cdot \ln 4 - 4\bigr)\cdot 2^{n-x-3}. \tag{8.37}$$

Since $\ln 4 > 1$ and $n\ge 5$, for $x\in[1,\infty)$ we have that $\bigl(2(n+x)-3\bigr)\cdot\ln 4 - 4 \ge (2\cdot 6-3)\cdot 1 - 4 = 5 > 0$. Hence, $u_n'(x)$ is positive and so $u_n(x)$ is strictly increasing in the interval $[1,\infty)$. Thus, for $x \geq 1$, $u_n(x) \geq 0$ and $u_n(x) = 0 \iff x = 1$. Therefore, taking (8.34) into account,

$$(8.38) \qquad \begin{cases} 2 \cdot g_{\mathrm{sb}}(n-m) + (4m-2) \cdot 2^{n-m-2} \leq g_{\mathrm{sb}}(n), \text{ and this} \\ \text{inequality turns to an equality if and only if } m = 1. \end{cases}$$

Combining (8.33) and (8.38), we obtain that $\mathrm{CE}(L) \leq g_{\mathrm{sb}}(n)$, as required.

Next but still in the scope of Case 1, assume that $\mathrm{CE}(L) = g_{\mathrm{sb}}(n)$. Then (8.33) and (8.38) give that $m = 1$ and the inequality in (8.33) is an equality. Since (8.33) was obtained from the inequalities (8.15), $|\mathrm{Con}(L')| \leq 2^{n-m-2}$, and $\mathrm{CE}(L') \leq g_{\mathrm{sb}}(n-m)$, these three inequalities are also equalities. In particular, $\mathrm{CE}(L') = g_{\mathrm{sb}}(n-m) = g_{\mathrm{sb}}(n-1) = g_{\mathrm{sb}}(|L'|)$, and the induction hypothesis implies that L' is of the form $L' = C^* +_{\mathrm{glu}} B_4' +_{\mathrm{glu}} C^{**}$ where C^* and C^{**} are finite chains and B_4' is isomorphic to B_4. By Lemma 8.2, there are $p, q \in L$ such that $p \prec q$ and $X := \{p, q\} = [p, q]$ is the only non-singleton block of α. Note that $p = 0_X$ and $q = 1_X$. Denote by C' and C'' the sets $\{y \in L : y/\alpha \in C^*\}$ and $\{y \in L : y/\alpha \in C^{**}\}$, respectively. Observe that C' and C'' are chains. Indeed, if $x, y \in C'$, then either both x/α and y/α are singletons and their comparability in C^* gives that $x \nparallel y$, or one of them is a singleton, the other one is $X = \{p, q\}$, and (8.4) yields that $x \nparallel y$. Since C' and C'' are chains, we can assume that $X \in B_4'$ since otherwise $L = C' +_{\mathrm{glu}} B_4 +_{\mathrm{glu}} C''$ is clear. If X is the bottom element of B_4', then B_4' is of the form $B_4' = \{X, a/\alpha = \{a\}, b/\alpha = \{b\}, v/\alpha = \{v\}\}$ with top element $\{v\}$, (8.3) gives that $a \wedge b = 1_X = q$, and we conclude that $\{q, a, b, v\}$ is sublattice of L, this sublattice is isomorphic to B_4, and $L = C' +_{\mathrm{glu}} B_4 +_{\mathrm{glu}} C''$ again, as required. By duality, L is also of the required form $C' +_{\mathrm{glu}} B_4 +_{\mathrm{glu}} C''$ if X is the largest element of B_4'. We are left with the possibility that

$$(8.39) \qquad X \in B_4' \text{ is neither the bottom, nor the top of } B_4'.$$

Then $B_4' = \{\{u\}, \{a\}, X, \{v\}\}$ such that $\{u\}$ and $\{v\}$ are the smallest element and the largest element of B_4', respectively. Using (8.3), we have that $a \vee p = v$ and $a \wedge q = u$. Hence, $\{u, a, p, q, v\}$ is (isomorphic to) N_5; see Figure 2. Using that C' and C'' are chains, it follows that L is of the form $L = C' +_{\mathrm{glu}} N_5 +_{\mathrm{glu}} C''$. Hence, (8.19) and (8.32) yield that $\mathrm{CE}(L) = g_{\mathrm{pn}}(n) < g_{\mathrm{sb}}(n)$, contradicting our assumption. This excludes (8.39) and completes Case 1 by having proved that

$$(8.40) \qquad \begin{cases} \text{if } \mathrm{CE}(L) < g_{\mathrm{mx}}(n) \text{ and } L/\alpha \text{ is not a chain for some } \alpha \in \mathrm{At}(\mathrm{Con}(L)), \\ \text{then } \mathrm{CE}(L) \leq g_{\mathrm{sb}}(n) \text{ and, furthermore, } \mathrm{CE}(L) = g_{\mathrm{sb}}(n) \text{ implies that} \\ L = C' +_{\mathrm{glu}} B_4 +_{\mathrm{glu}} C'' \text{ for some chains } C' \text{ and } C''. \end{cases}$$

