COMETIC FUNCTORS AND REPRESENTING ORDER-PRESERVING MAPS BY PRINCIPAL LATTICE CONGRUENCES

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Dedicated to the memory of E. Tamás Schmidt

ABSTRACT. Let $\mathbf{Lat}_5^{\mathrm{sd}}$ and $\mathbf{Pos}_{01}^{+}$ denote the category of selfdual bounded lattices of length 5 with $\{0,1\}$-preserving lattice homomorphisms and that of bounded ordered sets with $\{0,1\}$-preserving isotone maps, respectively. For an object L in $\mathbf{Lat}_5^{\mathrm{sd}}$, the ordered set of principal congruences of the lattice L is denoted by $\mathrm{Princ}(L)$. By means of congruence generation, $\mathrm{Princ}\colon \mathbf{Lat}_5^{\mathrm{sd}} \to \mathbf{Pos}_{01}^{+}$ is a functor. We prove that if $\mathbf{A}$ is a small subcategory of $\mathbf{Pos}_{01}^{+}$ such that every morphism of $\mathbf{A}$ is a monomorphism, understood in $\mathbf{A}$, then $\mathbf{A}$ is the Princ-image of an appropriate subcategory of $\mathbf{Lat}_5^{\mathrm{sd}}$. This result extends G. Grätzer's earlier theorems where $\mathbf{A}$ consisted of one or two objects and at most one non-identity morphism, and the author's earlier result where all morphisms of $\mathbf{A}$ were 0-separating and no hom-set had more the two morphisms. Furthermore, as an auxiliary tool, we derive some families of maps, also known as functions, from injective maps and surjective maps; this can be useful in various fields of mathematics, not only in lattice theory. Namely, for every small concrete category $\mathbf{A}$, we define a functor F_{com}, called *cometic functor*, from $\mathbf{A}$ to the category $\mathbf{Set}$ of sets and a natural transformation $\boldsymbol{\pi}^{\mathrm{com}}$, called *cometic projection*, from F_{com} to the forgetful functor of $\mathbf{A}$ into $\mathbf{Set}$ such that the F_{com}-image of every monomorphism of $\mathbf{A}$ is an injective map and the components of $\boldsymbol{\pi}^{\mathrm{com}}$ are surjective maps.

1. Prerequisites and outline

This paper consists of an easy category theoretical part followed by a more involved lattice theoretical part.

The *category theoretical* first part, which consists of Sections 2 and 3, is devoted to certain families of *maps*, also known as *functions*. Only some easy concepts are needed from category theory; their definitions will be recalled in the paper. Hence, there is no prerequisite for this part. Our purpose is to

2000 *Mathematics Subject Classification.* 06B10, 18B05. July 20, 2018.

Key words and phrases. Cometic functor, cometic projection, natural transformation, injective map, monomorphism, principal congruence, lattice congruence, lifting diagrams, ordered set, poset, quasi-colored lattice, preordering, quasiordering, isotone map, monotone map.

This research was supported by NFSR of Hungary (OTKA), grant number K 115518.

derive some families of maps from injective maps and surjective maps. This part can be interesting in various fields of algebra and even outside algebra.

The *lattice theoretical* second part is built on the first part. The readers of the second part are not assumed to have deep knowledge of lattice theory; a little part of any book on lattices, including Grätzer [8] and Nation [15], is sufficient.

Outline. The paper contains two theorems and it is structured as follows. Section 2 recalls some basic concepts from category theory. In Section 3, we introduce cometic functors and cometic projections, and prove Theorem 3.6 on them. In Section 4, we formulate Theorem 4.7 on the representation of families of isotone maps by principal lattice congruences. The rest of the sections are devoted to the proof of this theorem. First, Section 5 gives a heuristic overview of the proof. Section 6 tailors the toolkit developed for quasi-colored lattices in Czédli [4] to the present environment; when reading this section, [4] should be nearby. In Section 7, we prove a lemma that allows us to work with certain homomorphisms efficiently. Finally, with the help of cometic functors and cometic projections, Section 8 completes the proof of Theorem 4.7.

2. Introduction to the category theory part

2.1. **Notation, terminology, and the rudiments.** Recall that a *category* $\mathbf{A}$ is a system $\langle \mathrm{Ob}(\mathbf{A}), \mathrm{Mor}(\mathbf{A}), \circ \rangle$ formed from a class $\mathrm{Ob}(\mathbf{A})$ of *objects*, a class $\mathrm{Mor}(\mathbf{A})$ of *morphisms*, and a partially defined binary operation $\circ$ on $\mathrm{Mor}(\mathbf{A})$ such that $\mathbf{A}$ satisfies certain axioms. Each $f \in \mathrm{Mor}(\mathbf{A})$ has a *source object* $X \in \mathrm{Ob}(\mathbf{A})$ and a *target object* $Y \in \mathrm{Ob}(\mathbf{A})$; the collection of morphisms with source object X and target object Y is denoted by $\mathrm{Mor}(X, Y)$ or $\mathrm{Mor}_{\mathbf{A}}(X, Y)$. The axioms require that $\mathrm{Mor}(X, Y)$ is a set for all $X, Y \in \mathrm{Ob}(\mathbf{A})$, every $\mathrm{Mor}(X, X)$ contains a unique *identity morphism* $\mathbf{1}_X$, $f \circ g$ is defined and belongs to $\mathrm{Mor}(X, Z)$ iff $f \in \mathrm{Mor}(Y, Z)$ and $g \in \mathrm{Mor}(X, Y)$, this multiplication is associative, and the identity morphisms are left and right units with respect to the multiplication. Note that $\mathrm{Mor}(X, Y)$ is often called a *hom-set* of $\mathbf{A}$ and $\mathrm{Mor}(\mathbf{A})$ is the disjoint union of the hom-sets of $\mathbf{A}$. If $\mathbf{A}$ and $\mathbf{B}$ are categories such that $\mathrm{Ob}(\mathbf{A}) \subseteq \mathrm{Ob}(\mathbf{B})$ and $\mathrm{Mor}(\mathbf{A}) \subseteq \mathrm{Mor}(\mathbf{B})$, then $\mathbf{A}$ is a *subcategory* of $\mathbf{B}$. If $\mathbf{A}$ is a category and $\mathrm{Ob}(\mathbf{A})$ is a set, then $\mathbf{A}$ is said to be a *small category.*

Definition 2.1. If $\mathbf{A}$ is a category such that

(i) every object of $\mathbf{A}$ is a set, possibly with a structure on it,
(ii) for all $X, Y \in \mathrm{Ob}(\mathbf{A})$ and $f \in \mathrm{Mor}(X, Y)$, f is a map from X to Y, and
(iii) the operation is the usual composition of maps,

then $\mathbf{A}$ is a *concrete category.* Note the rule $(f \circ g)(x) = f\big(g(x)\big)$, that is, we compose maps from right to left. Note also that $\mathrm{Mor}(X, Y)$ does not have to contain all possible maps from X to Y. The category of all sets with all maps between sets will be denoted by **Set**.

Remark 2.2. In category theory, the concept of concrete categories is usually based on forgetful functors and it has a more general meaning. Since this paper is not only for category theorists, we adopt Definition 2.1, which is conceptually simpler but, apart from mathematically insignificant technicalities, will not reduce the generality of our result, Theorem 3.6.

For an arbitrary category $\mathbf{A}$ and $f \in \mathrm{Mor}(\mathbf{A})$, if $f \circ g_1 = f \circ g_2$ implies $g_1 = g_2$ for all $g_1, g_2 \in \mathrm{Mor}(\mathbf{A})$ such that both $f \circ g_1$ and $f \circ g_2$ are defined, then f is a *monomorphism* in $\mathbf{A}$. Note that if $\mathbf{A}$ is a subcategory of $\mathbf{B}$, then a monomorphism of $\mathbf{A}$ need not be a monomorphism of $\mathbf{B}$. In a concrete category, an injective morphism is always a monomorphism but not conversely. The opposite (that is, left-right dual) of the concept of monomorphisms is that of *epimorphisms*. We say that $f \in \mathrm{Mor}(\mathbf{A})$ is an *isomorphism* in $\mathbf{A}$ if there is a $g \in \mathrm{Mor}(\mathbf{A})$ such that both $f \circ g$ and $g \circ f$ are identity morphisms. Every isomorphism is both a monomorphism and epimorphism. Next, let $\mathbf{A}$ and $\mathbf{B}$ be categories. An assignment $F\colon \mathbf{A} \to \mathbf{B}$ is a *functor* if $F(X) \in \mathrm{Ob}(\mathbf{B})$ for every $X \in \mathrm{Ob}(\mathbf{A})$, $F(f) \in \mathrm{Mor}_{\mathbf{B}}(F(X), F(Y))$ for every $f \in \mathrm{Mor}_{\mathbf{A}}(X, Y)$, F commutes with $\circ$, and F maps the identity morphisms to identity morphisms. If $F(f) = F(g)$ implies $f = g$ for all $X, Y \in \mathrm{Ob}(\mathbf{A})$ and all $f, g \in \mathrm{Mor}_{\mathbf{A}}(X, Y)$, then F is called a *faithful functor*. Although category theory seems to avoid talking about equality of objects, to make our theorems stronger, we introduce the following concept.

Definition 2.3. For categories $\mathbf{A}$ and $\mathbf{B}$ and a functor $F\colon \mathbf{A} \to \mathbf{B}$, F is a *totally faithful functor* if, for all $f, g \in \mathrm{Mor}(\mathbf{A})$, $F(f) = F(g)$ implies that $f = g$.

Remark 2.4. Let $F\colon \mathbf{A} \to \mathbf{B}$ be a functor. Then F is totally faithful iff it is faithful and, for all $X, Y \in \mathrm{Ob}(\mathbf{A})$, $F(X) = F(Y)$ implies that $X = Y$.

Proof. First, assume that F is totally faithful. Clearly, F is faithful. Let $X, Y \in \mathrm{Ob}(\mathbf{A})$ such that $F(X) = F(Y)$. Then $F(\mathbf{1}_X) = \mathbf{1}_{F(X)} = \mathbf{1}_{F(Y)} = F(\mathbf{1}_Y)$. Using that F is totally faithful, we obtain that $\mathbf{1}_X = \mathbf{1}_Y$, whereby $X = Y$. Second, to see the converse implication, assume that F is faithful and, in addition, it satisfies the implication from Remark 2.4. Let $f_1 \in \mathrm{Mor}_{\mathbf{A}}(X_1, Y_1)$ and $f_2 \in \mathrm{Mor}_{\mathbf{A}}(X_2, Y_2)$ such that $F(f_1) = F(f_2)$. Then $F(f_1) = F(f_2)$ belongs to $\mathrm{Mor}_{\mathbf{B}}(F(X_1), F(Y_1)) \cap \mathrm{Mor}_{\mathbf{B}}(F(X_2), F(Y_2))$, so this intersection is not empty. Since $\mathrm{Mor}(\mathbf{B})$ is the *disjoint* union of the hom-sets of $\mathbf{B}$, we obtain that $\langle F(X_1), F(Y_1)\rangle = \langle F(X_2), F(Y_2)\rangle$. Hence, by our additional assumption on F, $\langle X_1, Y_1\rangle = \langle X_2, Y_2\rangle$. This allows us to apply that F is faithful, and we conclude that $f_1 = f_2$, showing that F is totally faithful. $\square$

For a concrete category $\mathbf{A}$, the well-known

$$\textit{forgetful functor} \quad G^{\mathbf{A}}_{\mathrm{forg}}\colon \mathbf{A} \to \mathbf{Set} \quad \text{will often be denoted by} \quad G_{\mathrm{forg}} \tag{2.1}$$

if the superscript $\mathbf{A}$ is understood from the context. (The mnemonic in the subscript comes from "forgetful".) This functor sends objects, which are structures, to their underlying sets and acts identically on morphisms, which are maps. For a functor $F\colon \mathbf{A} \to \mathbf{B}$, the F-*image* of $\mathbf{A}$ is the category

$$F(\mathbf{A}) = \langle \{F(X) : X \in \mathrm{Ob}(\mathbf{A})\},\ \{F(f) : f \in \mathrm{Mor}(\mathbf{A})\}, \circ\rangle. \tag{2.2}$$

Next, let F and G be functors from a category $\mathbf{A}$ to a category $\mathbf{B}$. A *natural transformation* $\boldsymbol{\kappa}\colon F \to G$ is a system $\langle \boldsymbol{\kappa}_X : X \in \mathrm{Ob}(\mathbf{A})\rangle$ of morphisms of $\mathbf{B}$ such that the *component* $\boldsymbol{\kappa}_X$ of $\boldsymbol{\kappa}$ at X belongs to $\mathrm{Mor}_{\mathbf{B}}(F(X), G(X))$ for every $X \in \mathrm{Ob}(\mathbf{A})$, and for every $X, Y \in \mathrm{Ob}(\mathbf{A})$ and every $f \in \mathrm{Mor}_{\mathbf{A}}(X, Y)$, the diagram

$$\begin{array}{ccc} F(X) & \xrightarrow{F(f)} & F(Y) \\ {\scriptstyle\boldsymbol{\kappa}_X}\big\downarrow & & {\scriptstyle\boldsymbol{\kappa}_Y}\big\downarrow \\ G(X) & \xrightarrow{G(f)} & G(Y) \end{array}$$

commutes, that is, $\boldsymbol{\kappa}_Y \circ F(f) = G(f) \circ \boldsymbol{\kappa}_X$. If all the components $\boldsymbol{\kappa}_X$ of $\boldsymbol{\kappa}$ are isomorphisms in $\mathbf{B}$, then $\boldsymbol{\kappa}$ is a *natural isomorphism*. If there is a natural isomorphism $\boldsymbol{\kappa}\colon F \to G$, then F and G are *naturally isomorphic functors*. Naturally isomorphic functors are, sometimes, also called *naturally equivalent*.

3. Cometic functors and projections

Our purpose is to derive some families of maps from injective and surjective maps. In order to do so, we introduce some concepts. The third component of an arbitrary triplet $\langle x, y, z\rangle$ is obtained by the *third projection* $\mathrm{pr}^{(3)}$, in notation, $\mathrm{pr}^{(3)}(\langle x, y, z\rangle) = z$.

Definition 3.1. Given a small concrete category $\mathbf{A}$, a triplet $c = \langle f, x, y\rangle$ is an *eligible triplet* of $\mathbf{A}$ if there exist $X, Y \in \mathrm{Ob}(\mathbf{A})$ such that $f \in \mathrm{Mor}_{\mathbf{A}}(X, Y)$, $x \in X$, $y \in Y$, and $f(x) = y$. The third component of $c = \langle f, x, y\rangle$ will also be denoted by

$$\boldsymbol{\pi}^{\mathrm{com}}_Y(\langle f, x, y\rangle) := \mathrm{pr}^{(3)}(\langle f, x, y\rangle) = y = f(x),\ \text{ provided that } y \in Y.$$

For $x \in X \in \mathrm{Ob}(\mathbf{A})$,

$$\vec{v}^{\,\mathrm{triv}}(x) = \vec{v}^{\,\mathrm{triv}}_X(x) \ \text{ denotes } \ \langle \mathbf{1}_X, x, x\rangle,$$

the *trivial triplet* at x. Note the obvious rule

$$\boldsymbol{\pi}^{\mathrm{com}}_X(\vec{v}^{\,\mathrm{triv}}_X(x)) = x,\ \text{ for } x \in X. \tag{3.1}$$

Definition 3.2. Given a small concrete category $\mathbf{A}$ (see Definition 2.1), we define the *cometic functor*

$$F_{\mathrm{com}} = F^{\mathbf{A}}_{\mathrm{com}}\colon \mathbf{A} \to \mathbf{Set}$$

associated with $\mathbf{A}$ as follows. For each $Y \in \mathrm{Ob}(\mathbf{A})$, we let

$$F_{\mathrm{com}}(Y) := \{\langle f, x, y\rangle : \langle f, x, y\rangle \text{ is an eligible triplet of } \mathbf{A} \text{ and } y \in Y\}.$$

For $Y, Z \in \mathrm{Ob}(\mathbf{A})$ and $g \in \mathrm{Mor}_{\mathbf{A}}(Y, Z)$, we define $F_{\mathrm{com}}(g)$ as the map

$$F_{\mathrm{com}}(g)\colon F_{\mathrm{com}}(Y) \to F_{\mathrm{com}}(Z), \text{ defined by}$$
$$\langle f, x, y\rangle \mapsto \langle g \circ f, x, g(y)\rangle.$$

The map $X \to F_{\mathrm{com}}(X)$, defined by $x \mapsto \vec{v}^{\,\mathrm{triv}}(x)$, will be denoted by $\vec{v}^{\,\mathrm{triv}}_X$.

We could also denote an eligible triplet $\langle f, x, y\rangle$ by $x \overset{f}{\mapsto} y$, but technically the triplet is a more convenient notation than the f-labeled "\mapsto" arrow. However, in this paragraph, let us think of eligible triplets as arrows. The trivial arrows $\vec{v}^{\,\mathrm{triv}}_X(x)$ with $x \in X$ correspond to the elements of X. Besides these arrows, $F_{\mathrm{com}}(X)$ can contain many other arrows, which are of different lengths and of different directions in space but with third components in X. This geometric interpretation of $F_{\mathrm{com}}(X)$ resembles a real comet; the trivial arrows form the nucleus while the rest of arrows the coma and the tail. This explains the adjective "cometic".

