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Complete congruence lattices of two related modular lattices

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Dedicated to George Grätzer on his eightieth birthday

Abstract. By a 1991 result of R. Freese, G. Grätzer, and E. T. Schmidt, every complete lattice A is isomorphic to the lattice $\mathrm{Com}(K)$ of complete congruences of a strongly atomic, 3-distributive, complete modular lattice K. In 2002, Grätzer and Schmidt improved 3-distributivity to 2-distributivity. Here, we represent morphisms between two complete lattices with complete lattice congruences in three ways. Namely, for $i \in \{1,2,3\}$, let A_i and A'_i be arbitrary complete lattices and let $f_i\colon A_i \to A'_i$ be maps such that (i) f_1 is $(\bigvee, 0)$-preserving and 0-separating, (ii) f_2 is $(\bigwedge, 0, 1)$-preserving, and (iii) f_3 is $(\bigvee, 0)$-preserving. We prove that, for $i \in \{1,2,3\}$, there exist strongly atomic, 2-distributive, complete modular lattices K_i and K'_i such that $A_i \cong \mathrm{Com}(K_i)$, $A'_i \cong \mathrm{Com}(K'_i)$, and, in addition, (i) K_1 is a principal ideal of K'_1 and f_1 is represented by complete congruence extension, (ii) K'_2 is a sublattice of K_2 and f_2 is represented by restriction, and (iii) f_3 is represented as the composite of a map naturally induced by a complete lattice homomorphism from K_3 to K'_3 and the complete congruence generation in K'_3. Also, our approach yields a relatively easy construction that proves the above-mentioned 2002 result of Grätzer and Schmidt.

1. Introduction

The congruence lattice $\mathrm{Con}(L)$ of a lattice L is always a distributive algebraic lattice (but not conversely by Wehrung [38]). Hence, it was a surprise when Wille [39] discovered that the lattice $\mathrm{Com}(K)$ of *complete* congruences of a complete lattice K need not be distributive. It was another surprise that this result and also the representability of all finite lattices in Teo [37] were proved by means of concept lattices. Soon afterwards, Grätzer [7] announced that every complete lattice A is isomorphic to the complete congruence lattice $\mathrm{Com}(K)$ of a complete lattice K; he outlined his approach in [8], and the first complete proof appeared in Grätzer and Lakser [16]. (Actually, [16] proves more, because $\mathrm{Com}(K)$ and the automorphism group of K are shown to be independent.) Later, Freese, Grätzer, and Schmidt [6] proved that

$$\text{every complete lattice } A \text{ is isomorphic to } \mathrm{Com}(K) \text{ for a suitable strongly atomic, 3-distributive, complete modular lattice } K. \tag{1.1}$$

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Furthermore, Grätzer and Schmidt [30] improved this result by adding that

$$\text{in (1.1), 3-distributivity can be replaced with 2-distributivity,} \tag{1.2}$$

but the present paper is motivated by (1.1) rather than (1.2). We know from the last sentence of Freese, Grätzer, and Schmidt [6] that

$$\begin{gathered}\text{if } K \text{ is a complete } \textit{distributive} \text{ lattice with a}\\ \text{prime interval, then } \mathrm{Com}(K) \text{ has an atom.}\end{gathered} \tag{1.3}$$

Hence, if we want a strongly atomic K, then we cannot replace 3-distributivity with distributivity in (1.1). However, there are several additional results on representations with complete (or $\mathfrak{m}$-complete) lattice congruences. These results were proved by Grätzer and Lakser [17], Grätzer, Lakser, and Wolk [23], Grätzer, Johnson, and Schmidt [13], and Grätzer and Schmidt [25], [26], [27], and [28]. The study of lattices of complete congruences culminated in Grätzer and Schmidt [29], where every complete lattice was represented as $\mathrm{Com}(D)$ for a suitable complete *distributive* lattice D, and an $\mathfrak{m}$-complete version was also given. By (1.3), this D is necessarily very complicated in general.

Representing a *finite distributive* lattice as the congruence lattice of a finite lattice is a similar task with many results; here we mention only Grätzer and Schmidt [24], where the first proof of Dilworth's theorem appeared, Grätzer and Knapp [14], Grätzer, Lakser, and Schmidt [20], Grätzer, Quackenbush, and Schmidt [22], and Grätzer, Schmidt, and Thomsen [31]; see also Grätzer [12] for a more extensive bibliography. Also, there are several results on the congruence lattices of *two related finite* lattices; we mention only Czédli [3], Grätzer and Lakser [15], and Grätzer, Lakser, and Schmidt [18, 19, 21]; see again Grätzer [12] for more references. Besides (1.1), the main motivation, the present paper is motivated also by these results on two related lattices.

2. Our results and the outline of the paper

Our starting point was to modify the construction given by Freese, Grätzer, and Schmidt [6] to obtain the following result, which is now a corollary of each of the three main theorems of the paper, to be mentioned soon, and also of a more general but quite technical statement, Lemma 4.12, which is postponed to Section 4. Due to Huhn [35], lattices satisfying the identity

$$x \wedge \bigvee_{0\leq i\leq n} y_i = \bigvee_{0\leq j\leq n} \Bigl(x \wedge \bigvee_{0\leq i\leq n,\ i\neq j} y_i\Bigr)$$

are called *n-distributive*. Distributive lattices are the 1-distributive ones. The variety generated by the subspace lattices of vector spaces over the two-element field will be denoted by $\mathcal{L}(\mathbb{Z}_2\text{-Mod})$; it is a minimal modular non-distributive congruence variety from Freese [5]. We will add the adjective "modular" to its members only for emphasis. A lattice is *strongly atomic* if each of its non-singleton intervals contains an atom (with respect to the interval).

Corollary 2.1. *Every complete lattice A is isomorphic to the complete congruence lattice* $\mathrm{Com}(K)$ *of a suitable strongly atomic, 2-distributive, complete modular lattice $K \in \mathcal{L}(\mathbb{Z}_2\text{-Mod})$.*

The first argument right after Lemma 3 in Freese, Grätzer, and Schmidt [6] says that the lattice they construct also belongs to $\mathcal{L}(\mathbb{Z}_2\text{-Mod})$. Hence, Corollary 2.1 adds only 2-distributivity to (1.1). Note that, for an arbitrary lattice, 2-distributivity implies 3-distributivity but not conversely. Because of $K \in \mathcal{L}(\mathbb{Z}_2\text{-Mod})$, Corollary 2.1 is slightly stronger than (1.2). Our construction for Corollary 2.1 is simpler than the constructions in Freese, Grätzer, and Schmidt [6] and Grätzer and Schmidt [27] for (1.1) and (1.2), respectively.

Section 3 describes the construction for Corollary 2.1. For a first impression on it, note in advance that N_5, the five-element nonmodular lattice, is represented with $\mathrm{Com}(K)$, where K is given in Figure 1. This figure is integral as far as an infinite lattice can be diagrammed in a readable way. For any finite lattice A, an appropriate K with $A \cong \mathrm{Com}(K)$ can be diagrammed similarly.

Section 4 proves Lemma 4.12, which immediately implies Corollary 2.1 and, less immediately, three theorems that we are going to formulate below.

Let L be a complete lattice. For $X \subseteq L^2$, $\mathrm{com}(X) = \mathrm{com}_L(X)$ denotes the *complete congruence generated* by X. If $X = \{\langle a, b\rangle\}$ consists of a single pair, then $\mathrm{com}(X) = \mathrm{com}(a, b)$ is a *principal complete congruence.* A nonempty subset $S \subseteq L$ is a *complete sublattice* if $\bigvee_L X \in S$ and $\bigwedge_L X \in S$ hold for all $\varnothing \neq X \subseteq S$. For example, $S = G(x \leq z)$ in Figure 2 is a sublattice but not a complete sublattice of $G(x \leq z)^{\mathrm{cl}}$, although S is a complete lattice. Maps preserving arbitrary *nonempty* joins are called $\bigvee$*-preserving*; similarly for meets. The following statement will be proved in Section 4.

Observation 2.2. If K is a complete sublattice of a complete lattice K', then the *extension map* $\mathrm{ext}_{KK'}$ from $\mathrm{Com}(K)$ to $\mathrm{Com}(K')$, defined by $\Theta \mapsto \mathrm{com}_{K'}(\Theta)$, is a $(\bigvee, 0)$-preserving map. It is also a 0*-separating map*, that is, $0 \in \mathrm{Com}(K)$ is the only preimage of $0 \in \mathrm{Com}(K')$.

In a reasonable sense, the converse also holds by our first theorem.

Theorem 2.3. *Let A and A' be complete lattices and let $f\colon A \to A'$ be a $(\bigvee, 0)$-preserving and 0-separating map. Then there exist a strongly atomic, 2-distributive, complete modular lattice $K' \in \mathcal{L}(\mathbb{Z}_2\text{-Mod})$, a principal ideal K of K', and lattice isomorphisms $\xi\colon A \to \mathrm{Com}(K)$ and $\xi'\colon A' \to \mathrm{Com}(K')$ such that every member of* $\mathrm{Com}(K) \cup \mathrm{Com}(K')$ *is a* principal *complete congruence and the diagram*

$$\begin{array}{ccc} A & \xrightarrow{\ f\ } & A' \\ {\scriptstyle\xi}\big\downarrow & & \big\uparrow{\scriptstyle\xi'^{-1}} \\ \mathrm{Com}(K) & \xrightarrow{\mathrm{ext}_{KK'}} & \mathrm{Com}(K') \end{array}$$

commutes, that is, $f = \xi'^{-1} \circ \mathrm{ext}_{KK'} \circ \xi$. (Equivalently, $\xi' \circ f = \mathrm{ext}_{KK'} \circ \xi$.)

Next, let K' be a sublattice of a complete lattice K such that K' is a complete lattice but not necessarily a complete sublattice of K. For Θ in $\mathrm{Com}(K)$, the *restriction* $\Theta \cap (K' \times K')$ of Θ to K' will be denoted by $\Theta\rceil_{K'}$. Note that $\Theta\rceil_{K'}$ is a congruence but need not be a complete congruence of K'. For example, if $K = G(x \le z)^{\mathrm{cl}}$ and $K' = G(x \le z)$ in Figure 2 and the Θ-blocks are $\downarrow u = \{w \in K : w \le u\}$ and $\uparrow v = \{w \in K : w \ge v\}$, then $\Theta \in \mathrm{Com}(K)$ but $\Theta\rceil_{K'} \notin \mathrm{Com}(K')$. If K and K' are chosen so that $\Theta\rceil_{K'} \in \mathrm{Com}(K')$ for all $\Theta \in \mathrm{Com}(K)$, then we say that the *restriction map* $\mathrm{res}_{KK'} \colon \mathrm{Com}(K) \to \mathrm{Com}(K')$, defined by $\Theta \mapsto \Theta\rceil_{K'}$, *preserves completeness.* In this case, $\mathrm{res}_{KK'}$ is a $(\bigwedge, 0, 1)$-preserving map. Below, we state that every such map between two complete lattices can be represented in this form.

Theorem 2.4. *Assume that A and A' are complete lattices and $f \colon A \to A'$ is a $(\bigwedge, 0, 1)$-preserving map. Then there exist a strongly atomic, 2-distributive, complete modular lattice $K \in \mathcal{L}(\mathbb{Z}_2\text{-Mod})$, a sublattice K' of K that is a strongly atomic complete lattice, and lattice isomorphisms $\xi \colon A \to \mathrm{Com}(K)$ and $\xi' \colon A' \to \mathrm{Com}(K')$ such that the restriction map $\mathrm{res}_{KK'}$ from $\mathrm{Com}(K)$ to $\mathrm{Com}(K')$ preserves completeness, every member of $\mathrm{Com}(K) \cup \mathrm{Com}(K')$ is a* principal *complete congruence, and the diagram*

$$\begin{array}{ccc} A & \xrightarrow{\ f\ } & A' \\ \xi\big\downarrow & & \big\uparrow\xi'^{-1} \\ \mathrm{Com}(K) & \xrightarrow{\mathrm{res}_{KK'}} & \mathrm{Com}(K') \end{array}$$

commutes, that is, $f = \xi'^{-1} \circ \mathrm{res}_{KK'} \circ \xi$ or, equivalently, $\xi' \circ f = \mathrm{res}_{KK'} \circ \xi$.

The proof of the following observation is postponed to Section 4.

Observation 2.5. Let K and K' be complete lattices, and let $g \colon K \to K'$ be a complete lattice homomorphism, that is, a $(\bigvee, \bigwedge)$-map. (Neither $g(0) = 0$, nor $g(1) = 1$ is assumed.) Define a map $g^* \colon \mathrm{Com}(K) \to \mathrm{Com}(K')$ by

$$g^*(\Theta) := \mathrm{com}_{K'}\big(g(\Theta)\big) = \mathrm{com}_{K'}\big(\{\langle g(x), g(y)\rangle : \langle x, y\rangle \in \Theta\}\big). \tag{2.1}$$

Then g^* is a $(\bigvee, 0)$-preserving map.

By the following theorem, every $(\bigvee, 0)$-preserving map between two complete lattices can be represented as g^* from Observation 2.5.

Theorem 2.6. *Let A and A' be complete lattices, and let $f \colon A \to A'$ be a $(\bigvee, 0)$-preserving map. Then there exist strongly atomic, 2-distributive, complete modular lattices K and K' in $\mathcal{L}(\mathbb{Z}_2\text{-Mod})$, a complete lattice homomorphism $g \colon K \to K'$, and lattice isomorphisms $\xi \colon A \to \mathrm{Com}(K)$ and $\xi' \colon A' \to \mathrm{Com}(K')$ such that, with g^* defined in (2.1), the diagram*

$$\begin{array}{ccc} A & \xrightarrow{\ f\ } & A' \\ \xi\big\downarrow & & \big\uparrow\xi'^{-1} \\ \mathrm{Com}(K) & \xrightarrow{\ g^*\ } & \mathrm{Com}(K') \end{array}$$

commutes, that is, $f = \xi'^{-1} \circ g^* \circ \xi$ *or, equivalently,* $\xi' \circ f = g^* \circ \xi$*; furthermore, every member of* $\mathrm{Com}(K) \cup \mathrm{Com}(K')$ *is a* principal *complete congruence.*

Note that by letting A' and f be A and the identity map, each of Theorems 2.3, 2.4, and 2.6 implies Corollary 2.1.

2.1. Our method. Theorems 2.3, 2.4, and 2.6 will be proved in Section 5. Roughly saying, the ideas of their proofs are the following. First, the construction for Corollary 2.1 is based on *open gadgets*, which are similar to but simpler than those used by Freese, Grätzer, and Schmidt [6]. In particular, our gadgets do not contain monochromatic squares. Hence, as opposed to [6], neither a monochromatic cube, nor the subspace lattice of the projective Fano plane occurs here. Figure 2 indicates what sort of gadgets we need, and Figure 1 shows how these gadgets are used to represent a *single* complete lattice as $\mathrm{Com}(K)$. The role of an open gadget is to force a J-constraint $x \leq \bigvee Y$; "J" stands for "join". The second idea is that the edges of K and K' are colored and we work with *color-preserving* complete congruences. It is only in the last step of the construction when we make all complete congruences color-preserving. Third, if we represent a *map*, some J-constraints should be deleted and some others should be inserted when passing from K to K'. Deletion can be represented in two different ways: either by *adding* a lock element to "deactivate" the corresponding open gadget, or by *deleting* the open gadget. Insertion goes in the opposite way. Fourth, we work with systems of J-constraints called *presentations* rather than directly with the lattices K and K'. Finally, by a particular case of the Adjoint Functor Theorem, $(\bigwedge, 0, 1)$-preserving maps have left (in other words, lower) adjoints; they show us how to use our gadgets.

3. Our toolkit and the construction for a single lattice

Since we intend to represent maps, not only a single complete lattice, the tools we introduce in this section are more general than those needed to prove Corollary 2.1. However, it would not be hard to extract a new and relatively easy construction and a proof from this paper for just Corollary 2.1.

A lattice is *nontrivial* if it has at least two elements. Corollary 2.1 is obvious for the trivial lattice. Clearly, Corollary 2.1 implies Theorems 2.3, 2.4, and 2.6 in the particular case when one of A and A' is trivial. Therefore, unless otherwise stated explicitly, every lattice in the rest of the paper is assumed to be *nontrivial* and *complete*, even if this is not always emphasized. For a set X, we use the notation $P(X) := \{Y : Y \subseteq X\}$ and $P^{-\varnothing}(X) := P(X) \setminus \{\varnothing\}$.

3.1. Presenting a complete join-semilattice. Basically, we are going to follow the well-known way how algebras are (finitely) presented. Hence, not all the (easy) statements of this subsection will be proved here. The following definition deals with *complete join-semilattices with* 0, $(\bigvee, 0)$-semilattices in short. (Here and in similar situations later, the usage of the big operation sign

$\bigvee$ rather than its binary variant, $\vee$, indicates that the semilattice is *complete.*) As opposed to sets, the elements of a *system* have multiplicities, which are nonzero cardinals. For example, the elements of a sequence form a system.

Definition 3.1. Let X be a set such that $0 \in X$. A *J-constraint* over X is a pair $\langle x, Y\rangle \in X \times P^{-\varnothing}(X)$ such that $\{x\} \cap Y \subseteq \{0\}$. ("J" comes from "join".) A *presentation* over X is a pair $\langle X; R\rangle$ where R is a system (or, in particular, a set) of J-constraints over X. By a $(\bigvee, 0)$-semilattice *presented by* $\langle X; R\rangle$ we mean a pair $\langle A; \iota\rangle$ such that the following three conditions hold.

(i) $\iota\colon X \to A$ is a 0-preserving map, and A is a $(\bigvee, 0)$-semilattice generated by $\iota(X)$.

(ii) For every $\langle x, Y\rangle \in R$, we have that $\iota(x) \leq \bigvee_{y\in Y} \iota(y)$. (In other words, ι *preserves* R.)

(iii) For every $(\bigvee, 0)$-semilattice B and every 0-preserving map $\lambda\colon X \to B$, if $\lambda(x) \leq \bigvee_{y\in Y} \lambda(y)$ holds for all $\langle x, Y\rangle \in R$ (that is, if λ preserves R), then there exists a $(\bigvee, 0)$-homomorphism $\pi\colon A \to B$ such that $\lambda = \pi \circ \iota$.

We say that $\langle A; \iota\rangle$ is *surjectively presented* by $\langle X; R\rangle$ and $\langle X; R\rangle$ is a *surjective presentation* if, in addition to (i)–(iii), the following condition also holds.

(iv) $\iota(X) = A$.

We often drop ι from the notation and say that A is presented by $\langle X; R\rangle$. We will write $\langle x \leq \bigvee Y\rangle$ and $\langle x \leq y_1 \vee \cdots \vee y_n\rangle$ in compound notation rather than $\langle x, Y\rangle$ and $\langle x, \{y_1, \dots, y_n\}\rangle$, respectively, and we abridge $\langle x, \{y\}\rangle$ to $\langle x, y\rangle$.

