CD-INDEPENDENT SUBSETS IN MEET-DISTRIBUTIVE LATTICES

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Abstract. A subset X of a finite lattice L is CD-independent if the meet of any two incomparable elements of X equals 0. In 2009, Czédli, Hartmann and Schmidt proved that any two maximal CD-independent subsets of a finite distributive lattice have the same number of elements. In this paper, we prove that if L is a finite meet-distributive lattice, then the size of every CD-independent subset of L is at most the number of atoms of L plus the length of L. If, in addition, there is no three-element antichain of meet-irreducible elements, then we give a recursive description of maximal CD-independent subsets. Finally, to give an application of CD-independent subsets, we give a new approach to count islands on a rectangular board.

1. Introduction and the main result

1.1. **Outline and goals.** The concept of CD-independent subsets in lattices was introduced in Czédli, Hartmann and Schmidt [12]. The primary purpose of the present paper is to generalize the main result of [12] from distributive lattices to meet-distributive ones. The secondary goal is to give a combinatorial application of the lattice-theoretical paper [12] by counting islands. Since the cross-reference between the combinatorial part, Section 4, and the lattice theoretical part, Sections 1–3, is minimal, see Definition 1.2 and Proposition 1.3, readers interested mainly in combinatorics can start directly with Section 4.

After recalling some lattice theoretical concepts, the present section formulates the main result, Theorem 1.4. Its proof is given in Section 2, while Section 3 gives some examples that rule out certain generalizations.

1.2. **Basic concepts from Lattice Theory.** All lattices in the present paper are assumed to be finite, even if this is not repeated all the times. For $u \neq 0$ in a (finite) lattice L, let u_* denote the meet of all lower covers of u. If the interval $[u_*, u]$ is a distributive lattice for every $u \in L \setminus \{0\}$, then L is *meet-distributive.* This concept goes back to Dilworth [19] but there are more than a dozen equivalent definitions. In fact, meet-distributivity or its dual is one of the most often rediscovered concepts in Lattice Theory; see Adaricheva [1], Adaricheva, Gorbunov and Tumanov [3],

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Monjardet [31], and Caspard and Monjardet [7]; see also Czédli [9, Proposition 2.1 and Remark 2.2] and Adaricheva and Czédli [2] for recent surveys.

As usual, a finite lattice L is *lower semimodular* if whenever $a, b \in L$ such that a is covered by $a \vee b$, in notation $a \prec a \vee b$, then $a \wedge b \prec b$. Equivalently, if the implication $a \prec b \Rightarrow a \wedge c \preceq b \wedge c$ holds for all $a, b, c \in L$. We will often use the fact, without further reference, that finite meet-distributive lattices are lower semimodular; see Dilworth [19] and Monjardet [7], or see also the dual of Czédli [9, Proposition 2.1 and Remark 2.2] for an overview.

An element of L is *meet-irreducible* if it has exactly one cover. The set of meet-irreducible elements of L is denoted by $\mathrm{Mi}(L)$. The set $\mathrm{Ji}(L)$ of *join-irreducible* elements is defined dually. Following Grätzer and Knapp [23] and, in the present form, Czédli and Schmidt [15], L is *dually slim* if $\mathrm{Mi}(L)$ contains no three-element antichain. Due to Czédli [10], and to the dual of results in Czédli and Grätzer [11] and Czédli and Schmidt [15], [16], and [17], dually slim lower semimodular lattices are understood quite well.

Remark 1.1. It follows from the dual of Czédli, Ozsvárt, and Udvari [14] and Czédli and Schmidt [18, Corollary 3.5] that, for a finite lattice $L = \langle L; \leq \rangle$, the following three conditions are equivalent:

- L is meet-distributive and dually slim;
- L is lower semimodular and dually slim;
- there exist an $n \in \mathbb{N}_0$, a finite group G, and composition series $\{1\} = H_0 \triangleleft H_1 \triangleleft \cdots \triangleleft H_n = G$ and $\{1\} = K_0 \triangleleft K_1 \triangleleft \cdots \triangleleft K_n = G$ such that $\langle L; \leq \rangle$ is isomorphic to $\langle \{H_i \cap K_j : 0 \leq i, j \leq n\}; \subseteq \rangle$.

For a lattice L, let $\mathrm{Atoms}(L)$ and $\mathrm{length}(L)$ stand for the set of atoms of L and the length of L, respectively. Since we only deal with lower semimodular, finite lattices, $\mathrm{length}(L)$ equals the size of any maximal chain minus 1.

Definition 1.2 (Czédli, Hartmann and Schmidt [12])**.** A subset X of a lattice L is *CD-independent* if for any $x, y \in X$ such that x and y are incomparable (in notation, $x \parallel y$), we have $x \wedge y = 0$. In other words, if any two elements of X are either Comparable, or Disjoint; this is were the acronym CD comes from. Note that CD-independence is also known as *laminarity*, see Pach, Pluhár, Pongrácz, and Szabó [32].

The main result of [12] is the following:

Proposition 1.3 ([12])**.** *Let L, X, and C be a finite, lower semimodular lattice, a maximal CD-independent subset of L, and a maximal chain of L, respectively. Then the following two assertions hold.*

- $C \cup \mathrm{Atoms}(L)$ *is a maximal CD-independent subset of L.*
- *If, in addition, L is distributive, then $|X| = |C \cup \mathrm{Atoms}(L)|$, that is, the size of every maximal CD-independent subset is* $\mathrm{length}(L) + |\mathrm{Atoms}(L)|$.