Lemma 3.3. *$F_{\mathrm{com}} = F^{\mathbf{A}}_{\mathrm{com}}$ from Definition 3.2 is a totally faithful functor.*

Proof. First, we prove that $F_{\mathrm{com}} := F^{\mathbf{A}}_{\mathrm{com}}$ is a functor. Obviously, the F_{com}-image of an identity morphism is an identity morphism. Assume that $X, Y, Z \in \mathrm{Ob}(\mathbf{A})$, $f \in \mathrm{Mor}_{\mathbf{A}}(X, Y)$, $g \in \mathrm{Mor}_{\mathbf{A}}(Y, Z)$, $c = \langle h, x, y\rangle \in F_{\mathrm{com}}(X)$, and let us compute:

$$\begin{aligned}\big(F_{\mathrm{com}}(g) \circ F_{\mathrm{com}}(f)\big)(c) &= F_{\mathrm{com}}(g)\big(F_{\mathrm{com}}(f)(c)\big)\\ &= F_{\mathrm{com}}(g)\big(\langle f \circ h, x, f(y)\rangle\big) = \langle g \circ (f \circ h), x, g(f(y))\rangle\\ &= \langle (g \circ f) \circ h, x, (g \circ f)(y)\rangle = F_{\mathrm{com}}(g \circ f)(c).\end{aligned}$$

Hence, $F_{\mathrm{com}}(g) \circ F_{\mathrm{com}}(f) = F_{\mathrm{com}}(g \circ f)$ and F_{com} is a functor. In order to prove that F_{com} is faithful, assume that $X, Y \in \mathrm{Ob}(\mathbf{A})$, $f, g \in \mathrm{Mor}_{\mathbf{A}}(X, Y)$, and $F_{\mathrm{com}}(f) = F_{\mathrm{com}}(g)$; we have to show that $f = g$. This is clear if $X = \varnothing$. Otherwise, for $x \in X$,

$$\langle f \circ \mathbf{1}_X, x, f(x)\rangle = F_{\mathrm{com}}(f)(\vec{v}^{\,\mathrm{triv}}(x)) = F_{\mathrm{com}}(g)(\vec{v}^{\,\mathrm{triv}}(x)) = \langle g \circ \mathbf{1}_X, x, g(x)\rangle.$$

Comparing either the third components (for all $x \in X$), or the first components, we conclude that $f = g$. Thus, F_{com} is faithful. Finally, if $X, Y \in \mathrm{Ob}(\mathbf{A})$ and $X \nsubseteq Y$, then there is an $x \in X \setminus Y$. Since $\vec{v}^{\,\mathrm{triv}}(x) \in F_{\mathrm{com}}(X) \setminus F_{\mathrm{com}}(Y)$, we conclude that F_{com} is totally faithful. □

Definition 3.4. Let $\mathbf{A}$ be a small concrete category, let $G^{\mathbf{A}}_{\mathrm{forg}}\colon \mathbf{A} \to \mathbf{Set}$ be the forgetful functor, see (2.1), and keep Definition 3.2 in mind. Then the transformation

$$\boldsymbol{\pi}^{\mathrm{com}} = \boldsymbol{\pi}^{\mathrm{com},\mathbf{A}}\colon F_{\mathrm{com}} \to G^{\mathbf{A}}_{\mathrm{forg}}$$

whose components are defined by

$$\boldsymbol{\pi}^{\mathrm{com}}_X\colon F_{\mathrm{com}}(X) \to X \quad \text{ and } \quad \boldsymbol{\pi}^{\mathrm{com}}_X(c) := \mathrm{pr}^{(3)}(c),$$

for $X \in \mathrm{Ob}(\mathbf{A})$ and $c \in F_{\mathrm{com}}(X)$, is the *cometic projection* associated with $\mathbf{A}$. (Note that $\boldsymbol{\pi}^{\mathrm{com}}_X$ is simply the restriction of the third projection $\mathrm{pr}^{(3)}$ to $F_{\mathrm{com}}(X)$.)

Lemma 3.5. *The cometic projection defined above is a natural transformation and its components are surjective maps.*

Proof. Let $X, Y \in \mathbf{A}$ and $f \in \mathrm{Mor}(X, Y)$. We have to prove that the diagram

$$\begin{array}{ccc} F_{\mathrm{com}}(X) & \xrightarrow{F_{\mathrm{com}}(f)} & F_{\mathrm{com}}(Y) \\ {\scriptstyle \boldsymbol{\pi}_X^{\mathrm{com}}}\Big\downarrow & & {\scriptstyle \boldsymbol{\pi}_Y^{\mathrm{com}}}\Big\downarrow \\ X & \xrightarrow{\quad f \quad} & Y \end{array} \tag{3.2}$$

commutes. For an arbitrary triplet $c = \langle h, x, y\rangle \in F_{\mathrm{com}}(X)$, we have that

$$\begin{aligned}\big(\boldsymbol{\pi}_Y^{\mathrm{com}} \circ F_{\mathrm{com}}(f)\big)(c) &= \boldsymbol{\pi}_Y^{\mathrm{com}}\big(F_{\mathrm{com}}(f)(c)\big) = \boldsymbol{\pi}_Y^{\mathrm{com}}\big(\langle f \circ h, x, f(y)\rangle\big) \\ &= f(y) = f\big(\boldsymbol{\pi}_X^{\mathrm{com}}(c)\big) = (f \circ \boldsymbol{\pi}_X^{\mathrm{com}})(c),\end{aligned}$$

which proves the commutativity of (3.2). Finally, for $X \in \mathrm{Ob}(\mathbf{A})$ and $x \in X$, $x = \boldsymbol{\pi}_X^{\mathrm{com}}(\vec{v}^{\mathrm{triv}}(x))$. Thus, the components of $\boldsymbol{\pi}^{\mathrm{com}}$ are surjective. $\square$

Now, we are in the position to state the main result of this section; it also summarizes Lemmas 3.3 and 3.5.

Theorem 3.6. *Let* $\mathbf{A}$ *be a small concrete category.*

(A) *For the cometic functor* $F_{\mathrm{com}} = F_{\mathrm{com}}^{\mathbf{A}}$ *and the cometic projection* $\boldsymbol{\pi}^{\mathrm{com}} = \boldsymbol{\pi}^{\mathrm{com},\mathbf{A}}$ *associated with* $\mathbf{A}$*, the following hold.*
   - (i) $F_{\mathrm{com}}\colon \mathbf{A} \to \mathbf{Set}$ *is a totally faithful functor and* $\boldsymbol{\pi}^{\mathrm{com}}\colon F_{\mathrm{com}} \to G_{\mathrm{forg}}^{\mathbf{A}}$ *is a natural transformation whose components are surjective maps.*
   - (ii) *For every* $f \in \mathrm{Mor}(\mathbf{A})$*,* f *is a monomorphism in* $\mathbf{A}$ *if and only if* $F_{\mathrm{com}}(f)$ *is an injective map.*

(B) *Whenever* $F\colon \mathbf{A} \to \mathbf{Set}$ *is a functor and* $\boldsymbol{\kappa}\colon F \to G_{\mathrm{forg}}^{\mathbf{A}}$ *is a natural transformation whose components are surjective maps, then for every morphism* $f \in \mathrm{Mor}(\mathbf{A})$*, if* $F(f)$ *is an injective map, then* f *is a monomorphism in* $\mathbf{A}$*.*

By part (B), we cannot "translate" more morphisms to injective maps than those translated by F_{com}. In this sense, part (B) is the converse of part (A) (with less assumptions on the functor). A category $\mathbf{A}$ is *finite* if both $\mathrm{Ob}(\mathbf{A})$ and $\mathrm{Mor}(\mathbf{A})$ are finite sets. The following remark will automatically follow from the proof of Theorem 3.6.

Remark 3.7. If $\mathbf{A}$ in Theorem 3.6 is a *finite* concrete category, then so is its F_{com}-image, $F_{\mathrm{com}}(\mathbf{A})$; see (2.2).

Proof of Theorem 3.6. (Ai) is the conjunction of Lemmas 3.3 and 3.5.

In order to prove part (B), let $\mathbf{A}$ be a small concrete category, let $F\colon \mathbf{A} \to \mathbf{Set}$ be a functor, and let $\boldsymbol{\kappa}\colon F \to G_{\mathrm{forg}}^{\mathbf{A}}$ be a natural transformation with surjective components. Assume that $Y, Z \in \mathrm{Ob}(\mathbf{A})$ and $f \in \mathrm{Mor}_{\mathbf{A}}(Y, Z)$ such that $F(f)$ is injective. In order to prove that f is a monomorphism in $\mathbf{A}$, let $X \in \mathrm{Ob}(\mathbf{A})$ and $g_1, g_2 \in \mathrm{Mor}_{\mathbf{A}}(X, Y)$ such that $f \circ g_1 = f \circ g_2$; we have to show that $g_1 = g_2$. That is, we have to show that, for an arbitrary $x \in X$, $g_i(x)$ does not depend on $i \in \{1,2\}$. By the surjectivity of $\boldsymbol{\kappa}_X$, we can pick an element $a \in F_{\mathrm{com}}(X)$ such that $x = \boldsymbol{\kappa}_X(a)$. Since $f \circ g_1 = f \circ g_2$,

$$F(f)\bigl(F(g_i)(a)\bigr) = \bigl(F(f) \circ F(g_i)\bigr)(a) = F(f \circ g_i)(a)$$

does not depend on $i \in \{1,2\}$. Hence, the injectivity of $F(f)$ yields that $F(g_i)(a)$ does not depend on $i \in \{1,2\}$. Since $\boldsymbol{\kappa}$ is a natural transformation,

$$\begin{array}{ccc} F(X) & \xrightarrow{F(g_i)} & F(Y) \\ {\scriptstyle\boldsymbol{\kappa}_X}\Big\downarrow & & {\scriptstyle\boldsymbol{\kappa}_Y}\Big\downarrow \\ X & \xrightarrow{g_i} & Y \end{array}$$

is a commutative diagram, and we obtain that

$$g_i(x) = g_i\bigl(\boldsymbol{\kappa}_X(a)\bigr) = (g_i \circ \boldsymbol{\kappa}_X)(a) = \bigl(\boldsymbol{\kappa}_Y \circ F(g_i)\bigr)(a) = \boldsymbol{\kappa}_Y\bigl(F(g_i)(a)\bigr).$$

Hence, $g_i(x)$ does not depend on $i \in \{1,2\}$, because neither does $F(g_i)(a)$. Consequently, $g_1 = g_2$. Thus, f is a monomorphism, proving part (B).

In order to prove the "only if" direction of (Aii), assume that $X, Y \in \mathrm{Ob}(\mathbf{Y})$ and $f \in \mathrm{Mor}_{\mathbf{A}}(X,Y)$ is a monomorphism in the category $\mathbf{A}$. We have to show that $F_{\mathrm{com}}(f)$ is injective. In order to do so, let $c_i = \langle h_i, z_i, x_i \rangle \in F_{\mathrm{com}}(X)$ such that $F_{\mathrm{com}}(f)(c_1) = F_{\mathrm{com}}(f)(c_2)$. Since the middle components in

$$\langle f \circ h_1, z_1, f(x_1) \rangle = F_{\mathrm{com}}(f)(c_1) = F_{\mathrm{com}}(f)(c_2) = \langle f \circ h_2, z_2, f(x_2) \rangle$$

are equal, we have that $z_1 = z_2$. Since f is a monomorphism, the equality of the first components yields that $h_1 = h_2$. Since c_1 and c_2 are eligible triplets, the first two components determine the third. Hence, $c_1 = c_2$ and $F_{\mathrm{com}}(f)$ is injective, as required. This proves the "only if" direction of part (Aii).

Finally, the "if" direction of (Aii) follows from (Ai) and (B). $\square$

Remark 3.8. There are many examples of monomorphisms in small concrete categories that are not injective. For example, let $f \colon X \to Y$ be a non-injective map between two distinct sets. Consider the category $\mathbf{A}$ with $\mathrm{Ob}(\mathbf{A}) = \{X, Y\}$ and $\mathrm{Mor}(\mathbf{A}) = \{\mathbf{1}_X, \mathbf{1}_Y, f\}$; then f is a monomorphism in $\mathbf{A}$. For a bit more general example, see Example 4.10.

Remark 3.9. Let $\mathbf{A}$ be as in Theorem 3.6, $X, Y \in \mathrm{Ob}(\mathbf{A})$, and let f belong to $\mathrm{Mor}(X,Y)$. Since $\vec{v}_X^{\mathrm{triv}}$ from Definition 3.2 is a right inverse of $\boldsymbol{\pi}_X^{\mathrm{com}}$, the commutativity of (3.2) yields easily that $f = \boldsymbol{\pi}_Y^{\mathrm{com}} \circ F_{\mathrm{com}}(f) \circ \vec{v}_X^{\mathrm{triv}}$. Note, however, that $\vec{v}^{\mathrm{triv}}$ is not a natural transformation in general.

Remark 3.10. Let $\mathbf{A}$ be as in Theorem 3.6. As an easy consequence of the theorem, every monomorphism of $F_{\mathrm{com}}(\mathbf{A})$ is an injective map. In this sense, $F_{\mathrm{com}}(\mathbf{A})$ is "better" than $\mathbf{A}$. Since $F_{\mathrm{com}}(\mathbf{A})$ is obtained by the cometic functor, one might, perhaps, call it the *celestial category* associated with $\mathbf{A}$.

4. Introduction to the lattice theory part

From now on, the paper is mainly for lattice theorists. Motivated by the history of the congruence lattice representation problem, which culminated in Wehrung [17] and Růžička [16], Grätzer in [9] has recently started an analogous new topic of lattice theory. Namely, for a lattice L, let $\mathrm{Princ}(L) = \langle \mathrm{Princ}(L), \subseteq \rangle$ denote the ordered set of principal congruences of L. A congruence is *principal* if it is generated by a pair $\langle a, b\rangle$ of elements. Ordered sets (also called partially ordered sets or posets) and lattices with 0 and 1 are called *bounded*. If L is a bounded lattice, then $\mathrm{Princ}(L)$ is a bounded ordered set. Conversely, Grätzer [9] proved that every bounded ordered set P is isomorphic to $\mathrm{Princ}(L)$ for an appropriate bounded lattice L of length 5. The ordered sets $\mathrm{Princ}(L)$ of countable but not necessarily bounded lattices L were characterized in Czédli [3]. There are also results that represent two or more bounded ordered sets together with some isotone maps simultaneously by means of principal congruences of lattices; the present paper extends these results. In order to review these earlier results in an economic way and to formulate our theorem later, we need the following definition.

Definition 4.1. We define the following four categories.

- (i) $\mathbf{Lat}^{+}_{01}$ is the category of at least 2-element bounded lattices with $\{0,1\}$-preserving lattice homomorphisms.
- (ii) $\mathbf{Lat}_5$ is the category of lattices of length 5 with $\{0,1\}$-preserving lattice homomorphisms.
- (iii) $\mathbf{Lat}^{\mathrm{sd}}_5$ is the category of selfdual bounded lattices of length 5 with $\{0,1\}$-preserving lattice homomorphisms.
- (iv) $\mathbf{Pos}^{+}_{01}$ is the category of at least 2-element bounded ordered sets with $\{0,1\}$-preserving isotone (that is, order-preserving) maps.

The superscript $+$ above is to remind us that the least structures, the singleton ones, are excluded. Note that $\mathbf{Lat}^{\mathrm{sd}}_5$ is a subcategory of $\mathbf{Lat}_5$, which is a subcategory of $\mathbf{Lat}^{+}_{01}$. Note also that if X and Y are ordered sets and $|Y| = 1$, then $\mathrm{Mor}(X, Y)$ consists of the trivial map and $\mathrm{Mor}(Y, X) \neq \varnothing$ iff $|X| = 1$. Hence, we do not loose anything interesting by excluding the singleton ordered sets from $\mathbf{Pos}^{+}_{01}$. A similar comment applies for singleton lattices, which are excluded from $\mathbf{Lat}^{+}_{01}$.

For an algebra A and $x, y \in A$, the principal congruence generated by $\langle x, y\rangle$ is denoted by $\mathrm{con}(x, y)$ or $\mathrm{con}_A(x, y)$. For lattices, the following observation is due to Grätzer [10]; see also Czédli [2] for the injective case. Note that $\mathrm{Princ}(A)$ is meaningful for every algebra A.

Lemma 4.2. *If A and B are algebras of the same type and $f\colon A \to B$ is a homomorphism, then*

$$\begin{aligned} \mathrm{Princ}(f) = \zeta_{f,A,B}\colon \mathrm{Princ}(A) \to \mathrm{Princ}(B),\ \textit{defined by} \\ \mathrm{con}_A(x, y) \mapsto \mathrm{con}_B(f(x), f(y)), \end{aligned} \tag{4.1}$$

is a 0*-preserving isotone map. Thus, for every concrete category* $\mathbf{A}$ *of similar algebras with all homomorphisms as morphisms,* Princ *is a functor from* $\mathbf{A}$ *to the category of ordered sets having* 0 *with* 0*-preserving isotone maps.*

Proof. We only have to prove that $\zeta_{f,A,B}$ is a well-defined map, since the rest of the statement is obvious. That is, we have to prove that if $\mathrm{con}_A(a,b) = \mathrm{con}_A(c,d)$, then $\mathrm{con}_B(f(a), f(b)) = \mathrm{con}_B(f(c), f(d))$. Clearly, it suffices to prove that if $a, b, c, d \in A$ such that $\langle a, b\rangle \in \mathrm{con}_A(c,d)$, then $\langle f(a), f(b)\rangle \in \mathrm{con}_B(f(c), f(d))$. According to a classical lemma of Mal'cev [14], see also Fried, Grätzer and Quackenbush [5, Lemma 2.1], the containment $\langle a, b\rangle \in \mathrm{con}_A(c,d)$ is witnessed by a system of certain equalities among terms applied for certain elements of A. Since f preserves these equalities, $\langle f(a), f(b)\rangle \in \mathrm{con}_B(f(c), f(d))$, as required. ☐

It follows from Lemma 4.2 that

$$\begin{aligned} &\mathrm{Princ}\colon \mathbf{Lat}_5^{\mathrm{sd}} \to \mathbf{Pos}_{01}^{+}, \text{defined by} \\ &\qquad X \mapsto \mathrm{Princ}(X) \text{ for } X \in \mathrm{Ob}(\mathbf{Lat}_5^{\mathrm{sd}}) \text{ and} \\ &\qquad f \mapsto \zeta_{f,X,Y} \text{ for } f \in \mathrm{Mor}(X,Y), \end{aligned} \tag{4.2}$$

is a functor. Note that Princ could similarly be defined with $\mathbf{Lat}_{01}^{+}$ or $\mathbf{Lat}_5$ as its domain category. Prior to Definition 4.4, we observe the following.