For example, N_5 from Figure 1 is surjectively presented by

$$\big\langle \{0, x, y, z, 1\}; \{\langle 1 \leq x \vee y\rangle, \langle x \leq z\rangle, \langle z \leq 1\rangle, \langle y \leq 1\rangle\}\big\rangle; \tag{3.1}$$

in this case ι is defined by $0 \mapsto 0$, $x \mapsto a$, $y \mapsto b$, $z \mapsto c$, and $1 \mapsto 1$. Note that

$$\big\langle \{0, x, y, z, 1\}; \{\langle 1 \leq x \vee y\rangle, \langle z \leq x \vee y\rangle, \langle x \leq z\rangle, \langle x \leq 1\rangle, \langle y \leq 1\rangle\}\big\rangle \tag{3.2}$$

is also a surjective presentation of N_5. (3.1) and (3.2) are minimal presentations of N_5 in the sense that none of their J-constraints can be omitted. There are non-surjective presentations like $\big\langle \{0, x, y, z\}; \{\langle y \leq x \vee z\rangle\}\big\rangle$. Note that since the equality $\lambda = \pi \circ \iota$ determines $\pi\rceil_{\iota(X)}$, 3.1(i) implies that

$$\text{if } \langle A; \iota\rangle \text{ is presented by } \langle X; R\rangle, \text{ then } \pi \text{ in 3.1(iii) is uniquely determined.} \tag{3.3}$$

If $\langle A_1, \iota_1\rangle$ and $\langle A_2, \iota_2\rangle$ are presented by $\langle X; R\rangle$, then they are called *isomorphic* if there is a $(\bigvee, 0)$-semilattice isomorphism $\varphi\colon A_1 \to A_2$ such that $\iota_2 = \varphi \circ \iota_1$. We know from Hales [32, Theorem 1] that for $|X| \geq 3$, there exists no free complete lattice generated by X. However, the free $(\bigvee, 0)$-semilattice over X exists and is isomorphic to $\langle P(X \setminus \{0\}); \bigvee, \varnothing\rangle$. (Remember that $0 \in X$.) This fact (or the standard universal algebraic technique of taking the "diagonal" in an appropriate direct product) and (3.3) give in a routine way that, for every presentation $\langle X; R\rangle$, $\langle A; \iota\rangle$ presented by $\langle X; R\rangle$ exists and it is unique up to isomorphism. For a nontrivial $(\bigvee, 0)$-semilattice A, let $X := A$, let $\iota\colon X \to A$

be the identity map id_A, and let R be the set of all J-constraints $\langle x, Y\rangle$ over X such that $x \neq 0$, $Y \neq \{0\}$, and $\langle x, Y\rangle$ holds in A. We say that $\langle X; R\rangle$ is the *canonical presentation* of $\langle A; \iota\rangle$. Clearly,

every nontrivial $(\bigvee, 0)$-semilattice is surjectively presented by its canonical presentation. (3.4)

Note that the canonical presentation is far from being optimal in general. For a presentation $\langle X; R\rangle$, let R^- denote the set $\{\langle x, Y\rangle \in R : x \neq 0\}$. It follows from Definition 3.1(i)–(ii) that for every $\langle A; \iota\rangle$,

$\langle A; \iota\rangle$ is (surjectively) presented by $\langle X; R\rangle$ iff it is (surjectively) presented by $\langle X; R^-\rangle$. (3.5)

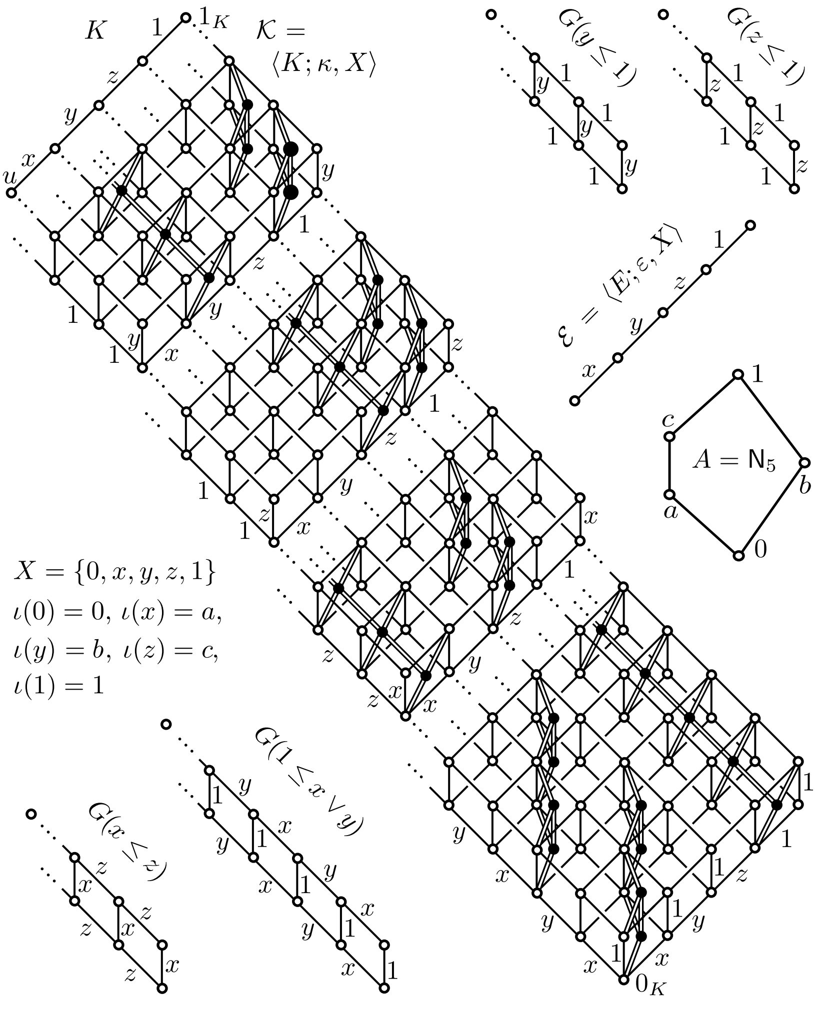

FIGURE 1. Representing N_5 as $\mathrm{Com}(K)$

3.2. Colored complete lattices. Coloring, which goes back to Jakubík [36], is a standard tool in studying congruence lattices of lattices; for example, see the papers (co)authored by Grätzer or Schmidt in the bibliographic section of the present paper. Our version is the following.

Definition 3.2 (CC-lattices and CCC-lattices)**.** The set of *prime intervals* (also called *edges*) of a lattice L will be denoted by $\mathrm{Ip}(L)$. Adding all singleton intervals to $\mathrm{Ip}(L)$, we obtain $\mathrm{Ip}^{00}(L) = \{[a,a] : a \in L\} \cup \mathrm{Ip}(L)$. If X is a set with 0 and $\gamma\colon \mathrm{Ip}^{00}(L) \to X$ is a surjective map such that for all $a, b \in L$, $\gamma([a,b]) = 0 \iff a = b$, then γ is a *coloring* of L. Here X is the *color set* and for a prime interval $\mathfrak{p} = [a,b]$, $\gamma(\mathfrak{p}) = \gamma([a,b])$ is the *color* of $\mathfrak{p}$. (The color of a singleton interval, which is always 0, deserves less attention.) The triplet $\langle L;\gamma,X\rangle$ is *a colored complete lattice*, in short, a *CC-lattice*, if L is a complete lattice and $\gamma\colon \mathrm{Ip}^{00}(L) \to X$ is a coloring of L. A coloring $\gamma\colon \mathrm{Ip}^{00}(L) \to X$ is a *cofinal coloring* and $\langle L;\gamma,X\rangle$ is a *cofinally colored complete lattice*, in short, a *CCC-lattice*, if it is a CC-lattice and for all $x \in X \setminus \{0\}$ and $u \in L \setminus \{1\}$, there exist infinitely many $\langle v,w\rangle \in \mathrm{Ip}(L)$ such that $u \le v \prec w$ and $\gamma([v,w]) = x$. Whenever more than one color set occurs, we always assume that

$$\text{any two color sets have the same } 0. \tag{3.6}$$

For a CC-lattice $\mathcal{L} = \langle L;\gamma,X\rangle$, a complete congruence Θ of L is a *color-preserving complete congruence*, in short a *CPC-congruence* of $\mathcal{L}$, if for any two $\mathfrak{p},\mathfrak{q} \in \mathrm{Ip}(L)$, if $\gamma(\mathfrak{p}) = \gamma(\mathfrak{q})$ and Θ collapses $\mathfrak{p}$, then Θ also collapses $\mathfrak{q}$.

Since the intersection of CPC-congruences is a CPC-congruence again, the CPC-congruences of $\mathcal{L} = \langle L;\gamma,X\rangle$ form a complete lattice, which is denoted by $\mathrm{Cpcc}(\mathcal{L}) = \mathrm{Cpcc}(\langle L;\gamma,X\rangle)$. For a relation $\rho \subseteq L^2$ and a pair $\langle u,v\rangle \in L^2$,

$$\mathrm{cpcc}(\rho) = \mathrm{cpcc}_{\mathcal{L}}(\rho) \quad \text{and} \quad \mathrm{cpcc}(\langle u,v\rangle) = \mathrm{cpcc}_{\mathcal{L}}(\langle u,v\rangle) \tag{3.7}$$

will denote the least CPC-congruence including ρ and containing $\langle u,v\rangle$, respectively. Similarly, for a color $x \in X$ and $Y \subseteq X$, we let

$$\begin{aligned} &\mathrm{cpcc}(x) = \mathrm{cpcc}(\langle u,v\rangle), \text{ where } [u,v] \in \mathrm{Ip}^{00}(L) \text{ with } \gamma([u,v]) = x,\\ &\mathrm{cpcc}(Y) = \bigvee\{\mathrm{cpcc}(x) : x \in Y\}; \end{aligned} \tag{3.8}$$

$[u,v] \in \mathrm{Ip}^{00}(L)$ above exists since γ is surjective. A *colored complete sublattice*, in short a *CC-sublattice*, of $\mathcal{K} = \langle K;\gamma,X\rangle$ is a CC-lattice $\mathcal{K}' = \langle K';\gamma',X'\rangle$ such that K' is a complete sublattice of K, $\{0\} \subseteq X' \subseteq X$, $\mathrm{Ip}(K') \subseteq \mathrm{Ip}(K)$, and γ' is the restriction $\gamma\rceil_{\mathrm{Ip}^{00}(K')}$ of γ to $\mathrm{Ip}^{00}(K')$. If so, then $\mathcal{K}$ is a (*colored*) *extension* of $\mathcal{K}'$. If, in addition, $X' = X$ and

$$\mathrm{cpext}_{\mathcal{K}'\mathcal{K}}\colon \mathrm{Cpcc}(\mathcal{K}') \to \mathrm{Cpcc}(\mathcal{K}), \text{ defined by } \Theta \ \mapsto \mathrm{cpcc}_{\mathcal{K}}(\Theta), \tag{3.9}$$

is an isomorphism, then $\mathcal{K}$ is a *CPC-congruence-preserving extension* of $\mathcal{K}'$. Our plan is to represent A as $\mathrm{Cpcc}(\langle K';\gamma',X\rangle)$ with $X = A$ first, and then to find a CPC-congruence-preserving extension $\langle K;\gamma,X\rangle$ of $\langle K';\gamma',X\rangle$ such that every complete congruence of K is a CPC-congruence.

Definition 3.3. Let $\mathcal{L}_i = \langle L_i; \gamma_i, X_i\rangle$, $i = 1, 2$, be CC-lattices. Their *direct product* is the CC-lattice $\mathcal{L}_1 \times \mathcal{L}_2 = \langle L_1 \times L_2; \gamma_1 \times \gamma_2, X_1 \cup X_2\rangle$, where

$$(\gamma_1 \times \gamma_2)(\mathfrak{p}) = \begin{cases} \gamma_1([u_1, u_2]), & \text{if } \mathfrak{p} = [\langle u_1, v\rangle, \langle u_2, v\rangle] \text{ with } u_1 \prec_{L_1} u_2, \\ \gamma_2([v_1, v_2]), & \text{if } \mathfrak{p} = [\langle u, v_1\rangle, \langle u, v_2\rangle] \text{ with } v_1 \prec_{L_2} v_2 \end{cases}$$

for $\mathfrak{p} \in \mathrm{Ip}(L_1 \times L_2)$ and $(\gamma_1 \times \gamma_2)([u, u]) = 0$ for $u \in L_1 \times L_2$.

3.3. Gadgets. Assume that A is a complete lattice that we want to represent, up to isomorphism, as $\mathrm{Com}(K)$. We illustrate the idea of our construction with the particular case $A = \mathsf{N}_5$; see Figure 1. The n-element chain will be denoted by C_n; in particular, $\mathsf{C}_2 = \{0, 1\}$. A *bounded well-ordered chain* is a well-ordered chain with a largest element. Before the following definition, the reader may want to see Figure 2, where $G(x \le z)^{\mathrm{cl}}$ is a closed gadget of type $\langle x, \{z\}\rangle$ while $G(x \le z)$, depicted three times, is an open gadget of the same type. Also, in Figure 1, $G(1 \le x \vee y)$ is an open gadget of type $\langle 1, \{x, y\}\rangle$.

Definition 3.4 (Gadgets)**.** (A) Let $\langle x, Y\rangle$ be a J-constraint over a set X with 0. The *target chain* for x is the unique CC-lattice $\langle T; \gamma_T, \{0, x\}\rangle$ such that $T = \mathsf{C}_2$ for $x \ne 0$ and $T = \mathsf{C}_1$ (the singleton lattice) for $x = 0$. A *work chain* for Y is a CCC-lattice $\langle W; \gamma_W, Y \cup \{0\}\rangle$ such that W is a bounded well-ordered chain. Then the direct product

$$\mathcal{V} = \langle V; \delta, \{x, 0\} \cup Y\rangle = \langle W; \gamma_W, Y \cup \{0\}\rangle \times \langle T; \gamma_T, \{0, x\}\rangle \tag{3.10}$$

is a *closed gadget* of type $\langle x, Y\rangle$. Its CC-sublattices $W \times \{0\}$ and $W \times \{1\}$ are the *closed lower chain* and the *closed upper chain*, respectively. (3.10) is a *closed ladder gadget* if $x \ne 0$ and $Y \ne \{0\}$ (equivalently, if neither the work chain, nor the target chain is a singleton); otherwise it is a (*closed*) *chain gadget.* The closed upper and lower chains in a closed ladder gadget are disjoint, but they are the same in a chain gadget. In (3.10), $\langle 1, 0\rangle = \langle 1_W, 0_T\rangle$ is the *lock element*; removing the lock element from the closed lower chain, we obtain the *open lower chain* $(W \times \{0\})^{\mathrm{op}} = W^{\mathrm{op}} \times \{0\}$, which might be the empty chain.

(B) An *open* (*ladder*) *gadget* is what we obtain from a closed ladder gadget by omitting its lock element. That is, starting from a closed ladder gadget $\mathcal{V} = \langle V; \delta, \{x, 0\} \cup Y\rangle$ and taking the sublattice $V^{\mathrm{op}} := V \setminus \{\text{lock element}\}$ and $\delta^{\mathrm{op}} := \delta\rceil_{\mathrm{Ip}^{00}(V^{\mathrm{op}})}$, the open (ladder) gadget corresponding to $\mathcal{V}$ is

$$\mathcal{U} = \langle U; \gamma, \{x, 0\} \cup Y\rangle = \mathcal{V}^{\mathrm{op}} := \langle V^{\mathrm{op}}; \delta^{\mathrm{op}}, \{x, 0\} \cup Y\rangle. \tag{3.11}$$

It is also of type $\langle x, Y\rangle$, and its lattice part is the disjoint union of the open lower chain and the closed upper chain. Also, it is a CCC-lattice.

(C) Only ladder gadgets have both closed and open variants; chain gadgets are always closed. This allows us to drop “ladder” and “closed” from “open ladder gadget” and “closed chain gadget”, respectively. For ladder gadgets, *closing* and *opening* a gadget mean adding a lock element and removing a lock element, respectively. The notation for opening is given in (3.11); to define that for closing, we mention that (3.11) will be equivalent to

$$\langle V; \delta, \{x, 0\} \cup Y\rangle = \mathcal{U}^{\mathrm{cl}} := \langle U^{\mathrm{cl}}; \gamma^{\mathrm{cl}}, \{x, 0\} \cup Y\rangle.$$

(D) The *singleton gadget* is also called the *trivial gadget*. It is a rather special (closed) chain gadget, and it is the only gadget of type $\langle 0, \{0\}\rangle$. If we want to exclude types of the form $\langle x, \{0\}\rangle$, we will speak of *infinite gadgets*.

(E) The term *gadget* without an adjective stands for any of the above; it can be a chain gadget, a closed ladder gadget, or an open (ladder) gadget.

(F) $G(x \le Y)$ will always denote a fixed *open* (ladder) gadget of type $\langle x, Y\rangle$.

The idea is that

if a complete congruence collapses the open lower chain of an open (ladder) gadget, then it collapses the whole gadget, (3.12)

but this is not so for the corresponding closed ladder gadget. Closed (ladder or chain) gadgets are only necessary technicalities for the proof of Theorem 2.6. For intervals $[u, v]$ and $[s, t]$, $[u, v]$ *transposes up* to $[s, t]$, in notation $[u, v] \nearrow [s, t]$, if $v \vee s = t$ and $v \wedge s = u$. In this case, $[s, t]$ *transposes down* to $[u, v]$, written as $[s, t] \searrow [u, v]$, and the two intervals are *transposed* or, in other words, *perspective*. *Projectivity* is the reflexive-transitive closure of perspectivity. The following lemma is trivial since work chains are cofinally colored; see Definitions 3.2 and 3.4.

Lemma 3.5. *The work chain, and thus the closed upper chain and the closed lower chain, of a closed gadget has no coatom. Hence, these chains are either infinite or they are singletons, and the open lower chain has no largest element. In a closed or open gadget, if two prime intervals are transposed, then they have the same color. Every open gadget is a CCC-lattice* (*not only a CC-lattice*), *but it is not a CC-sublattice of the corresponding closed ladder gadget.*

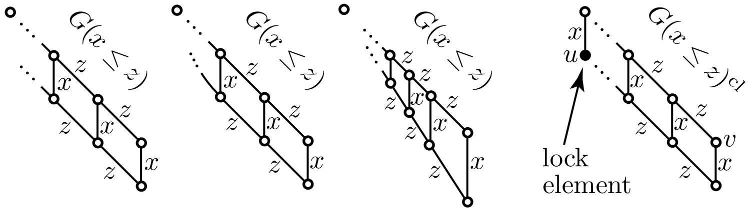

FIGURE 2. $G(x \le z)$ in three styles and $G(x \le z)^{\mathrm{cl}}$

3.4. Multi-gadgets. Following Czédli [2], where the term "special sum" was used, glued sums (of arbitrary many lattices) are defined as follows. Let I be a well-ordered index set and, for each $i \in I$, let L_i be a bounded lattice. The *ordinal sum* $K := \sum_{i\in I} L_i$ is defined in the usual way: it is the lattice whose base set is the disjoint union of the L_i, $i \in I$, and $u \le v$ if either $u \in L_i$,

$v \in L_j$, and $i < j$, or $u, v \in L_i$ and $u \leq_{L_i} v$. Let Θ be the smallest equivalence relation on K such that, for all $i \prec j \in I$, Θ collapses the top element 1_{L_i} of L_i with the bottom element 0_{L_j} of L_j. (The notation $i \prec j$ means that j covers i; here j is the unique cover of i.) Then Θ is a lattice congruence. The quotient lattice K/Θ is called the *glued sum* of the lattices L_i; it is denoted by $\sum'_{i \in I} L_i$. If $I = \{0 < \dots < n-1\}$, then we can write $L_0 +' \dots +' L_{n-1}$. For example, if $I = \{0 < 1 < 2\}$ and $L_0 = L_1 = L_2 = \mathsf{C}_2$, the two-element chain, then $\sum'_{i \in I} L_i = L_0 +' L_1 +' L_2$ is C_4, the four-element chain.

Definition 3.6. Let I be a bounded well-ordered index set. For each $i \in I$, let $\mathcal{U}_i = \langle U_i; \gamma_i, X_i \rangle$ be a gadget; (3.6) applies. Then the glued sum

$$\sum_{i \in I}{}' \mathcal{U}_i := \langle \sum_{i \in I}{}' L_i; \bigcup_{i \in I} \gamma_i, \bigcup_{i \in I} X_i \rangle \tag{3.13}$$

is called a *multi-gadget*. The gadgets $\mathcal{U}_i$, $i \in I$, are the (*gadget*) *summands* of the multi-gadget while the non-singleton summands are its *components*. We always assume that whenever a multi-gadget is considered, then

$$\text{its decomposition into gadget summands, see (3.13), is also given;} \tag{3.14}$$

this convention allows us to speak of the summands and the components of multi-gadgets.

Without convention (3.14), the singleton summands would never be determined, because their presence or absence does not influence the glued sum (3.13). Furthermore, even the chain components would not be determined without (3.14) sometimes, because, say, for $\varnothing \neq Y \subseteq Z \subseteq X$, the glued sum $G(0 \leq \bigvee Y) +' G(0 \leq \bigvee Y) +' G(0 \leq \bigvee Z)$ is a chain gadget itself.

If v and w are two distinct covers of an element u in a distributive CC-lattice, then $[u, v \vee w] = \{u, v, w, v \vee w\} := S$ is called a *cell*. Besides $0_S := u$ and $1_S := v \vee w$, the cell S has two *corners*, denoted by $S^{(\ell)}$ and $S^{(r)}$. (In figures, $S^{(\ell)}$ denotes the left corner but $S^{(\ell)}$ and $S^{(r)}$ play symmetric roles in our computations.) If $\gamma([u,v]) = \gamma([u,w])$, then S is *monochromatic cell* and its color is $\gamma(S) := \gamma([u,v]) = \gamma([u,w])$. The following lemma is trivial; note that it explains why we stipulate $\{x\} \cap Y \subseteq \{0\}$ in Definition 3.1.

Lemma 3.7. *The ladder components of a multi-gadget $\mathcal{U}$ can be recognized without* (3.14)*, $\mathcal{U}$ does not contain a monochromatic cell, and its lattice part is a strongly atomic, distributive, complete lattice. Every summand of $\mathcal{U}$ is a CC-sublattice.*

Remark 3.8. We will rely on the fact that, based on (3.14), the summands of a multi-gadget form a *well-ordered sequence* (which is a system) in a natural way.