Note that it is also possible to define the concept of CD-independent subsets of posets, see Horváth and Radeleczki [27], but the present paper is restricted to lattices. In view of further results in [12], we cannot expect that the second part of this proposition extends to a significantly larger class of lattices. However, replacing distributivity by meet-distributivity, which is a weaker assumption, the theorem below still shows some property of CD-independent subsets.

For a poset H, let $\max(H)$ stand for the set of maximal elements of H. If u is an element of a lattice L, then the principal ideal $\{x \in L : x \leq u\}$ is denoted by $\downarrow u$. For $a, b \in L$, $\langle a, b\rangle$ is a *complemented pair* if $a \wedge b = 0$ and $a \vee b = 1$. Given $a \in L$, if c is the largest element of L such that $a \wedge c = 0$, then c is the *pseudocomplement* of a. Note that a need not have a pseudocomplement, but if it has, then its pseudocomplement is uniquely determined. If a and b are mutually pseudocomplements of each other, then $\langle a, b\rangle$ is a *pseudocomplemented pair*.

Since the concept of pseudocomplemented pairs seems to be new, some comments are appropriate here. The five-element nonmodular lattice, N_5, witnesses that a complemented pair need not be a pseudocomplemented pair. The lattice obtained from N_5 by adding a new top element shows that a pseudocomplemented pair need not be a complemented pair. However,

(1.1) if L is a distributive lattice, then every complemented pair $\langle a, b\rangle$ of L is a pseudocomplemented pair

since if $a \wedge x = 0$, then $x = 1 \wedge x = (a \vee b) \wedge x = (a \wedge x) \vee (b \wedge x) = 0 \vee (b \wedge x) = b \wedge x$ implies $x \leq b$, whence b is the pseudocomplement of a.

1.3. **Main result.** For a subset X of L, we let

$$\mathrm{ComplP}(L) = \{\langle a, b\rangle \in L^2 : \langle a, b\rangle \text{ is a complemented pair of } L\},$$
$$\mathrm{PseudCP}(L) = \{\langle a, b\rangle \in L^2 : \langle a, b\rangle \text{ is a pseudocomplemented pair of } L\}.$$

Since $\langle 0, 1\rangle \in \mathrm{PseudCP}(L) \cap \mathrm{ComplP}(L)$ but we always want to exclude this possibility, we shall stipulate $a \parallel b$ or $\{a, b\} \subseteq \mathrm{Mi}(L)$. Our main goal is to prove the following result. While its Part (1) is relatively simple, Parts (2) and (3) give a lot of additional information on maximal CD-independent subsets for two important subclasses of meet-distributive lattices. Obviously, if X is a *maximal* CD-independent subset of L, then $\{0, 1\} \cup \mathrm{Atoms}(L) \subseteq X$. Therefore, to characterize maximal CD-independent subsets in the theorem below, it suffices to consider those subsets X of L that extend $\{0, 1\} \cup \mathrm{Atoms}(L)$.

Theorem 1.4. *Let L be a finite lattice consisting of at least two elements.*

(1) *If L is meet-distributive and Y is a CD-independent subset of L, then we have* $|Y| \leq \mathrm{length}(L) + |\mathrm{Atoms}(L)|$.

In the rest of the theorem, let X be a subset of L such that $\{0, 1\} \cup \mathrm{Atoms}(L) \subseteq X$. We denote $|\max(X \setminus \{1\})|$ by k, and we let $\max(X \setminus \{1\}) = \{a_1, \dots, a_k\}$.

(2) *If L is dually slim and lower semimodular (equivalently, dually slim and meet-distributive), then the following two conditions are equivalent.*
   (a) *X is a maximal CD-independent subset of L.*
   (b) (i) *Either $k = 1$, a_1 is a coatom of L, and $X \setminus \{1\}$ is a maximal CD-independent subset of $\downarrow a_1$,*
   (ii) *or $k = 2$, $\langle a_1, a_2\rangle \in \mathrm{ComplP}(L) \cap \mathrm{PseudCP}(L)$, $a_1 \parallel a_2$, and $X \cap \downarrow a_i$ is a maximal CD-independent subset of $\downarrow a_i$ for $i = 1, 2$.*

   Furthermore, if Part (2(b)ii) holds, then $\{a_1, a_2\} \subseteq \mathrm{Mi}(L)$.

(3) *If L is distributive, then the following two conditions are equivalent.*
   (a) *X is a maximal CD-independent subset of L.*
   (b) (i) *Either $k = 1$, a_1 is a coatom of L, and $X \setminus \{1\}$ is a maximal CD-independent subset of $\downarrow a_1$,*

(ii) *or* $k = 2$, $\langle a_1, a_2\rangle \in \mathrm{ComplP}(L)$, $a_1 \parallel a_2$, *and* $X \cap {\downarrow} a_i$ *is a maximal CD-independent subset of* ${\downarrow} a_i$ *for* $i = 1, 2$.

Remark 1.5. By Remark 1.1 and Proposition 1.3, $C \cup \mathrm{Atoms}(L)$ is always a maximal CD-independent subset, provided L is meet-distributive and C is a maximal chain of L. Furthermore, $|C \cup \mathrm{Atoms}(L)| = \mathrm{length}(L) + |\mathrm{Atoms}(L)|$ by lower semimodularity. Hence, the upper bound in Part (1) is sharp. Note also that Parts (2) and (3) give recursive descriptions for maximal CD-independent subsets.

The following statement will be derived from Part (2) of Theorem 1.4.

Corollary 1.6. *If* L *is a finite, dually slim, lower semimodular lattice,* $a_1, a_2 \in L \setminus \{0, 1\}$, *and* $\langle a_1, a_2\rangle \in \mathrm{ComplP}(L) \cap \mathrm{PseudCP}(L)$, *then* $a_1, a_2 \in \mathrm{Mi}(L)$.