Lemma 4.3. *In the category* $\mathbf{Pos}_{01}^{+}$*, the monomorphisms, the epimorphisms, and the isomorphisms are exactly the injective* $\{0,1\}$*-preserving isotone maps, the surjective* $\{0,1\}$*-preserving isotone maps, and the order isomorphisms, respectively.*

Proof. All maps in the proof are assumed to be $\{0,1\}$-preserving and isotone. It is well-known that an injective map is a monomorphism and a surjective map is an epimorphism. In order to prove the converse, assume that $f\colon X \to Y$ is a non-injective morphism in $\mathbf{Pos}_{01}^{+}$. Pick $x_1 \neq x_2 \in X$ such that $f(x_1) = f(x_2)$, and let $Z = \{0 \prec z \prec 1\}$ be a three-element chain. Define the $\{0,1\}$-preserving isotone map $g_i\colon Z \to X$ by the rule $g_i(z) = x_i$. Since $g_1 \neq g_2$ but $f \circ g_1 = f \circ g_2$, f is not injective. Next, assume that $f\colon X \to Y$ is a non-surjective morphism of $\mathbf{Pos}_{01}^{+}$, pick a $y \in Y \setminus f(X)$, and pick two elements, y_1 and y_2, outside Y. On the set $Y' := (Y \setminus \{y\}) \cup \{y_1, y_2\}$, define the ordering relation by the rule $u < v$ iff either $\{u, v\} \cap \{y_1, y_2\} = \varnothing$ and $u <_Y v$, or $u = y_i$ and $y <_Y v$, or $v = y_i$ and $u <_Y y$ for some $i \in \{1, 2\}$. Note that y_1 and y_2 are incomparable. Let $g_i\colon Y \to Y'$ be defined by $u \mapsto u$ if $u \neq y$ and $y \mapsto y_i$. Then $g_1, g_2 \in \mathrm{Mor}(\mathbf{Pos}_{01}^{+})$, $g_1 \circ f = g_2 \circ f$ but $g_1 \neq g_2$, showing that f is not an epimorphism. Finally, if $h\colon X \to Y$ is an order isomorphism, then it is an isomorphism in category theoretical sense. Conversely, if $h \in \mathrm{Mor}_{\mathbf{Pos}_{01}^{+}}(X,Y)$ is an isomorphism in category theoretical sense, then it has an inverse in $\mathrm{Mor}_{\mathbf{Pos}_{01}^{+}}(Y,X)$, whereby h is an order isomorphism. ☐

Definition 4.4. Let $\mathbf{A}$ be a small category and let $F_{\mathrm{pos}}\colon \mathbf{A} \to \mathbf{Pos}^+_{01}$ be a functor. Following Gillibert and Wehrung [6], we say that a functor

$$E_{\mathrm{Lift}}\colon \mathbf{A} \to \mathbf{Lat}^{\mathrm{sd}}_5 \qquad \text{or} \qquad E_{\mathrm{Lift}}\colon \mathbf{A} \to \mathbf{Lat}_5$$

lifts the functor F_{pos} *with respect to the functor* Princ, if F_{pos} is naturally isomorphic to the composite functor $\mathrm{Princ} \circ E_{\mathrm{Lift}}$.

Note that the existence of $E_{\mathrm{Lift}}\colon \mathbf{A} \to \mathbf{Lat}^{\mathrm{sd}}_5$ above is a stronger requirement than the existence of $E_{\mathrm{Lift}}\colon \mathbf{A} \to \mathbf{Lat}_5$. Every ordered set $\langle P; \leq\rangle$ can be viewed as a small category whose objects are the elements of P and, for $X, Y \in P$, $|\mathrm{Mor}(X, Y)| = 1$ for $X \leq Y$ and $|\mathrm{Mor}(X, Y)| = 0$ for $X \not\leq Y$. Small categories obtained in this way are called *categorified posets*. Based on Lemma 4.3, the known results on representations of isotone maps by principal congruences can be stated in the following two propositions. A map is 0-*separating* if the only preimage of 0 with respect to this map is 0.

Proposition 4.5 (Czédli [4])**.** *Assume that* $\mathbf{A}$ *is a categorified poset. If* $F_{\mathrm{pos}}\colon \mathbf{A} \to \mathbf{Pos}^+_{01}$ *is a functor such that* $F_{\mathrm{pos}}(f)$ *is* 0*-separating for all* $f \in \mathrm{Mor}(\mathbf{A})$*, then there exists a functor* $E_{\mathrm{Lift}}\colon \mathbf{A} \to \mathbf{Lat}^{\mathrm{sd}}_5$ *that lifts* F_{pos} *with respect to* Princ*.*

Note that [4] extends the result of Czédli [2], in which $\mathbf{A}$ is the categorified two-element chain but $F(f)$ is still 0-separating. As another extension of [2], Grätzer dropped the injectivity in the following statement, which we translate to our terminology as follows.

Proposition 4.6 (Grätzer [10])**.** *If* $\mathbf{A}$ *is the categorified two-element chain, then for every functor* $F_{\mathrm{pos}}\colon \mathbf{A} \to \mathbf{Pos}^+_{01}$*, there exists a functor* $E_{\mathrm{Lift}}\colon \mathbf{A} \to \mathbf{Lat}_5$ *that lifts* F_{pos} *with respect to* Princ*.*

Equivalently, in a simpler language and using the notation given in (4.1), Proposition 4.6 asserts that if X_1 and X_2 are nontrivial bounded ordered sets and $f\colon X_1 \to X_2$ is a $\{0, 1\}$-preserving isotone map, then there exist lattices L_1 and L_2 of length 5, order isomorphisms $\boldsymbol{\kappa}_i\colon \mathrm{Princ}(L_i) \to X_i$ for $i \in \{1, 2\}$, and a $\{0, 1\}$-preserving lattice homomorphism $g\colon L_1 \to L_2$ such that the diagram

$$\begin{array}{ccc}
\mathrm{Princ}(L_1) & \xrightarrow{\zeta_{g,L_1,L_2}} & \mathrm{Princ}(L_2) \\
{\scriptstyle \boldsymbol{\kappa}_1}\big\downarrow & & {\scriptstyle \boldsymbol{\kappa}_2}\big\downarrow \\
X_1 & \xrightarrow{f} & X_2
\end{array}$$

is commutative, that is, $f = \boldsymbol{\kappa}_2 \circ \zeta_{g,L_1,L_2} \circ \boldsymbol{\kappa}_1^{-1}$.

Now we are in the position to formulate the second theorem of the paper.

Theorem 4.7. *Let* $\mathbf{A}$ *be a small category such that*

$$\textit{every } f \in \mathrm{Mor}(\mathbf{A}) \textit{ is a monomorphism in } \mathbf{A}. \tag{4.3}$$

Then for every faithful functor $F_{\text{pos}}\colon \mathbf{A} \to \mathbf{Pos}_{01}^{+}$*, there exists a faithful functor*

$$E_{\text{Lift}}\colon \mathbf{A} \to \mathbf{Lat}_5^{\text{sd}}$$

that lifts F_{pos} *with respect to* Princ. *Furthermore, if* F_{pos} *is totally faithful, then there exists a totally faithful* E_{Lift} *that lifts* F_{pos} *with respect to* Princ.

Apart from some remarks and examples at the end of the present section, the rest of the paper is devoted to the proof of this theorem. After the necessary constructions and preparatory statements given in Sections 6 and 7, the proof is completed in Section 8 right after Corollary 8.2.

Remark 4.8. Based on Wehrung [18], an anonymous referee of Czédli [4] has pointed out that a faithful functor from an *arbitrary* small category to $\mathbf{Pos}_{01}^{+}$ cannot be lifted with respect to Princ in general; see [4, Observation 6.5] for details. Therefore, assumption (4.3) cannot be omitted from Theorem 4.7

Remark 4.9. Subsection 2.3 of [4], which is due to the above-mentioned referee, can be adopted to the present paper. That is, if $\mathbf{PLat}_5$ denotes the category of *polarity lattices* of length 5 with polarity-preserving lattice homomorphisms, then Theorem 4.7 remains valid if we replace by $\mathbf{Lat}_5^{\text{sd}}$ by $\mathbf{PLat}_5$. (Keeping the size limited, we do not elaborate the straightforward details.)

Observe that Propositions 4.5 and 4.6 are particular cases of Theorem 4.7, since every morphism of a categorified poset is a monomorphism and the functors in these statements are automatically faithful. In order to avoid the feeling that Proposition 4.6 or similar situations are the only cases where Theorem 4.7 takes care of non-injective isotone maps, we give an example.

Example 4.10. Let $D_1, D_2 \subseteq \text{Ob}(\mathbf{Pos}_{01}^{+})$ such that D_1 and D_2 are disjoint sets and D_1 is nonempty. We define a small category $\mathbf{A} = \mathbf{A}(\mathbf{Pos}_{01}^{+}, D_1, D_2)$ by the equalities $\text{Ob}(\mathbf{A}) = D_1 \cup D_2$ and

$$\begin{aligned} \text{Mor}(\mathbf{A}) = \{f \in \text{Mor}_{\mathbf{Pos}_{01}^{+}}(X,Y) : {}&\text{either } X, Y \in D_1 \text{ and } f \text{ is a} \\ &\text{monomorphism in } \mathbf{Pos}_{01}^{+},\ \text{or } X \in D_2 \text{ and } Y \in D_1, \\ &\text{or } X = Y \in D_2 \text{ and } f = \mathbf{1}_X\}. \end{aligned} \tag{4.4}$$

Then all morphisms in $\mathbf{A}$ are monomorphisms in $\mathbf{A}$ but, clearly, many of them are not injective in general. (The same is true for all subcategories of $\mathbf{A}$. Also, the same holds even if we start from a variety of general algebras rather than from $\mathbf{Pos}_{01}^{+}$. By Lemma 4.3, we can replace "monomorphism" by "injective" in the second line of (4.4).) Now if $F_{\text{pos}}\colon \mathbf{A} \to \mathbf{Pos}_{01}^{+}$ is the *inclusion functor* defined by $X \mapsto X$ for objects and $f \mapsto f$ for morphisms, then Theorem 4.7 yields a totally faithful functor $E_{\text{Lift}}\colon \mathbf{A} \to \mathbf{Lat}_5^{\text{sd}}$ that lifts F_{pos} with respect to Princ.

Proof. We prove that all morphisms in $\mathbf{A}$ above are monomorphisms in $\mathbf{A}$. Let $f \in \mathrm{Mor}_{\mathbf{A}}(X,Y)$, and assume that $g_1, g_2 \in \mathrm{Mor}_{\mathbf{A}}(Z,X)$ such that $f \circ g_1 = f \circ g_2$. If $X \in D_2$, then $Z = X$ and $g_1 = \mathbf{1}_Z = g_2$. Otherwise $X, Y \in D_1$ and f is a monomorphism in $\mathbf{Pos}^+_{01}$, whence we conclude the equality $g_1 = g_2$ again. Thus, f is a monomorphism in $\mathbf{A}$. □

Example 4.11. In a self-explanatory (simpler but less precise) language, we mention two particular cases of Example 4.10. First, we can represent all automorphisms of a bounded ordered set simultaneously by principal congruences. Second, if we are given two distinct bounded ordered sets X and Y, then we can simultaneously represent all $\{0,1\}$-preserving isotone $X \to Y$ maps by principal congruences.

5. The main ideas for the proof of Theorem 4.7

5.1. **Outlining the role of gadgets and quasi-colored lattices.** In order to construct a lattice L with a given $\mathrm{Princ}(L)$ (up to isomorphism), we will use uniform building blocks, which are called *gadgets*; see Grätzer [7, 9] for this terminology, and see Czédli [2, 3, 4] and Grätzer [9, 10] based on similar gadgets. These gadgets and those in the present paper serve the following purpose. Assume that we want to construct a lattice $L = \bigcup_{\iota<\kappa} L_\iota$ as a directed union of a well-ordered system of sublattices L_ι to represent an ordered set $\langle X; \le\rangle$ as $\mathrm{Princ}(L)$. Let $\mathrm{con}_{L_\iota}(a_x, b_x)$ and $\mathrm{con}_{L_\iota}(a_y, b_y)$ be incomparable congruences of L_ι corresponding to x and y, respectively, such that $x < y$ in X. Then we merge L_ι and a copy G of our gadget to obtain $L_{\iota+1}$ such that $\{a_x, b_x, a_y, b_y\} \subseteq L_\iota \cap G$ and $\mathrm{con}_G(a_x, b_x) \le \mathrm{con}_G(a_y, b_y)$ forces that $\mathrm{con}_{L_{\iota+1}}(a_x, b_x) \le \mathrm{con}_{L_{\iota+1}}(a_y, b_y)$. In order to avoid that undesired inequalities among principal congruences of $L_{\iota+1}$ enter, we need some insight into the transition from $\mathrm{Princ}(L_\iota)$ to $\mathrm{Princ}(L_{\iota+1})$. This insight will be provided by *quasi-colorings*, which were introduced in Czédli [1] and were successfully used for principal lattice congruences in Czédli [2, 3, 4]. Besides that quasi-colorings conveniently determine the principal congruences (this will be precisely formulated in Lemma 8.1 and Corollary 8.2), there is a natural way to merge them when the corresponding lattices are merged. See Subsection 1.5 in Czédli [3] for an alternative introduction to these ideas.

5.2. **On the rest of the ideas.** This subsection is not necessary for the rest of the paper, but it gives information for those who want to understand the rest of ideas without reading the rigorous and long proofs and definitions that we present in the remaining part of the paper.

Examples 2.2 and 3.1 of Czédli [4] (with Figures 1–4 there) show most of the ideas needed in the particular case where $\mathbf{A}$ is a categorified poset and our isotone maps are 0-separating; see Proposition 4.5 here. Since we do not assume 0-separation, we also need the quotients (see Figures 2 and 3 here) of our gadgets, see Figure 1. (Of course, these quotients will be merged with their duals to turn them selfdual quasi-colored lattices.) The isotone map ψ_{31} in [4, Figure 1] is not injective since $\psi_{31}(q_3) = \psi_{31}(r_3) = q_1$. It is described below [4, Figure 3] how the lattices L_i are obtained from the auxiliary lattices W_i in [4, Figure 3]. Let ψ^*_{31} denote the lattice homomorphism $L_3 \to L_1$ that corresponds to ψ_{31} in the sense that $\mathrm{Princ}(\psi^*_{31})$ will represent ψ_{31}. Observe that ψ^*_{31} is injective on the set of (thin) basic edges; otherwise the method of [4] would collapse.

In order to prove Theorem 4.7, we have a lot of isotone maps ψ and we have to make them injective maps ψ^* on the sets of basic edges. We apply the cometic functor to obtain injective maps that can used to define these maps ψ^*. Armed with these ψ^*, Figures 1, 2, and 3 (here), and the above-mentioned ideas taken from [4], we have a rough idea how to prove Theorem 4.7.

6. Gadgedts, quasi-colored lattices and a toolkit for them

6.1. **Gadgets and basic facts.** We follow the terminology of Czédli [4]. If ν is a quasiorder, that is, a reflexive transitive relation, then $\langle x, y\rangle \in \nu$ will occasionally be abbreviated as $x \leq_\nu y$. For a lattice or ordered set $L = \langle L; \leq\rangle$ and $x, y \in L$, $\langle x, y\rangle$ is called an *ordered pair* of L if $x \leq y$. If $x = y$, then $\langle x, y\rangle$ is a *trivial ordered pair*. The set of ordered pairs of L is denoted by $\mathrm{Pairs}^{\leq}(L)$. If $X \subseteq L$, then $\mathrm{Pairs}^{\leq}(X)$ will stand for $X^2 \cap \mathrm{Pairs}^{\leq}(L)$. We also need the notation $\mathrm{Pairs}^{\prec}(L) := \{\langle x, y\rangle \in \mathrm{Pairs}^{\leq}(X) : x \prec y\}$ for the set of *covering pairs*. By a *quasi-colored lattice* we mean a structure

$$\mathcal{L} = \langle L, \leq; \gamma; H, \nu\rangle$$

where $\langle L; \leq\rangle$ is a lattice, $\langle H; \nu\rangle$ is a quasiordered set, $\gamma\colon \mathrm{Pairs}^{\leq}(L) \to H$ is a *surjective* map, and for all $\langle u_1, v_1\rangle, \langle u_2, v_2\rangle \in \mathrm{Pairs}^{\leq}(L)$,

(C1) if $\gamma(\langle u_1, v_1\rangle) \leq_\nu \gamma(\langle u_2, v_2\rangle)$, then $\mathrm{con}(u_1, v_1) \leq \mathrm{con}(u_2, v_2)$ and
(C2) if $\mathrm{con}(u_1, v_1) \leq \mathrm{con}(u_2, v_2)$, then $\gamma(\langle u_1, v_1\rangle) \leq_\nu \gamma(\langle u_2, v_2\rangle)$.