Lemma 3.9. *Let $\mathcal{V}$ be an infinite gadget of type $\langle x, Y \rangle$ such that $\mathcal{V}$ is not a closed ladder gadget. If $\mathcal{V}$ is a CC-sublattice of a CC-lattice $\mathcal{K} = \langle K; \kappa, X \rangle$, then* $\operatorname{cpcc}_{\mathcal{K}}(x) \leq \bigvee_{y \in Y} \operatorname{cpcc}_{\mathcal{K}}(y)$ *holds in* $\operatorname{Cpcc}(\mathcal{K})$*; see* (3.7) *for the notation.*

Proof. If $\mathcal{V}$ is a chain gadget, then $x = 0$ and the statement clearly holds. Hence, we can assume that $\mathcal{V}$ is an open ladder gadget $G(x \le \bigvee Y)$. Let $\Theta = \bigvee_{y\in Y} \mathrm{cpcc}_{\mathcal{K}}(y)$. Since Θ is color-preserving, it collapses the edges of the lower chain of $G(x \le \bigvee Y)$. The join of this lower chain is the same in the *complete* sublattice $G(x \le \bigvee Y)$ as in K. Hence, by (3.12), Θ collapses the target chain of $G(x \le \bigvee Y)$, which is x-colored. Thus, $\mathrm{cpcc}_{\mathcal{K}}(x) \le \Theta$. $\square$

Definition 3.10. Keeping (3.14) in mind, for a multi-gadget $\mathcal{U} = \langle U; \gamma, X\rangle$, let $R := \{\langle x, Y\rangle : \mathcal{U}$ has an infinite gadget summand of type $\langle x, Y\rangle$ and this component is not a closed ladder gadget$\}$. Then $\langle X; R\rangle$ is the *presentation determined by* the multi-gadget. If it is a surjective presentation, then $\mathcal{U}$ is a *surjective multi-gadget* and $\langle A; \iota\rangle$ (or just simply the complete lattice A) from Definition 3.1 is the *fundamental lattice* of the multi-gadget $\mathcal{U}$. By Remark 3.8, we often consider R a *well-ordered system* in the natural way; in this case, a J-constraint can occur more than once in R, and the well-ordering of the J-constraints corresponds to the well-ordering of the components.

For example, the principal ideal $\downarrow u$ (with the inherited coloring) in Figure 1 is a multi-gadget, and (3.1) is the presentation it determines. Hence, $\downarrow u$ (with the inherited coloring) is a surjective multi-gadget and its fundamental lattice is $\langle \mathsf{N}_5; \iota\rangle$, where $\iota(0) = 0$, $\iota(x) = a$, $\iota(y) = b$, $\iota(z) = c$, and $\iota(1) = 1$. If we do not use chain gadgets to build $\mathcal{U}$, that is, in most of the cases, then R in Definition 3.10 is $\{\langle x, Y\rangle : \mathcal{U}$ has an open gadget component of type $\langle x, Y\rangle\}$.

Remark 3.11. As opposed to the presentation $\langle X; R\rangle$ of a multi-gadget $\mathcal{U}$, it follows from (3.5) and the first part of Lemma 3.7 that the fundamental lattice of $\mathcal{U}$ does not depend on convention (3.14). So, the surjectivity of $\mathcal{U}$ does not depend on (3.14) either.

Definition 3.12. Let X be a color set, that is, $0 \in X$. For each $x \in X \setminus \{0\}$, color the only edge of C_2 with x. Let $\mathcal{E} = \langle E; \varepsilon, X\rangle$ be the glued sum of these colored two-element chains according to a bounded well-ordered index set. Then $\mathcal{E}$ is a colored bounded well-ordered chain such that each color occurs exactly once. This colored chain is called an *equalizer chain* for X.

For example, an equalizer chain $\mathcal{E} = \langle E; \varepsilon, X\rangle$ for $X = \{0, x, y, z, 1\}$ is given in Figure 1. Generally, neither the equalizer chain, nor its order type is uniquely determined by X, but we always think of a fixed equalizer chain.

Definition 3.13. Let $\mathcal{U} = \langle U; \gamma, X\rangle$ be a multi-gadget and let $\mathcal{E} = \langle E; \varepsilon, X\rangle$ be an equalizer chain for X. Then the CC-lattice $\mathcal{U} \times \mathcal{E}$, see Definition 3.3, is an (*unsaturated*) *grid* associated with the multi-gadget. According to the canonical embedding $U \to U \times E$, defined by $u \mapsto \langle u, 0\rangle$, we consider $\mathcal{U}$ a CC-sublattice of $\mathcal{U} \times \mathcal{E}$.

By Lemma 3.5, for $\mathcal{U} \times \mathcal{E}$ as in Definition 3.13 above and $\mathfrak{p}, \mathfrak{q} \in \mathrm{Ip}(U \times E)$,

$$\text{if } \mathfrak{p} \text{ and } \mathfrak{q} \text{ are perspective intervals, then } (\gamma \times \varepsilon)(\mathfrak{p}) = (\gamma \times \varepsilon)(\mathfrak{q}). \tag{3.15}$$

Definition 3.14. To saturate the grid $\mathcal{U} \times \mathcal{E} = \langle U \times E; \gamma \times \varepsilon, X\rangle$ above, we turn each of its monochromatic cells into an M_3 by adding a new element into the interval spanned by the cell. The new elements are called *eyes.* (In Figures 1, 3, 4, and 6, the eyes are the black-filled elements.) The *cell of an eye* u, denoted by $\mathrm{cell}(u)$, is the cell of $U \times E$ into which u was inserted. Let K be the set we obtain after adding an eye to every monochromatic cell. The *set of eyes*, that is, the set of new elements, is $\mathrm{Eye}(K) := K \setminus (U \times E)$, and $\mathrm{Old}(K) := U \times E$ denotes the *set of old elements.* Each eye u has a unique lower cover $u_* := 0_{\mathrm{cell}(u)}$ and a unique upper cover $u^* := 1_{\mathrm{cell}(u)}$. The corners of $\mathrm{cell}(u)$ are denoted by $u^{(\ell)}$ and $u^{(r)}$. Letting $u_* = u^* := u$ for $u \in U \times E$,

$$\mathrm{Eye}(K) = \{u \in K : u_* \prec u \prec u^*\},\ \mathrm{Old}(K) = \{u \in K : u_* = u = u^*\}. \quad (3.16)$$

Using the notation following (3.12), the ordering $\leq$ on K, also written as $\leq_K$, is the reflexive-transitive closure of (that is, the quasiorder generated by)

$$\begin{aligned}\{\langle u, v\rangle \in \mathrm{Old}(K)^2 : u \leq_{\mathrm{Old}(K)} v\}&\\ \cup \{\langle u_*, u\rangle : u \in \mathrm{Eye}(K)\} \cup \{\langle u, u^*\rangle : u \in \mathrm{Eye}(K)\}&\\ \cup \{\langle u, v\rangle : u, v \in \mathrm{Eye}(K) \text{ and } [u_*, u^*] \nearrow [v_*, v^*]\}&.\end{aligned}$$

Now that the ordering has been defined, we have that

$$\mathrm{Old}(K) \cap {\uparrow_K}\, u = {\uparrow_{\mathrm{Old}(K)}}\, u^*, \quad \mathrm{Old}(K) \cap {\downarrow_K}\, u = {\downarrow_{\mathrm{Old}(K)}}\, u_*. \quad (3.17)$$

We will show soon that $K = \langle K; \leq_K\rangle$ is a modular lattice. We can uniquely extend $\gamma \times \varepsilon$ to a coloring $\kappa\colon \mathrm{Ip}^{00}(K) \to X$ by the rule that, for $\mathfrak{p} \in \mathrm{Ip}(K)$,

$$\kappa(\mathfrak{p}) = \begin{cases} (\gamma \times \varepsilon)(\mathfrak{p}), & \text{if } \mathfrak{p} \in \mathrm{Ip}(U \times E),\\ (\gamma \times \varepsilon)(S), & \text{if } \mathfrak{p} = [u_*, u] \text{ and } S = \mathrm{cell}(u),\\ (\gamma \times \varepsilon)(S), & \text{if } \mathfrak{p} = [u, u^*] \text{ and } S = \mathrm{cell}(u),\\ (\gamma \times \varepsilon)([u_*, v_*]), & \text{if } \mathfrak{p} = [u, v],\ u, v \in \mathrm{Eye}(K),\\ & \quad \text{and } [u_*, u^*] \nearrow [v_*, v^*]. \end{cases} \quad (3.18)$$

The CC-lattice $\mathcal{K} = \langle K; \kappa, X\rangle$ is the *saturated grid induced by* the multi-gadget $\mathcal{U} = \langle U; \gamma, X\rangle$. The *fundamental lattice* of $\mathcal{K}$ is that of $\mathcal{U}$. CC-sublattices of $\mathcal{U} \times \mathcal{E}$ can be saturated analogously. Single gadgets are particular multi-gadgets, so they also induce saturated grids.

For example, $\mathcal{K}$ in Figure 1 is a saturated grid; its fundamental lattice is $\langle A; \iota\rangle = \langle \mathsf{N}_5; \iota\rangle$. We will show that $\mathrm{Com}(K) \cong A$ always holds.

4. Auxiliary lemmas and a key lemma

The finitary counterparts of Observations 2.2 and 2.5, with Con( ) instead of Com( ), are well known; they follow easily from known descriptions of the join of two congruences and that of congruence generation by means of finite sequences of elements. This method is not applicable now, but the easy proofs below work also for these counterparts if we replace Com( ) with Con( ).

Proof of Observation 2.5. Clearly, g^* is a 0-preserving map. Let $\alpha_i \in \mathrm{Com}(K)$ for $i \in I$, $\beta := \bigvee\{\alpha_i : i \in I\} \in \mathrm{Com}(K)$, $\beta' := g^*(\beta) \in \mathrm{Com}(K')$, $\alpha_i' := g^*(\alpha_i) \in \mathrm{Com}(K')$, and $\gamma' := \bigvee\{\alpha_i' : i \in I\} \in \mathrm{Com}(K')$. We have to show that $\beta' = \gamma'$. By its definition, (2.1), g^* is order-preserving. Hence, $\beta' \geq \alpha_i'$ for all $i \in I$, and we obtain that $\beta' \geq \gamma'$.

To verify the converse inequality, let $\gamma := \{\langle u, v\rangle \in K^2 : \langle g(u), g(v)\rangle \in \gamma'\}$. We claim that $\gamma \in \mathrm{Com}(K)$. It is an equivalence, because so is γ'. To show that γ preserves arbitrary meets, assume that $\langle u_i, v_i\rangle \in \gamma$ for $i \in I$. Then $\langle g(u_i), g(v_i)\rangle \in \gamma'$ for $i \in I$. Since $\gamma' \in \mathrm{Com}(K')$, we obtain that $\langle \bigwedge_{i\in I} g(u_i), \bigwedge_{i\in I} g(v_i)\rangle \in \gamma'$. Using that g is $\bigwedge$-preserving, we conclude $\langle g(\bigwedge_{i\in I} u_i), g(\bigwedge_{i\in I} v_i)\rangle \in \gamma'$, which yields that $\langle \bigwedge_{i\in I} u_i, \bigwedge_{i\in I} v_i\rangle \in \gamma$. By duality, $\langle \bigvee_{i\in I} u_i, \bigvee_{i\in I} v_i\rangle \in \gamma$ also holds. Thus, $\gamma \in \mathrm{Com}(K)$.

Clearly, $\alpha_i \leq \gamma$ for $i \in I$. This yields that $\beta \leq \gamma$, and we also have $\beta' = g^*(\beta) \leq g^*(\gamma)$ since g^* is order-preserving. For all $\langle u, v\rangle \in \gamma$, the pair $\langle g(u), g(v)\rangle$, which is in the generating set of $g^*(\gamma)$, belongs to γ' by the definition of γ. Hence, $g^*(\gamma) \leq \gamma'$, and we obtain $\beta' \leq \gamma'$, as required. □

Proof of Observation 2.2. Clearly, $\mathrm{ext}_{KK'}$ is 0-separating. With $g\colon K \to K'$, defined by $x \mapsto x$, Observation 2.5 applies. □

The following statement follows from Definition 1.1 (including its third line after (2)), Statement 1.4, and Subsection 6.4 in Herrmann and Huhn [34].

Lemma 4.1 (Herrmann and Huhn [34]). *A strongly atomic, complete, modular lattice is* not 2*-distributive if and only if there exists a* covering 3*-frame in* K*, that is, if there are* $u, v, w_0, \dots, w_3 \in K$ *such that, for every three-element subset* $\{i, j, k\}$ *of* $\{0, 1, 2, 3\}$,

(i) $w_i \vee w_j \vee w_k = v$,
(ii) $w_i \wedge (w_j \vee w_k) = u$, *and*
(iii) *each of* $w_0, w_1, \dots, w_3$ *covers* u.

Lemma 4.2. *If* $\langle K; \kappa, X\rangle$ *is a saturated grid, then* $K \in \mathcal{L}(\mathbb{Z}_2\text{-}\mathrm{Mod})$ *and* K *is a strongly atomic,* 2*-distributive, complete modular lattice.*

Proof. We keep the notation of Definition 3.14. Basically, we follow the argument of Freese, Grätzer, and Schmidt [6] in a simpler setting. Observe that

$$K \text{ is a lattice,} \tag{4.1}$$

because it can be obtained by forming direct products of either a (not necessarily finite) chain and M_3, or three chains, taking gluings, forming directed unions, taking gluings again, and forming a directed union. (A more detailed explanation will be given soon; until then, the reader can consider (4.1) only a hypothesis.) Since the summands in (3.13) are strongly atomic complete distributive lattices, so are U and $U \times E$. Accepting that K is a lattice, it is clearly strongly atomic and complete, since saturation preserves these properties.

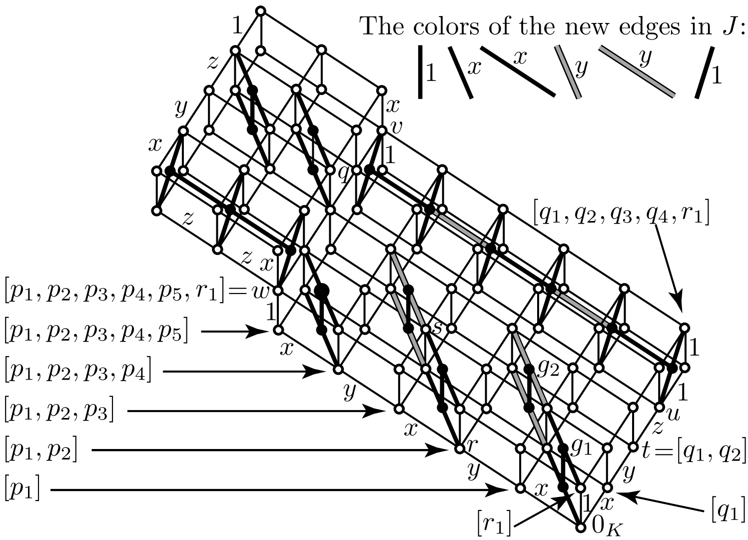

FIGURE 3. A finite sublattice S of K and $J = \downarrow v$ in S

A typical finite sublattice of K is given in Figure 3, which is drawn in a bit different style than Figure 1. (Open ladder gadgets and chain gadget components can be disregarded, because otherwise we can embed the multi-gadget into a larger one whose components are closed ladder gadgets.) In general, we obtain a finite sublattice S of K by taking finite sublattices of closed ladder gadgets, forming the direct products of these sublattices and a finite subchain of E, saturating these direct products in the sense of Definition 3.14 to obtain finite lattices S_i, and gluing these S_i. For example, if S is the lattice of Figure 3, which is a sublattice of the infinite lattice K given in Figure 1, then S_0 is the ideal $\downarrow v$, S_1 is the filter $\uparrow w$, and S is a gluing of S_0 and S_1. The S_i are always lattices, because they can be obtained as repeated gluings of suitable intervals, each of them being the direct product either of a finite chain and M_3, or three finite chains. For instance, the intervals $[u, v]$, $[r, s]$, and $[t, q]$ in Figure 3 are (isomorphic to) $\mathsf{C}_6 \times \mathsf{M}_3$, $\mathsf{M}_3 \times \mathsf{C}_2$, and $\mathsf{C}_6 \times \mathsf{C}_2 \times \mathsf{C}_2$, respectively.

Almost the same argument showed the validity of (4.1) earlier; the only difference is that then we had to allow infinite chains. Alternatively, K is a lattice, because it can be obtained as a directed union of finite lattices S described above. So, from now on, K is a lattice. Since gluing and directed union preserve modularity, K is a modular lattice. As a directed union of finite lattices, K is *locally finite*, that is, each of its finitely generated sublattice is finite. Before continuing the proof, we mention other possibilities of proving modularity and local finiteness. Since every maximal complemented interval of S_i is either a boolean cube, or of the form $\mathsf{M}_3 \times \mathsf{C}_2$, we can apply Herrmann [33] to conclude modularity.

An alternative argument for local finiteness runs as follows. Assume that H is a finite subset of K, and keep the notation from (3.16) in mind. Let S_1 be the sublattice of $\mathrm{Old}(K)$ generated by $\{h_* : h \in H\} \cup \{h^* : h \in H\}$; it is finite by distributivity. Since every element of $\mathrm{Old}(K) = U \times E$ has at most three covers and at most three lower covers in $\mathrm{Old}(K)$, there are only finitely many cells of $U \times E$ whose intersection with S_1 is nonempty. Adding the elements of these cells to S_1 we obtain a finite subset S_2 of $U \times E$, which generates a finite sublattice S_3 of $U \times E$. Finally, let $S_4 = S_3 \cup \{e \in \mathrm{Eye}(K) : \mathrm{cell}(e) \subseteq S_3\}$. Since S_4 is finite and includes H, we conclude that

$$K \text{ is locally finite.} \tag{4.2}$$

Since K is locally finite, to show that $K \in \mathcal{L}(\mathbb{Z}_2\text{-Mod})$, it suffices to show that every finite sublattice of K belongs to $\mathcal{L}(\mathbb{Z}_2\text{-Mod})$. This follows from the fact that every finite sublattice S of K can be embedded into the subspace lattice of a finite vector space over the two-element field. We show this only for S in Figure 3; the general case is quite similar. Let U_0 and E_0 be the images of S under the first projection to U and the second projection to E. First, we pick a finite set X and embed $U_0 \times E_0$ in the powerset lattice $\langle P(X); \cap, \cup\rangle$. Actually, we choose $X = \{p_1, \dots, p_7, q_1, \dots, q_4, r_1, r_2\}$; the notation is in accordance with the "coordinate axes" suggested by the figure. From now on, we can assume that $U_0 \times E_0$ is a sublattice of $P(X)$. Let V be the vector space over the two-element field with basis X, and let $\mathrm{Sub}(V)$ be its subspace lattice. Defining $g(Y)$ as the subspace $[Y]$ spanned by Y, we obtain a lattice embedding $g\colon U_0 \times E_0 \to \mathrm{Sub}(V)$. Observe that an arbitrary cell S of $U_0 \times E_0$ is of the form $\{[Y], [Y \cup \{x\}], [Y \cup \{y\}], [Y \cup \{x, y\}]\}$ where $Y \subset X$, $x, y \in X \setminus Y$, and $x \neq y$. If S contains an eye inserted in this cell, then we let $[Y \cup \{x - y\}]$ be the g-image of this eye. For example, the g-image of the big eye (big black-filled circle) in the figure is $[p_1, \dots, p_4, r_1, p_5 - q_1]$. In this way, extending g to all eyes belonging to S, we obtain a map $S \to \mathrm{Sub}(V)$, which is also denoted by g. The straightforward details showing that g is a lattice embedding will be omitted; note that this is a particular case of a more involved embeddability statement used in Freese, Grätzer, and Schmidt [6].

Finally, suppose for a contradiction that K and, consequently, a finite sublattice S of K are not 2-distributive. Then an appropriate element u has four covers, $w_0, \dots, w_3$, according to Lemma 4.1. It is clear from the construction of S, see Figure 3, that three of these four covers generate an M_3 sublattice. This is a contradiction, since these three covers fail to satisfy 4.1(ii). Consequently, K is 2-distributive, as required. $\square$

Lemma 4.3 (Grätzer and Lakser [16])**.** *In a strongly atomic lattice, each complete congruence Θ is determined by the prime intervals it collapses.*

Note that Freese, Grätzer, and Schmidt [6, Lemma 1] also used this statement. We need it mainly in the following particular form; to formulate it, we define the *color set* of a CPC-congruence Θ of a CC-lattice $\langle U;\gamma,X\rangle$ as

$$\mathrm{Cols}(\Theta) := \{\gamma([u,v]) : [u,v] \in \mathrm{Ip}^{00}(U) \text{ and } \langle u,v\rangle \in \Theta\}. \tag{4.3}$$

Corollary 4.4. *For every CPC-congruence Θ in a strongly atomic CC-lattice,*

$$\begin{aligned}\Theta &= \mathrm{cpcc}(\{\langle u,v\rangle : u \prec v,\ \langle u,v\rangle \in \Theta\}) = \bigvee\{\mathrm{cpcc}(x) : x \in \mathrm{Cols}(\Theta)\}\\ &= \mathrm{cpcc}(\mathrm{Cols}(\Theta)) = \{\langle u,v\rangle : \mathrm{Cols}([u\wedge v, u\vee v]) \subseteq \mathrm{Cols}(\Theta)\}.\end{aligned} \tag{4.4}$$

Proof. Let $\Psi := \mathrm{cpcc}(\{\langle u,v\rangle : u \prec v,\ \langle u,v\rangle \in \Theta\})$, and consider the binary relation $\Gamma := \{\langle u,v\rangle : \mathrm{Cols}([u\wedge v, u\vee v]) \subseteq \mathrm{Cols}(\Theta)\}$ occurring in (4.4). Since $\Theta \geq \Psi$ is clear, Ψ cannot collapse more prime intervals than Θ. Hence, Θ and Ψ collapse the same prime intervals, and Lemma 4.3 yields $\Theta = \Psi$, which is the first equality in (4.4). The next two equalities (together with the notation) follow from (3.8). If $\langle u,v\rangle$ belongs to Θ, then Θ collapses the interval $[u\wedge v, u\vee v]$. This yields that $\Theta \subseteq \Gamma$. Conversely, let $\langle u,v\rangle \in \Gamma$ and denote the interval $[u\wedge v, u\vee v]$ by I. Clearly, I is a strongly atomic CC-lattice with respect to the restriction of the original coloring map. The restriction $\Theta\rceil_I$ is a CPC-congruence on I. Since $\langle u,v\rangle \in \Gamma$, $\Theta\rceil_I$ collapses all prime intervals of I. So does $I\times I$, the largest complete congruence on I. Thus, by Lemma 4.3, $\Theta\rceil_I = I\times I$. Hence, $\langle u,v\rangle \in I\times I \subseteq \Theta$, and we obtain that $\Gamma \subseteq \Theta$. Therefore, $\Theta = \Gamma$, which proves the third equality in (4.4). □

Next, we recall a part of Freese, Grätzer, and Schmidt [6, Lemma 2].