In view of the fact that the analogous statement fails in the eight-element boolean lattice, this corollary is a bit surprising.

2. Circles and the proof of the main result

Before proving Theorem 1.4, we recall some results from Czédli [10]. Note that this will be the first application of the main result of [10]. As usual, a *circle* in the plane is a set $\{\langle x, y\rangle : (x - u)^2 + (y - v)^2 = r^2\}$ where $u, v, r \in \mathbb{R}$ and $r \geq 0$. Let F be a finite set of circles in the plane. A subset Y of F is *closed* if whenever $C \in F$ and C is in the convex hull of $\bigcup\{D : D \in Y\}$, then $C \in Y$. Less formally (but not quite precisely), if Y is closed with respect to the usual convex hull operation restricted to F. Let $\mathrm{Lat}(F)$ denote the set of closed subsets of F. With respect to inclusion, $\mathrm{Lat}(F)$ is a lattice, and $\varnothing, F \in \mathrm{Lat}(F)$. We call $\mathrm{Lat}(F)$ a *lattice of circles*. If the centers of the circles in F are on the same line, then F is *collinear*. In the collinear case, we always assume that the line containing the centers is the x axis. A collinear set F of circles is *separated* if no point of the x axis belongs to more than one member of F. For example, if we disregard the dotted arcs, then F depicted in Figure 1 is a separated collinear set of circles.

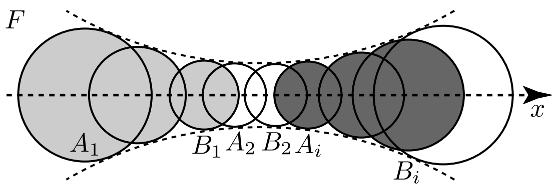

Figure 1. A separated concave set of collinear circles

Next, let F be a separated collinear set of circles. For $C \in F$, C is of the form $\{\langle x, y\rangle : (x - u)^2 + y^2 = r^2\}$, where $0 \leq r \in \mathbb{R}$. The points $\mathrm{LPt}(C) = \langle u - r, 0\rangle$ and $\mathrm{RPt}(C) = \langle u + r, 0\rangle$ on the x-axis are the *leftmost point* and the *rightmost point* of C, respectively. Note that for points $\langle a, 0\rangle$ and $\langle b, 0\rangle$ on the x-axis, $\langle a, 0\rangle \leq \langle b, 0\rangle$ is always understood as $a \leq b$. For $A, B \in F$, let

$$\mathrm{FromTo}_F(A, B) = \{C \in F : \mathrm{LPt}(A) \leq \mathrm{LPt}(C) \text{ and } \mathrm{RPt}(C) \leq \mathrm{RPt}(B)\}.$$

This set can be empty, and even if it is not empty, neither A, nor B has to belong to it. For example, if B is encapsulated in A, then $\mathrm{FromTo}_F(A, B)$ contains B but not

A. If $A, B \in \mathrm{FromTo}_F(A,B)$, then we write $\mathrm{HInt}_F(A,B)$ instead of $\mathrm{FromTo}_F(A,B)$, and we call $\mathrm{HInt}_F(A,B)$ a *horizontal interval* determined by A and B. That is,

$$\mathrm{HInt}_F(A,B) = \{C \in F : \mathrm{LPt}(A) \le \mathrm{LPt}(C) \text{ and } \mathrm{RPt}(C) \le \mathrm{RPt}(B)\},$$
$$\text{provided } \mathrm{LPt}(A) \le \mathrm{LPt}(B) \text{ and } \mathrm{RPt}(A) \le \mathrm{RPt}(B).$$

For example, $\mathrm{HInt}_F(A_1, B_1)$ in Figure 1 consists of the light grey circles while $\mathrm{HInt}_F(A_i, B_i)$ consists of the dark grey ones. Note that, for a pair $\langle A, B\rangle \in F^2$, the horizontal interval $\mathrm{HInt}_F(A,B)$ is not necessarily defined. If we want to emphasize that $\mathrm{HInt}_F(A,B)$ exists, then we write $\mathrm{FromTo}_F(A,B) = \mathrm{HInt}_F(A,B)$. We say that F is a *concave set* of collinear circles if for all $C_1, C_2, C_3 \in F$, the conjunction of $\mathrm{LPt}(C_1) \le \mathrm{LPt}(C_2)$ and $\mathrm{RPt}(C_2) \le \mathrm{RPt}(C_3)$ implies that the smallest closed subset of F that contains C_1 and C_3 also contains C_2. In other words, F is concave if for all $X \in \mathrm{Lat}(F)$ and $A, B \in X$ such that $\mathrm{HInt}_F(A,B)$ defined, $\mathrm{HInt}_F(A,B) \subseteq X$. In Figure 1, the circles determined by dotted arcs do not belong to F; their purpose is to indicate what concavity means. However, this figure does not reflect generality since F in Figure 2 with encapsulated circles is also a concave set of collinear circles.

Using a result of Edelman [20], see also [10, Lemma 3.5], one can translate some results of [10] to the language of Lattice Theory as follows.

Proposition 2.1 ([10, Proposition 2.1, Theorem 2.2, and Lemma 3.1])**.**

(A) *If F is a finite set of circles in the plane, then* $\mathrm{Lat}(F)$ *is a meet-distributive lattice.*

(B) *Dually slim, lower semimodular lattices are, up to isomorphism, characterized as lattices* $\mathrm{Lat}(F)$ *where F is a separated concave set of collinear circles.*

(C) *If F is a separated concave set of collinear circles, then we have* $\mathrm{Lat}(F) = \{\varnothing\} \cup \{\mathrm{HInt}_F(A,B) : A, B \in F\}$. *Furthermore, for each $\varnothing \ne X \in \mathrm{Lat}(F)$, there are a unique $A \in X$ and a unique $B \in X$ such that $X = \mathrm{HInt}_F(A,B)$.*

Now, we are ready to prove our main result.