This concept is taken from Czédli [4]; see Grätzer, Lakser, and Schmidt [13], Grätzer [7, page 39], and Czédli [1, 3] for the evolution of this concept. It follows easily from (C1), (C2), and the surjectivity of γ that if $\langle L, \leq; \gamma; H, \nu\rangle$ is a quasi-colored bounded lattice, then $\langle H; \nu\rangle$ is a quasiordered set with a least element and a greatest element; possibly with many least elements and many greatest elements. Let $U(H)$ stand for the set of greatest elements. For $\langle x, y\rangle \in \mathrm{Pairs}^{\leq}(L)$, $\gamma(\langle x, y\rangle)$ is called the *color* (rather than the quasi-color) of $\langle x, y\rangle$, and we say that $\langle x, y\rangle$ is colored (rather than quasi-colored) by $\gamma(\langle x, y\rangle)$. For $T \subseteq H$, we say that $\langle x, y\rangle$ is *T-colored* if $\gamma(\langle x, y\rangle) \in T$. Usually, the following convention applies to our figures of quasi-colored lattices that contain thick edges and, possibly, also thin edges: if γ is a quasi-coloring, then for an ordered pair $\langle x, y\rangle$,

$$\gamma(\langle x,y\rangle) = \begin{cases} 0, & \text{iff } x = y,\\ w, & \text{if } x \prec y \text{ is a thin edge labeled by } w,\\ u \in U(H), & \text{if the interval } [x,y] \text{ contains is a thick edge},\\ \gamma(\langle x',y'\rangle), & \text{if } [x,y] \text{ and } [x',y'] \text{ are transposed intervals.}\end{cases} \tag{6.1}$$

If H has exactly one largest element $1 = 1_H$ and so $U(H) = \{1\}$, then our figures determine the corresponding quasi-colorings by convention (6.1). Note, however, that this convention only partially applies to Figure 6, which is *not* a quasi-colored lattice. The quasi-colored lattice

$$\mathcal{G}_2^{\rm up}(p,q) := \langle G_2^{\rm up}(p,q), \lambda_{2pq}^{\rm up}; \gamma_{2pq}^{\rm up}; H_2(p,q), \nu_{2pq}\rangle$$

in Figure 1, taken from Czédli [4] where it was denoted by $\mathcal{G}^{\rm up}(p,q)$, is our *upward gadget of type* 2. Its quasi-coloring is defined by (6.1); note that $\gamma_{2pq}^{\rm up}(\langle c_4^{pq}, d_4^{pq}\rangle) = q$. Using the quotient lattices

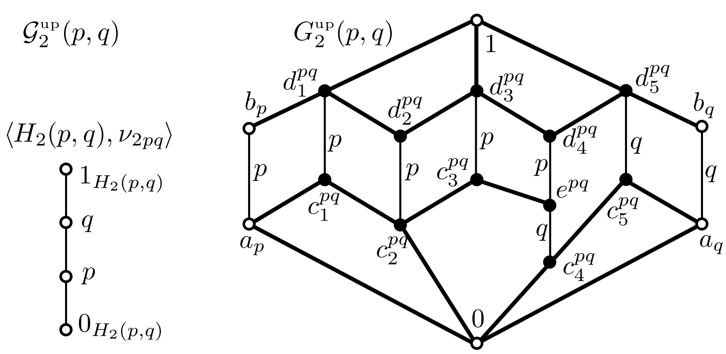

FIGURE 1. The upward gadget of rank 2

$$\begin{aligned} G_0^{\rm up}(p,q) &:= G_2^{\rm up}(p,q)/\mathrm{con}(a_q,b_q) \text{ and}\\ G_1^{\rm up}(p,q) &:= G_2^{\rm up}(p,q))/\mathrm{con}(a_p,b_p),\end{aligned} \tag{6.2}$$

we also define the gadgets

$$\begin{aligned} \mathcal{G}_0^{\rm up}(p,q) &:= \langle G_0^{\rm up}(p,q), \lambda_{0pq}^{\rm up}; \gamma_{0pq}^{\rm up}; H_0(p,q), \nu_{0pq}\rangle \text{ and}\\ \mathcal{G}_1^{\rm up}(p,q) &:= \langle G_1^{\rm up}(p,q), \lambda_{1pq}^{\rm up}; \gamma_{1pq}^{\rm up}; H_1(p,q), \nu_{1pq}\rangle\end{aligned}$$

of rank 0 and rank 1, respectively; see Figures 2 and 3. Note that the rank is $\mathrm{length}([a_p,b_p]) + \mathrm{length}([a_q,b_q])$. We obtain the *downward gadgets* $\mathcal{G}_2^{\rm dn}(p,q)$, $\mathcal{G}_1^{\rm dn}(p,q)$, and $\mathcal{G}_0^{\rm dn}(p,q)$ of ranks 2, 1, and 0 from the corresponding upward gadgets by dualizing; see Czédli [4, (4.3)]. Instead of d_{ij}^{pq} and, if applicable, c_{ij}^{pq} and e^{pq}, their elements are denoted by d_{pq}^{ij}, c_{pq}^{ij}, and e_{pq}; see [4]. By a *single gadget* we mean an upper or lower gadget. The adjective "upper" or "lower" is the *orientation* of the gadget. A single gadget of rank j without specifying its orientation is denoted by $G_j^{\forall}(p,q)$.

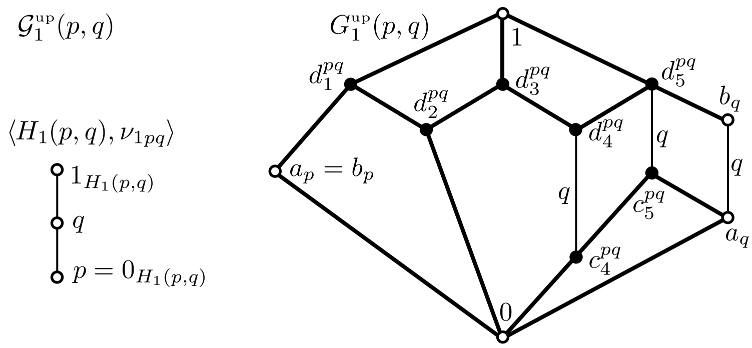

FIGURE 2. The upward gadget of rank 1

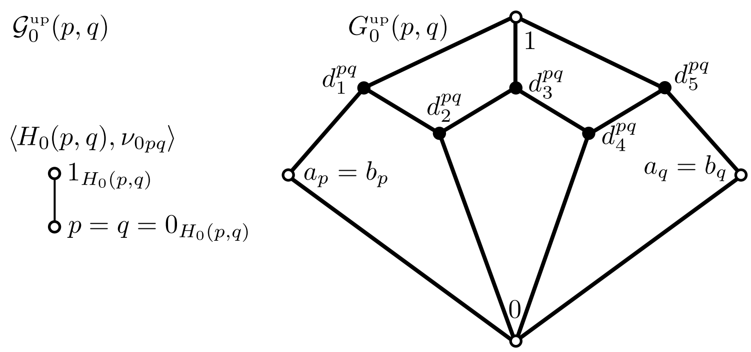

FIGURE 3. The upward gadget of rank 0

In case of all our gadgets $G_j^{\forall}(p,q)$, we automatically assume that $p \neq q$. Also, we always assume that for $i, j \in \{0,1,2\}$, the ordered pairs $\langle p,q\rangle$, $\langle u,v\rangle$, and the strings $\mathrm{s}, \mathrm{t} \in \{\mathrm{up}, \mathrm{dn}\}$ are such that $\langle p,q,i,\mathrm{s}\rangle \neq \langle u,v,j,\mathrm{t}\rangle$,

$$\text{the intersection of } G_i^{\mathrm{t}}(p,q) \text{ and } G_j^{\mathrm{s}}(u,v) \text{ is as small as it follows from the notation.} \tag{6.3}$$

This convention allows us to form the union $\mathcal{G}_i^{\mathrm{db}}(p,q)$ of $\mathcal{G}_i^{\mathrm{up}}(p,q)$ and $\mathcal{G}_i^{\mathrm{dn}}(p,q)$, for $i \in \{0,1,2\}$, which we call a *double gadget* of rank i. While $\mathcal{G}_1^{\mathrm{db}}(p,q)$ and $\mathcal{G}_0^{\mathrm{db}}(p,q)$ are given in Figures 4 and 5, the double gadget $\mathcal{G}_2^{\mathrm{db}}(p,q)$ of rank 2 is depicted in Czédli [4, Figure 4]. Observe that all the thin edges are q-colored in $\mathcal{G}_1^{\mathrm{db}}(p,q)$ and, in lack of thin edges, all the edges are 1-colored in $\mathcal{G}_0^{\mathrm{db}}(p,q)$. For $i \in \{0,1,2\}$, $G_i^{\mathrm{db}}(p,q)$ is a selfdual lattice; we will soon point out that $\mathcal{G}_i^{\mathrm{db}}(p,q)$ is a quasi-colored lattice. Note that

$$\text{In each of } G_j^{\forall}(p,q),\ \mathrm{con}(a_p,b_p) \leq \mathrm{con}(a_q,b_q)\text{; we will use our gadgets to force this inequality in larger lattices.} \tag{6.4}$$

Of course, the inequality in (6.4) is important only for $j = 2$, since it trivially holds for $j \in \{0,1\}$.

For $S \subseteq X \times X$, the least quasiorder including S is denoted by $\mathrm{quo}(S) = \mathrm{quo}_X(S)$; we write $\mathrm{quo}(a,b)$ rather than $\mathrm{quo}(\{\langle a,b\rangle\})$.

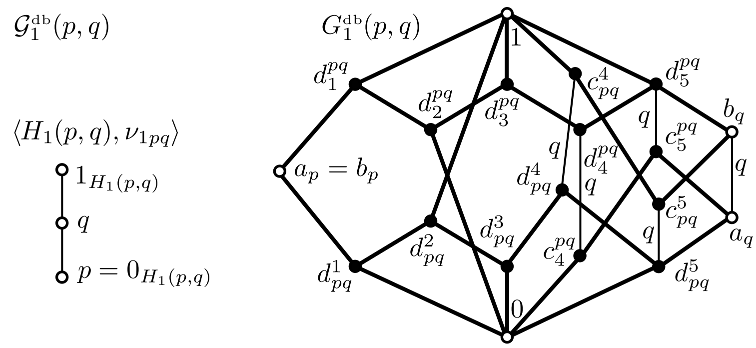

FIGURE 4. The double gadget of rank 1

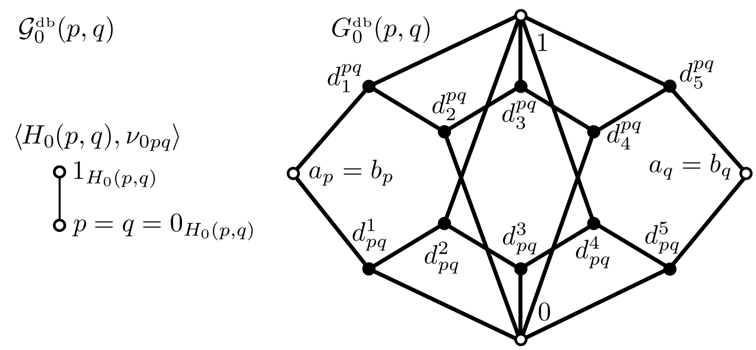

FIGURE 5. The double gadget of rank 0

Lemma 6.1. *Assume that* $L = \langle L; \le_L\rangle = \langle L; \lambda_L\rangle$ *is a lattice of length* 5, *and let* $0 < a_p \preceq b_p < 1$ *and* $0 < a_q \preceq b_q < 1$ *in* L *such that, with* $j := \mathrm{length}([a_p, b_p]) + \mathrm{length}([a_q, b_q])$,

$$a_p \vee_L a_q = 1, \quad b_p \wedge_L b_q = 0, \quad L \cap G_j^{\mathrm{up}}(p,q) = \{0, a_p, b_p, a_q, b_q, 1\},$$
$$0 \le \mathrm{length}([a_p, b_p]) \le \mathrm{length}([a_q, b_q]) \le 1,$$
$$\mathrm{length}([0, b_p]) \le 2 + \mathrm{length}([a_p, b_p]), \quad \mathrm{length}([a_p, 1]) \le 2 + \mathrm{length}([a_p, b_p]),$$
$$\mathrm{length}([0, b_q]) \le 2 + \mathrm{length}([a_q, b_q]), \quad \mathrm{length}([a_q, 1]) \le 2 + \mathrm{length}([a_q, b_q]).$$

Let

$$L^{\vartriangle} := L \cup G_j^{\mathrm{up}}(p,q) \ \text{ and } \ \lambda^{\vartriangle} := \mathrm{quo}(\lambda_L \cup \lambda_{jpq}^{\mathrm{up}});$$

see [4, Figure 8] *for* $j = 2$. *Then* $L^{\vartriangle} = \langle L^{\vartriangle}; \lambda^{\vartriangle}\rangle$, *also denoted by* $L^{\vartriangle}_{jpq}$ *or* $\langle L^{\vartriangle}_{jpq}; \le^{\vartriangle}\rangle$, *is a lattice of length* 5. *Also, both* L *and* $G_j^{\mathrm{up}}(p,q)$ *are* $\{0,1\}$-*sublattices of* $L^{\vartriangle}$.

Since the lattices required by Theorem 4.7 are selfdual, we will use selfdual gadgets, which are defined under the name "double gadgets" as follows.

Definition 6.2. Within $L^{\vartriangle}$, the (sublattice) $G_j^{\mathrm{up}}(p,q)$ is the *upper gadget from* $\langle a_p, b_p\rangle$ *to* $\langle a_q, b_q\rangle$. By duality, we can analogously glue the *lower gadget*

$G_j^{\rm dn}(p,q)$ into L *from* $\langle a_p,b_p\rangle$ *to* $\langle a_q,b_q\rangle$. Applying Lemma 6.1, its dual, and (6.3), we can glue the *double gadget* $G_j^{\rm db}(p,q)$ into L *from* $\langle a_p,b_p\rangle$ *to* $\langle a_q,b_q\rangle$.

Proof of Lemma 6.1. For $j=2$, the lemma coincides with [4, Lemma 4.5] while the case $j<2$ is analogous but simpler. Hence, it would suffice to say that the proof in [4] works without any essential modification. However, since we will need some formulas from the proof later, we give some details for $j\in\{0,1,2\}$. In order to simplify our equalities below, we denote $G_j^{\rm up}(p,q)$ by $G_j^{\rm up}$ and, in subscript position, by G. As in [4], we can still use the sublattice

$$B=B(p,q):=\{0,a_p,b_p,a_q,b_q,1\}=L\cap G_j^{\rm up}(p,q),$$

the closure operators

$$\begin{aligned}&{}^*\colon G_j^{\rm up}\to B,\text{ where } x^*\text{ is the smallest element of } B\cap{\uparrow_G}\,x,\\&{}^\bullet\colon L\to B,\text{ where } x^\bullet\text{ is the smallest element of } B\cap{\uparrow_L}\,x,\end{aligned}$$

and, dually, the interior operators

$$\begin{aligned}&{}_*\colon G_j^{\rm up}\to B,\text{ where } x_*\text{ is the largest element of } B\cap{\downarrow_G}\,x,\\&{}_\bullet\colon L\to B,\text{ where } x_\bullet\text{ is the largest element of } B\cap{\downarrow_L}\,x;\end{aligned}$$

which were introduced in [4, (4.9) and (4.10)]. Since our gadgets are "wide enough" in some geometric sense, the operators above are well-defined. As in [4, (4.11)],

$$\begin{aligned}&\lambda^\vartriangle\text{ is an ordering, } \lambda^\vartriangle\rceil_L=\lambda_L,\quad \lambda^\vartriangle\rceil_G=\lambda^{\rm up}_{pq},\\&\text{for } x\in L\text{ and } y\in G_j^{\rm up},\ \ x\leq^\vartriangle y\iff x^\bullet\leq_G y\iff x\leq_L y_*,\\&\text{for } x\in G_j^{\rm up}\text{ and } y\in L,\ \ x\leq^\vartriangle y\iff x^*\leq_L y\iff x\leq_G y_\bullet.\end{aligned}\tag{6.5}$$

Denote the lattice operations in L and $G_j^{\rm up}$ by $\vee_L$, $\wedge_L$, and $\vee_G$, $\wedge_G$, respectively. For $x,y\in L^\vartriangle$, we have that

$$\text{if } x\in L\setminus G_j^{\rm up}\text{ and } y\in G_j^{\rm up}\setminus L,\ \text{then } x\wedge^\vartriangle y=x\wedge_L y_*,\tag{6.6}$$

$$\text{if } x\in L\setminus G_j^{\rm up}\text{ and } y\in G_j^{\rm up}\setminus L,\ \text{then } x\vee^\vartriangle y=x^\bullet\vee_G y,\tag{6.7}$$

$$\text{if } x,y\in L,\ \text{then } x\wedge^\vartriangle y=x\wedge_L y,\text{ and } x\vee^\vartriangle y=x\vee_L y,\tag{6.8}$$

$$\text{if } x,y\in G_j^{\rm up},\ \text{then } x\wedge^\vartriangle y=x\wedge_G y,\text{ and } x\vee^\vartriangle y=x\vee_G y.\tag{6.9}$$

Based on (6.5), these equations are proved by exactly the same argument as their particular cases, [4, (4.12)–(4.15)] for $j=2$. It follows from (6.6)–(6.9) that $L^\vartriangle$ is a lattice. □

6.2. **Large lattices.** In this subsection and the next one, we use our double gadgets to build a "large" quasi-colored lattice for a given quasiordered set of colors; this immediate plan will be verified by (the proof of) Lemma 6.4. It will turn out later from Lemma 8.1 and Corollary 8.2 that Lemma 6.4 implies the representability of a given ordered set by principal congruences.