Lemma 4.5 ([6]). *A congruence Θ of a strongly atomic complete lattice is a complete congruence if and only if each Θ-block is an interval.*

Definition 4.6. The *color set* of a convex subset V of a CC-lattice $\mathcal{U} = \langle U;\gamma,X\rangle$ is $\mathrm{Cols}(V) = \mathrm{Cols}_\gamma(V) := \{\gamma([u,v]) : [u,v] \in \mathrm{Ip}^{00}(V)\}$. (The role of convexity is to guarantee that $\mathrm{Ip}^{00}(V) \subseteq \mathrm{Ip}^{00}(U)$.) If $\mathcal{U}$ happens to be a gadget of type $\langle x,Y\rangle$, then $t(\mathcal{U}) := x$ is its *target color* while $W(\mathcal{U}) := Y$ (the color set of the upper or, equivalently, lower chain) is its *work color set.* Intervals $[a,b]$ and principal filters $\uparrow b$ of a well-ordered chain will be called *segments.* If $\mathcal{U}_j$ is a summand, see (3.14), Θ is a CPC-congruence of a multi-gadget, and Y_Θ stands for $\mathrm{Cols}(\Theta)$, then we define

$$\mathrm{top}(\Theta,u,U_j) := \bigvee\{v \in U_j : u \leq v \text{ and } \mathrm{Cols}([u,v]) \subseteq Y_\Theta\} \in U_j \text{ and}$$

$$\mathrm{bot}(\Theta,u,U_j) := \bigwedge\{v \in U_j : v \leq u \text{ and } \mathrm{Cols}([v,u]) \subseteq Y_\Theta\} \in U_j;$$

clearly, they are the largest and the smallest element of $[u](\Theta\rceil_{U_j})$. As usual, for lattices L_i and $\Theta_i \in \mathrm{Con}(L_i)$, $i \in \{1,2\}$, the *product congruence* is

$$\Theta_1 \times \Theta_2 := \{\langle\langle u_1,v_1\rangle,\langle u_2,v_2\rangle\rangle : \langle u_1,u_2\rangle \in \Theta_1 \text{ and } \langle v_1,v_2\rangle \in \Theta_2\}.$$

To have a complete insight into CPC-congruences of multi-gadgets, we need the following lemma even if it is tedious.

Lemma 4.7. *Keeping* (3.14) *in mind, let* Θ *and* $\mathcal{U}_j$ *be a CPC-congruence and a component of a multi-gadget* $\mathcal{U} = \langle U; \gamma, X\rangle$*, respectively, and let* $u \in U_j$*. Assuming the notation of Definitions 3.6, 3.10 and 4.6, the following hold.*

(i) Θ *is determined by* $Y_\Theta := \mathrm{Cols}(\Theta)$.

(ii) Θ *collapses* U_j*, that is,* $U_j \times U_j \subseteq \Theta$*, if and only if* $\mathrm{Cols}(\mathcal{U}_j) \subseteq Y_\Theta$.

(iii) *If* $\mathcal{U}_j$ *is an open ladder gadget and* $W(\mathcal{U}_j) \subseteq Y_\Theta$*, then* $\mathrm{Cols}(\mathcal{U}_j) \subseteq Y_\Theta$.

(iv) *If* $W(\mathcal{U}_j) \nsubseteq Y_\Theta$*, then* $\mathrm{top}(\Theta, u, U_j) = 1_{U_j}$ *iff either* $u = 1_{U_j}$*, or* u *is a lock element and* $t(\mathcal{U}_j) \in Y_\Theta$.

(v) *If* $W(\mathcal{U}_j) \subseteq Y_\Theta$ *but* $t(\mathcal{U}_j) \notin Y_\Theta$*, then* $\mathrm{top}(\Theta, u, U_j) = 1_{U_j}$ *iff* u *is in the closed upper chain of* U_j*. (*$\mathcal{U}_j$ *is necessarily a closed ladder gadget.)*

(vi) *If* $\mathrm{top}(\Theta, u, U_j) \neq 1_{U_j}$*, then* $\mathrm{top}(\Theta, u, U_j)$ *is the largest element of* $[u]\Theta$*, that is,* $1_{[u]\Theta} = \mathrm{top}(\Theta, u, U_j)$*. If we have that* $\mathrm{top}(\Theta, u, U_j) = 1_{U_j}$ *and* $\mathrm{Cols}(\mathcal{U}_i) \subseteq Y_\Theta$ *for all* $i > j$*, then* $1_{[u]\Theta} = 1_U$*. The only remaining case is that* $\mathrm{top}(\Theta, u, U_j) = 1_{U_j}$ *and there is a smallest* i *such that* $i > j$ *and* $\mathrm{Cols}(\mathcal{U}_i) \nsubseteq Y_\Theta$*; then* $1_{[u]\Theta} = \mathrm{top}(\Theta, 0_{U_i}, U_i)$.

(vii) *If* $\mathrm{bot}(\Theta, u, U_j) \neq 0_{U_j}$*, then* $0_{[u]\Theta} = \mathrm{bot}(\Theta, u, U_j)$.

(viii) *Assume that* $\mathrm{bot}(\Theta, u, U_j) = 0_{U_j}$*, and let* i *be the smallest element of the set* $\{j\} \cup \{j' \in I : j' < j$ *and* $\mathrm{Cols}(\mathcal{U}_{j''}) \subseteq Y_\Theta$ *for all* j'' *such that* $j' \leq j'' < j\}$*. If* i *has no lower cover in* I*, then* $0_{[u]\Theta} = 0_{U_i}$*. If* i *has a (necessarily unique) lower cover* k*, then there are three subcases. First, if* $\mathcal{U}_k$ *is a closed ladder gadget and* $t(\mathcal{U}_k) \in Y_\Theta$*, then* $0_{[u]\Theta}$ *is the lock element of* $\mathcal{U}_k$*. Second, if* $\mathcal{U}_k$ *is a closed ladder gadget and* $W(\mathcal{U}_k) \subseteq Y_\Theta$*, then* $0_{[u]\Theta}$ *is the least element of the upper chain of* $\mathcal{U}_k$*. Third and otherwise,* $0_{[u]\Theta} = 1_{U_k} = 0_{U_i}$.

Besides (i)–(viii)*, the "intrinsic" behavior of* $\Theta\rceil_{U_j}$ *is also described as follows.*

(ix) *Assume that* $\mathrm{Cols}(\mathcal{U}_j) \nsubseteq Y_\Theta$*, and denote the target chain and the open lower chain of* U_j *by* T_j *and* B_j*, respectively. Then the* $\Theta\rceil_{B_j}$*-blocks are the maximal segments whose edges are colored by elements of* Y_Θ*,* T_j *is collapsed by* Θ *iff* $t(\mathcal{U}_j) \in Y_\Theta$*, and* $\Theta\rceil_{B_j \times T_j} = \Theta\rceil_{B_j} \times \Theta\rceil_{T_j}$.

Finally, we have also the following description of Θ*: for arbitrary* $u, v \in U$*,*

(x) $\langle u, v\rangle \in \Theta$ *if and only if* $\mathrm{Cols}[u \wedge v, u \vee v] \subseteq \mathrm{Cols}(\Theta)$.

For a *surjective* multi-gadget $\mathcal{U}$, the possible color sets $Y_\Theta = \mathrm{Cols}(\Theta)$ above will be described by Lemma 4.8.

Proof of Lemma 4.7. (i), (ii), and (x) follow from Corollary 4.4. Lemma 3.9 implies (iii). For the rest of the proof, note that $\mathcal{U}_j$ is not a singleton by Definition 3.6. The case $|U_j| = 2$ is trivial and will not be considered. The case of chain gadgets is trivial again by the cofinality of their colorings and Corollary 4.4. Even if $\mathcal{U}_j$ is a closed ladder gadget, cofinality applies since $\mathcal{U}_j^{\mathrm{op}}$ is a CCC-lattice. (In other words, the open lower chain and the open upper chain of $\mathcal{U}_j$ are cofinally colored.) Hence, we can conclude (iv)–(viii) in a straightforward way. (ix) follows from (i)–(viii) and the fact that the direct product of two lattices has no skew congruence; see Fraser and Horn [4]. □

Lemma 4.8. *Let $\mathcal{U} = \langle U; \gamma, X\rangle$ be a surjective multi-gadget with fundamental lattice $\langle A; \iota\rangle$; see Definition* 3.10 *and Remark* 3.11 *and keep* (3.8) *in mind. Then the maps*

$$f_{4.5}\colon \mathrm{Cpcc}(\mathcal{U}) \to A, \text{ defined by } \Theta \mapsto \bigvee \iota(\mathrm{Cols}(\Theta)), \text{ and} \tag{4.5}$$

$$f_{4.6}\colon A \to \mathrm{Cpcc}(\mathcal{U}), \text{ defined by } a \mapsto \mathrm{cpcc}_{\mathcal{U}}(\iota^{-1}(\downarrow a)), \tag{4.6}$$

are reciprocal lattice isomorphisms. Furthermore, for every $x \in X$ and $a \in A$,

$$f_{4.6}(\iota(x)) = \mathrm{cpcc}_{\mathcal{U}}(x) \text{ and} \tag{4.7}$$

$$\mathrm{Cols}(f_{4.6}(a)) = \iota^{-1}(\downarrow a) := \{y \in X : \iota(y) \le a\}. \tag{4.8}$$

Finally, a subset Y of X is the color set $\mathrm{Cols}(\Theta)$ *of some* $\Theta \in \mathrm{Cpcc}(\mathcal{U})$ *if and only if $Y = \iota^{-1}(\downarrow a)$ holds form some $a \in A$.*

Proof. As in Definition 3.10, let $\langle X; R\rangle$ denote the surjective presentation determined by the multi-gadget $\mathcal{U}$. Consider the map

$$\lambda\colon X \to \mathrm{Cpcc}(\mathcal{U}), \text{ defined by } x \mapsto \mathrm{cpcc}_{\mathcal{U}}(x).$$

By Lemma 3.9 and the last sentence of Lemma 3.7, λ preserves R. Hence, by Definition 3.1, there is a $(\bigvee, 0)$-homomorphism $\pi\colon A \to \mathrm{Cpcc}(\mathcal{U})$ such that $\lambda = \pi \circ \iota$. This π is uniquely determined by the rule

$$\pi(\iota(x)) = \mathrm{cpcc}_{\mathcal{U}}(x), \quad \text{for } x \in X, \tag{4.9}$$

since $\iota\colon X \to A$ is surjective. We claim that π and $f_{4.6}$ are the same maps. To show this, let $a \in A$ and pick an $x \in X$ such that $\iota(x) = a$. Since $x \in \iota^{-1}(\downarrow a)$, we have that $\pi(a) = \mathrm{cpcc}_{\mathcal{U}}(x) \le f_{4.6}(a)$. Conversely, let $y \in \iota^{-1}(\downarrow a)$, that is, $\iota(y) \le a$. Since π is order-preserving, $\mathrm{cpcc}_{\mathcal{U}}(y) = \pi(\iota(y)) \le \pi(a) = \mathrm{cpcc}_{\mathcal{U}}(x)$. Since $f_{4.6}(a)$ is the join of these $\mathrm{cpcc}_{\mathcal{U}}(y)$, see (3.8) and (4.6), $f_{4.6}(a) \le \pi(a)$. Thus, $f_{4.6} = \pi$. In view of (4.9), this proves (4.7), and we also obtain that

$$f_{4.6} \text{ is a } (\bigvee, 0)\text{-homomorphism from } A \text{ to } \mathrm{Cpcc}(\mathcal{U}). \tag{4.10}$$

Hence, for an arbitrary $\Theta \in \mathrm{Cpcc}(\mathcal{U})$, we can compute as follows.

$$f_{4.6}(f_{4.5}(\Theta)) \overset{(4.5)}{=} f_{4.6}\big(\bigvee\{\iota(x) : x \in \mathrm{Cols}(\Theta)\}\big)$$
$$\overset{(4.10)}{=} \bigvee\{f_{4.6}(\iota(x)) : x \in \mathrm{Cols}(\Theta)\} \overset{(4.7)}{=} \bigvee\{\mathrm{cpcc}_{\mathcal{U}}(x) : x \in \mathrm{Cols}(\Theta)\} \overset{(4.4)}{=} \Theta.$$

This yields that

$$f_{4.6} \circ f_{4.5} \text{ is the identity map on } \mathrm{Cpcc}(\mathcal{U}). \tag{4.11}$$

Next, in order to show that $f_{4.5}$ is surjective map, let $a \in A$. We define $Y := \{x \in X : \iota(x) \le a\}$. By Lemma 4.7(ii)–(ix), considered as conditions depending on $Y_\Theta := Y$, define an equivalence relation Θ on U. With emphasis on 4.7(ix), it follows in a straightforward way that the restriction of Θ to every component is a lattice congruence. Hence, Θ is a congruence on U, which is a glued sum of its components. We obtain from Lemma 4.5 that Θ is a complete congruence. The edges and the colors of Θ come from the components of $\mathcal{U}$.

Hence, it follows from the definition of Θ and, mainly, from 4.7(ix) that Θ is color-preserving, $\mathrm{Cols}(\Theta) = Y$, and $\Theta \in \mathrm{Cpcc}(\mathcal{U})$. Thus, since $\mathcal{U}$ and ι are surjective,

$$f_{4.5}(\Theta) \overset{(4.5)}{=} \bigvee\{\iota(x) : x \in \mathrm{Cols}(\Theta)\} = \bigvee\{\iota(x) : x \in X,\ \iota(x) \le a\} = a,$$

proving the surjectivity of $f_{4.5}$. On the other hand, (4.11) yields that $f_{4.5}$ is injective, whence it is a bijection. Multiplying (4.11) with $f_{4.5}^{-1}$ from the right, we obtain $f_{4.6} = f_{4.5}^{-1}$, and it follows that $f_{4.5}$ and $f_{4.6}$ are reciprocal bijections. It is clear by (4.10) that $f_{4.6}$ is order-preserving. Suppose, for a contradiction, that $f_{4.5}$ is not order-preserving, and pick $\Theta, \Psi \in \mathrm{Cpcc}(\mathcal{U})$ such that $\Theta \le \Psi$ but $f_{4.5}(\Theta) \not\le f_{4.5}(\Psi)$. Then $f_{4.5}(\Theta) \vee f_{4.5}(\Psi) \neq f_{4.5}(\Psi)$. As a bijection, $f_{4.6}$ preserves non-equalities and, by (4.10), it also preserves joins. Hence,

$$\begin{aligned}\Psi = \Theta \vee \Psi = f_{4.6}(f_{4.5}(\Theta)) \vee f_{4.6}(f_{4.5}(\Psi))&\\ = f_{4.6}\big(f_{4.5}(\Theta) \vee (f_{4.5}(\Psi)\big) \neq f_{4.6}(f_{4.5}(\Psi)) = \Psi,&\end{aligned}$$

which is a contradiction. Thus, $f_{4.5}$ is order-preserving. Hence, $f_{4.5}$ and $f_{4.6}$ are reciprocal lattice isomorphisms because they are reciprocal order-preserving bijections.

To prove (4.8), let $a \in A$ and define $\Theta := f_{4.6}(a)$. Assume that $y \in \iota^{-1}(\downarrow a)$. Since γ is surjective, there is an interval $[u, v] \in \mathrm{Ip}^{00}(U)$ with $\gamma([u, v]) = y$. By (4.6), Θ collapses this interval. Hence, $y \in \mathrm{Cols}(\Theta)$ and $\iota^{-1}(\downarrow a) \subseteq \mathrm{Cols}(\Theta)$. Conversely, if $y \in \mathrm{Cols}(\Theta)$, then $\iota(y) \le \bigvee \iota(\mathrm{Cols}(\Theta))$, which is $f_{4.5}(\Theta) = a$ by (4.5). Hence, $y \in \iota^{-1}(\downarrow a)$, proving $\mathrm{Cols}(\Theta) \subseteq \iota^{-1}(\downarrow a)$ and (4.8).

Finally, we deal with the last sentence of the lemma. If $Y = \mathrm{Cols}(\Theta)$, then we let $a := f_{4.5}(\Theta)$ and (4.8) yields that $Y = \iota^{-1}(\downarrow a)$. If $Y = \iota^{-1}(\downarrow a)$, then letting $\Theta := f_{4.6}(a)$, we conclude from (4.8) that Y of the form $\mathrm{Cols}(\Theta)$. $\square$

Given a surjective multi-gadget $\mathcal{U} = \langle U; \gamma, X\rangle$ with fundamental lattice $\langle A; \iota\rangle$ and equalizer chain $\mathcal{E} = \langle E; \varepsilon, X\rangle$ (see Definitions 3.10, 3.12, and 3.13 and Remark 3.11), let

$$f_{4.12}(a) := \{\langle u, v\rangle \in E^2 : \mathrm{Cols}_\varepsilon([u \wedge v, u \vee v]) \subseteq \iota^{-1}(\downarrow a)\},\ \text{for } a \in A. \quad (4.12)$$

Lemma 4.9. *Let $\mathcal{U} = \langle U; \gamma, X\rangle$ be a surjective multi-gadget with fundamental lattice $\langle A; \iota\rangle$ and equalizer chain $\mathcal{E} = \langle E; \varepsilon, X\rangle$. Then the grid $\mathcal{U} \times \mathcal{E}$ from Definition* 3.13 *is a CPC-congruence-preserving extension of $\mathcal{U}$ and*

$$f_{4.13} \colon A \to \mathrm{Cpcc}(\mathcal{U} \times \mathcal{E}),\ \textit{defined by } \iota(x) \mapsto \mathrm{cpcc}_{\mathcal{U}\times\mathcal{E}}(x) \quad (4.13)$$

is a lattice isomorphism. Also, for $a \in A$, we have that $f_{4.12}(a) \in \mathrm{Cpcc}(\mathcal{E})$,

$$f_{4.13}(a) = f_{4.6}(a) \times f_{4.12}(a),\ \textit{and} \quad (4.14)$$

$$\mathrm{Cols}(f_{4.13}(a)) = \iota^{-1}(\downarrow a). \quad (4.15)$$

Proof. For $a \in A$, $f_{4.12}(a)$ is an equivalence on the chain E and its blocks are convex sublattices. Hence, it is a lattice congruence. Each block has a smallest element, because E is well-ordered. If a block B of $f_{4.12}(a)$ does not contain 1_E, the top element of E, then there is a smallest $e \in E$ such that $\varepsilon([e, e^*]) \notin \iota^{-1}(\downarrow a)$, where e^* stands for the unique cover of e. Clearly, e is the top element of B. So the blocks of $f_{4.12}(a)$ are intervals, and we obtain from Lemma 4.5 that $f_{4.12}(a)$ is a complete congruence on E. It is color-preserving, since each color occurs only once. Since $f_{4.6}(a) \in \mathrm{Cpcc}(\mathcal{U})$ by Lemma 4.8 and $f_{4.12}(a) \in \mathrm{Cpcc}(\mathcal{E})$, it follows that

$$g(a) := f_{4.6}(a) \times f_{4.12}(a) \in \mathrm{Cpcc}(\mathcal{U} \times \mathcal{E}). \tag{4.16}$$

Clearly, the map $g\colon A \to \mathrm{Cpcc}(\mathcal{U} \times \mathcal{E})$, defined by (4.16), is order-preserving. If $g(a) \le g(b)$, then $f_{4.6}(a) \le f_{4.6}(b)$, and we conclude by Lemma 4.8 that $a \le b$. Hence g is an order embedding.