Proof of Theorem 1.4. We prove Part (1) by induction on $|L|$. For $|L| \le 4$, L is distributive and (1) follows from Proposition 1.3.

Assume that $|L| > 4$ and that Part (1) holds for all lattices of smaller size. Let $k = |\max(Y \setminus \{1\})|$, and let $\max(Y \setminus \{1\}) = \{a_1, \dots, a_k\}$. We can assume that Y is a maximal CD-independent subset of L; this assumption implies $\mathrm{Atoms}(L) \subseteq Y$.

Assume first that $k = 1$, and let $Y_1 = Y \cap {\downarrow}a_1 = Y \setminus \{1\}$. Clearly, Y_1 is a CD-independent subset of ${\downarrow}a_1$. By the induction hypothesis, $|Y_1| \le \mathrm{length}({\downarrow}a_1) + |\mathrm{Atoms}({\downarrow}a_1)|$. This, together with $\mathrm{Atoms}({\downarrow}a_1) \subseteq \mathrm{Atoms}(L)$ and $1 + \mathrm{length}({\downarrow}a_1) \le \mathrm{length}(L)$, yields

$$|Y| = 1 + |Y_1| \le 1 + \mathrm{length}({\downarrow}a_1) + |\mathrm{Atoms}({\downarrow}a_1)| \le \mathrm{length}(L) + |\mathrm{Atoms}(L)|,$$

as desired.

Next, we assume $k \ge 2$. Since $\mathrm{Atoms}(L) \subseteq Y \setminus \{1\} \subseteq {\downarrow}a_1 \cup \dots \cup {\downarrow}a_k$, we conclude $\mathrm{Atoms}(L) = \mathrm{Atoms}({\downarrow}a_1) \cup \dots \cup \mathrm{Atoms}({\downarrow}a_k)$. Here the union is disjoint since $\{a_1, \dots, a_k\}$ is an antichain and $a_i \wedge a_j = 0$ for $i \ne j$ by the CD-independence of Y. Consequently,

$$|\mathrm{Atoms}(L)| = |\mathrm{Atoms}({\downarrow}a_1)| + \dots + |\mathrm{Atoms}({\downarrow}a_k)|. \tag{2.1}$$

For $1 \le i < j \le k$, we have $\mathrm{Ji}({\downarrow}a_i) \cap \mathrm{Ji}({\downarrow}a_j) = \varnothing$ since $a_i \wedge a_j = 0$. On the other hand, $\mathrm{Ji}({\downarrow}a_t) \subseteq \mathrm{Ji}(L)$ for $t = 1, \dots, k$. Hence, $|\mathrm{Ji}({\downarrow}a_1)| + \dots + |\mathrm{Ji}({\downarrow}a_k)| \le |\mathrm{Ji}(L)|$.

We know from Stern [34, Theorem 7.2.27], who attributes it to Avann [4] and [5], that $|\mathrm{Ji}(K)| = \mathrm{length}(K)$ for every meet-distributive lattice; see also the dual of Czédli [9, Proposition 2.1(iii)⇔(v)] for more historical comments. Clearly, the ideals $\downarrow a_i$ are meet-distributive. Thus the last inequality turns into

$$\mathrm{length}(\downarrow a_1) + \cdots + \mathrm{length}(\downarrow a_k) \leq \mathrm{length}(L). \tag{2.2}$$

Next, for $i = 1, \dots, k$, let $Y_i = Y \cap \downarrow a_i$. Since Y_i is clearly a CD-independent subset of $\downarrow a_i$, the induction hypothesis gives

$$|Y_i| \leq |\mathrm{Atoms}(\downarrow a_i)| + \mathrm{length}(\downarrow a_i) \text{ for } i = 1, \dots, k. \tag{2.3}$$

Now, using the previous formulas, $Y_i \cap Y_j = \{0\}$ for $i \neq j$, and $k \geq 2$, we can compute as follows; note that 2 at the beginning counts $0 = 0_L$ and $1 = 1_L$.

$$\begin{aligned} |Y| &= 2 + \sum_{i=1}^{k} |Y_i \setminus \{0\}| \overset{(2.3)}{\leq} 2 + \sum_{i=1}^{k} \bigl(|\mathrm{Atoms}(\downarrow a_i)| + \mathrm{length}(\downarrow a_i) - 1\bigr) \\ &\leq \sum_{i=1}^{k} |\mathrm{Atoms}(\downarrow a_i)| + \sum_{i=1}^{k} \mathrm{length}(\downarrow a_i) \overset{(2.1),(2.2)}{\leq} |\mathrm{Atoms}(L)| + \mathrm{length}(L). \end{aligned}$$

This completes the induction step and proves Part (1).

Next, we prove Part (2). Proposition 2.1(B), together with Remark 1.1, allows us to assume that $L = \mathrm{Lat}(F)$, where F is a separated concave set of collinear circles. Since F is separated, it contains a unique *leftmost circle*, $C_{\ell\mathrm{m}}$. That is, we have $\mathrm{LPt}(C_{\ell\mathrm{m}}) < \mathrm{LPt}(D)$ for all $D \in F \setminus \{C_{\ell\mathrm{m}}\}$. Similarly, we have a unique *rightmost circle* $C_{\mathrm{rm}} \in F$ with the property $\mathrm{RPt}(D) < \mathrm{RPt}(C_{\mathrm{rm}})$ for all $D \in F \setminus \{C_{\mathrm{rm}}\}$.