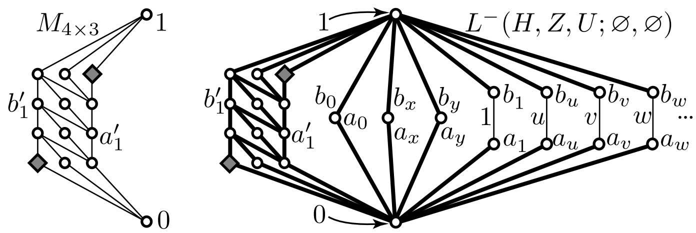

FIGURE 6. $M_{4\times 3}$ and $L^-(H,Z,U;\varnothing,\varnothing)$, which is *not* quasi-colored

Moreover, Lemma 6.4 gives us even more; it gives sufficient flexibility, which is needed to *simultaneously* represent many ordered sets and isotone maps.

Let H be a set and $Z, U \subset H$ such that

$$0 \in Z, \quad 1 \in U, \quad \text{and} \quad Z \cap U = \varnothing. \tag{6.10}$$

This notation is explained by our intention: Z and U will be the set of "zeros" (least elements) and that of "units" (largest elements) of H somewhat later. The selfdual simple lattice on the left of Figure 6 is denoted by $M_{4\times 3}$; see also [4, Figure 9] for another diagram. (The two square-shaped gray-filled elements will play a special role in Lemma 7.2.) Also, we denote by

$$L^-(H,Z,U;\varnothing,\varnothing) = \langle L^-(H,Z,U;\varnothing,\varnothing); \lambda_{L^-(H,Z,U;\varnothing,\varnothing)}\rangle \tag{6.11}$$

the lattice on the right, where $Z = \{0, x, y \dots\}$ and $H \setminus Z = \{1, u, v, w, \dots\}$. Of course, $1 \in U \subseteq H \setminus Z$. The lattice given in (6.11) is almost the same as that on the right of [4, Figure 9]. Note, however, that $|Z|$ and $|U|$ can be arbitrarily large cardinals. Note also that for $z \in Z$, $a_z = b_z$. The role of $M_{4\times 3}$ in the construction is two-fold. First, it is a simple lattice and it guarantees that all the thick edges are 1-colored, that is, they generate the largest congruence, even if $|H| = 2$. Second, $M_{4\times 3}$ guarantees that $L^-(H,Z,U;\varnothing,\varnothing)$ is of length 5. Since $\langle a_1, b_1\rangle$ is 1-colored according to labeling but this edge does not generate the largest congruence, $L^-(H,Z,U;\varnothing,\varnothing)$ is not a quasi-colored lattice (at least, not if 1 is intended to be a largest elements in H). So we cannot be satisfied yet. In order to make this edge and all the $\langle a_r, b_r\rangle$, for $r \in U$, generate the largest congruence, Definition 6.2 allows us

$$\begin{array}{l}\text{to glue, for each } r \in U\text{, a distinct copy of } G_2^{\mathrm{db}}(p,q)\\ \text{into } L^-(H,Z,U;\varnothing,\varnothing) \text{ from } \langle a_1', b_1'\rangle \text{ to } \langle a_r, b_r\rangle.\end{array} \tag{6.12}$$

(No matter if we glue the gadgets one by one by a transfinite induction or glue them simultaneously, we obtain the same.) It follows from Lemma 6.1 that we obtain a lattice in this way; we denote this lattice by

$$L(H,Z,U;\varnothing,\varnothing) = \langle L(H,Z,U;\varnothing,\varnothing); \lambda_{H,Z,U;\varnothing,\varnothing}\rangle.$$

Note that after adding the above-mentioned gadgets to $L^-(H,Z,U;\varnothing,\varnothing)$,

$$\text{all edges of the gadgets in (6.12) become thick;} \tag{6.13}$$

this follows from (6.1) and (6.4). Let

$$\nu_{H,Z,U;\varnothing,\varnothing} = \mathrm{quo}((Z \times H) \cup (H \times U)),$$

and define $\gamma_{H,Z,U;\varnothing,\varnothing}$ by convention (6.1). It is straightforward to see that

$$\begin{aligned} &\mathcal{L}(H,Z,U;\varnothing,\varnothing) = \\ &\quad \langle L(H,Z,U;\varnothing,\varnothing), \lambda_{H,Z,U;\varnothing,\varnothing}; \gamma_{H,Z,U;\varnothing,\varnothing}; H, \nu_{H,Z,U;\varnothing,\varnothing}\rangle \end{aligned} \tag{6.14}$$

is a quasi-colored lattice.

Next, to obtain larger lattices, we are going to insert gadgets into the lattice $L(H,Z,U;\varnothing,\varnothing)$ in a certain way. It will prompt follow Lemma 6.1 that we obtain lattices; in particular, $\lambda_{H,Z,U;I,J}$ in (6.18) will be a lattice order. Assume that

$$\begin{gathered} I \text{ and } J \text{ are subsets of } H \times H \text{ such that } p \neq q \text{ and the} \\ \text{implications } (q \in Z \Rightarrow p \in Z) \text{ and } (p \in U \Rightarrow q \in U) \\ \text{hold for every } \langle p,q\rangle \in I \cup J. \end{gathered} \tag{6.15}$$

With this assumption, we define the *rank of a pair* $\langle p,q\rangle \in I \cup J$ as follows:

$$r(\langle p,q\rangle) := \begin{cases} 0, & \text{if } p,q \in Z, \\ 1, & \text{if } p \in Z \text{ and } q \in H \setminus Z, \\ 2, & \text{if } p,q \in H \setminus Z. \end{cases} \tag{6.16}$$

Let us agree that, for every $\langle p,q\rangle \in I \cup J$ and $j := r(\langle p,q\rangle)$,

$$\begin{aligned} &G_j^{\mathrm{up}}(p,q) \cap L(H,Z,U;\varnothing,\varnothing) = \{0,a_p,b_p,a_q,b_q,1\} \text{ and} \\ &G_j^{\mathrm{dn}}(p,q) \cap L(H,Z,U;\varnothing,\varnothing) = \{0,a_p,b_p,a_q,b_q,1\}. \end{aligned} \tag{6.17}$$

Taking Conventions (6.3) and (6.17) into account, we define

$$\begin{aligned} L(H,Z,U;I,J) &:= L(H,Z,U;\varnothing,\varnothing) \cup \bigcup_{\langle p,q\rangle \in I} G^{\mathrm{up}}_{r(\langle p,q\rangle)}(p,q) \\ &\qquad \cup \bigcup_{\langle p,q\rangle \in J} G^{\mathrm{dn}}_{r(\langle p,q\rangle)}(p,q), \quad \text{and} \\ \lambda_{H,Z,U;I,J} &:= \mathrm{quo}\Big(\lambda_{H,Z,U;\varnothing,\varnothing} \cup \bigcup_{\langle p,q\rangle \in I} \lambda^{\mathrm{up}}_{r(\langle p,q\rangle)pq} \\ &\qquad \cup \bigcup_{\langle p,q\rangle \in J} \lambda^{\mathrm{dn}}_{r(\langle p,q\rangle)pq}\Big). \end{aligned} \tag{6.18}$$

Based on Lemma 6.1 and its dual, a trivial transfinite induction yields that

$$L(H,Z,U;I,J) = \langle L(H,Z,U;I,J); \lambda_{H,Z,U;I,J}\rangle$$

is a lattice of length 5. Clearly, if $I = J$, then this lattice is selfdual. Let us emphasize that whenever we use the notation $L(H,Z,U;I,J)$, (6.15) is assumed.

Remark 6.3. For later reference, we note that for lattices of the form (6.18), we treat a_p, b_p, c_{ij}^{pq}, d_{ij}^{pq}, c_{pq}^{ij}, etc. as if they were tuples $\langle a,p\rangle$, $\langle b,p\rangle$, $\langle c,p,q,i,j\rangle$, $\langle d,p,q,i,j\rangle$, $\langle c^{\text{dual}},p,q,i,j\rangle$, etc.. Therefore,

$$\begin{aligned} L(H_1,Z_1,U_1;I_1,J_1) = L(H_2,Z_2,U_2;I_2,J_2) &\text{ iff} \\ \langle H_1,Z_1,U_1,I_1,J_1\rangle = \langle H_2,Z_2,U_2,I_2,J_2\rangle&. \end{aligned}$$

6.3. **Large quasi-colored lattices.** Assuming (6.10), let $H^{-ZU} := H \setminus (Z\cup U)$. Also, let $\nu_{H,Z,U;\varnothing,\varnothing} = \operatorname{quo}((Z\times H)\cup(H\times U))$. Note that each $z\in Z$ is a least element of $\langle H;\nu_{H,Z,U;\varnothing,\varnothing}\rangle$ and each $u\in U$ is a largest element. Also, for any two distinct $p,q\in H^{-ZU}$, p and q are incomparable, that is, none of $\langle p,q\rangle$ and $\langle q,p\rangle$ belongs to $\nu_{H,Z,U;\varnothing,\varnothing}$. With convention (6.15), let

$$\begin{aligned} \nu_{H,Z,U;I,J} &:= \operatorname{quo}_H(\nu_{H,Z,U;\varnothing,\varnothing}\cup I\cup J) \\ &= \operatorname{quo}((Z\times H)\cup(H\times U)\cup I\cup J). \end{aligned}$$

Based on (6.17), it is easy to see that

$$\gamma_{H,Z,U;I,J} := \gamma_{H,Z,U;\varnothing,\varnothing}\cup \bigcup_{\langle p,q\rangle\in I}\gamma^{\text{up}}_{r(\langle p,q\rangle)pq}\cup\bigcup_{\langle p,q\rangle\in J}\gamma^{\text{dn}}_{r(\langle p,q\rangle)pq} \tag{6.19}$$

is a well-defined map from $\operatorname{Pairs}^{\leq}(L(H,Z,U;I,J))$ to H.

Lemma 6.4. *Assume* (6.15)*. Then*

$$\begin{aligned} &\mathcal{L}(H,Z,U;I,J) \\ &\quad := \langle L(H,Z,U;I,J),\lambda_{H,Z,U;I,J};\gamma_{H,Z,U;I,J};H,\nu_{H,Z,U;I,J}\rangle \end{aligned} \tag{6.20}$$

is a quasi-colored lattice of length 5. If $I=J$, then it is a selfdual lattice.

Proof. If $Z=\{0\}$ and $r(\langle p,q\rangle)=2$ for all $\langle p,q\rangle\in I\cup J$, then the statement is practically the same as [4, Lemma 4.6]. (Although $1\notin U=\varnothing$ in [4, Lemma 4.6], this does not make any difference.) As in [4], the only nontrivial task is to show (C2). This argument in [4] has two ingredients, and these ingredients also work in the present situation.

First, let $\boldsymbol{\alpha}$ be the equivalence on $L(H,Z,U;I,J)$ whose non-singleton equivalence classes are the $[a_p,b_p]$ for $p\in H^{-ZU}$, the $[c_i^{pq},d_i^{pq}]$ for $\langle p,q\rangle\in I$ and $i\in\{1,\dots,5\}$, and the $[d_{pq}^i,c_{pq}^i]$ for $\langle p,q\rangle\in J$ and $i\in\{1,\dots,5\}$. Using the Technical Lemma from Grätzer [11], cited in [4, Lemma 4.1], it is straightforward to see that $\boldsymbol{\alpha}$ is a congruence. Clearly, $\boldsymbol{\alpha}$ is distinct from $\nabla_{L(H,Z,U;I,J)}$, the largest congruence of $L(H,Z,U;I,J)$. Like in [4, (4.28)], this implies easily that, for any $\langle x,y\rangle\in\operatorname{Pairs}^{\leq}(L(H,Z,U;I,J))$,

$$\gamma_{H,Z,U;I,J}(\langle x,y\rangle)=1 \iff \operatorname{con}(x,y)=\nabla_{L(H,Z,U;I,J)}.$$

The second ingredient of the proof is to show that

$$\begin{aligned} &\text{if } p,q\in H^{-ZU},\ \operatorname{con}(a_p,b_p)\leq\operatorname{con}(a_q,b_q)\neq\nabla_{L(H,Z,U;I,J)}, \text{ and} \\ &p\neq q, \text{ then } \langle p,q\rangle = \langle\gamma_{H,Z,U;I,J}(\langle a_p,b_p\rangle),\gamma_{H,Z,U;I,J}(\langle a_q,b_q\rangle)\rangle \\ &\text{belongs to } \nu_{H,Z,U;I,J}; \end{aligned} \tag{6.21}$$

compare this with [4, (4.29)]. The inequality $\mathrm{con}(a_p, b_p) \leq \mathrm{con}(a_q, b_q)$ is equivalent to the containment $\langle a_p, b_p\rangle \in \mathrm{con}(a_q, b_q)$. This containment is witnessed by a *shortest* sequence of consecutive prime intervals in the sense of the Prime-projectivity Lemma of Grätzer [12]; note that this lemma is cited in [4, Lemma 4.2]. If one of the prime intervals in the sequence generates $\nabla_{L(H,Z,U;I,J)}$, then the easy direction of the Prime-projectivity Lemma yields that $\mathrm{con}(a_q, b_q) = \nabla_{L(H,Z,U;I,J)}$, a contradiction. Hence, none of these prime intervals generates $\nabla_{L(H,Z,U;I,J)}$. Thus, since (C1) is easily verified in the same way as in [4], none of these prime intervals is 1-colored. In other words, all prime intervals of the sequence are thin edges. Gadgets of rank 0 contain no thin edges, so the sequence avoids them. The same holds for the gadgets mentioned in (6.12) and (6.13). Gadgets of rank 1 contain too few thin edges, so the sequence can only make a loop in them; this is impossible since we consider the shortest sequence. Thus, the sequence goes in the sublattice that we obtain by omitting all gadgets of rank less than 2, all gadgets occurring in (6.13), and all elements $a_z = b_z$ for $z \in Z$. So we can work in this sublattice, which is the same as the lattice considered in [4, (4.29)]. Consequently, the proof of [4, (4.29)] yields (6.21). Thus, (C2) holds. □

7. From quasiorders to homomorphisms

For a quasiordered set $\langle H; \nu\rangle$, we define

$$\begin{aligned} Z(H) = Z(H, \nu) &:= \{x \in H : (\forall y \in H)\ (\langle x, y\rangle \in \nu)\} \text{ and} \\ U(H) = U(H, \nu) &:= \{x \in H : (\forall y \in H)\ (\langle y, x\rangle \in \nu)\}. \end{aligned} \tag{7.1}$$

These are the set of *smallest elements* (the notation comes from "zeros") and that of *largest elements* ("units"). If ν is clear from the context, we prefer the notations $Z(H)$ and $U(H)$ to $Z(H, \nu)$ and $U(H, \nu)$, respectively. In this section, we are only interested in the following particular case of the quasi-colored lattices $\mathcal{L}(H, Z, U; I, J)$.

Definition 7.1. For a quasiordered set $H = \langle H; \nu\rangle$, assume that

$$0 \in Z(H), \quad 1 \in U(H), \quad \text{and} \quad 0 \neq 1. \tag{7.2}$$

With this assumption, we define

$$\mathcal{L}(H, \nu) = \langle L(H, \nu), \lambda_{H,\nu}; \gamma_{H,\nu}; H, \nu\rangle \ \text{ as } \ \mathcal{L}(H, Z(H), U(H); \nu, \nu) \tag{7.3}$$

according to (6.20); this is possible since (6.15) clearly holds. Let us note that $\nu = \nu_{H,Z(H),U(H);\nu,\nu}$ and, clearly, $L(H, \nu)$ is a selfdual lattice of length 5.

For quasiordered sets $\langle H_1; \nu_1\rangle$ and $\langle H_2; \nu_2\rangle$, a map $f\colon H_1 \to H_2$ is *isotone* if $\langle x, y\rangle \in \nu_1$ implies $\langle f(x), f(y)\rangle \in \nu_2$ for all $x, y \in H_1$. Now, we are in the position to state the main lemma of this subsection. By (6.18) and (7.3), our lattices are extensions of lattices of the form given in Figure 6.

So the parenthetical sentence above (6.11) explains what the distinguished elements are in the following lemma.

Lemma 7.2. *Let $\langle H_1;\nu_1\rangle$ and $\langle H_2;\nu_2\rangle$ be quasiordered sets, both with* 0 *and* 1 *such that* $0\neq 1$. *If $f\colon H_1\to H_2$ is an* injective *isotone map such that $f(Z(H_1))\subseteq Z(H_2)$ and $f(U(H_1))\subseteq U(H_2)$, then there exists a* unique *$\{0,1\}$-preserving lattice homomorphism $g\colon L(H_1,\nu_1)\to L(H_2,\nu_2)$ such that*

$$g(a_p)=a_{f(p)}\ \text{ and }\ g(b_p)=b_{f(p)},\ \text{ for all } p\in H_1, \tag{7.4}$$

and the g-image of the square-shaped gray-filled atom and coatom, see Figure 6, is the square-shaped gray-filled atom and coatom, respectively.