Next, consider an arbitrary $\Theta \in \mathrm{Cpcc}(\mathcal{U} \times \mathcal{E})$. By the Fraser-Horn property of lattices, see [4], there are unique $\Theta_1 \in \mathrm{Con}(U)$ and $\Theta_2 \in \mathrm{Con}(E)$ such that $\Theta = \Theta_1 \times \Theta_2$. Clearly, the Θ_i are complete congruences and, using Definition 3.13, it follows that they are color-preserving. That is, $\Theta_1 \in \mathrm{Cpcc}(\mathcal{U})$ and $\Theta_2 \in \mathrm{Cpcc}(\mathcal{E})$. We claim that

$$\mathrm{Cols}(\Theta) = \mathrm{Cols}(\Theta_1) = \mathrm{Cols}(\Theta_2). \tag{4.17}$$

To see this, let $y \in X$. Since γ is surjective, there is an $[u, v] \in \mathrm{Ip}^{00}(U)$ with $\gamma([u, v]) = y$. By Definition 3.13, $(\gamma \times \varepsilon)([\langle u, 0\rangle, \langle v, 0\rangle]) = y$. This interval is collapsed by Θ iff Θ_1 collapses $[u, v]$. Thus, since both Θ and Θ_1 are color-preserving, $\mathrm{Cols}(\Theta_1) = \mathrm{Cols}(\Theta)$. Using the surjectivity of ε, a similar argument yields that $\mathrm{Cols}(\Theta_2) = \mathrm{Cols}(\Theta)$, proving (4.17).

Now, to prove the surjectivity of g, Lemma 4.8 gives a unique $a \in A$ such that $\Theta_1 = f_{4.6}(a)$. Then $\mathrm{Cols}(\Theta_2) = \mathrm{Cols}(\Theta_1) = \iota^{-1}(\downarrow a)$ by (4.8) and (4.17). By the surjectivity of ε, $\mathrm{Cols}(f_{4.12}(a))$ is also $\iota^{-1}(\downarrow a)$. Since Θ_2 and $f_{4.12}(a)$ are determined by their color sets, see (4.4), it follows that $\Theta_2 = f_{4.12}(a)$. Hence, $\Theta = \Theta_1 \times \Theta_2 = f_{4.6}(a) \times f_{4.12}(a) = g(a)$. Thus, as a surjective order-embedding, g is a lattice isomorphism. Combining $\Theta_2 = f_{4.12}(a)$ with (4.17), we also obtain that

$$\text{for every } a \in A, \quad \mathrm{Cols}(g(a)) = \iota^{-1}(\downarrow a). \tag{4.18}$$

Next, for an $x \in X$, let $a := \iota(x)$, $\Theta := g(a)$ as above, and let $\Psi := f_{4.13}(a) = \mathrm{cpcc}_{\mathcal{U} \times \mathcal{E}}(x)$. Consider $\Psi_1 := \{\langle u, v\rangle \in U^2 : \langle\langle u, 0\rangle, \langle v, 0\rangle\rangle \in \Psi\}$; clearly, $\Psi_1 \in \mathrm{Cpcc}(\mathcal{U})$. Consider an arbitrary $y \in \mathrm{Cols}(\Theta)$. By (4.7), (4.8), and (4.18), $y \in \iota^{-1}(\downarrow a) = \mathrm{Cols}(f_{4.6}(a)) = \mathrm{Cols}(\mathrm{cpcc}_{\mathcal{U}}(x))$. Since γ is surjective, there is an interval $[u, v] \in \mathrm{Ip}^{00}(U)$ such that $\gamma([u, v]) = x$. By Definition 3.13, $(\gamma \times \varepsilon)([\langle u, 0\rangle, \langle v, 0\rangle]) = x$, which shows that Ψ collapses the interval $[\langle u, 0\rangle, \langle v, 0\rangle]$. Hence, $\langle u, v\rangle \in \Psi_1$, and we conclude that $\mathrm{cpcc}_{\mathcal{U}}(x) \subseteq \Psi_1$. Therefore, $y \in \mathrm{Cols}(\mathrm{cpcc}_{\mathcal{U}}(x)) \subseteq \mathrm{Cols}(\Psi_1)$. Hence, we can take a y-colored $[u', v'] \in \mathrm{Ip}^{00}(U)$ that is collapsed by Ψ_1. Since $[\langle u', 0\rangle, \langle v', 0\rangle]$ is also y-colored and it is collapsed by Ψ, we have that $y \in \mathrm{Cols}(\Psi)$. This shows the inclusion $\mathrm{Cols}(\Theta) \subseteq \mathrm{Cols}(\Psi)$. Conversely, (4.18) gives that $x \in \iota^{-1}(\downarrow a) = \mathrm{Cols}(\Theta)$, which yields that $\Psi := \mathrm{cpcc}_{\mathcal{U}\times\mathcal{E}}(x) \subseteq \Theta$ and thus $\mathrm{Cols}(\Psi) \subseteq \mathrm{Cols}(\Theta)$. So $\mathrm{Cols}(\Psi) = \mathrm{Cols}(\Theta)$, and (4.4) yields that $f_{4.13}(a) = \Psi = \Theta = g(a)$.

We have just seen that $f_{4.13}$ and g are the same maps. Hence, $f_{4.13}$ is an isomorphism. Also, (4.16) and (4.18) imply (4.14) and (4.15), respectively.

Finally, we are going to show that the grid is a CPC-congruence-preserving extension of the multi-gadget. To do so, let $x \in X$, $a := \iota(x)$, $\Psi := f_{4.13}(a) = \mathrm{cpcc}_{\mathcal{U}\times\mathcal{E}}(x)$, and $\Theta := f_{4.6}(a) = \mathrm{cpcc}_{\mathcal{U}}(x)$. As in the paragraph following (4.18), we have that $\mathrm{cpcc}_{\mathcal{U}}(x) \subseteq \Psi_1$. Now we consider the multi-gadget a CC-sublattice of the corresponding grid, whence $\Theta = \mathrm{cpcc}_{\mathcal{U}}(x) \subseteq \Psi_1 \subseteq \Psi$. Hence, $\mathrm{cpcc}_{\mathcal{U}\times\mathcal{E}}(\Theta) \subseteq \Psi$. Conversely, since Θ and thus $\mathrm{cpcc}_{\mathcal{U}\times\mathcal{E}}(\Theta)$ collapse the x-colored edges, $\Psi \subseteq \mathrm{cpcc}_{\mathcal{U}\times\mathcal{E}}(\Theta)$. Hence, $\Psi = \mathrm{cpcc}_{\mathcal{U}\times\mathcal{E}}(\Theta)$. Therefore, by (3.9), all we have to show is that the map $\Theta \mapsto \Psi$ is an isomorphism. By Lemma 4.8 and the already established part of Lemma 4.9, this map is the composite $f_{4.13} \circ f_{4.6}^{-1} = f_{4.13} \circ f_{4.5}$ of two isomorphisms, whence it is also an isomorphism. This completes the proof of Lemma 4.9 . $\square$

In the following lemma, L is the Hall–Dilworth gluing of I and F.

Lemma 4.10. *Let I be an ideal and F be a filter of a modular lattice L such that $I \cap F \neq \varnothing$. Assume that $\mathfrak{p} \in \mathrm{Ip}(I)$ and $\mathfrak{q} \in \mathrm{Ip}(F)$ such that $\mathfrak{p}$ is perspective to $\mathfrak{q}$. Then there exists a prime interval $\mathfrak{r} \in \mathrm{Ip}(I \cap F)$ such that $\mathfrak{p} \nearrow \mathfrak{r} \nearrow \mathfrak{q}$.*

Proof. Clearly, $\mathfrak{p} = [a_1, b_1]$ is up-perspective to $\mathfrak{q} = [a_2, b_2]$. Take an element $c \in I \cap F$. Replacing it with $a_1 \vee c$ if necessary, we can assume that $a_1 \leq c$. Define $a_3 := a_2 \wedge c$; it is in $I \cap F$ since $a_2 \wedge c \in F$ and $a_3 \leq c \in I$. Note that $a_1 \leq a_3 \leq a_2$. Since $a_1 \leq a_3 \wedge b_1 \leq a_2 \wedge b_1 = a_1$, it follows that $a_3 \not\geq b_1$. Hence, $a_3 < a_3 \vee b_1 := b_3 \in I \cap F$. By (semi)modularity, $a_3 \prec b_3$, that is, $\mathfrak{r} := [a_3, b_3] \in \mathrm{Ip}(I \cap F)$. Clearly, $b_1 \leq b_3 \leq b_2$ and $\mathfrak{p} \nearrow \mathfrak{r}$. Since $b_2 \geq a_2 \vee b_3 \geq a_2 \vee b_1 = b_2$, we have that $a_2 \not\geq b_3$. Hence, $a_3 \leq a_2 \wedge b_3 < b_3$. This and $a_3 \prec b_3$ give that $a_3 = a_2 \wedge b_3$, whence $\mathfrak{r} \nearrow \mathfrak{q}$. $\square$

For a lattice L and $\mathfrak{q}, \mathfrak{q}' \in \mathrm{Ip}(L)$, we say that $\mathfrak{q}$ is *prime-projective* to $\mathfrak{q}'$ if there is an $n \in \mathbb{N}_0 = \{0, 1, 2, \dots\}$ and there are *prime* intervals $\mathfrak{p}_i \in \mathrm{Ip}(L)$ such that $\mathfrak{p}_0 = \mathfrak{q}$, $\mathfrak{p}_n = \mathfrak{q}'$, and $\mathfrak{p}_{i-1}$ is perspective to (in other words, transposed to) $\mathfrak{p}_i$ for $i \in \{1, \dots, n\}$. This terminology is taken from Grätzer [11].

Lemma 4.11. *If $\langle K; \kappa, X\rangle$ is a surjective saturated grid, $\mathfrak{p}, \mathfrak{q} \in \mathrm{Ip}(K)$, and $\mathfrak{p}$ is prime-projective to $\mathfrak{q}$, then $\kappa(\mathfrak{p}) = \kappa(\mathfrak{q})$.*

Proof. Clearly, we can assume that $\mathfrak{p}$ is perspective to $\mathfrak{q}$. By (4.2), $\mathfrak{p}$ and $\mathfrak{q}$ belong to a finite sublattice S that occurs in the proof of Lemma 4.2; for example, see Figure 3. Remember that the eyes, that is, the elements of $K \setminus (U \times E)$, are black-filled. The thick and the double-lined edges are new in the sense that they do not belong to $\mathrm{Ip}(U \times E)$. This finite sublattice is a gluing of finitely many "large intervals" obtained as sublattices of saturated gadgets. For example, the lattice of Figure 3 is glued from two such "large" intervals, $[0, v]$ and $[w, 1]$. We can assume that $\mathfrak{p}$ and $\mathfrak{q}$ belong to the same

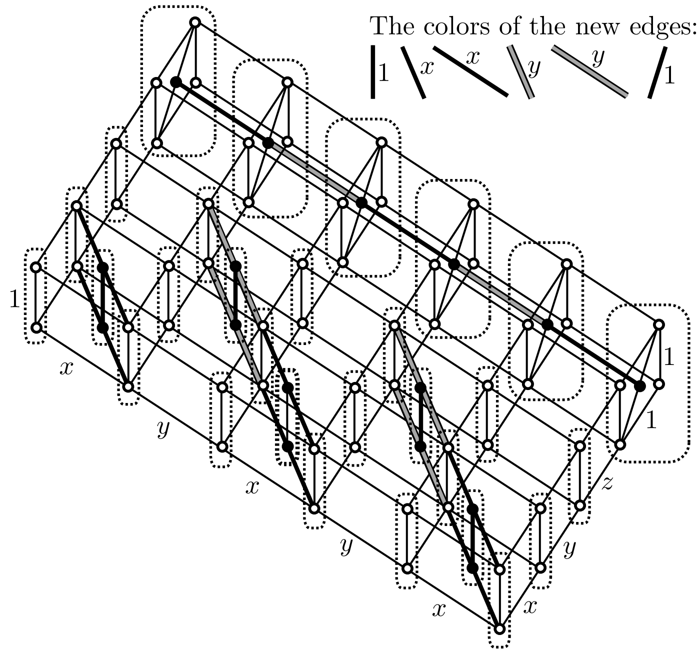

FIGURE 4. Ψ on J

"large interval" J, because otherwise Lemma 4.10 would give a sequence $\mathfrak{r}_0 = \mathfrak{p}$, $\mathfrak{r}_1, \ldots, \mathfrak{r}_n = \mathfrak{q}$ of prime intervals such that, for all $i \in \{1, \ldots, n\}$, $\mathfrak{r}_{i-1}$ and $\mathfrak{r}_i$ are perspective and belong to the same "large interval". Since J need not be a cover-preserving sublattice of K, note that, for $\mathfrak{r} \in \mathrm{Ip}(J)$, $\kappa(\mathfrak{r})$ is defined only if $\mathfrak{r} \in \mathrm{Ip}(K)$. It suffices to prove that

$$\text{if } \mathfrak{p}, \mathfrak{q} \in \mathrm{Ip}(J) \cap \mathrm{Ip}(K) \text{ are perspective, then } \kappa(p) = \kappa(q). \tag{4.19}$$

If $\mathfrak{p}, \mathfrak{q} \in \mathrm{Ip}(U \times E)$ are old prime intervals, then $\kappa(p) = \kappa(q)$ holds by (3.15). Let $J = \downarrow v$ in Figure 3; it is sufficiently general to indicate the general case without difficult technicalities. In this case, the target color is 1. (It is irrelevant that 1 is the top of A). Let $J^- := J \cap (U \times E)$; it is the sublattice of J formed by the empty-filled elements. Then $J^- = \mathsf{C}_k \times \mathsf{C}_2 \times \mathsf{C}_m$, where C_2 is the target chain of the gadget whose saturated grid includes J, and C_m, a subchain of the equalizer chain, contains each color at most once. The *lower* and *upper layers* of J^- are $\mathsf{C}_k \times \{0\} \times \mathsf{C}_m$ and $\mathsf{C}_k \times \{1\} \times \mathsf{C}_m$, respectively. There are two cases. First, assume that $\kappa(\mathfrak{p})$ is the target color of the gadget whose saturated grid includes J. Let Ψ denote the smallest congruence of J that collapses the (old and new) target-colored edges of J; see Figure 4. It follows in a straightforward way from Grätzer [10] that no other edge of J is collapsed by Ψ. On the other hand, Ψ collapses $\mathfrak{q}$, because $\mathfrak{q}$ is perspective to $\mathfrak{p}$. Hence, $\kappa(\mathfrak{p}) = \kappa(\mathfrak{q})$. Second, assume that $\kappa(\mathfrak{p})$ is not the target color. Let, say, $\kappa(\mathfrak{p}) = x$; see J in Figure 3. Let Γ_0 be the congruence of the lower layer given in Figure 5, and denote Δ_{C_2}

the smallest congruence on C_2. Then $\Gamma_1 = \Gamma_0 \times \Delta_{\mathsf{C}_2}$ is a congruence of J^-. To obtain a congruence Γ on J, add the rest of x-colored new edges (and the $\langle e, e\rangle$ pairs) to Γ_1. By Grätzer [10] again, Γ is a congruence on J and all edges it collapses are x-colored. Since Γ collapses $\mathfrak{p}$, it collapses $\mathfrak{q}$, and we obtain that $\kappa(\mathfrak{q}) = x = \kappa(\mathfrak{p})$ again. This proves (4.19) and the lemma. $\square$

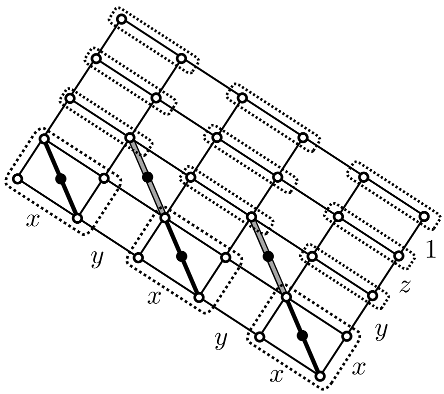

FIGURE 5. Collapsing the x-colored edges on the lower layer

Lemma 4.12 (Key Lemma). *Let $\mathcal{U} = \langle U; \gamma, X\rangle$ be a surjective multi-gadget with fundamental lattice $\langle A; \iota\rangle$ and saturated grid $\mathcal{K} = \langle K; \kappa, X\rangle$, see Definition 3.14, and keep the notation of Lemma 4.9 valid. Then $\mathcal{K}$ is a CPC-congruence-preserving extension of the unsaturated grid $\mathcal{U} \times \mathcal{E}$ and also of $\mathcal{U}$. Furthermore,*

$$\begin{aligned} &f_{4.20}\colon A \to \mathrm{Cpcc}(\mathcal{K}), \text{ defined by} \\ &\qquad a \mapsto \bigl\{\langle u, v\rangle \in K^2 : \mathrm{Cols}[u \wedge v, u \vee v] \subseteq \iota^{-1}(\downarrow a)\bigr\}, \end{aligned} \tag{4.20}$$

is a lattice isomorphism and

$$f_{4.20} = \mathrm{cpext}_{\mathcal{U}\times\mathcal{E},\mathcal{K}} \circ f_{4.13}. \tag{4.21}$$

The blocks of $f_{4.20}(a)$ are exactly the following:

(i) *the intervals $[u, v]_K$ of K for blocks $[u, v]_{U\times E}$ of $f_{4.13}(a)$ and*
(ii) *the intervals $[e, h]_K$ such that $e, h \in K\setminus(U\times E)$, $e_* = e^*\wedge h_*$, $h^* = e^*\vee h_*$, $\langle e_*, h_*\rangle \in f_{4.13}(a)$, $\langle e^*, h^*\rangle \in f_{4.13}(a)$, h_* is the top of an $f_{4.13}(a)$-block, and e^* is the bottom of an $f_{4.13}(a)$-block.*

Moreover,

$$\begin{aligned} &\mathrm{Cpcc}(\mathcal{K}) = \mathrm{Com}(K), \text{ whence } f_{4.20} \text{ is} \\ &\text{also an } A \to \mathrm{Com}(K) \text{ isomorphism.} \end{aligned} \tag{4.22}$$

Finally, for every $[u,v] \in \mathrm{Ip}^{00}(K)$ *and* $x \in X$,

$$f_{4.20}\big(\iota\big(\kappa([u,v])\big)\big) = \mathrm{cpcc}_{\mathcal{K}}(\langle u,v\rangle) = \mathrm{com}_K(\langle u,v\rangle), \tag{4.23}$$
$$f_{4.20}(\iota(x)) = \mathrm{cpcc}_{\mathcal{K}}(x), \text{ and} \tag{4.24}$$
$$\text{every } \Theta \in \mathrm{Com}(K) \text{ is a principal complete congruence.} \tag{4.25}$$

Remark 4.13. In (i), we give the convex closures of the $f_{4.13}(a)$-blocks in K. The interval $[e,h]_K$ in (ii) will be called an *induced segment of eyes*; it is always a chain. For example, the two big eyes in Figure 1, that is the two big black-filled elements, constitute an induced segment of eyes, which is an $f_{4.20}(b)$-block. In general, a maximal convex sublattice that is a chain and consists of eyes is called a *line of eyes*; the induced segment of eyes above is an interval of a line of eyes. The equations in (ii) mean that $[e_*,e^*]$ transposes up to $[h_*,h^*]$. Note that since $[e_*,h_*]$ and $[e^*,h^*]$ are perspective intervals, $\langle e_*,h_*\rangle \in f_{4.13}(a)$ if and only if $\langle e^*,h^*\rangle \in f_{4.13}(a)$.

Proof of Lemma 4.12. By Lemma 4.9, an arbitrary $\Theta \in \mathrm{Cpcc}(\mathcal{U}\times\mathcal{E})$ is of the unique form $f_{4.13}(a)$. By (4.4) and (4.15),

$$\Theta = f_{4.13}(a) = \{\langle u,v\rangle \in (U\times E)^2 : \mathrm{Cols}([u\wedge v, u\vee v]) \subseteq \iota^{-1}(\downarrow a)\}. \tag{4.26}$$

Let $\widehat{\Theta}$ denote the (not necessarily complete) lattice congruence generated by the relation $f_{4.13}(a)$ in K. We claim that

$$\mathrm{Cols}(\widehat{\Theta}) = \mathrm{Cols}(\Theta). \tag{4.27}$$

The inclusion "$\supseteq$" is clear. To show the converse inclusion, assume that x is in $\mathrm{Cols}(\widehat{\Theta})$, witnessed by an x-colored $\mathfrak{p} \in \mathrm{Ip}(K)$ collapsed by $\widehat{\Theta}$. (For illustration, we can think of $\mathfrak{p}$ as the $x = x_1$-colored thick double-lined edge in Figure 6.) Since K is modular, we conclude from Grätzer [9, Theorem 230 and Lemma 247] that $\mathfrak{p}$ is prime-projective to a prime interval collapsed by Θ in $U\times E$. Hence, Lemma 4.11 yields the validity of (4.27).