Case 2. We assume that for every atom $\alpha \in \mathrm{Con}(L)$, L/α is a chain. Let α denote a fixed atom of $\mathrm{Con}(L)$. Similarly to the first part of Case 1 concluding with (8.33) and using the same notation, $|L'| = n - m$, $|\mathrm{Con}(L')| = 2^{n-m-1}$ by Lemma 8.1(b), and $\mathrm{CE}(L') \leq g_{\mathrm{sb}}(n-m)$ by the induction hypothesis. Thus, (8.15) yields that

$$(8.41) \qquad \mathrm{CE}(L) \leq 2g_{\mathrm{sb}}(n-m) + (4m-2) \cdot 2^{n-m-1}.$$

Since L' is a chain but L is not, (8.28) implies that $m = h_{\mathrm{eq}}(\alpha) \geq 2$. Let

$$v_n(x) := g_{\mathrm{sb}}(n) - \big(2g_{\mathrm{sb}}(n-x) + (4x-2)\cdot 2^{n-x-1}\big). \tag{8.42}$$

With this auxiliary real function, computation shows that

$$v_n(x) = (2n-1)\cdot 2^{n-3} - (4n+12x-10)\cdot 2^{n-x-3},$$

$$v_n(2) = (2n-9)\cdot 2^{n-4} > 0, \quad \text{since } n \geq 5, \text{ and} \tag{8.43}$$

$$v_n'(x) = ((4n+12x-10)\cdot \ln 2 - 12)\cdot 2^{n-x-3}. \tag{8.44}$$

Since $n \geq 5$ and $x = m \geq 2$, we have that $(4n+12x-10)\cdot\ln 2 - 12 \geq 34\cdot\ln 2 - 12 = 17\cdot\ln 4 - 12 \geq 17 - 12 > 0$. Hence, $v_n'(x) > 0$ and $v_n(x)$ is strictly increasing in $[2,\infty)$. This fact, $m \geq 2$, and (8.43) yield that $v_n(m) > 0$. Therefore, (8.42) gives that $2g_{\mathrm{sb}}(n-m) + (4m-2)\cdot 2^{n-m-1} < g_{\mathrm{sb}}(n)$, whereby (8.41) implies that

$$\begin{cases} \text{if } \mathrm{CE}(L) < g_{\mathrm{mx}}(n) \text{ and } L/\alpha \text{ is a chain for each} \\ \alpha \in \mathrm{At}(\mathrm{Con}(L)), \text{ then } \mathrm{CE}(L) < g_{\mathrm{sb}}(n), \end{cases} \tag{8.45}$$

completing the argument in Case 2.

Next, we are going to prove by induction on $k = |K|$ that

$$\begin{cases} \text{if } K \text{ is a } k\text{-element lattice of the form } C' +_{\mathrm{glu}} B_4 +_{\mathrm{glu}} C'' \\ \text{with chains } C' \text{ and } C'', \text{ then } \mathrm{CE}(K) = g_{\mathrm{sb}}(k). \end{cases} \tag{8.46}$$

The smallest possible value of k is 4, for which Lemma 8.2 yields easily that $\mathrm{CE}(K) = \mathrm{CE}(B_4) = 14 = g_{\mathrm{sb}}(4)$. So let $k > 4$. Duality allows us to assume that $|C'| \geq 2$ and K has a unique atom b. Like in the argument proving (8.19), $\gamma := \mathrm{con}(b,1)$ is a complement of $\beta := \mathrm{equ}(0,b) = \mathrm{con}(0,b) \in \mathrm{At}(\mathrm{Con}(K))$ and K/β is also of the form mentioned in (8.46). By Lemma 8.1(c), $|\mathrm{Con}(K/\beta)| = 2^{k-1-2}$. Thus, (6.2), (8.17), the induction hypothesis, and (8.31) give that

$$\mathrm{CE}(K) = 2\mathrm{CE}(K/\beta) + 2|\mathrm{Con}(K/\beta)| = 2g_{\mathrm{sb}}(k-1) + 2\cdot 2^{k-1-2} = g_{\mathrm{sb}}(K),$$

proving (8.46). Finally, (8.40), (8.45), and (8.46) imply part (c) of the theorem. □

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