Assume Part (2a), that is, let X be a maximal CD-independent subset of $L = \mathrm{Lat}(F)$. By Proposition 2.1(C), there exist unique A_j and B_j in F such that we have $a_j = \mathrm{HInt}_F(A_j, B_j)$, for $j \in \{1, \dots, k\}$. For example, in Figure 1, where the label of a circle is always below its center, a_1 consists of the light grey circles while a_i consists of the dark grey ones. Since $\mathrm{LPt}(A_j) \leq \mathrm{RPt}(A_j)$, $\mathrm{LPt}(B_j) \leq \mathrm{RPt}(B_j)$, and $A_j, B_j \in a_j = \mathrm{HInt}_F(A_j, B_j)$, we know that

$$\mathrm{LPt}(A_j) \leq \mathrm{RPt}(A_j) \leq \mathrm{RPt}(B_j) \text{ and } \mathrm{LPt}(A_j) \leq \mathrm{LPt}(B_j) \leq \mathrm{RPt}(B_j), \tag{2.4}$$

for $j \in \{1, \dots, k\}$. However, note that $\mathrm{LPt}(B_j) \leq \mathrm{RPt}(A_j)$ or even $A_j = B_j$ can occur.

If we had $\mathrm{LPt}(A_i) = \mathrm{LPt}(A_j)$ for some $i \neq j$, then we would obtain $A_i = A_j$ since F is separated, and $A_i = A_j \in a_i \cap a_j = \varnothing$ would be a contradiction. Thus the $\mathrm{LPt}(A_i)$, $i \in \{1, \dots, k\}$, are pairwise distinct, and so are the $\mathrm{RPt}(B_i)$. Hence, we can choose the indices such that

$$\mathrm{LPt}(A_1) < \mathrm{LPt}(A_2) < \cdots < \mathrm{LPt}(A_k). \tag{2.5}$$

If we had $\mathrm{RPt}(B_i) \geq \mathrm{RPt}(B_j)$ for some $1 \leq i < j \leq k$, then

$$\mathrm{LPt}(A_i) \overset{(2.5)}{<} \mathrm{LPt}(A_j) \text{ and } \mathrm{RPt}(A_j) \overset{(2.4)}{\leq} \mathrm{RPt}(B_j) \leq \mathrm{RPt}(B_i)$$

would give $A_j \in a_i \cap a_j$, a contradiction. This shows

$$\mathrm{RPt}(B_1) < \mathrm{RPt}(B_2) < \cdots < \mathrm{RPt}(B_k). \tag{2.6}$$

Next, for the sake of contradiction, suppose $k \geq 3$. Now, (2.4), (2.5), (2.6), and $k \geq 3$ easily imply $a_1 \vee a_2 \subseteq \mathrm{HInt}_F(A_1, B_2)$, $B_2 \notin a_1$, $A_1 \notin a_2$, and $B_k \notin \mathrm{HInt}_F(A_1, B_2)$. Hence, $a_1 \vee a_2$ is neither the largest element of $\mathrm{Lat}(F)$, nor it is one of a_1 and a_2. Therefore, X is a proper subset of $X' = X \cup \{a_1 \vee a_2\}$. To obtain a contradiction, it suffices to prove that X' is CD-independent. It is sufficient to show that $(a_1 \vee a_2) \wedge a_i = 0$ for $i \le 3$ since X is CD-independent. Hence, for $i \ge 3$, it suffices to prove $\mathrm{HInt}_F(A_1, B_2) \cap \mathrm{HInt}_F(A_i, B_i) = \varnothing$. Suppose for (an encapsulated) contradiction that a circle $C \in F$ belongs to this intersection. This gives

$$\mathrm{LPt}(A_2) \overset{(2.5)}{<} \mathrm{LPt}(A_i) \le \mathrm{LPt}(C) \text{ and } \mathrm{RPt}(C) \le \mathrm{RPt}(B_2),$$

implying $C \in \mathrm{HInt}_F(A_2, B_2) \cap \mathrm{HInt}_F(A_i, B_i) = a_2 \cap a_i = \varnothing$. This is a contradiction proving that X' is CD-independent. However, this is impossible since X was a maximal CD-independent subset of $\mathrm{Lat}(F)$. This proves $k \le 2$.

Armed with $k \le 2$, first we deal with the case $k = 2$. Since $\max(X \setminus \{1\})$ is an antichain, $a_1 \parallel a_2$. Their meet is 0_L, that is, $a_1 \cap a_2 = \varnothing$, since X is CD-independent. If we had $a_1 \vee a_2 < 1$, then $X \cup \{a_1 \vee a_2\}$ would also be CD-independent, in contradiction with the maximality of X. Hence, $\langle a_1, a_2 \rangle \in \mathrm{ComplP}(L)$.

Suppose for contradiction that $\mathrm{LPt}(C_{\ell\mathrm{m}}) < \mathrm{LPt}(A_1)$. It follows from (2.4), (2.5), and (2.6) that $\mathrm{FromTo}_F(A_1, B_2) = \mathrm{HInt}_F(A_1, B_2)$, $a_1, a_2 \subseteq \mathrm{HInt}_F(A_1, B_2)$, but $C_{\ell\mathrm{m}} \notin \mathrm{HInt}_F(A_1, B_2)$, contradicting $a_1 \vee a_2 = 1_L$. This proves the first half of

$$\begin{aligned} &a_1 = \mathrm{HInt}_F(A_1, B_1) = \mathrm{HInt}_F(C_{\ell\mathrm{m}}, B_1) \text{ and, in particular, } A_1 = C_{\ell\mathrm{m}} \\ &a_2 = \mathrm{HInt}_F(A_2, B_2) = \mathrm{HInt}_F(A_2, C_{\mathrm{rm}}) \text{ and, in particular, } B_2 = C_{\mathrm{rm}}; \end{aligned} \tag{2.7}$$

its second half follows similarly.