By (7.1), $0\in Z(H_i)$, $1\in U(H_i)$, and $Z(H_i)\cap U(H_i)=\varnothing$ hold for $i\in\{1,2\}$. The assumption of injectivity cannot be omitted from this lemma, because if f is a non-injective $\{0,1\}$-preserving homomorphism, then (7.4) yields that the kernel of g collapses some $a_p\neq a_q$, so this kernel is the largest congruence, contradicting $g(0)=0\neq 1=g(1)$.

Proof of Lemma 7.2. First, we deal with the uniqueness of g. Since $g(0)=0\neq 1=g(1)$, the kernel congruence $\ker(g)$ of g cannot collapse a thick (that is, a $U(H_1)$-colored) edge. Since all edges of $M_{4\times 3}$ are thick, the restriction $g\rceil_{M_{4\times 3}}$ of g to $M_{4\times 3}$ is injective. Since no other sublattice of L_2 than $M_{4\times 3}$ itself is isomorphic to $M_{4\times 3}$, it follows that $g(M_{4\times 3})$ is the unique $M_{4\times 3}$ sublattice of $L(H_2;\nu_2)$. Observe that except for the two doubly irreducible atoms and the two doubly irreducible coatoms, each element of $M_{4\times 3}$ is a fixed point of all automorphisms of $M_{4\times 3}$. Therefore, since g preserves the “square-shaped gray-filled” property, we conclude that $g\rceil_{M_{4\times 3}}$ is uniquely determined. The g-images of the a_p and b_p, $p\in H_1$, are determined by the assumption on g. Observe that an upper gadget $G_2^{\mathrm{up}}(p,q)$ has exactly two non-trivial congruences, $\mathrm{con}(a_p,b_p)$ and $\mathrm{con}(a_q,b_q)$; $G_1^{\mathrm{up}}(p,q)$ has only $\mathrm{con}(a_q,b_q)$, and $G_0^{\mathrm{up}}(p,q)$ has none. The same holds for lower gadgets. Therefore, since $\ker(g)$ cannot collapse a thick edge, it follows easily that the restriction of g to any gadget is uniquely determined. Therefore, g is unique.

In the rest of the proof, we intend to show the existence of g. We will define an appropriate g as the union of some partial maps. Let $g_{M_{4\times 3}}$ denote the unique isomorphism from the $M_{4\times 3}$ sublattice of $L(H_1,\nu_1)$ onto the $M_{4\times 3}$ sublattice of $L(H_2,\nu_2)$ such that $g_{M_{4\times 3}}$ preserves the “square-shaped gray-filled” property. For $i\in\{1,2\}$, we denote $\nu_i\setminus\{\langle x,x\rangle : x\in H_i\}$ by ν_i^+. Next, let $\langle p,q\rangle\in\nu_1^+$ and $j:=r(\langle p,q\rangle)$; according to (6.16) with $Z:=Z(H_1,\nu_1)$. By the construction of $L(H_1,\nu_1)$, see (6.18), (7.3), and Definition 6.2, the gadget $G_j^{\mathrm{up}}(p,q)$ is a $\{0,1\}$-sublattice of $L(H_1,\nu_1)$ from $\langle a_p,b_p\rangle$ to $\langle a_q,b_q\rangle$. Let $p'=f(p)$, $q'=f(q)$, and $j'=r(\langle p',q'\rangle)$. Besides that f is isotone, we frequently need the assumption that it is injective; at present, we conclude $\langle p',q'\rangle\in\nu_2^+$ from these assumptions. (Later, we will not always emphasize similar consequences of these assumptions.) It follows from $\langle p',q'\rangle\in\nu_2^+$ and the construction of $L(H_2,\nu_2)$ that $G^{\mathrm{up}}_{j'}(p',q')$ is a gadget in $L(H_2,\nu_2)$ from $\langle a_{p'},b_{p'}\rangle$ to $\langle a_{q'},b_{q'}\rangle$. We obtain from $f(Z(H_1))\subseteq Z(H_2)$ that

$$j'\leq j. \tag{7.5}$$

According to (6.2), we can take the unique surjective $\{0,1\}$-preserving lattice homomorphism $g^{\mathrm{up}}_{pq}\colon G^{\mathrm{up}}_{j}(p,q)\to G^{\mathrm{up}}_{j'}(p',q')$ such that $g^{\mathrm{up}}_{pq}(a_p)=a_{p'}$, $g^{\mathrm{up}}_{pq}(b_p)=b_{p'}$, $g^{\mathrm{up}}_{pq}(a_q)=a_{q'}$, and $g^{\mathrm{up}}_{pq}(b_q)=b_{q'}$. We take the $\{0,1\}$-preserving lattice homomorphism $g^{\mathrm{dn}}_{pq}\colon G^{\mathrm{dn}}_{j}(p,q)\to G^{\mathrm{dn}}_{j'}(p',q')$ analogously. Note that $g_{M_{4\times3}}$ maps $a'_1\in L(H_1,\nu_1)$ onto $a'_1\in L(H_2,\nu_2)$, and the same is true for b'_1. For $u\in U(H_1)$, we know that $f(u)\in U(H_2)$. By construction, there is an upper gadget of rank 2 from $\langle a'_1,b'_1\rangle$ to $\langle a_u,b_u\rangle$ in $L(H_1,\nu_1)$, and we have an upper gadget of rank 2 from $\langle a'_1,b'_1\rangle$ to $\langle a_{f(u)},b_{f(u)}\rangle$ in $L(H_2,\nu_2)$. The unique isomorphism from the first gadget to the second such that $a'_1\mapsto a'_1$, $b'_1\mapsto b'_1$, $a_u\mapsto a_{f(u)}$, and $b_u\mapsto b_{f(u)}$ is denoted by $g^{\mathrm{up}}_{1'u}$. Here $1'$ in the subscript is only a symbol, which does not belong to $H_1\cup H_2$. We define the isomorphism $g^{\mathrm{dn}}_{1'u}$ between the corresponding lower gadgets similarly. For $\langle p_1,q_1\rangle,\langle p_2,q_2\rangle\in\nu_1^+$ and $u\in U(H_1)$, any two of the homomorphisms $g_{M_{4\times3}}$, $g^{\mathrm{up}}_{p_1q_1}$, $g^{\mathrm{dn}}_{p_1q_1}$, $g^{\mathrm{up}}_{p_2q_2}$, $g^{\mathrm{dn}}_{p_2q_2}$, $g^{\mathrm{up}}_{1'u}$, and $g^{\mathrm{dn}}_{1'u}$ agree on the intersection of their domains. Therefore,

$$g:=g_{M_{4\times3}}\cup\bigcup_{\langle p,q\rangle\in\nu_1^+}g^{\mathrm{up}}_{pq}\ \cup\bigcup_{\langle p,q\rangle\in\nu_1^+}g^{\mathrm{dn}}_{pq}\ \cup\bigcup_{u\in U(H_1)}g^{\mathrm{up}}_{1'u}\ \cup\bigcup_{u\in U(H_1)}g^{\mathrm{dn}}_{1'u}$$

is a well-defined $\{0,1\}$-preserving map from $L(H_1,\nu_1)$ to $L(H_2,\nu_2)$.

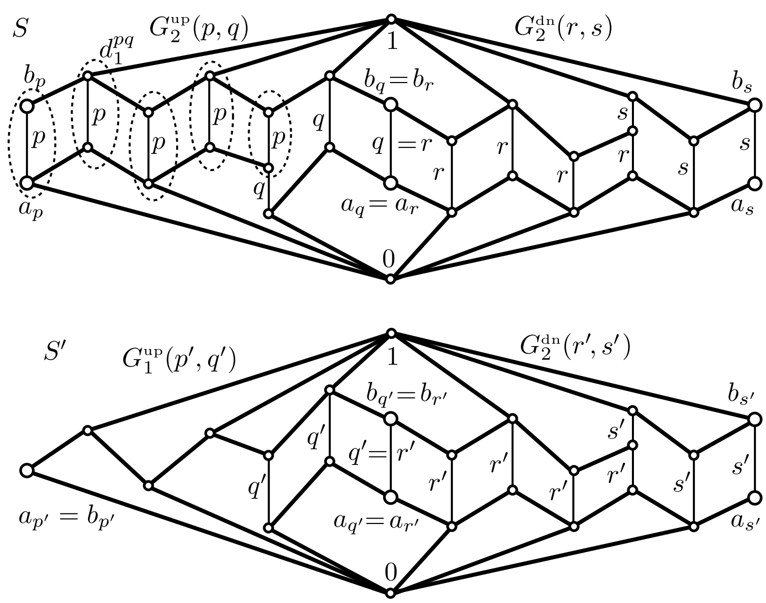

FIGURE 7. $\langle\mathrm{up},\mathrm{dn}\rangle$, $q=r$, and $\langle j,j',k,k'\rangle=\langle 2,1,2,2\rangle$

Next, we are going to show that, for all $x, y \in L(H_1; \nu_1)$,

$$g(x \vee y) = g(x) \vee g(y) \text{ and } g(x \wedge y) = g(x) \wedge g(y). \tag{7.6}$$

Clearly, we can assume that $\{x, y\} \cap M_{4\times 3} = \varnothing$ and no single gadget contains both x and y. Therefore, $\{0, 1\} \cap \{x, y\} = \varnothing$ and there are single gadgets $G^{\forall}_j(p, q)$ and $G^{\forall}_k(r, s)$ containing x and y, respectively. Of course, $p \neq q$ and $r \neq s$; however, we do not know more than $|\{p, q, r, s\}| \in \{2, 3, 4\}$. (It may even happen that $\langle r, s\rangle = \langle q, p\rangle$.) We can work in the union $S := G^{\forall}_j(p, q) \cup G^{\forall}_k(r, s)$, which is a sublattice by (6.6)–(6.9); see also the upper parts of Figures 7, 8, and 9. Alternatively, S is a sublattice by Lemma 6.1. Let $p' := f(p)$, $q' := f(q)$, $r' := r(p)$, $s' := f(s)$, and $S' := G^{\forall}_{j'}(p', q') \cup G^{\forall}_{k'}(r', s')$; see the lower parts of Figures 7, 8, and 9, where $g(x)$ is geometrically below x for every $x \in S$. Again, S' is a sublattice by (6.6)–(6.9). (Note that if $\langle p, q\rangle$ or $\langle r, s\rangle$ is of the form $\langle 1', u\rangle$ with $u \in U(H_1)$, then we have to extend f by $1' \mapsto 1'$, since $1' \notin H_1$.) Let $j := r(\langle p, q\rangle)$, $k := r(\langle r, s\rangle)$, $j' := r(\langle p', q'\rangle)$, and $k' := r(\langle r', s'\rangle)$.

We know from (7.5) that $j' \leq j$ and, similarly, $k' \leq k$. Hence, by the definition of our gadgets of rank less than 2, there are congruences $\boldsymbol{\alpha}_1$ and $\boldsymbol{\alpha}_2$ of $G^{\forall}_j(p, q)$ and $G^{\forall}_k(r, s)$ and surjective homomorphisms (namely, the natural projections) $g_1 \colon G^{\forall}_j(p, q) \to G^{\forall}_{j'}(p', q')$ and $g_2 \colon G^{\forall}_k(r, s) \to G^{\forall}_{k'}(r', s')$ such that $\boldsymbol{\alpha}_1$ is the kernel of g_1 and $\boldsymbol{\alpha}_2$ is the kernel of g_2. In Figures 7, 8, and 9, the nontrivial $\boldsymbol{\alpha}_1$-blocks and nontrivial $\boldsymbol{\alpha}_2$-blocks are indicated by dotted lines.

By the definition of g, $g_1 \cup g_2$ is the restriction $g\rceil_S$ of g to S. Thus, to verify (7.6), we need to show that $g_1 \cup g_2 \colon S \to S'$ is a homomorphism.

$$\text{It suffices to show that } \boldsymbol{\alpha}_1 \cup \boldsymbol{\alpha}_2 \text{ is a congruence of } S, \tag{7.7}$$

because then S' is the quotient lattice of S modulo $\boldsymbol{\alpha}_1 \cup \boldsymbol{\alpha}_2$ and $g_1 \cup g_2$ is the natural projection homomorphism of S to this quotient lattice. There are several cases but all of them can be settled similarly. We only discuss those given by Figures 7, 8, and 9. By Grätzer [11], each of these cases would be quite easy, although a bit tedious. However, to indicate that the rest of cases are similar, we give slightly more sophisticated arguments for them. Note that these figures also use the injectivity of f; for example, this is why $p' \neq s'$ and $q' \neq r'$ in Figure 8.

In case of Figure 7, let $H = \{0, p, q, r, s, 1\}$ and

$$\nu = \mathrm{quo}\big(\{\langle p, q\rangle, \langle q, r\rangle, \langle r, q\rangle, \langle r, s\rangle\} \cup (\{0\} \times H) \cup (H \times \{1\})\big).$$

(In general, the quasiordered set $\langle H; \nu\rangle$ is quite different from $\langle H_1; \nu_1\rangle$ and $\langle H_2; \nu_2\rangle$.) Using that S is a sublattice of the quasi-colored lattice $\mathcal{L}(H, \nu)$, see Lemma 6.4 and Definition 7.1, it is easy to see that $\boldsymbol{\alpha}_1 \cup \boldsymbol{\alpha}_2$ is a congruence of S. Namely, we can quite easily show that $\boldsymbol{\alpha}_1 \cup \boldsymbol{\alpha}_2 = \mathrm{con}_S(a_p, b_p)$. Clearly, $\mathrm{con}_S(a_p, b_p)$ collapses the p-colored edges. If it collapsed a t-colored edge for some $t \in \{q, r, s, 1\}$ in S, then it would collapse the same edge (with the same color) in $L(H, \nu)$, but then (C2) would give $t \leq_\nu p$, a contradiction.

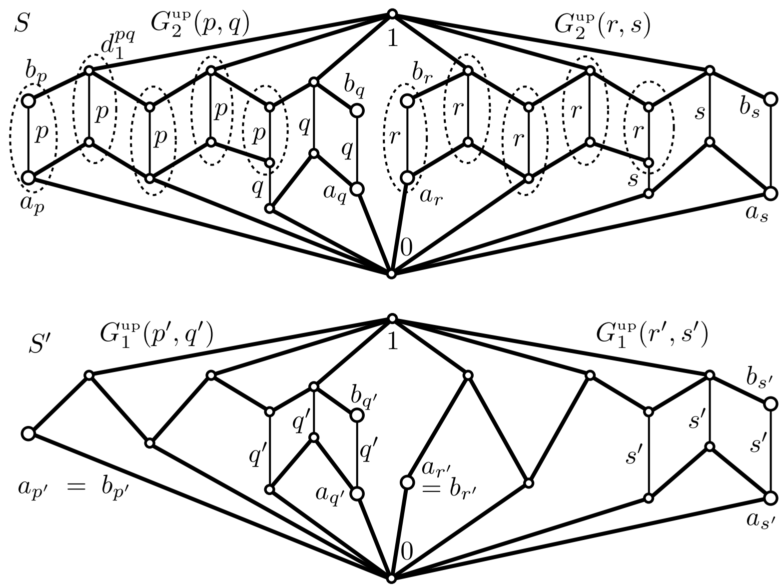

FIGURE 8. $\langle \mathrm{up}, \mathrm{up}\rangle$, $\{p, q, r, s\}| = 4$, and $\langle j, j', k, k'\rangle = \langle 2, 1, 2, 1\rangle$

In case of Figure 8, let $\langle H; \nu\rangle$ be the six element lattice in which there are exactly two maximal chains, $\{0 \prec p \prec q \prec 1\}$ and $\{0 \prec r \prec s \prec 1\}$. The same argument as above shows that $\mathrm{con}_S(a_p, b_p)$ collapses the p-colored edges and only those, while $\mathrm{con}_S(a_r, b_r)$ collapses exactly the r-colored edges. In order to see that $\boldsymbol{\alpha}_1 \cup \boldsymbol{\alpha}_2$ is a congruence, it suffices to show that $\boldsymbol{\alpha}_1 \cup \boldsymbol{\alpha}_2 = \mathrm{con}_S(a_p, b_p) \vee \mathrm{con}_S(a_r, b_r)$. Clearly, $\boldsymbol{\alpha}_1 \cup \boldsymbol{\alpha}_2 \subseteq \mathrm{con}_S(a_p, b_p) \vee \mathrm{con}_S(a_r, b_r)$. Assume that $\langle x, y\rangle \in \mathrm{Pairs}^{\prec}(S)$ such that $\langle x, y\rangle \in \mathrm{con}_S(a_p, b_p) \vee \mathrm{con}_S(a_r, b_r)$. In other words, $\mathrm{con}_S(x, y) \leq \mathrm{con}_S(a_p, b_p) \vee \mathrm{con}_S(a_r, b_r)$. Since a covering pair of a lattice always generates a join-irreducible congruence and the congruence lattice of a lattice is distributive, it follows that $\mathrm{con}_S(x, y) \leq \mathrm{con}_S(a_p, b_p)$ or $\mathrm{con}_S(x, y) \leq \mathrm{con}_S(a_r, b_r)$. Hence, $\langle x, y\rangle \in \boldsymbol{\alpha}_1$ or $\langle x, y\rangle \in \boldsymbol{\alpha}_2$, and we obtain the required inclusion, $\boldsymbol{\alpha}_1 \cup \boldsymbol{\alpha}_2 \supseteq \mathrm{con}_S(a_p, b_p) \vee \mathrm{con}_S(a_r, b_r)$.