As a counterpart of Lemma 4.7(x), now we claim that for $u,v\in K$,

$$\langle u,v\rangle \in \widehat{\Theta} \iff \mathrm{Cols}([u\wedge v,u\vee v]) \subseteq \mathrm{Cols}(\widehat{\Theta}) \overset{(4.27)}{=} \mathrm{Cols}(\Theta). \tag{4.28}$$

Clearly, it suffices to verify (4.28) for $u<v$. The "$\Rightarrow$" part follows trivially from the convexity of $\widehat{\Theta}$-blocks. To prove the converse implication, assume that $\mathrm{Cols}([u,v]) \subseteq \mathrm{Cols}(\Theta)$. The first case is that $u^* \le v_*$. By (3.16), $[u^*,v_*] \subseteq [u,v]$, both in K and in $\mathrm{Old}(K) = U\times E$, and we have that $\mathrm{Cols}_{U\times E}([u^*,v_*]) \subseteq \mathrm{Cols}_K([u^*,v_*]) \subseteq \mathrm{Cols}_K([u,v]) \subseteq \mathrm{Cols}(\Theta)$. By Lemma 4.7(x), $\langle u^*,v_*\rangle \in \Theta \subseteq \widehat{\Theta}$. If $\langle u,v\rangle = \langle u^*,v_*\rangle$, then we have already arrived at $\langle u,v\rangle \in \widehat{\Theta}$, as required. If, say, $u^*\neq u$, then $u_* \prec u \prec u^*$, whence $\mathrm{Cols}_{U\times E}([u_*,u^*]) = \mathrm{Cols}_K([u,u^*]) \subseteq \mathrm{Cols}_K([u,v]) \subseteq \mathrm{Cols}(\Theta)$ and Lemma 4.7(x) yield that $\langle u_*,u^*\rangle \in \Theta \subseteq \widehat{\Theta}$. Hence, $\langle u_*,v_*\rangle \in \widehat{\Theta}$ by transitivity. After using the same trick at v, transitivity gives that $\langle u_*,v^*\rangle \in \widehat{\Theta}$. Hence, the convexity of $\widehat{\Theta}$-blocks yields that $\langle u,v\rangle \in \widehat{\Theta}$.

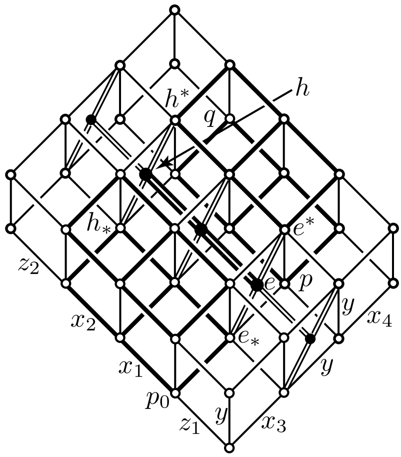

FIGURE 6. Illustration for the proof of Lemma 4.12

Second, assume that $u^* \nleq v_*$. Since $u < v$, at least one of $u = u^*$ and $v = v_*$ fails. Let, say, $v \neq v_*$; the other possibility is similar. Since $u < v$, we have that either $u \leq v_*$, or there is an eye e in the line T of eyes through v such that $u \leq e < v$. The first case is excluded because it would lead to $u^* \leq v_*$ by (3.17). We have that $u \in T$, since otherwise we would obtain $u^* \leq v_*$. So $[u, v]$ is a segment of eyes. By Definition 3.14, $\mathrm{Cols}_{U\times E}([u_*, v_*]) = \mathrm{Cols}_K([u, v]) \subseteq \mathrm{Cols}(\Theta)$. Hence, Lemma 4.7(x) gives that $\langle u_*, v_*\rangle \in \Theta$. Since $[u_*, v_*]$ is up-perspective to $[u, v]$, we conclude that $\langle u, v\rangle \in \widehat{\Theta}$, as required. Thus, (4.28) holds. (4.4), (4.27), and (4.28) imply that

$$\text{the restriction } \widehat{\Theta}\rceil_{U\times E} \text{ of } \widehat{\Theta} \text{ to } U \times E \text{ is } \Theta. \tag{4.29}$$

Observe that, for $u \leq v$ in $U \times E$,

$$\text{if } [u, v]_{U\times E} \text{ is a block of } \Theta, \text{ then } [u, v]_K \text{ is a block of } \widehat{\Theta}. \tag{4.30}$$

To see this, first we show that u is the smallest element of $[u]\widehat{\Theta}$. Suppose, for a contradiction, that $w < u$ but $w \in [u]\widehat{\Theta}$. Since u is the smallest element of its Θ-block, (4.29) excludes that $w \in U \times E$. Hence, $w \in [u]\widehat{\Theta}$ is an eye and, by (3.17), $w \prec w^* \leq u$. By the convexity of $\widehat{\Theta}$-blocks, $\langle w^*, u\rangle \in \widehat{\Theta}$. Since $w^* \in U \times E$, the already excluded case yields that $w^* = u$. Thus, $\widehat{\Theta}$ collapses $[w, w^*] = [w, u] \in \mathrm{Ip}(K)$, and so $\kappa([w, w^*]) \in \mathrm{Cols}(\widehat{\Theta})$. Taking $\kappa([w^{(\ell)}, u] = \kappa([w^{(\ell)}, w^*]) = \kappa([w, w^*]) \in \mathrm{Cols}(\widehat{\Theta})$, (4.28), and (4.29) into account, we obtain that $\langle w^{(\ell)}, u\rangle \in \Theta$. This is a contradiction since u is the smallest element of $[u]\Theta$. Therefore, u is also the smallest element of $[u]\widehat{\Theta}$. We obtain similarly that v is its top element, proving (4.30).

Next, we claim that, for every $\widehat{\Theta}$-block T,

$$\text{if } T \text{ has an old element, then } T \text{ is of the form (4.30).} \tag{4.31}$$

To see (4.31), pick an element w in $T' := T \cap (U \times E)$. (4.29) easily yields that T' is a Θ-block. Both the convex closure of T', which is a $\widehat{\Theta}$-block by (4.30), and T contain w. Thus, these two $\widehat{\Theta}$-blocks are the same, proving (4.31).

Next, let T be a $\widehat{\Theta}$-block consisting only of new elements. Clearly, T is a convex subset of a line F of eyes; see Remark 4.13. Fix an element $w \in T$. For $u \in T$, we have that $u \le w$ if $u_* \le w_*$. By (4.29), since $[u_*, w_*]$ and $[u, w]$ are perspective intervals, $\langle u, w\rangle \in \widehat{\Theta}$ iff $[u_*, w_*] \in \widehat{\Theta}$ iff $[u_*, w_*] \in \Theta$. Since Θ is a complete congruence and $T_* := \{u_* : u \in T\}$ is a $\bigwedge$-closed subset of $U \times E$ (because meets are formed componentwise), there is a smallest element $u \in T$ with $\langle u_*, w_*\rangle \in \Theta$, that is, $\langle u, w\rangle \in \widehat{\Theta}$. So T has a smallest element e. The dual argument with $T^* := \{u^* : u \in T\}$ yields that T has a largest element h. Both $e = 0_T$ and $h = 1_T$ are eyes, and it follows that $T = [e, h]$; see Remark 4.13. Repeating the last equality for later reference, we claim that

$$\begin{aligned} &T = [e, h],\ h_* \text{ is the largest element of } [h_*]\Theta,\\ &\text{and } e^* \text{ is the smallest element of } [e^*]\Theta; \end{aligned} \tag{4.32}$$

see Figure 6. (In the figure, only a part of K is drawn, $x_1, \dots, x_4$ belong to $\mathrm{Cols}(\Theta)$ but y, z_1, z_2 do not. Two elements are connected by a path consisting of thick lines iff they are collapsed by $\widehat{\Theta}$. For example, $\langle p, q\rangle \in \widehat{\Theta}$.) We give the details only for e^*; the case of h_* is similar. (Actually, h_* can be handled in an easier way, since K is strongly atomic.) Suppose, for a contradiction that there is a $u \in [e^*]\Theta$ such that $u < e^*$. Then either one of $u \le e$, $u \le e^{(\ell)}$, and $u \le e^{(r)}$ holds, or there exists an eye $f \in F$ with $u \le f^*$ and $f < e$. In case of the first alternative, the color of the monochromatic cell of e would belong to $\mathrm{Cols}(\widehat{\Theta})$ by convexity, and we would have $\langle e, e^*\rangle \in \widehat{\Theta}$ by (4.28), so T would contain an old element, which would be a contradiction. The second alternative leads to $\langle e^*, f^*\rangle \in \widehat{\Theta}$, implying $\langle e, f\rangle \in \widehat{\Theta}$ by perspectivity, which contradicts the fact that e is the smallest element of T. This proves (4.32). We will also need the converse:

$$\text{if an interval } [e, h]_K \text{ is of the form 4.12(ii), then it is a } \widehat{\Theta}\text{-block.} \tag{4.33}$$

The equations in 4.12(ii) mean that $[e_*, e^*] \nearrow [h_*, h^*]$, see Remark 4.13. Hence, it is straightforward to see that $[e, h]$ is a line T of eyes. Since $\langle e_*, h_*\rangle$ is in $f_{4.13}(a) = \Theta \subseteq \widehat{\Theta}$ and $[e_*, h_*] \nearrow [e, h]$, we have that $\langle e, h\rangle \in \widehat{\Theta}$. Hence, it suffices to show that e and h are the smallest and the largest elements of their $\widehat{\Theta}$-blocks, respectively. Suppose, for a contradiction, that $u < e$ and $\langle u, e\rangle \in \widehat{\Theta}$ for some $u \in K$. Then either $u \le e_*$, or $u \le w < e$ for some eye $w \in T$. In the first case, $\widehat{\Theta}$ collapses $[e_*, e] \in \mathrm{Ip}(K)$, which has the same color as $[e^{(\ell)}, e^*]$ is in $\mathrm{Ip}(K)$. Hence, $(\gamma \times \varepsilon)([e^{(\ell)}, e^*]) = \kappa([e^{(\ell)}, e^*]) = \kappa([e_*, e]) \in \mathrm{Cols}(\widehat{\Theta})$, that is, $(\gamma \times \varepsilon)([e^{(\ell)}, e^*]) \in \mathrm{Cols}(\Theta)$ by (4.27). Thus, $\Theta = f_{4.13}(a)$, (4.15), and (4.26) give that $\langle e^{(\ell)}, e^*\rangle \in \Theta$, which is a contradiction since e^* is the smallest element in its Θ-block. In the second case, $[w, e] \nearrow [w^*, e^*]$ and $\langle w, e\rangle \in \widehat{\Theta}$ yield that $\langle w^*, e^*\rangle \in \widehat{\Theta}$, whence $\langle w^*, e^*\rangle \in \Theta$ by (4.29), which is the same sort of contradiction since $w^* < e^*$. So e^* is the least element of its $\widehat{\Theta}$-block. By a dual argument, h_* is the largest element in its $\widehat{\Theta}$-block. Thus, (4.33) holds.

Combining (4.30), (4.31), (4.32), and Lemma 4.5, we obtain that $\widehat{\Theta}$ is a complete congruence. Thus, taking $\Theta = f_{4.13}(a)$ from (4.26) into account,

$$\begin{aligned} \widehat{\Theta} &\overset{(4.4)}{=} \{\langle u, v\rangle : \mathrm{Cols}([u \wedge v, u \vee v]) \subseteq \mathrm{Cols}(\widehat{\Theta})\} \\ &\overset{(4.27)}{=} \{\langle u, v\rangle : \mathrm{Cols}([u \wedge v, u \vee v]) \subseteq \mathrm{Cols}(\Theta)\} \\ &\overset{(4.15)}{=} \{\langle u, v\rangle : \mathrm{Cols}([u \wedge v, u \vee v]) \subseteq \iota^{-1}({\downarrow} a)\} = f_{4.20}(a). \end{aligned} \tag{4.34}$$

Since $\widehat{\Theta}$ is a complete congruence and $f_{4.20}(a)$ is clearly color-preserving, it follows from (4.34) that $f_{4.20}(a) = \widehat{\Theta}$ is a CPC-congruence. Since it is also the lattice congruence generated by Θ, it is the smallest CPC-congruence including Θ. Hence, $f_{4.20}(a) = \widehat{\Theta} = \mathrm{cpext}_{\mathcal{U}\times\mathcal{E},\mathcal{K}}(\Theta) = \mathrm{cpext}_{\mathcal{U}\times\mathcal{E},\mathcal{K}}(f_{4.13}(a))$, proving (4.21). Letting $a = \iota(x)$, (4.24) follows from

$$\begin{aligned} f_{4.20}(\iota(x)) &\overset{(4.21)}{=} \mathrm{cpext}_{\mathcal{U}\times\mathcal{E},\mathcal{K}}(f_{4.13}(\iota(x))) \overset{(3.9)}{=} \mathrm{cpcc}_{\mathcal{K}}(f_{4.13}(\iota(x))) \\ &\overset{(4.13)}{=} \mathrm{cpcc}_{\mathcal{K}}(\mathrm{cpcc}_{\mathcal{U}\times\mathcal{E}}(x)) = \mathrm{cpcc}_{\mathcal{K}}(x). \end{aligned}$$

Next, to prove that $f_{4.20}$ is an isomorphism, let $a_1, a_2 \in A$. For $i \in \{1, 2\}$, let $\Theta_i = f_{4.13}(a_i)$, and let $\widehat{\Theta}_i$ be the lattice congruence generated by Θ_i in K. By (4.34), $\widehat{\Theta}_i = f_{4.20}(a_i)$. If $a_1 \le a_2$, then $\Theta_1 \le \Theta_2$ by Lemma 4.9, whence $f_{4.20}(a_1) = \widehat{\Theta}_1 \le \widehat{\Theta}_2 = f_{4.20}(a_2)$. Conversely, if $f_{4.20}(a_1) = \widehat{\Theta}_1 \le \widehat{\Theta}_2 = f_{4.20}(a_2)$, then $\Theta_1 \le \Theta_2$ by (4.29), and we conclude $a_1 = f_{4.13}^{-1}(\Theta_1) \le f_{4.13}^{-1}(\Theta_2) = a_2$ from Lemma 4.9. Hence, $f_{4.20}$ is an order embedding. To prove that it is surjective, let $\Psi \in \mathrm{Cpcc}(\mathcal{K})$. Define $\Theta = \Psi\rceil_{U\times E}$. Since $U \times E$ is a complete sublattice of K, Θ is a complete congruence of $U \times E$. By Lemma 4.9, $\Theta = f_{4.13}(a)$ for a unique element $a \in A$. Since the coloring $\gamma \times \varepsilon$ is surjective and Ψ is color-preserving, it follows that $\mathrm{Cols}(\Theta) = \mathrm{Cols}(\Psi)$. Combining this with (4.4) and (4.27), we obtain that $\Psi = \widehat{\Theta}$. Hence, by (4.34), $\Psi = f_{4.20}(a)$. This proves that $f_{4.20}$ is an order isomorphism and, thus, a lattice isomorphism.

We obtain 4.12(i) and 4.12(ii) of the lemma from (4.30), (4.31), (4.32), and (4.33). To prove (4.22), let $x \in X$ and let $\mathfrak{p}$ be an x-colored edge of the grid $\langle U \times E; \gamma \times \varepsilon, X\rangle$. That is, $(\gamma \times \varepsilon)(\mathfrak{p}) = x$. By construction, $\mathfrak{p}$ is perspective to an edge of a monochromatic cell. After saturation, all edges of this cell are projective to the only x-colored edge of $\{0\} \times \mathcal{E}$. So are the new x-colored edges of $\mathcal{K}$. Hence, any two edges of the same color are projective and so they are congruence-equivalent. This implies (4.22). The first equality in (4.23) follows from (4.24), while the second is clear since $\mathrm{cpcc}_{\mathcal{K}}$ and com_K are the same operators by (4.22). Finally, (4.25) follows from (4.22), (4.24), and the fact that $f_{4.20}$ is surjective. This completes the proof of Lemma 4.12. □

Proof of Corollary 2.1. For $\langle A; \bigvee, 0\rangle$, let $\langle X; R\rangle$ be the presentation from (3.4). After taking a bounded well-ordering of R and choosing an open ladder gadget $G(x \le \bigvee Y)$ of type $\langle x, Y\rangle$ for each $\langle x, Y\rangle \in R$, the glued sum of these open gadgets is a surjective multi-gadget $\mathcal{U}$ with fundamental lattice $\langle A;\iota\rangle$. Thus, Lemma 4.2 and (4.22) from Lemma 4.12 apply. $\square$

5. Representing morphisms

In its full generality, the following lemma is needed only when proving Theorem 2.6; a particular case will suffice to prove Theorem 2.3. If $\langle x,Y\rangle$ is a J-constraint over X and $Z\subseteq X$, then $\langle x,Y\rangle_{Z\leftarrow 0}$ denotes the J-constraint obtained from $\langle x,Y\rangle$ by substituting 0 for every element of Z. For example,

$$\langle x_1,\{x_2,x_3,x_4,x_5\}\rangle_{\{x_3,x_5,x_6\}\leftarrow 0}=\langle x_1,\{x_2,0,x_4\}\rangle,$$
$$\langle x_1,\{x_2,x_3\}\rangle_{\{x_2,x_3,x_4\}\leftarrow 0}=\langle x_1,\{0\}\rangle,\text{ and}$$
$$\langle x_1,\{x_2,x_3\}\rangle_{\{x_1,x_3\}\leftarrow 0}=\langle 0,\{x_2,0\}\rangle.$$

For a system (or set) R of J-constraints over X and $Z\subseteq X$, we let

$$R_{Z\leftarrow 0}:=\{\langle x,Y\rangle_{Z\leftarrow 0}:\langle x,Y\rangle\in R\}\setminus\{\langle 0,0\rangle\},\text{ which is a system.}\tag{5.1}$$

Note that $\langle 0,0\rangle$ is an abbreviation for $\langle 0,\{0\}\rangle$. For a map $f\colon X\to X'$, we use the notation $f^{-1}(0)=\{x\in X: f(x)=0\}$.

Lemma 5.1. *Let A and A' be complete lattices such that $A\cap A'=\{0\}$, and let $f\colon A\to A'$ be a $(\bigvee,0)$-preserving map. Let $\langle X;R\rangle$ and $\langle\tilde X;\tilde R\rangle$ be the canonical presentations of $\langle A;\iota\rangle$ and $\langle A';\tilde\iota\rangle$, respectively. (In particular, $X=A$, $\tilde X=A'$, and $\iota\colon X\to A$ and $\tilde\iota\colon\tilde X\to A'$ are the identity maps* id_A *and* $\mathrm{id}_{A'}$*.) With $\widehat X:=(X\setminus f^{-1}(0))$, let $\ddot f$ denote the “back and forth graph” of $f\rceil_{\widehat X}$, that is, $\ddot f:=\{\langle x,f(x)\rangle: x\in\widehat X\}\cup\{\langle f(x),x\rangle: x\in\widehat X\}$. We let $X':=\widehat X\cup\tilde X$, $\widehat R:=R_{f^{-1}(0)\leftarrow 0}$, $R':=\widehat R\cup\ddot f\cup\tilde R$, and define a map*

$$\iota'\colon X'\to A'\quad\text{by}\quad x\mapsto\begin{cases} f(x), & \text{if } x\in\widehat X,\\ x=\tilde\iota(x), & \text{if } x\in\tilde X.\end{cases}\tag{5.2}$$

Then $\langle A';\iota'\rangle$ is surjectively presented by $\langle R';X'\rangle$.

Proof. The condition stipulated in Definition 3.1(i) trivially holds. Let $\langle x,Y\rangle$ belong to R'; there are three cases to consider. First, assume that $\langle x,Y\rangle\in\widehat R$, that is, $\langle 0,\{0\}\rangle\neq\langle x,Y\rangle=\langle x_0,Y_0\rangle_{f^{-1}(0)\leftarrow 0}$ for some $\langle x_0,Y_0\rangle\in R$. Clearly,

$$f\text{ is order-preserving},\quad f(x)=f(x_0),\quad\text{and}\quad f(Y)=f(Y_0),\tag{5.3}$$

because $x\neq x_0$ implies that $f(x_0)=0=x=f(x)$, and similarly for the elements of Y. Since $\langle x_0,Y_0\rangle\in R$ implies $\iota(x_0)\leq\bigvee\iota(Y_0)$, we obtain that

$$\begin{aligned}\iota'(x)=f(x)=f(x_0)=f(\iota(x_0))&\quad\text{(by (5.2), (5.3), and }\iota=\mathrm{id}_A)\\ \leq f(\textstyle\bigvee\iota(Y_0))&\quad\text{(because of }\iota(x_0)\leq\textstyle\bigvee\iota(Y_0)\text{ and (5.3))}\\ =f(\textstyle\bigvee Y_0)=\textstyle\bigvee f(Y_0)&\quad\text{(since }\iota=\mathrm{id}_A\text{ and }f\text{ is }\textstyle\bigvee\text{-preserving)}\\ =\textstyle\bigvee f(Y)=\textstyle\bigvee\iota'(Y)&\quad\text{(by (5.3) and (5.2)).}\end{aligned}$$

This is what 3.1(ii) requires. Second, assume that $\langle x, Y\rangle \in \tilde{R}$. By (5.2), $\iota' \rceil_{\tilde{X}} = \tilde{\iota}$. Thus, since $\langle A'; \tilde{\iota}\rangle$ is presented by $\langle \tilde{X}; \tilde{R}\rangle$, it follows that $\iota'(x) = \tilde{\iota}(x) \le \bigvee \tilde{\iota}(Y) = \bigvee \iota'(Y)$. Third, let $\langle x, Y\rangle = \langle x, y\rangle \in \ddot{f}$. Assume that, say, $\langle x, y\rangle = \langle f(z), z\rangle$ with $z \in \widehat{X}$; the case $\langle x, y\rangle = \langle z, f(z)\rangle$ would be similar. Using (5.2), we obtain $\iota'(x) = \iota'(f(z)) = f(z) \le f(z) = \iota'(z) = \iota'(y) = \bigvee \iota'(Y)$. Thus, 3.1(ii) holds for $\langle A'; \iota'\rangle$ and $\langle R'; X'\rangle$. Since $\iota' \rceil_{\tilde{X}} = \tilde{\iota} = \mathrm{id}_{A'}$, 3.1(iv) also holds.