If we had an $x \in L$ such that $a_2 < x < 1$ and $a_1 \wedge x = 0$, then $X \cup \{x\}$ would be a CD-independent subset that is strictly larger than X. Thus, we conclude that

$$a_1 \wedge a_2 = 0 \text{ but, for every } x \in L,\ a_2 < x < 1 \text{ implies } a_1 \wedge x \ne 0. \tag{2.8}$$

Next, we show that, for every $V \in F$,

$$\text{if } a_1 \cap \mathrm{HInt}_F(V, V) = \varnothing \text{, then } a_1 \cap \mathrm{HInt}_F(V, C_{\mathrm{rm}}) = \varnothing. \tag{2.9}$$

Suppose the contrary, that is, let a circle $D \in F$ belong to $a_1 \cap \mathrm{HInt}_F(V, C_{\mathrm{rm}})$ such that $a_1 \cap \mathrm{HInt}_F(V, V) = \varnothing$. Since $V \in \mathrm{HInt}_F(V, V)$, we have $V \notin a_1 = \mathrm{HInt}_F(C_{\ell\mathrm{m}}, B_1)$, which gives $\mathrm{RPt}(B_1) < \mathrm{RPt}(V)$. Since we also have $\mathrm{RPt}(D) \le \mathrm{RPt}(B_1)$ by $D \in a_1$, we conclude $\mathrm{RPt}(D) \le \mathrm{RPt}(V)$ by transitivity. On the other hand, $D \in \mathrm{HInt}_F(V, C_{\mathrm{rm}})$ yields $\mathrm{LPt}(V) \le \mathrm{LPt}(D)$, and it follows that $D \in \mathrm{HInt}_F(V, V)$. Hence, $D \in a_1 \cap \mathrm{HInt}_F(V, V)$, which is a contradiction proving (2.9).

Now, we are in the position to prove that a_2 is a pseudocomplement of a_1. Assume that $x \in L \setminus \{0\}$ such that $a_1 \wedge x = 0$. Consider an arbitrary circle V in x. The obvious inequality $\mathrm{HInt}_F(V, V) \le x$ implies $a_1 \cap \mathrm{HInt}_F(V, V) = \varnothing$. Applying (2.9), we obtain $a_1 \cap \mathrm{HInt}_F(V, C_{\mathrm{rm}}) = \varnothing$. By (2.8), this rules out the inequality $a_2 < \mathrm{HInt}_F(V, C_{\mathrm{rm}})$, which is equivalent to $\mathrm{LPt}(V) < \mathrm{LPt}(A_2)$. Therefore, $\mathrm{LPt}(A_2) \le \mathrm{LPt}(V)$, and we have $V \in a_2$. Since $V \in x$ was arbitrary, we conclude $x \le a_2$. This proves that a_2 is a pseudocomplement of a_1. An analogous argument shows that a_1 is a pseudocomplement of a_2. Thus $\langle a_1, a_2 \rangle \in \mathrm{PseudCP}(L)$.

Finally, armed with $\langle a_1, a_2 \rangle \in \mathrm{ComplP}(L)$ and $a_1 \parallel a_2$, and using the maximality of X, it is straightforward to see that $X \cap {\downarrow} a_i$ is a maximal CD-independent set of ${\downarrow} a_i$, for $i = 1, 2$. This settles the case $k = 2$.

Since the case $k = 1$ is evident by the maximality of X, we have shown that (2a) implies (2b).

Next, to prove that Part (2(b)ii) implies $\{a_1, a_2\} \subseteq \mathrm{Mi}(L)$, assume Part (2(b)ii). Also, assume $a_1 = u_1 \wedge u_2$. We have $u_i = \mathrm{HInt}_F(U_i, W_i)$ for $i = 1, 2$ and appropriate circles $U_1, W_1, U_2, W_2 \in F$. Since $a_1 \leq u_i$, (2.7) yields $U_i = C_{\ell\mathrm{m}}$, for $i = 1, 2$. Thus, since $\mathrm{RPt}(W_1), \mathrm{RPt}(W_2) \in \mathbb{R}$ are comparable, u_1 and u_2 are comparable, and $a_1 = u_1 \wedge u_2 \in \{u_1, u_2\}$. This and an analogous argument for a_2 show that $\{a_1, a_2\} \subseteq \mathrm{Mi}(L)$. Hence, Part (2(b)ii) implies $\{a_1, a_2\} \subseteq \mathrm{Mi}(L)$.

Next, we prove that (2b) implies (2a). We can assume $k = 2$ since the case $k = 1$ is trivial. Since $\max(X \setminus \{1\}) = \{a_1, a_2\}$, we have $X = \{1\} \cup (X \cap {\downarrow}a_1) \cup (X \cap {\downarrow}a_2)$, and this is a disjoint union. It follows trivially from $\langle a_1, a_2\rangle \in \mathrm{ComplP}(L)$ that X is CD-independent. To prove that it is maximal, assume that $u \in L$ such that $X' = X \cup \{u\}$ is also CD-independent. Depending on the ordering on $\{a_1, u\}$, there are three cases.