For Figure 9, we use the same $\langle H; \nu\rangle$ as for Figure 7 and, practically, the same argument as for Figure 8 to show that $\boldsymbol{\alpha}_1 \cup \boldsymbol{\alpha}_2 = \mathrm{con}_S(a_r, b_r)$. By (7.7), this completes the proof of Lemma 7.2. $\square$

8. COMPLETING THE LATTICE THEORETICAL PART

For a quasiordered set $\langle H, \nu\rangle$, we let $\Theta_\nu = \nu \cap \nu^{-1}$. It is known that Θ_ν is an equivalence relation, and the definition

$$\langle x/\Theta_\nu, y/\Theta_\nu\rangle \in \nu/\Theta_\nu \overset{\mathrm{def}}{\iff} \langle x, y\rangle \in \nu \tag{8.1}$$

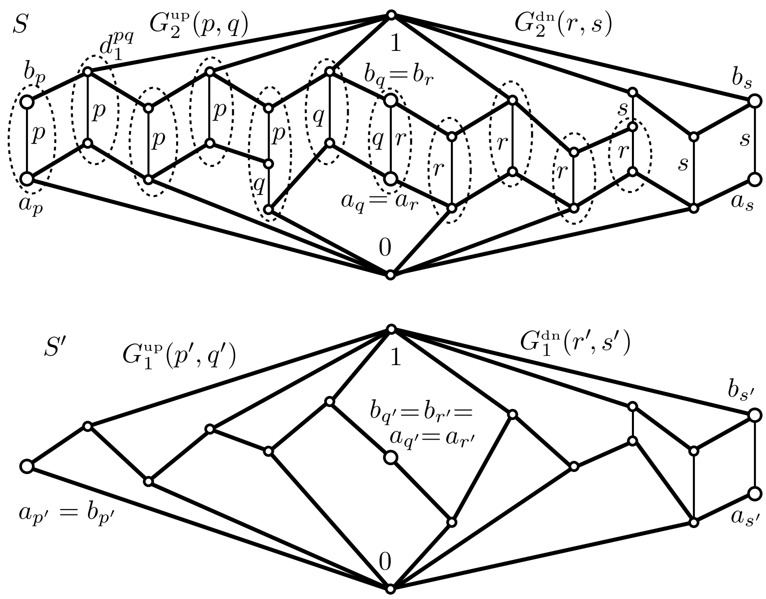

FIGURE 9. $\langle \mathrm{up}, \mathrm{dn}\rangle$, $q = r$, and $\langle j, j', k, k'\rangle = \langle 2, 0, 2, 1\rangle$

turns the quotient set H/Θ_ν into an ordered set $\langle H; \nu\rangle/\Theta_\nu$, which is also denoted by $\langle H/\Theta_\nu; \nu/\Theta_\nu\rangle$. The following lemma is a straightforward consequence of (C1) and (C2), see [2, Lemma 3.1], [3, Lemma 2.1], or [4, Lemma 4.7], where the inverse isomorphism is considered. Although the lemma was only formulated for the particular quasi-colored lattices constructed in these papers, its easy proof makes it valid for every quasi-colored lattice, so it is time to formulate it more generally.

Lemma 8.1. *For every quasi-colored lattice $\langle L, \leq; \gamma; H, \nu\rangle$,* $\mathrm{Princ}(L)$ *is isomorphic to $\langle H; \nu\rangle/\Theta_\nu$ and the map $\langle \mathrm{Princ}(L); \subseteq\rangle \to \langle H; \nu\rangle/\Theta_\nu$, defined by $\mathrm{con}(x, y) \mapsto \gamma(\langle x, y\rangle)/\Theta_\nu$, is an order isomorphism.*

As a consequence of this lemma and our construction, or (the proof of) [4, Lemma 4.7], we obtain the following corollary.

Corollary 8.2. *If $\langle H; \nu\rangle$ is a quasiordered set satisfying* (7.2)*, then the map*

$$\zeta_{H,\nu} \colon \langle H; \nu\rangle/\Theta_\nu \to \langle \mathrm{Princ}(L(H, \nu)); \subseteq\rangle$$

defined by $p/\Theta_\nu \mapsto \mathrm{con}(a_p, b_p)$ is an order isomorphism.

Proof of Theorem 4.7. Let $F_{\mathrm{pos}} \colon \mathbf{A} \to \mathbf{Pos}^+_{01}$ be a faithful functor as in the theorem, and let

$$\mathbf{B} := F_{\mathrm{pos}}(\mathbf{A}).$$

For $X \in \mathrm{Ob}(\mathbf{A})$ and $f \in \mathrm{Mor}(\mathbf{A})$, $F_{\mathrm{pos}}(X)$ is an ordered set and $F_{\mathrm{pos}}(f)$ is an isotone map; we will use the notation

$$\langle \overline{X}; \leq_X\rangle := F_{\mathrm{pos}}(X) \text{ and } \overline{f} := F_{\mathrm{pos}}(f).$$

In $\mathbf{B}$, two ordered sets with the same underlying set but different orderings are two distinct objects. Since we do not want to identify distinct objects when we forget their orderings, we index the underlying sets as follows. For $\langle Y;\nu\rangle \in \mathrm{Ob}(\mathbf{B})$, we let $G'_{\mathrm{forg}}(\langle Y;\nu\rangle) := Y \times \{\nu\}$. For $g \in \mathrm{Mor}(\langle Y_1;\nu_1\rangle, \langle Y_2;\nu_2\rangle)$, we let

$$g' = G'_{\mathrm{forg}}(g)\colon Y_1 \times \{\nu_1\} \to Y_2 \times \{\nu_2\}, \text{ defined by } \langle u,\nu_1\rangle \mapsto \langle g(u),\nu_2\rangle.$$

In this way, we have defined a totally faithful functor $G'_{\mathrm{forg}}\colon \mathbf{B} \to \mathbf{Set}$; the subscript comes from "forgetful" and the prime reminds us that G'_{forg} is slightly different from the forgetful functor G_{forg}. For $\langle X;\nu\rangle \in \mathrm{Ob}(\mathbf{B})$ and $u \in X$, if ν is understood, we often write X' and u' instead of $X \times \{\nu\}$ and $\langle u,\nu\rangle$. With this abbreviation, $g' = G'_{\mathrm{forg}}(g)\colon Y'_1 \to Y'_2$ is defined by $u' \mapsto (g(u))'$. Hence, for $X, Y \in \mathbf{A}$, $f \in \mathrm{Mor}_{\mathbf{A}}(X,Y)$, and $u \in \overline{X}$,

$$\begin{aligned} &\overline{X}' = \overline{X} \times \{\leq_X\} = G'_{\mathrm{forg}}(F_{\mathrm{pos}}(X)), \ \ u' = \langle u, \leq_X\rangle \in \overline{X}', \\ &\overline{f}' = G'_{\mathrm{forg}}(F_{\mathrm{pos}}(f))\colon \overline{X}' \to \overline{Y}', \ \text{ and } \ \overline{f}'(u') = (\overline{f}(u))' = \langle \overline{f}(u), \leq_Y\rangle. \end{aligned} \tag{8.2}$$

The image

$$\mathbf{C} := G'_{\mathrm{forg}}(\mathbf{B}) = (G'_{\mathrm{forg}} \circ F_{\mathrm{pos}})(\mathbf{A})$$

is a small concrete category, a subcategory of $\mathbf{Set}$; its objects and morphisms are the $\overline{X}'$ for $X \in \mathrm{Ob}(\mathbf{A})$ and the $\overline{f}'$ for $f \in \mathrm{Mor}(\mathbf{A})$, respectively, as described in (8.2). We claim that

$$\text{all morphisms of } \mathbf{C} \text{ are monomorphisms.} \tag{8.3}$$

Since F_{pos} is assumed to be faithful and G'_{forg} is obviously faithful, (8.3) will follow from the following trivial observation.

$$\begin{gathered} \text{If } F\colon \mathbf{U} \to \mathbf{V} \text{ is a faithful functor, } \mathbf{V} = F(\mathbf{U}), \\ \text{and } f_1 \in \mathrm{Mor}(\mathbf{U}) \text{ is a monomorphism, then} \\ F(f_1) \text{ is a monomorphism in } \mathbf{V}. \end{gathered} \tag{8.4}$$

In order to show this, assume that $f_1 \in \mathrm{Mor}_{\mathbf{U}}(X,Y)$ is a monomorphism and $f_2^*, f_3^* \in \mathrm{Mor}_{\mathbf{V}}(Z^*, F(X))$ such that $F(f_1) \circ f_2^* = F(f_1) \circ f_3^*$. Since $\mathbf{V}$ is the F-image of $\mathbf{U}$, there exist $Z \in \mathrm{Ob}(\mathbf{U})$ and $f_2, f_3 \in \mathrm{Mor}_{\mathbf{U}}(Z,X)$ such that $Z^* = F(Z)$, $f_2^* = F(f_2)$, and $f_3^* = F(f_3)$. Since $F(f_1 \circ f_2) = F(f_1) \circ F(f_2) = F(f_1) \circ f_2^* = F(f_1) \circ f_3^* = F(f_1) \circ F(f_3) = F(f_1 \circ f_3)$ and F is faithful, $f_1 \circ f_2 = f_1 \circ f_3$. Using that f_1 is a monomorphism in $\mathbf{U}$, we obtain that $f_2 = f_3$. Hence, $f_2^* = F(f_2) = F(f_3) = f_3^*$, showing that $F(f_1)$ is a monomorphism. This proves (8.4) and, consequently, (8.3).

Although $\overline{X}' = G'_{\mathrm{forg}}(\langle \overline{X}; \leq_X\rangle) = G'_{\mathrm{forg}}(F_{\mathrm{pos}}(X))$ is only a set for $X \in \mathrm{Ob}(\mathbf{A})$, we shall use the ordering $\leq'_X$ induced by $\leq_X$ on it as follows: for $x, y \in \overline{X}$,

$$\langle x, \leq_X\rangle \leq'_X \langle y, \leq_X\rangle \overset{\mathrm{def}}{\iff} x \leq_X y, \text{ that is, } x' \leq'_X y' \overset{\mathrm{def}}{\iff} x \leq_X y. \tag{8.5}$$

The least element and the largest element of $\langle \overline{X}; \leq_X\rangle = F_{\mathrm{pos}}(X)$ will be denoted by $0'_X = \langle 0_{\langle \overline{X};\leq_X\rangle}, \leq_X\rangle$ and $1'_X = \langle 1_{\langle \overline{X};\leq_X\rangle}, \leq_X\rangle$. By (8.5),

$$0'_X \text{ resp. } 1'_X \text{ are the least resp. greatest element of } \langle \overline{X}', \leq'_X\rangle. \tag{8.6}$$

Next, denoting the cometic functor $F^{\mathbf{C}}_{\mathrm{com}}$ by F_{com}, see Definition 3.2, we let

$$\mathbf{D} := F^{\mathbf{C}}_{\mathrm{com}}(\mathbf{C}).$$

By (8.3) and Theorem 3.6,

$$\text{all morphisms of } \mathbf{D} \text{ are injective maps;} \tag{8.7}$$

this is why we can apply Lemma 7.2 soon. Since we have three functors already, it is worth defining their composite,

$$G_{\mathrm{prod}} := F_{\mathrm{com}} \circ G'_{\mathrm{forg}} \circ F_{\mathrm{pos}}, \quad \text{from } \mathbf{A} \text{ to } \mathbf{D}.$$

For $X \in \mathrm{Ob}(\mathbf{A})$, the cometic projection from Definition 3.4 allows us to define a relation ν_X on the set $G_{\mathrm{prod}}(X) = F_{\mathrm{com}}(\overline{X}')$, as follows: for eligible triplets $c_1, c_2 \in G_{\mathrm{prod}}(X) = F_{\mathrm{com}}(\overline{X}')$,

$$\langle c_1, c_2\rangle \in \nu_X \overset{\text{def}}{\iff} \boldsymbol{\pi}^{\mathrm{com}}_{\overline{X}'}(c_1) \leq'_X \boldsymbol{\pi}^{\mathrm{com}}_{\overline{X}'}(c_1). \tag{8.8}$$

Clearly, ν_X is a quasiorder. The set of least elements of $\langle G_{\mathrm{prod}}(X); \nu_X\rangle$ will be denoted by $Z(G_{\mathrm{prod}}(X))$. Similarly, $U(G_{\mathrm{prod}}(X))$ will stand for the set of largest elements. (8.6) and (8.8) make it clear that

$$\begin{aligned} Z(G_{\mathrm{prod}}(X)) &= \{c \in G_{\mathrm{prod}}(X) : \boldsymbol{\pi}^{\mathrm{com}}_{\overline{X}'}(c) = 0'_X\}, \text{ and} \\ U(G_{\mathrm{prod}}(X)) &= \{c \in G_{\mathrm{prod}}(X) : \boldsymbol{\pi}^{\mathrm{com}}_{\overline{X}'}(c) = 1'_X\}. \end{aligned} \tag{8.9}$$

It also follows from (8.8) that these sets are nonempty, because

$$\begin{aligned} \vec{v}^{\,\mathrm{triv}}(0') &= \langle \mathbf{1}_{\overline{X}'}, 0'_X, 0'_X\rangle \in Z(G_{\mathrm{prod}}(X)), \text{ and} \\ \vec{v}^{\,\mathrm{triv}}(1') &= \langle \mathbf{1}_{\overline{X}'}, 1'_X, 1'_X\rangle \in U(G_{\mathrm{prod}}(X)). \end{aligned}$$

Note the notational difference: $\mathbf{1}_{\overline{X}'}$ is the identity morphism on the set $\overline{X}'$, which is the support set of $\langle \overline{X}; \leq_X\rangle$, while $1'_X$ is the top element of the ordered set $\langle \overline{X}'; \leq'_X\rangle$, which is isomorphic to $\langle \overline{X}; \leq_X\rangle = F_{\mathrm{pos}}(X)$. Since the ordered set $F_{\mathrm{pos}}(X) \in \mathbf{Pos}^{+}_{01}$ consists of at least two elements, we obtain that $0'_X \neq 1'_X$ and the distinguished eligible triplets

$$\vec{v}^{\,\mathrm{triv}}(0'_X) \in Z(G_{\mathrm{prod}}(X)) \text{ and } \vec{v}^{\,\mathrm{triv}}(1'_X) \in U(G_{\mathrm{prod}}(X)) \text{ are distinct.} \tag{8.10}$$

Hence, for $X \in \mathrm{Ob}(\mathbf{A})$, Definition 7.1 allows us to consider the quasi-colored lattice

$$\begin{aligned} &\mathcal{L}(G_{\mathrm{prod}}(X), \nu_X) = \\ &\quad \langle L(G_{\mathrm{prod}}(X), \nu_X), \lambda_{G_{\mathrm{prod}}(X),\nu_X}; \gamma_{G_{\mathrm{prod}}(X),\nu_X}; G_{\mathrm{prod}}(X), \nu_X\rangle. \end{aligned} \tag{8.11}$$

We are going to turn the assignment given in (8.11) functorial. For f in $\mathrm{Mor}(\mathbf{A})$, $\overline{f}' = (G'_{\mathrm{forg}} \circ F_{\mathrm{pos}})(f)$ and $G_{\mathrm{prod}}(f)$ are only maps between two sets. However, (8.5) and (8.8), respectively, allow us to guess that these maps are isotone; these properties are conveniently formulated in the form (8.12) below and (8.13) later. We claim that for $X, Y \in \mathrm{Ob}(\mathbf{A})$ and $f \in \mathrm{Mor}_{\mathbf{A}}(X, Y)$,

$$\overline{f}' = (G'_{\mathrm{forg}} \circ F_{\mathrm{pos}})(f) \colon \langle \overline{X}'; \leq'_X\rangle \to \langle \overline{Y}'; \leq'_Y\rangle \text{ is an isotone map.} \tag{8.12}$$

In order to show this, assume that $x_1, x_2 \in \overline{X}$ such that $x_1' \le_X' x_2'$. By (8.5), $x_1 \le_X x_2$. Since $\overline{f} = F_{\text{pos}}(f)$ is an isotone map, $\overline{f}(x_1) \le_Y \overline{f}(x_2)$. Hence, by (8.5) again, $(\overline{f}(x_1))' \le_Y' (\overline{f}(x_2))'$. Thus, applying (8.2),

$$\overline{f}'(x_1') = (\overline{f}(x_1))' \le_Y' (\overline{f}(x_2))' = \overline{f}'(x_2'),$$

which proves (8.12). Next, we are going to show that for $X, Y \in \mathrm{Ob}(\mathbf{A})$ and $f \in \mathrm{Mor}_{\mathbf{A}}(X, Y)$,

$$G_{\text{prod}}(f) \colon \langle G_{\text{prod}}(X); \nu_X \rangle \to \langle G_{\text{prod}}(Y); \nu_Y \rangle \text{ is an isotone map.} \tag{8.13}$$

So let $X, Y \in \mathrm{Ob}(\mathbf{A})$ and $f \in \mathrm{Mor}_{\mathbf{A}}(X, Y)$. Since $\boldsymbol{\pi}^{\text{com}}$ is a natural transformation by Theorem 3.6 and $\mathbf{C}$, which is the domain of $F_{\text{com}} = F_{\text{com}}^{\mathbf{C}}$, is a subcategory of $\mathbf{Set}$, the diagram

$$\begin{array}{ccc} G_{\text{prod}}(X) = F_{\text{com}}(\overline{X}') & \xrightarrow{G_{\text{prod}}(f)} & G_{\text{prod}}(Y) = F_{\text{com}}(\overline{Y}') \\ \boldsymbol{\pi}^{\text{com}}_{\overline{X}'} \Big\downarrow & & \boldsymbol{\pi}^{\text{com}}_{\overline{Y}'} \Big\downarrow \\ \overline{X}' & \xrightarrow{\overline{f}'} & \overline{Y}' \end{array} \tag{8.14}$$

commutes. That is, for every eligible triplet $c \in G_{\text{prod}}(X)$,

$$\boldsymbol{\pi}^{\text{com}}_{\overline{Y}'}(G_{\text{prod}}(f)(c)) = \overline{f}'(\boldsymbol{\pi}^{\text{com}}_{\overline{X}'}(c)). \tag{8.15}$$

Assume that $\langle c_1, c_2 \rangle \in \nu_X$. By (8.8), $\boldsymbol{\pi}^{\text{com}}_{\overline{X}'}(c_1) \le_X' \boldsymbol{\pi}^{\text{com}}_{\overline{X}'}(c_2)$. By (8.12), this gives that $\overline{f}'(\boldsymbol{\pi}^{\text{com}}_{\overline{X}'}(c_1)) \le_Y' \overline{f}'(\boldsymbol{\pi}^{\text{com}}_{\overline{X}'}(c_2))$. Combining this inequality with (8.8) and (8.15), we obtain that $\langle G_{\text{prod}}(f)(c_1), G_{\text{prod}}(f)(c_2) \rangle \in \nu_Y$, proving (8.13).