Next, let B be a $(\bigvee, 0)$-semilattice and let $\lambda\colon X' \to B$ be a map preserving R' in the sense of 3.1(iii). Since $\lambda\rceil_{\tilde{X}}$ preserves $\tilde{R}$ and $\langle A'; \tilde{\iota}\rangle$ is presented by $\langle \tilde{X}; \tilde{R}\rangle$, there exists a $(\bigvee, 0)$-homomorphism $\pi\colon A' \to B$ such that $\lambda\rceil_{\tilde{X}} = \pi \circ \tilde{\iota}$. To see that $\lambda = \pi \circ \iota'$, let us compute based on (5.2). For $x \in \tilde{X}$, $\lambda(x) = \lambda\rceil_{\tilde{X}}(x) = (\pi \circ \tilde{\iota})(x) = \pi(\tilde{\iota}(x)) = \pi(\iota'(x)) = (\pi \circ \iota')(x)$, as required. For $x \in \widehat{X}$, $\lambda(x) \le \lambda(f(x))$ and $\lambda(f(x)) \le \lambda(x)$, since λ preservers the J-constraints $\langle x, f(x)\rangle, \langle f(x), x\rangle \in \ddot{f} \subseteq R'$. Hence, with $y := f(x) \in A' = \tilde{X}$, we have $\lambda(x) = \lambda(y)$. Thus, by the previous case, $\lambda(x) = \lambda(y) = (\pi \circ \iota')(y) = \pi(\iota'(y)) = \pi(y) = \pi(\iota'(x)) = (\pi \circ \iota')(x)$, as required. Hence, $\lambda = \pi \circ \iota'$, and 3.1(iii) holds. That is, $\langle A'; \iota'\rangle$ is surjectively presented by $\langle X'; R'\rangle$. $\square$

Proof of Theorem 2.3. Using lattice isomorphisms and their natural extensions to complete congruence lattices, it is routine to derive the theorem from the particular case $A \cap A' = \{0\}$. Therefore, we assume that $A \cap A' = \{0\}$. We will use the notation given in Lemma 5.1; in particular, $\langle A; \iota\rangle$ and $\langle A'; \iota'\rangle$ are surjectively presented by $\langle X; R\rangle$ and $\langle X'; R'\rangle$. Since f is a 0-separating map, $\widehat{X} = X \setminus \{0\}$. This, together with $X \cap X' = A \cap A' = \{0\}$, yields that $X = \widehat{X} \cup \{0\} \subseteq X'$. Since $R_{f^{-1}(0)\leftarrow 0} = R_{\{0\}\leftarrow 0} = R$, we obtain that $R \subseteq R'$. Also, $x \ne 0$ and $Y \ne \{0\}$ hold for all $\langle x, Y\rangle \in R'$. Therefore, we can pick an open ladder gadget $G(x \le \bigvee Y)$ for each $\langle x, Y\rangle \in R'$ and bounded well-orderings of X' and R' such that X and R are principal ideals according to these well-orderings. Construct the respective multi-gadgets $\langle U; \gamma, X\rangle$ and $\langle U'; \gamma', X'\rangle$ and equalizer chains $\langle E; \varepsilon, X\rangle$ and $\langle E'; \varepsilon', X'\rangle$ according to these well-orderings. For example, the glued sum of the open gadgets $G(x \le \bigvee Y)$, $\langle x, Y\rangle \in R$, is $\langle U; \gamma, X\rangle$. These multi-gadgets are surjective, since so are the presentations $\langle X; R\rangle$ and $\langle X'; R'\rangle$, and the corresponding fundamental lattices are $\langle A; \iota\rangle$ and $\langle A'; \iota'\rangle$. Let $\mathcal{K} = \langle K; \kappa, X\rangle$ and $\mathcal{K}' = \langle K'; \kappa', X'\rangle$ be the corresponding saturated grids. By construction, K is a principal ideal of K' and $\mathcal{K}$ is a CC-sublattice of $\mathcal{K}'$. Let $\xi\colon A \to \mathrm{Com}(K)$ and $\xi'\colon A' \to \mathrm{Com}(K)'$ be the lattice isomorphisms given by (4.24); see also (4.22). In particular, we will rely on the equalities $\mathrm{Com}(K) = \mathrm{Cpcc}(\mathcal{K})$ and $\mathrm{Com}(K') = \mathrm{Cpcc}(\mathcal{K}')$. Hence, the generation with respect to $\mathrm{Com}(K')$ and that with respect to $\mathrm{Cpcc}(\mathcal{K}')$ are the same and, consequently, $\mathrm{ext}_{KK'}$ coincides with $\mathrm{cpext}_{\mathcal{K}\mathcal{K}'}$. Let $a = \iota(x)$ be an arbitrary element of A, actually, $x = a$. We have to show that $\xi' \circ f = \mathrm{cpext}_{\mathcal{K}\mathcal{K}'} \circ \xi$. Since $\mathrm{cpcc}_{\mathcal{K}'}(x)\rceil_K \in \mathrm{Cpcc}(\mathcal{K})$ collapses an x-colored edge of $\mathcal{K}$, $\mathrm{cpcc}_{\mathcal{K}}(x) \subseteq \mathrm{cpcc}_{\mathcal{K}'}(x)$, implying $\mathrm{cpext}_{\mathcal{K}\mathcal{K}'}(\mathrm{cpcc}_{\mathcal{K}}(x)) \subseteq \mathrm{cpcc}_{\mathcal{K}'}(x)$. The converse inclusion also holds, since $\mathrm{cpcc}_{\mathcal{K}'}(x)$ is the smallest member of

$\mathrm{Cpcc}(\mathcal{K}') = \mathrm{Com}(K')$ that collapses an x-colored edge. Hence,

$$\begin{aligned}(\mathrm{cpext}_{\mathcal{K}\mathcal{K}'} \circ \xi)(a) &= \mathrm{cpext}_{\mathcal{K}\mathcal{K}'}(\xi(\iota(x)) \overset{(4.24)}{=} \mathrm{cpext}_{\mathcal{K}\mathcal{K}'}(\mathrm{cpcc}_{\mathcal{K}}(x))\\ &= \mathrm{cpcc}_{\mathcal{K}'}(x) \overset{(4.24)}{=} \xi'(\iota'(x)) \overset{(5.2)}{=} \xi'(f(x)) = (\xi' \circ f)(a),\end{aligned}$$

and we conclude that $\xi' \circ f = \mathrm{cpext}_{\mathcal{K}\mathcal{K}'} \circ \xi = \mathrm{ext}_{KK'} \circ \xi$. Finally, (4.25) implies that every complete congruence of K and K' is principal. $\square$

Proof of Theorem 2.4. Again, we can assume that $A \cap A' = \{0\}$. It is well known from category theory that $\{\bigwedge, 1\}$-preserving maps have left adjoints; see, for example, Borceaux [1, Example 3.3.9.e]. Hence, f has a unique *left adjoint* $g\colon A' \to A$. More elementarily, we define

$$g\colon A' \to A \text{ by the rule } g(b) = \bigwedge\{a \in A : b \le f(a)\}, \tag{5.4}$$

and it is straightforward to verify that, for $b \in A'$ and $a \in A$,

$$b \le f(a) \iff g(b) \le a. \tag{5.5}$$

Note that g is also called the *lower adjoint* of f. Note also that f is the right adjoint of g. This fact and the Adjoint Functor Theorem (see, for example, Borceaux [1, Theorem 3.3.3]) yield the following observation; however, for the reader's convenience, we prove it in an elementary, easy way. We claim that

$$g \text{ defined in (5.4) is a } \bigvee\text{-preserving map and } g(b) = 0 \iff b = 0. \tag{5.6}$$

Since $0 \le f(0)$ in A', $g(0) \le 0$ by (5.5), so $g(0) = 0$. Conversely, if $g(b) = 0$, then $g(b) \le 0$ and (5.5) give that $b \le f(0) = 0$, that is, $b = 0$. By (5.4), g is order-preserving. Let $\{b_i : i \in I\} \subseteq A'$, and define $a := \bigvee\{g(b_i) : i \in I\}$ and $b := \bigvee\{b_i : i \in I\}$; we have to show that $g(b) = a$. For every $i \in I$, $g(b_i) \le a$ and (5.5) give that $b_i \le f(a)$. Hence, $b \le f(a)$. Using (5.5) again, we have that $g(b) \le a$. On the other hand, $g(b) \ge g(b_i)$ for all $i \in I$ since g is order-preserving, and we obtain that $g(b) \ge a$. Thus, $g(b) = a$, proving (5.6).

Let $\tilde X = A$, $X' = A'$, and let $\tilde\iota\colon X \to A$ and $\iota'\colon X' \to A'$ be the identity maps on A and A', respectively. Then the canonical presentations $\langle \tilde X; \tilde R\rangle$ of $\langle A; \tilde\iota\rangle$ and $\langle X'; R'\rangle$ of $\langle A'; \iota'\rangle$ are surjective by (3.4). Let $\ddot g$ be the "positive back and forth graph" of g, that is,

$$\ddot g := (\{\langle x, g(x)\rangle : x \in X'\} \cup \{\langle g(x), x\rangle : x \in X'\}) \setminus \{\langle 0, 0\rangle\}.$$

By (5.6), if $\langle x, y\rangle \in \ddot g$, then $x \ne 0 \ne y$; this explains the adjective "positive". We let $X := \tilde X \cup X' = A \cup A'$, $R := \tilde R \cup \ddot g$, and we define a map

$$\iota\colon X \to A \text{ by } \iota(x) = \tilde\iota(x) = x \text{ for } x \in \tilde X \text{ and } \iota(x) = g(x) \text{ for } x \in X'. \tag{5.7}$$

Since $\tilde X \cap X' = \{0\}$ and $g(0) = 0$, ι is well-defined. We claim that

$$\langle X; R\rangle \text{ is a surjective presentation of } \langle A; \iota\rangle. \tag{5.8}$$

Clearly, 3.1(i) holds. For $\langle x, Y\rangle \in \tilde R$, the inequality of 3.1(ii) follows from the fact that ι extends $\tilde\iota$. For $\langle x, y\rangle \in \ddot g$, say $\langle x, y\rangle = \langle g(y), y\rangle$ with $y \in X' \setminus \{0\}$, we have that $\iota(g(y)) = g(y) = \iota(y)$. Hence, 3.1(ii) holds for $\langle X; R\rangle$ and $\langle A; \iota\rangle$.

To verify 3.1(iii), let B be a $(\bigvee,0)$-semilattice and let $\lambda\colon X\to B$ be a 0-preserving map that preserves R. Since $\tilde\lambda := \lambda\rceil_{\tilde X}$ preserves $\tilde R$, there exists a $(\bigvee,0)$-homomorphism $\pi\colon A\to B$ such that $\tilde\lambda=\pi\circ\tilde\iota$. Let $x\in X$. If $x\in\tilde X=A$, then $\lambda(x)=\tilde\lambda(x)=(\pi\circ\tilde\iota)(x)=\pi(\tilde\iota(x))=\pi(\iota(x))=(\pi\circ\iota)(x)$. If $0\neq x\in X'=A'$, then using that λ preserves the J-constraints $\langle x,g(x)\rangle,\langle g(x),x\rangle\in\ddot g\subseteq R$, we obtain that $\lambda(x)=\lambda(g(x))=\tilde\lambda(g(x))=(\pi\circ\tilde\iota)(g(x))=\pi\bigl(\tilde\iota(g(x))\bigr)=\pi(g(x))=\pi(\iota(x))=(\pi\circ\iota)(x)$. This shows that $\lambda=\pi\circ\iota$, and 3.1(iii) holds. The validity of 3.1(iv) is trivial, since $\iota\rceil_{\tilde X}=\tilde\iota=\mathrm{id}_A$. Thus, (5.8) holds.

If $\langle x,Y\rangle\in R\cup R'$, then $x\neq 0$, $Y\neq\{0\}$ and so $G(x\leq Y)$ will denote a fixed open gadget of type $\langle x,Y\rangle$. Take a bounded well-ordering of X such that X' is a principal ideal, and construct the equalizer chains $\mathcal E$ and $\mathcal E'$ accordingly. Then E' is a principal ideal of E. Take a bounded well-ordering of R and that of R'. With respect to these well-orderings, we let

$$\mathcal U'=\langle U';\kappa',X'\rangle:={\sum}'\{G(x\leq Y):\langle x,Y\rangle\in R'\}.$$

The "closed variant" of the multi-gadget $\mathcal U'$ is

$$(\mathcal U')^{\mathrm{cl}}=\langle (U')^{\mathrm{cl}};(\kappa')^{\mathrm{cl}},X'\rangle:={\sum}'\{G(x\leq Y)^{\mathrm{cl}}:\langle x,Y\rangle\in R'\}.$$

Using the binary and the $|R|$-ary glued sum constructions, we let

$$\mathcal U=\langle U;\kappa,X\rangle:=(\mathcal U')^{\mathrm{cl}}+'{\sum}'\{G(x\leq Y):\langle x,Y\rangle\in R\}.$$

Both $\mathcal U'$ and $\mathcal U$ are multi-gadgets. Let $\mathcal K'=\langle K';\kappa',X'\rangle$ and $\mathcal K=\langle K;\kappa,X\rangle$ denote the corresponding saturated grids. By construction, the (surjective) presentations they determine are exactly $\langle R';X'\rangle$ and $\langle R;X\rangle$; see Definitions 3.10 and 3.14.

Since U' is a sublattice of U and E' is a sublattice of E, it follows that K' is a sublattice of K. However, by Lemma 3.5, U' is not a complete sublattice of U in general. Note that

$$\mathrm{Ip}^{00}(K')\subseteq\mathrm{Ip}^{00}(K)\text{ and }\kappa'\text{ is the restriction of }\kappa. \tag{5.9}$$

By (5.8) and since $\langle R';X'\rangle$ is a surjective presentation of $\langle A';\iota'\rangle$, the fundamental lattices of $\mathcal U'$ and $\mathcal U$ are $\langle A';\iota'\rangle$ and $\langle A;\iota\rangle$, respectively.

For a $\Theta\in\mathrm{Com}(K)$, let $\Theta':=\mathrm{res}_{KK'}(\Theta)=\Theta\rceil_{K'}$. By Lemma 4.5, in order to show that the congruence Θ' is complete, it suffices to show that its blocks are intervals. As a first step, we deal with the restrictions of Θ to U' and $(U')^{\mathrm{cl}}$ rather than to K'. We will think of U' and $(U')^{\mathrm{cl}}$ as the $U'\times\{0\}$ part and the $(U')^{\mathrm{cl}}\times\{0\}$ part of the corresponding unsaturated grids, respectively. The blocks of Θ', $\Theta\rceil_{U'}$, and $\Theta\rceil_{(U')^{\mathrm{cl}}}$ are exactly the nonempty intersections of Θ-blocks with K', U', and $(U')^{\mathrm{cl}}$, respectively. Let B be an arbitrary Θ-block such that $B\cap U'\neq\varnothing$. Clearly, $B\cap U'$ is a convex sublattice of U'. By our construction based on well-ordered chains, there is no infinite descending chain in U'. Thus, $B\cap U'$ has a least element.

Next, we show that $B\cap U'$ has a largest element. Since $(U')^{\mathrm{cl}}$ is a complete sublattice of K and, thus, $\Theta\rceil_{(U')^{\mathrm{cl}}}$ is a complete congruence, $B\cap(U')^{\mathrm{cl}}$ has a largest element q. It suffices to show that $q \in U'$, because then q is clearly the largest element of $B \cap U'$. Suppose, for a contradiction, that $q \notin U'$. Then q is the lock element of a closed ladder gadget $\mathcal{U}_j^{\mathrm{cl}}$, which is an interval of $(U')^{\mathrm{cl}}$. Let the corresponding open gadget, $\mathcal{U}_j$, be of type $\langle x, Y\rangle$. By construction, $\langle x, Y\rangle \in R'$. Since $B \cap U' \neq \varnothing$ and $q \notin U'$, the $\Theta\rceil_{(U')^{\mathrm{cl}}}$-block $B \cap (U')^{\mathrm{cl}}$, that is, $[q]\Theta\rceil_{(U')^{\mathrm{cl}}}$, is not a singleton. By (4.22), $\Theta \in \mathrm{Com}(K) = \mathrm{Cpcc}(\mathcal{K})$ is color-preserving. Thus, since the coloring of $\mathcal{U}_j$ is cofinal, it follows from Lemma 4.7 that $\Theta\rceil_{(U')^{\mathrm{cl}}}$ collapses the closed lower chain of U_j. Since q is the largest element of $[q]\Theta\rceil_{(U')^{\mathrm{cl}}}$, $[q, 1_{U_j}]$ is not collapsed. The target chain of U_j is not collapsed either, because it is perspective to $[q, 1_{U_j}]$. These facts yield that $Y \subseteq \mathrm{Cols}_{\mathcal{K}}(\Theta)$ but $x \notin \mathrm{Cols}_{\mathcal{K}}(\Theta)$. Note that $x \in X' = A'$ and $Y' \subseteq X' = A'$, since $\langle x, Y\rangle \in R'$. Since $\langle A'; \iota'\rangle$ is presented by R', the J-constraint $\langle x, Y\rangle$ is preserved by the identity map ι'. Hence, $x \leq \bigvee Y$ holds in $A' = X'$. By (5.6), $g(x) \leq \bigvee\{g(y) : y \in Y\}$ holds in A. By Lemma 4.12, $f_{4.20}$ preserves this inequality and the join in it. Referring to this fact by $\leq^*$,

$$\begin{aligned}\mathrm{cpcc}_{\mathcal{K}}(x) &\overset{(4.24)}{=} f_{4.20}(\iota(x)) \overset{(5.7)}{=} f_{4.20}(g(x)) \leq^* \bigvee\{f_{4.20}(g(y)) : y \in Y\}\\ &\overset{(5.7)}{=} \bigvee\{f_{4.20}(\iota(y)) : y \in Y\} \overset{(4.24)}{=} \bigvee\{\mathrm{cpcc}_{\mathcal{K}}(y) : y \in Y\} = \mathrm{cpcc}_{\mathcal{K}}(Y).\end{aligned}$$

Hence, $x \in \mathrm{Cols}_{\mathcal{K}}(\mathrm{cpcc}_{\mathcal{K}}(x)) \subseteq \mathrm{Cols}_{\mathcal{K}}(\mathrm{cpcc}_{\mathcal{K}}(Y))$. By $Y \subseteq \mathrm{Cols}_{\mathcal{K}}(\Theta)$ and (4.4), $\mathrm{cpcc}_{\mathcal{K}}(Y) \subseteq \Theta$ and $\mathrm{Cols}_{\mathcal{K}}(\mathrm{cpcc}_{\mathcal{K}}(Y)) \subseteq \mathrm{Cols}_{\mathcal{K}}(\Theta)$. Thus, $x \in \mathrm{Cols}_{\mathcal{K}}(\Theta)$, which is a contradiction. Now that $B \cap U'$ has least and largest elements, the congruence $\Theta\rceil_{U'}$ is complete by Lemma 4.5. Observe that $\Theta\rceil_{E'}$ is also a complete congruence, since E' is a complete sublattice of E and K.

By the Fraser–Horn property from [4], the congruences of the unsaturated grid $U' \times E'$ are product congruences. In particular, $\Theta\rceil_{U' \times E'} = \Theta\rceil_{U'} \times \Theta\rceil_{E'}$, and we obtain that $\Theta\rceil_{U' \times E'}$ is a complete congruence. So, if B is a Θ-block with $B \cap (U' \times E') \neq \varnothing$, then this intersection is an interval in $U' \times E'$. Finally, the complete congruences and their blocks are well-described in Lemma 4.12, the Key Lemma, for both K' and K. Using this description, we conclude in a straightforward (but tedious) way that $B \cap K'$ has a largest and a least element in K'. Consequently, $\Theta' = \mathrm{res}_{KK'}(\Theta)$ is a complete congruence and the restriction map $\mathrm{res}_{KK'} \colon \mathrm{Com}(K) \to \mathrm{Com}(K')$ preserves completeness.