First, consider the case $u \parallel a_1$. Then the CD-independence of X' gives $a_1 \wedge u = 0$, and we obtain $u \leq a_2$ from $\langle a_1, a_2\rangle \in \mathrm{PseudCP}(L)$. That is, $u \in X' \cap {\downarrow}a_2$. Clearly, $X' \cap {\downarrow}a_2$ is CD-independent in ${\downarrow}a_2$ and it includes $X \cap {\downarrow}a_2$. The maximality of $X \cap {\downarrow}a_2$ yields $X' \cap {\downarrow}a_2 = X \cap {\downarrow}a_2$, and we conclude $u \in X' \cap {\downarrow}a_2 = X \cap {\downarrow}a_2 \subseteq X$, that is, $u \in X$. Second, if we had $a_1 < u < 1$, then $a_1 \not\leq a_2$ would exclude $u \leq a_2$, $\langle a_1, a_2\rangle \in \mathrm{PseudCP}(L)$ would exclude $u \parallel a_2$, and $u < 1 = a_1 \vee a_2$ would exclude $u \geq a_2$. Thus this case cannot occur. Third, the case $u \leq a_1$ implies $u \in X$, because $X \cap {\downarrow}a_1$ is a maximal CD-independent subset of ${\downarrow}a_1$. Therefore, $X' \subseteq X$ and X is a maximal CD-independent subset. This shows that (2b) implies (2a), completing the proof of Part (2).

Now, we deal with Part (3). Assume that L is a finite distributive lattice and that (3a) holds. The first paragraph in the proof of the Main Theorem of Czédli, Hartmann and Schmidt [12] explicitely says that $k = |\max(X \setminus \{1\})|$ is at most 2. Hence, using the maximality of X, (3b) follows in an obvious way.

Conversely, assume that (3b) holds. In virtue of (1.1), we conclude the validity of (3a) by that same argument that proved the implication (2b) $\Rightarrow$ (2a). This completes the proof of Theorem 1.4. □

Proof of Corollary 1.6. Assume that $a_1, a_2 \in L \setminus \{0, 1\}$ such that $\langle a_1, a_2\rangle$ belongs to $\mathrm{ComplP}(L) \cap \mathrm{PseudCP}(L)$. For $i \in \{1, 2\}$, let X_i be a maximal CD-independent subset of ${\downarrow}a_i$, and let $X = X_1 \cup X_2 \cup \{1\}$. Clearly, X is a CD-independent subset of L since $\langle a_1, a_2\rangle \in \mathrm{ComplP}(L)$. Hence, we can extend X to a maximal CD-independent subset X' of L.

For the sake of contradiction, suppose $X' \neq X$, and pick an element $u \in X' \setminus X$. Since $u \notin X$, we have $u \not\geq a_1 \vee a_2 = 1$. Hence, $u \not\geq a_1$ or $u \not\geq a_2$. Let, say, $u \not\geq a_1$. If we had $u \leq a_1$, then $X_1 \cup \{u\}$, which is strictly larger than X_1, would be a CD-independent subset of ${\downarrow}a_1$, contradicting the maximality of X_1. Thus $a_1 \parallel u$, and the CD-independence of X' yields $a_1 \wedge u = 0$. This, together with $\langle a_1, a_2\rangle \in \mathrm{PseudCP}(L)$, yields $u \leq a_2$, which clearly contradicts the maximality of X_2. It follows that $X = X'$ is a maximal CD-independent subset of L. Hence, clearly, $\{0, 1\} \cup \mathrm{Atoms}(L) \subseteq X$. Finally, since the antichain $\{a_1, a_2\}$ equals $\max(X \setminus \{1\})$, Part (2) of Theorem 1.4 implies $\{a_1, a_2\} \subseteq \mathrm{Mi}(L)$. □

3. Examples and comments

The proof of Part (2) of Theorem 1.4 was based on Proposition 2.1. Clearly, there exists a purely lattice theoretical proof of Part (2) since, in the worst case, we can repeat several parts from the proof of Proposition 2.1, given in Czédli [10].

However, the present approach based on circles gives more visual insight and it is much more economic; once we have [10], it is natural to use.

The examples given in this section show that the assumptions stipulated in Theorem 1.4 are relevant. In fact, we do not see any straightforward way of reasonable generalizations even if Y in Part (1) is assumed to be maximal. Note that it was already proved in Czédli, Hartmann and Schmidt [12] that distributivity in Proposition 1.3 cannot be replaced by a weaker lattice identity.

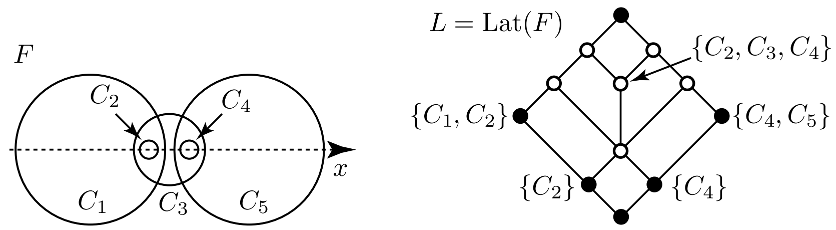

FIGURE 2. A maximal CD-independent set in L

Example 3.1. By Proposition 2.1(B), the lattice $L = \mathrm{Lat}(F)$ in Figure 2 is dually slim and meet-distributive. (It also follows from Czédli and Schmidt [16, Theorem 12] that L has these properties.) We have $\mathrm{length}(L) + |\mathrm{Atoms}(L)| = 7$. The black-filled elements form a maximal CD-independent subset of size 6. This shows that in Part (1) of Theorem 1.4, the inequality can be proper even if L is dually slim and Y is a maximal CD-independent subset.