Our next task is to show that, for every $f \in \mathrm{Mor}_{\mathbf{A}}(X, Y)$,

$$\begin{aligned} &G_{\text{prod}}(f)(Z(G_{\text{prod}}(X))) \subseteq Z(G_{\text{prod}}(Y)) \text{ and} \\ &G_{\text{prod}}(f)(U(G_{\text{prod}}(X))) \subseteq U(G_{\text{prod}}(Y)). \end{aligned} \tag{8.16}$$

Let $c \in Z(G_{\text{prod}}(X))$. By (8.9), $\boldsymbol{\pi}^{\text{com}}_{\overline{X}'}(c) = 0_X'$. Since $\overline{f} = F_{\text{pos}}(f)$ belongs to $\mathrm{Mor}(\mathbf{B}) \subseteq \mathrm{Mor}(\mathbf{Pos}^{+}_{01})$, $\overline{f}$ is 0-preserving. Hence, by (8.2) and (8.15),

$$\boldsymbol{\pi}^{\text{com}}_{\overline{Y}'}(G_{\text{prod}}(f)(c)) = \overline{f}'(\boldsymbol{\pi}^{\text{com}}_{\overline{X}'}(c)) = \overline{f}'(0_X') = (\overline{f}(0_{\langle \overline{X}, \le_X \rangle}))' = (0_{\langle \overline{Y}, \le_Y \rangle})' = 0_Y'.$$

By (8.9), this means that $G_{\text{prod}}(f)(c) \in Z(G_{\text{prod}}(Y))$. This proves the first half of (8.16); the second half follows in the same way.

Now, we are in the position to define a functor $E_{\text{Lift}} \colon \mathbf{A} \to \mathbf{Lat}^{\text{sd}}_5$ as follows. For $X \in \mathrm{Ob}(\mathbf{A})$ and $f \in \mathrm{Mor}_{\mathbf{A}}(X, Y) \subseteq \mathrm{Mor}(\mathbf{A})$, we let

$$\begin{aligned} &E_{\text{Lift}}(X) := L(G_{\text{prod}}(X); \nu_X), \text{ see (8.11)}, \\ &E_{\text{Lift}}(f) := \text{the unique } \{0,1\}\text{-preserving lattice homomorphism} \\ &\qquad\qquad \text{that Lemma 7.2 associates with } G_{\text{prod}}(f); \end{aligned} \tag{8.17}$$

it follows from (8.7), (8.10), (8.13), and (8.16) that Lemma 7.2 is applicable. We are going to show that E_{Lift} is a functor from $\mathbf{A}$ to $\mathbf{Lat}^{\text{sd}}_5$. By Lemma 6.4, Definition 7.1, and (8.11), we have that $E_{\text{Lift}}(X) \in \mathrm{Ob}(\mathbf{Lat}^{\text{sd}}_5)$. By Lemma 7.2, $G_{\text{prod}}(f) \in \mathrm{Mor}(\mathbf{Lat}^{\text{sd}}_5)$. If $f = 1_X \in \mathrm{Mor}_{\mathbf{A}}(X, X)$, then

$G_{\mathrm{prod}}(f)$ is the identity map since G_{prod} is a functor, and it follows from (7.4) and the uniqueness part of Lemma 7.2 that $E_{\mathrm{Lift}}(f)$ is the identity map $1_{E_{\mathrm{Lift}}(X)}$. Finally, assume that $X, Y, Z \in \mathrm{Ob}(\mathbf{A})$, $f_1 \in \mathrm{Mor}_{\mathbf{A}}(Y, Z)$, and $f_2 \in \mathrm{Mor}_{\mathbf{A}}(X, Y)$. We have to show that $E_{\mathrm{Lift}}(f_1 \circ f_2) = E_{\mathrm{Lift}}(f_1) \circ E_{\mathrm{Lift}}(f_2)$. By (7.4) and the uniqueness part of Lemma 7.2, it suffices to show that

$$E_{\mathrm{Lift}}(f_1 \circ f_2)(a_p) = (E_{\mathrm{Lift}}(f_1) \circ E_{\mathrm{Lift}}(f_2))(a_p) \tag{8.18}$$

for all eligible triplets $p \in G_{\mathrm{prod}}(X)$, and similarly for b_p. It suffices to deal with a_p. By (7.4) and (8.17), we have the following rule of computation:

$$E_{\mathrm{Lift}}(f)(a_p) = a_{G_{\mathrm{prod}}(f)(p)}. \tag{8.19}$$

We know that G_{prod}, as a composite of three functors, is a functor. Therefore, $G_{\mathrm{prod}}(f_1 \circ f_2) = G_{\mathrm{prod}}(f_1) \circ G_{\mathrm{prod}}(f_2)$. Using this equality and (8.19), we have

$$\begin{aligned} E_{\mathrm{Lift}}(f_1 \circ f_2)(a_p) &= a_{G_{\mathrm{prod}}(f_1 \circ f_2)(p)} = a_{(G_{\mathrm{prod}}(f_1) \circ G_{\mathrm{prod}}(f_2))(p)} \\ &= a_{G_{\mathrm{prod}}(f_1)(G_{\mathrm{prod}}(f_2)(p))} = E_{\mathrm{Lift}}(f_1)(a_{G_{\mathrm{prod}}(f_2)(p)}) \\ &= E_{\mathrm{Lift}}(f_1)(E_{\mathrm{Lift}}(f_2)(a_p)) = (E_{\mathrm{Lift}}(f_1) \circ E_{\mathrm{Lift}}(f_2))(a_p). \end{aligned}$$

Thus, (8.18) holds, and $E_{\mathrm{Lift}} \colon \mathbf{A} \to \mathbf{Lat}_5^{\mathrm{sd}}$ is a functor, as required.

Clearly, the composite of faithful or totally faithful functors is a faithful or totally faithful functor, respectively. By Theorem 3.6, F_{com} is totally faithful. So is G'_{forg}. Therefore, $G_{\mathrm{prod}} = F_{\mathrm{com}} \circ G'_{\mathrm{forg}} \circ F_{\mathrm{pos}}$ is faithful, and it is totally faithful if so is F_{pos}. Hence, it follows from (8.19) that E_{Lift} is faithful. Furthermore, if F_{pos} is totally faithful and $X \neq Y \in \mathrm{Ob}(\mathbf{A})$, then the same property of G_{prod} gives that $\{a_p : p \in G_{\mathrm{prod}}(X)\}$ is distinct from $\{a_p : p \in G_{\mathrm{prod}}(Y)\}$. Hence, it follows from Remark 6.3 and (8.17) that $E_{\mathrm{Lift}}(X) \neq E_{\mathrm{Lift}}(Y)$. Consequently, E_{Lift} is totally faithful if so is F_{pos}.

Finally, we are going to prove that E_{Lift} lifts F_{pos} with respect to Princ. The isomorphism provided by Corollary 8.2 will be denoted by ζ_X. That is,

$$\begin{aligned} \zeta_X \colon \langle G_{\mathrm{prod}}(X); \nu_X \rangle / \Theta_{\nu_X} &\to \langle \mathrm{Princ}(L(G_{\mathrm{prod}}(X), \nu_X)); \subseteq \rangle \\ &\overset{(8.17)}{=} (\mathrm{Princ} \circ E_{\mathrm{Lift}})(X), \\ \text{defined by } q/\Theta_{\nu_X} &\mapsto \mathrm{con}(a_q, b_q), \end{aligned} \tag{8.20}$$

is an order isomorphism. For $X \in \mathrm{Ob}(\mathbf{A})$, the map from $\langle G_{\mathrm{prod}}(X); \nu_X \rangle = \langle F_{\mathrm{com}}(\overline{X}'); \nu_X \rangle$ to $\langle \overline{X}'; \leq'_X \rangle$, defined by $q \mapsto \boldsymbol{\pi}_X^{\mathrm{com}}(q)$, see Definition 3.1 or around (8.14), is isotone by (8.8). By Theorem 3.6 (or Lemma 3.5), this map is surjective. Furthermore, for $p', q' \in \overline{X}'$ (that is, for $p, q \in \overline{X}$), if $p' \leq'_X q'$, then $\langle \vec{v}^{\mathrm{triv}}(p'), \vec{v}^{\mathrm{triv}}(q') \rangle \in \nu_X$ by (8.8) and, in addition, $p' = \boldsymbol{\pi}_X^{\mathrm{com}}(\vec{v}^{\mathrm{triv}}(p'))$ and $q' = \boldsymbol{\pi}_X^{\mathrm{com}}(\vec{v}^{\mathrm{triv}}(q'))$. So, the ordering $\leq'_X$ equals the $\boldsymbol{\pi}_X^{\mathrm{com}}$-image of ν_X. Thus, using a well-known fact about orders induced by quasiorders, the map

$$\langle G_{\mathrm{prod}}(X); \nu_X \rangle / \Theta_{\nu_X} \to \langle \overline{X}'; \leq'_X \rangle, \text{ defined by } q/\Theta_{\nu_X} \mapsto \boldsymbol{\pi}_{\overline{X}'}^{\mathrm{com}}(q),$$

is an order isomorphism. So is its inverse map,

$$\langle \overline{X}'; \leq'_X \rangle \to \langle G_{\mathrm{prod}}(X); \nu_X \rangle / \Theta_{\nu_X} \text{ defined by } p' \mapsto \vec{v}^{\mathrm{triv}}(p')/\Theta_{\nu_X}.$$

Since $\langle \overline{X}; \leq_X\rangle \to \langle \overline{X}'; \leq'_X\rangle$, defined by $x \mapsto x' = \langle x, \leq_X\rangle$, is also an order isomorphism by (8.5), the composite

$$\xi_X : \langle \overline{X}; \leq_X\rangle \to \langle G_{\mathrm{prod}}(X); \nu_X\rangle/\Theta_{\nu_X}, \text{ defined by } p \mapsto \vec{v}^{\,\mathrm{triv}}(p')/\Theta_{\nu_X}, \tag{8.21}$$

of the two isomorphisms is also an order isomorphism. So we can let

$$\boldsymbol{\kappa}_X := \zeta_X \circ \xi_X, \text{ which is an order isomorphism} \tag{8.22}$$

from $F_{\mathrm{pos}}(X) = \langle \overline{X}; \leq_X\rangle$ to $(\mathrm{Princ} \circ E_{\mathrm{Lift}})(X)$ by (8.20) and (8.21). As the last part of the proof, we are going to show that $\boldsymbol{\kappa} : F_{\mathrm{pos}} \to \mathrm{Princ} \circ E_{\mathrm{Lift}}$ is a natural isomorphism. By (8.22), we have to show only that it is a natural transformation. In order to do so, assume that $X, Y \in \mathrm{Ob}(\mathbf{A})$ and $f \in \mathrm{Mor}_{\mathbf{A}}(X, Y)$. Besides $\overline{f} = F_{\mathrm{pos}}(f)$ and $\overline{f}' = (G'_{\mathrm{forg}} \circ F_{\mathrm{pos}})(f)$, we will use the notation $h := (\mathrm{Princ} \circ E_{\mathrm{Lift}})(f)$. We have to show that the diagram

$$\begin{array}{ccc} F_{\mathrm{pos}}(X) = \langle \overline{X}; \leq_X\rangle & \xrightarrow{\overline{f} = F_{\mathrm{pos}}(f)} & F_{\mathrm{pos}}(Y) = \langle \overline{X}; \leq_Y\rangle \\ {\scriptstyle \boldsymbol{\kappa}_X}\Big\downarrow & & {\scriptstyle \boldsymbol{\kappa}_Y}\Big\downarrow \\ (\mathrm{Princ} \circ E_{\mathrm{Lift}})(X) & \xrightarrow{h = (\mathrm{Princ} \circ E_{\mathrm{Lift}})(f)} & (\mathrm{Princ} \circ E_{\mathrm{Lift}})(Y) \end{array} \tag{8.23}$$

commutes. First, we investigate the map h. For a triplet $q \in G_{\mathrm{prod}}(X)$, we have that $E_{\mathrm{Lift}}(f)(a_q) = a_{G_{\mathrm{prod}}(f)(q)}$ by (8.19). Analogously, $E_{\mathrm{Lift}}(f)(b_q) = b_{G_{\mathrm{prod}}(f)(q)}$. Therefore, applying the definition of Princ to the $\{0, 1\}$-lattice homomorphism $E_{\mathrm{Lift}}(f) : E_{\mathrm{Lift}}(X) \to E_{\mathrm{Lift}}(Y)$, see (4.1) and (4.2), we have that

$$h(\mathrm{con}(a_q, b_q)) = \mathrm{con}(a_{G_{\mathrm{prod}}(f)(q)}, b_{G_{\mathrm{prod}}(f)(q)}). \tag{8.24}$$

Consider an arbitrary $p \in F_{\mathrm{pos}}(X)$. By (8.20), (8.21), and (8.22),

$$\boldsymbol{\kappa}_X(p) = \zeta_X(\xi_X(p)) = \zeta_X(\vec{v}^{\,\mathrm{triv}}(p')/\Theta_{\nu_X}) = \mathrm{con}(a_{\vec{v}^{\,\mathrm{triv}}(p')}, b_{\vec{v}^{\,\mathrm{triv}}(p')}). \tag{8.25}$$

Hence, (8.24) yields that

$$(h \circ \boldsymbol{\kappa}_X)(p) = \mathrm{con}(a_{G_{\mathrm{prod}}(f)(\vec{v}^{\,\mathrm{triv}}(p'))}, b_{G_{\mathrm{prod}}(f)(\vec{v}^{\,\mathrm{triv}}(p'))}). \tag{8.26}$$

On the other hand, using (8.25) for Y and $\overline{f}(p)$ instead of X and p,

$$(\boldsymbol{\kappa}_Y \circ \overline{f})(p) = \boldsymbol{\kappa}_Y(\overline{f}(p)) = \mathrm{con}(a_{\vec{v}^{\,\mathrm{triv}}(\overline{f}(p)')}, b_{\vec{v}^{\,\mathrm{triv}}(\overline{f}(p)')}). \tag{8.27}$$

We are going to verify that (8.26) and (8.27) give the same principal congruence. Motivated by (C1), we focus on the colors of the respective ordered pairs that generate these two principal congruences. By the construction of our quasi-colored lattices, see Figure 6 and (6.19), these colors are $c_1 := G_{\mathrm{prod}}(f)(\vec{v}^{\,\mathrm{triv}}(p'))$, in (8.26), and $c_2 := \vec{v}^{\,\mathrm{triv}}(\overline{f}(p)')$, in (8.27). By (3.1), (8.2), and (8.15),

$$\begin{aligned} \boldsymbol{\pi}^{\mathrm{com}}_{\overline{Y}'}(c_1) &= \boldsymbol{\pi}^{\mathrm{com}}_{\overline{Y}'}(G_{\mathrm{prod}}(f)(\vec{v}^{\,\mathrm{triv}}(p'))) \overset{(8.15)}{=} \overline{f}'(\boldsymbol{\pi}^{\mathrm{com}}_{\overline{X}'}(\vec{v}^{\,\mathrm{triv}}(p'))) \\ &\overset{(3.1)}{=} \overline{f}'(p') \overset{(8.2)}{=} \overline{f}(p)' \overset{(3.1)}{=} \boldsymbol{\pi}^{\mathrm{com}}_{\overline{Y}'}(\vec{v}^{\,\mathrm{triv}}(\overline{f}(p)')) = \boldsymbol{\pi}^{\mathrm{com}}_{\overline{Y}'}(c_2). \end{aligned}$$

Hence, (8.8) yields that $\langle c_1, c_2\rangle \in \nu_Y$ and $\langle c_2, c_1\rangle \in \nu_Y$. Thus, we conclude from (C1) that (8.26) and (8.27) are the same principal congruences, which means that the diagram given in (8.23) commutes. This proves that E_{Lift} lifts F_{pos} with respect to Princ, as required. The proof of Theorem 4.7 is complete. $\square$

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