Next, let $\xi \colon A \to \mathrm{Com}(K)$ and $\xi' \colon A' \to \mathrm{Com}(K)'$ be the maps given by (4.20); see also (4.22). It is clear by (4.20) that, for $a \in A$ and $a' \in A'$,

$$\mathrm{Cols}_{\kappa}(\xi(a)) = \iota^{-1}(\downarrow a) \text{ and } \mathrm{Cols}_{\kappa'}(\xi'(a')) = \iota'^{-1}(\downarrow a'). \tag{5.10}$$

By Lemma 4.12, ξ and ξ' are isomorphisms. We have to prove only that $\xi' \circ f = \mathrm{res}_{KK'} \circ \xi$. Pick an arbitrary $a \in A$; we have to show that $\xi'(f(a)) = \mathrm{res}_{KK'}(\xi(a))$. Since $\mathrm{res}_{KK'}$ preserves completeness and, by (5.9), colors, both

$\xi'(f(a))$ and $\mathrm{res}_{KK'}(\xi(a))$ are CPC-congruences of K'. Thus, since

$$\begin{aligned}\mathrm{Cols}_{\kappa'}(\mathrm{res}_{KK'}(\xi(a))) &\overset{(5.9)}{=} X' \cap \mathrm{Cols}_{\kappa}(\xi(a)) \overset{(5.10)}{=} \{x \in X' = A' : \iota(x) \le a\}\\ &\overset{(5.7)}{=} \{x \in X' : g(x) \le a\} \overset{(5.5)}{=} \{x \in X' : x \le f(a)\}\\ &= \{x \in X' : \iota'(x) \le f(a)\} \overset{(5.10)}{=} \mathrm{Cols}_{\kappa'}(\xi'(f(a))),\end{aligned}$$

we conclude from Corollary 4.4 that $\xi'(f(a)) = \mathrm{res}_{KK'}(\xi(a))$. Finally, every complete congruence of K and K' is principal by (4.25). □

Before proving Theorem 2.6, we deal with the following easy lemma.

Lemma 5.2. *Let Θ be a CPC-congruence of a saturated grid $\langle K; \kappa, X\rangle$; see Definition* 3.14. *Let $\widehat{K}$ denote the quotient lattice K/Θ. Let $[u]\Theta, [v]\Theta \in \widehat{K}$, that is, $u, v \in K$. Then the following hold.*

(i) *If $u \prec v$ in K, then $[u]\Theta \preceq [v]\Theta$ in $\widehat{K}$. If, in addition, $\langle u, v\rangle \notin \Theta$, then $[u]\Theta \prec [v]\Theta$.*
(ii) *If $[u]\Theta \prec [v]\Theta$, then there exist "representatives" $u_1 \in [u]\Theta$ and $v_1 \in [v]\Theta$ such that $u_1 \prec v_1$.*
(iii) *If $[u]\Theta \prec [v]\Theta$, $u_1, u_2 \in [u]\Theta$ and $v_1, v_2 \in [v]\Theta$ such that $u_1 \prec v_1$ and $u_2 \prec v_2$, then $\kappa([u_1, v_1]) = \kappa([u_2, v_2])$.*
(iv) *If $[u]\Theta \prec [v]\Theta$, $u_1 \in [u]\Theta$ and $v_1 \in [v]\Theta$ such that $u_1 < v_1$, then there exist $u_2 \in [u]\Theta$ and $v_2 \in [v]\Theta$ such that $u_1 \le u_2 \prec v_2 \le v_1$.*

Proof. If $u \prec v$ and $[u]\Theta < [w]\Theta < [v]\Theta$, then $w' := (u \vee w) \wedge v \in [w]\Theta$ gives that $w' \notin \{u, v\}$ but $u \le w' \le v$. This contradiction proves (i). In the rest of the proof, we assume that $[u]\Theta \prec [v]\Theta$. Since Θ is a complete congruence (or by Lemma 4.5), $[u]\Theta$ has a largest element u_0. Since the Θ-blocks are complete sublattices, $[v]\Theta \cap {\uparrow}u_0$ has a least element, v_0. Suppose, for a contradiction, that we have an element w with $u_0 < w < v_0$. By the choice of u_0 and v_0, $[w]\Theta$ is neither $[u_0]\Theta = [u]\Theta$, nor $[v_0]\Theta = [v]\Theta$. This is a contradiction, because $[u] = [u_0]\Theta < [w]\Theta < [v_0]\Theta = [v]\Theta$. Thus, (ii) holds. Next, assume that $u_i \in [u]\Theta$ and $v_i \in [v]\Theta$ such that $u_i \prec v_i$, for $i \in \{1, 2\}$. Since $u_0 \wedge v_i \in [u]\Theta \wedge [v]\Theta = [u]\Theta$, we have that $u_0 \wedge v_i \neq v_i$. By the definition of u_0, we obtain $u_i \le u_0 \wedge v_i < v_i$, which gives that $u_0 \wedge v_i = u_i$. By Lemma 4.2, K is a modular lattice. Hence, we obtain that $u_0 \prec u_0 \vee v_i \in [v]\Theta$. This, together with the definition of v_0 yields that $u_0 \vee v_i = v_0$. The last two equalities give that $[u_0, v_0] \searrow [u_i, v_i]$. Thus, by Lemma 4.11, $\kappa([u_1, v_1]) = \kappa([u_0, v_0]) = \kappa([u_2, v_2])$. This proves (iii). Finally, to prove (iv), consider the elements $u_2 := \bigvee([u]\Theta \cap {\downarrow}v_1) \in [u]\Theta$ and $v_2 := \bigwedge([v]\Theta \cap {\uparrow}u_2) \in [v]\Theta$. Let $w \in K$ such that $u_2 \le w \le v_2$. Then $[u]\Theta \le [w]\Theta \le [v]\Theta$ and $[u]\Theta \prec [v]\Theta$ give that $w \in [u]\Theta$ or $w \in [v]\Theta$. In the first case, $w \le v_2 \le v_1$ and the definition of u_2 yield that $w \le u_2$. In the second case, $u_2 \le w$ and the definition of v_2 imply that $v_2 \le w$. Hence, $w \in \{u_2, v_2\}$. Thus, $u_2 \prec v_2$, proving (iv). □

Proof of Theorem 2.6. As before, we can assume that $A \cap A' = \varnothing$. We will use the notation and assumptions of Lemma 5.1. Let $d = \bigvee\{a \in A : f(a) = 0\}$.

Since f is a $(\bigvee, 0)$-preserving map, $f(d) = 0$, $f^{-1}(0) = \downarrow d$, $\widehat{X} = X \setminus \downarrow d$, and $\widehat{R} = R_{\downarrow d \leftarrow 0}$. We can assume that $d \neq 1$, because otherwise the statement of the theorem would follow by applying Corollary 2.1 to A and A' independently and defining $g \colon K \to K'$ as the constant $K \to \{0\}$ map. We claim that

$$\langle x, Y \rangle \in R \cup R' \Longrightarrow Y \neq \{0\}. \tag{5.11}$$

If $\langle x, Y \rangle \in R \cup \ddot{f} \cup \tilde{R}$, then $Y \neq \{0\}$ by the definition of canonical presentations and that of $\ddot{f}$. Suppose, for a contradiction, that $\langle x, 0 \rangle = \langle x, \{0\} \rangle \in \widehat{R}$ for some $x \in \widehat{X} \cup \{0\}$. By (5.1), $0 \neq x \in \widehat{X}$ and $\langle x, \{0\} \rangle = \langle x, Z \rangle_{\downarrow d \leftarrow 0}$ for some $\langle x, Z \rangle \in R$. The substitution turns Z into $\{0\}$, whence $Z \subseteq \downarrow d$. Since $\iota = \mathrm{id}_A$ preserves the J-constraint $\langle x, Z \rangle$, $x \leq \bigvee Z \leq d$ in A, that is, x is in $\downarrow d$. Hence, $\langle x, \{0\} \rangle = \langle x, Z \rangle_{\downarrow d \leftarrow 0} = \langle 0, \{0\} \rangle$, which contradicts $x \neq 0$ and proves (5.11). Choose a bounded well-ordering of R and consider R a well-ordered system; see Remark 3.8. For $\langle x_1, Y_1 \rangle$ and $\langle x_2, Y_2 \rangle$ in R, let $\langle x_1, Y_1 \rangle_{\downarrow d \leftarrow 0}$ precede $\langle x_2, Y_2 \rangle_{\downarrow d \leftarrow 0}$ in $\widehat{R}$ iff $\langle x_1, Y_1 \rangle$ precedes $\langle x_2, Y_2 \rangle$ in R. In this way, we consider $\widehat{R}$ a well-ordered system, too. Choose an equalizer chain $\mathcal{E}$ for X, see Definition 3.12, and $e \in E$ such that $\mathrm{Cols}(\downarrow_E e) = \downarrow_A d$. Since $\langle X; R \rangle$ is the canonical representation of A, we can pick an open ladder gadget $G(x, Y)$ for each $\langle x, Y \rangle \in \langle X; R \rangle$; the glued sum of these gadgets is a surjective multi-gadget $\mathcal{U} = \langle U; \gamma, X \rangle$. The saturated grid obtained from the unsaturated grid $\mathcal{U} \times \mathcal{E}$ will be denoted by $\mathcal{K} = \langle K; \kappa, X \rangle$. It is clear by (3.14) that $\langle X; R \rangle$ is the presentation determined by $\mathcal{U}$. It is important that (4.22) allows us to work with CPC-congruences rather then complete congruences. Since $d \in A = X$, we can let

$$\Theta := f_{4.20}(d) \overset{(4.24)}{=} \mathrm{cpcc}_{\mathcal{K}}(d). \tag{5.12}$$

Since $d \neq 1$ implies that Θ is not the largest CPC-congruence, we can assume that the above-mentioned bounded well-ordering of R is chosen so that the last open ladder gadget component is not collapsed by Θ. Let $\widehat{K}$ denote the quotient lattice K/Θ. It is clear from (4.20) that

$$\begin{gathered} \mathrm{Cols}_\kappa(\Theta) = \mathrm{Cols}_\kappa(\mathrm{cpcc}_{\mathcal{K}}(d)) = \iota^{-1}(\downarrow d) = \downarrow d, \text{ whence} \\ \text{for } [u, v] \in \mathrm{Ip}(K),\ \langle u, v \rangle \notin \Theta \Longrightarrow \kappa([u, v]) \in \widehat{X}. \end{gathered} \tag{5.13}$$

Thus, we can consider the map $\widehat{\kappa} \colon \mathrm{Ip}^{00}(\widehat{K}) \to \widehat{X}$ defined by

$$\widehat{\kappa}\big([\,[u]\Theta, [v]\Theta]\big) = \begin{cases} 0, & \text{if } \langle u, v \rangle \in \Theta, \text{ that is, } [u]\Theta = [v]\Theta, \\ \kappa([u, v]), & \text{if } [u, v] \in \mathrm{Ip}^{00}(K) \text{ and } \langle u, v \rangle \notin \Theta; \end{cases} \tag{5.14}$$

this map is well-defined by Lemma 5.2. Let $\widehat{E} := E/(\Theta\rceil_E)$; it determines an equalizer chain $\widehat{\mathcal{E}}$ for $\widehat{X}$. Besides $\downarrow_E e$, which is a whole block, the $(\Theta\rceil_E)$-blocks are singletons and $\widehat{\mathcal{E}}$ is isomorphic to its subchain $(E \setminus \downarrow_E e) \cup \{0\}$. By Lemma 4.12, which describes $\Theta = f_{4.20}(d)$, it is a straightforward (but tedious) task to conclude that $\widehat{\mathcal{K}} = \langle \widehat{K}; \widehat{\kappa}, \widehat{X} \rangle$ is a CC-lattice and even a saturated grid. The multi-gadget part of $\widehat{\mathcal{K}}$ will be denoted by $\widehat{\mathcal{U}} = \langle \widehat{U}; \widehat{\gamma}, \widehat{X} \rangle$; note that $\widehat{U} = U/(\Theta\rceil_U)$, and we consider $\widehat{\mathcal{U}}$ a CC-sublattice of $\widehat{\mathcal{K}}$.

Next, for $\langle x, Y\rangle \in R$, let us focus on $\mathcal{U}_j := G(x \le Y)$, which is a summand of the glued sum defining $\mathcal{U}$. It is straightforward to conclude from (4.20) and $\Theta = f_{4.20}(d)$ that $\mathcal{U}_j/(\Theta\rceil_{U_j})$ is a gadget of type $\langle x, Y\rangle_{\downarrow d\leftarrow 0}$. By (5.11), there are two cases. First, $\langle x, Y\rangle_{\downarrow d\leftarrow 0} = \langle 0, 0\rangle$, which is not in $\widehat{R}$, and $\mathcal{U}_j/(\Theta\rceil_{U_j})$ is the singleton gadget, which is not a component of $\widehat{\mathcal{U}}$; see the paragraph after Definition (3.6). Second, $\mathcal{U}_j/(\Theta\rceil_{U_j})$ is an infinite gadget component of $\widehat{\mathcal{U}}$ and its type belongs to $\widehat{R}$. (By the choice of the bounded well-ordering of R, the last $\mathcal{U}_j$ is surely such; this is why the glued sum $\widehat{\mathcal{U}}$ of the "quotient components" $\mathcal{U}_j/(\Theta\rceil_{U_j})$ will have a largest element.) We conclude that

$$\begin{aligned}&\widehat{R} \text{ is the system of types of infinite components of } \widehat{\mathcal{U}},\\ &\text{and no component of } \widehat{\mathcal{U}} \text{ is a closed ladder gadget.}\end{aligned} \tag{5.15}$$

Since $\widehat{X} \subseteq X'$, we can pick an equalizer chain $\mathcal{E}'$ for X' such that $\widehat{E}$ becomes a principal ideal of E'. Define $\mathcal{U}' = \widehat{\mathcal{U}} +' \sum'\{G(x \le Y) : \langle x, Y\rangle \in \ddot{f} \cup \tilde{R}\}$; it is a multi-gadget. Since $\widehat{U}$ is a principal ideal in U', so is $\widehat{U} \times \widehat{E}$ in $U' \times E'$. Let $\mathcal{K}' = \langle K'; \kappa', X'\rangle$ be the saturated grid determined by the unsaturated grid $\mathcal{U}' \times \mathcal{E}'$. Since there is only one way to saturate $\widehat{U} \times \widehat{E}$, we obtain that $\widehat{K}$ is a principal ideal of K'. By Definition 3.10 and (5.15), it follows that $\langle X'; R'\rangle$, which is a surjective presentation by Lemma 5.1, is the presentation determined by $\mathcal{U}'$. Hence, the multi-gadget $\mathcal{U}'$ and the corresponding saturated grid $\mathcal{K}'$ are surjective. By the definition of $\widehat{\mathcal{K}}$ and $\mathcal{K}'$, κ' extends $\widehat{\kappa}$, and

$$g\colon K \to K', \quad \text{defined by} \quad u \mapsto [u]\Theta, \tag{5.16}$$

is a complete lattice homomorphism. Let $\xi\colon A \to \mathrm{Cpcc}(\mathcal{K}) = \mathrm{Com}(K)$ and $\xi'\colon A' \to \mathrm{Cpcc}(\mathcal{K}') = \mathrm{Com}(K')$ be the lattice isomorphisms given by (4.20). Using (4.24), we have that

$$\xi(\iota(x)) = \mathrm{cpcc}_{\mathcal{K}}(x) \quad \text{for } x \in X \text{ and } \iota(x) = x \in A, \text{ and} \tag{5.17}$$

$$\xi'(\iota'(x)) = \mathrm{cpcc}_{\mathcal{K}'}(x) \quad \text{for } x \in X'. \tag{5.18}$$

By (4.22), the operators $\mathrm{cpcc}_{\mathcal{K}'}$ and $\mathrm{com}_{K'}$ are equal; we will rely on this fact without further warning. We have to show only that $\xi' \circ f = g^* \circ \xi$, that is,

$$\xi'(f(\iota(x))) = g^*(\xi(\iota(x))) \quad \text{for every } x \in X. \tag{5.19}$$

First, we assume that $x \in \widehat{X}$. Then $f(\iota(x)) = f(x) = \iota'(x)$ by (5.2) and x belongs also to X'. Hence, (2.1), (5.17), and (5.18) turn (5.19) into

$$\mathrm{cpcc}_{\mathcal{K}'}(x) = \mathrm{cpcc}_{\mathcal{K}'}(g(\mathrm{cpcc}_{\mathcal{K}}(x))), \tag{5.20}$$

which we have to show. Take an x-colored edge in $\mathcal{K}$; its g-image is also x-colored by (5.13) and (5.14). This yields the "$\le$" part of (5.20). To show the converse inequality, assume that $\langle u', v'\rangle$ is an arbitrary pair in $g(\mathrm{cpcc}_{\mathcal{K}}(x))$, that is, $\langle u, v\rangle \in \mathrm{cpcc}_{\mathcal{K}}(x)$, $u' = g(u)$, and $v' = g(v)$ hold for some pair $\langle u, v\rangle$. Clearly, we can also assume that $u \le v$ and $u' \le v'$, since otherwise we could work with $u \wedge v$ and $u \vee v$. To obtain the "$\ge$" part of (5.20), it suffices to show that $\langle u', v'\rangle \in \mathrm{cpcc}_{\mathcal{K}'}(x)$. By (4.4), this membership is equivalent to $\mathrm{Cols}_{\kappa'}([u',v']) \subseteq \mathrm{Cols}_{\kappa'}(\mathrm{cpcc}_{\mathcal{K}'}(x))$, and we obtain from (4.20) and (4.24) that $\mathrm{Cols}_{\kappa'}(\mathrm{cpcc}_{\mathcal{K}'}(x)) = \iota'^{-1}(\downarrow(\iota'(x)))$. Hence, we have to show only that if $u' \leq p' \prec q' \leq v'$, then $\kappa'([p',q']) \in \iota'^{-1}(\downarrow(\iota'(x)))$. By the definition of $\widehat{K}$ and (5.16) and since $g(K) = \widehat{K}$ is an ideal of K', we can pick $p, q \in K$ such that $p' = g(p)$ and $q' = g(q)$. We can assume that $u \leq p < q \leq v$, because otherwise we can replace first p and then q with $(u \vee p) \wedge v$ and $(p \vee q) \wedge v$, respectively. Furthermore, Lemma 5.2(iv) allows us to assume even that $u \leq p \prec q \leq v$. As mentioned above (5.16), κ' extends $\widehat{\kappa}$. Hence, we obtain from (5.14) and (5.16) that $\kappa'([p',q']) = \kappa([p,q])$; so our task is to show that $y := \kappa([p,q])$ belongs to $\iota'^{-1}(\downarrow(\iota'(x)))$. Since $p' \neq q'$, $\langle p, q\rangle$ is not collapsed by Θ and (5.13) gives that $y \in \widehat{X} \subseteq X'$. On the other hand, $\mathrm{cpcc}_{\mathcal{K}}(x) = \xi(\iota(x)) = f_{4.20}(\iota(x))$, see (5.17), collapses $\langle p, q\rangle$ by $\langle u, v\rangle \in \mathrm{cpcc}_{K}(x)$ and convexity. Hence, (4.20) gives that $y \in \iota^{-1}(\downarrow\iota(x))$, that is, $\iota(y) \leq \iota(x)$ in A. Since $\iota\rceil_{\widehat{X}}$ is the identity map, this means that $y \leq x$ in A. Hence, using (5.2) and that f is order-preserving, we obtain that $\iota'(y) = f(y) \leq f(x) = \iota'(x)$, which yields the required $y \in \iota'^{-1}(\downarrow(\iota'(x)))$. This proves (5.20) and thus (5.19) for $x \in \widehat{X} \cup \{0\}$.

Second, let $x \in X \setminus \widehat{X} = \downarrow d$. Then $\iota(x) = x \leq d$. Since $\xi = f_{4.20}$ is an isomorphism and $\Theta = \xi(d)$ by (5.12), we have that $\xi(\iota(x)) \leq \xi(d) = \Theta$. Hence $g(\xi(\iota(x)))$ is a subset of the equality relation on K'. Thus, $g^*(\xi(\iota(x)))$ is the least congruence on K'. So is $\xi'(f(\iota(x))) = \xi'(f(x)) = \xi'(0)$ since ξ' is 0-preserving. Therefore, (5.19) holds again. Finally, (4.25) implies that every complete congruence of K and K' is principal. $\square$

Note that one could extract a proof of Theorem 2.3 from that of Theorem 2.6. We have given a separate proof for Theorem 2.3 simply because it requires a much simpler construction than the proof of Theorem 2.6 above.

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