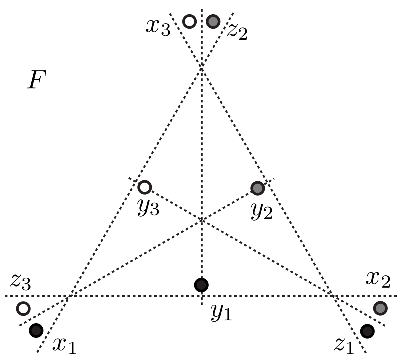

FIGURE 3. An atomistic example

A lattice is *atomistic* if each of its element is the join of some atoms. The following example indicates that atomicity would not improve Theorem 1.4(1).

Example 3.2. Let $L = \mathrm{Lat}(F)$, where F consists of the circles depicted in Figure 3. The dotted lines indicate how these nine little circles are positioned. (Note that smaller circles with radius 0, which are points, would also do.) Clearly, L is an atomistic lattice, and it is meet-distributive by Proposition 2.1(A). Since $\mathrm{Atoms}(L) = \big\{\{x\} : x \in F\big\}$, we have $|\mathrm{Atoms}(L)| = 9$. By an already mentioned result of Avann [4] and [5], see also Stern [34, Theorem 7.2.27], $\mathrm{length}(L) = |\mathrm{Ji}(L)| =$

$|\mathrm{Atoms}(L)|$. For $i = 1, 2, 3$, let $A_i = \{x_i, z_i\}$ and $B_i = \{x_i, y_i, z_i\}$; these subsets of F belong to L. Note that B_1, B_2, and B_3 consist of the black-filled circles, the grey-filled circles, and the empty circles, respectively. It is a straightforward but tedious task to verify that $Y = \mathrm{Atoms}(L) \cup \{\varnothing, F, A_1, B_1, A_2, B_2, A_3, B_3\}$ is a maximal CD-independent subset of L. Since $|Y| = 17$ and $\mathrm{length}(L) + |\mathrm{Atoms}(L)| = 18$, the equality in Part (1) of Theorem 1.4 is not an equality in this case.

4. An application to the theory of islands

The concept of islands appeared first in Czédli [8]. For definition, let m and n be natural numbers, and consider an m-by-n *rectangular board*, denoted by $\mathrm{Board}(m, n)$. It consists of little unite squares called cells, which are arranged in m columns and n rows. For example, $\mathrm{Board}(8, 8)$ is the chess-board and $\mathrm{Board}(8, 4)$ is depicted in Figure 4. Let $h\colon \mathrm{Board}(m, n) \to \mathbb{R}$ be a map, called *height function*. A nonempty set H of cells forming a rectangle is called a (*cellular*) *rectangular island* with respect to h if the minimum height of H is greater than the height of any cell around the perimeter of H. Let us emphasize that the empty set is never a cellular rectangle. The concept of islands was motivated by Foldes and Singhi [22], where cellular rectangular islands on $\mathrm{Board}(n, 1)$ played a key role in characterizing maximal instantaneous codes. The number of cellular rectangular islands of the system $\langle \mathrm{Board}(m, n); h\rangle$ depends on the height function, and it takes its maximum for some h. This maximum value, denoted by $f(m, n)$, is determined by the following result, where $\lfloor x \rfloor$ stands for the (lower) integer part of x.

Proposition 4.1 ([8])**.** *For $m, n \in \mathbb{N}$, $f(m, n) = \lfloor (mn + m + n - 1)/2 \rfloor$.*

This result was soon followed by many related ones, due to Barát, Foldes, E. K. Horváth, G. Horváth, Lengvárszky, Németh, Pach, Pluhár, Pongrácz, Šešelja, Szabó, and Tepavčević. The results of these authors, written alone or in various groups, range from triangular boards to the continuous case and from lattice theory to combinatorics, see [6], [21], [24], [25], [26], [28], [29], [30], [32], [33], and some further papers not referenced here. Since [21] and [24] give good overviews on islands, we do not go into further historical details. However, we mention the following feature of this research field. At the beginning, in [8] and also in [26] and [33], a lattice theoretical result of Czédli, Huhn, and Schmidt [13] on weakly independent subsets played the main role in proofs. Soon afterward, simpler approaches were discovered in [6], and Lattice Theory was more or less neglected thereafter.

By giving a new proof for Proposition 4.1 based on CD-independence, the goal of this section is to demonstrate that Lattice Theory is still competitive with other approaches. Note that, besides that this was the original motivation in Czédli, Hartmann and Schmidt [12], this task was also suggested by Horváth [24, Problem 9.1]. We only need Proposition 1.3, taken from [12], for this purpose.

Each cell of $\mathrm{Board}(m, n)$ has exactly four vertices. For a (cellular) rectangular subset X of $\mathrm{Board}(m, n)$, let $\mathrm{Grid}(X)$ denote the set of vertices of the cells of X. We call $\mathrm{Grid}(X)$ the *point rectangle* associated with the *cellular rectangle* X, while $\mathrm{Grid}(\mathrm{Board}(m, n))$ is the *grid* associated with $\mathrm{Board}(m, n)$. In general, a grid is a set $\{0, 1, \dots, i\} \times \{0, 1, \dots, j\}$ of points for some $i, j \in \mathbb{N} = \{1, 2, 3, \dots\}$, shifted to any location in the plane. For a set $\mathcal{H}$ of cellular rectangular subsets of $\mathrm{Board}(m, n)$, we let $\mathrm{SGrid}(\mathcal{H}) = \{\mathrm{Grid}(X) : X \in \mathcal{H}\}$. (The letter "S" in the mnemonic will remind us "set".) The idea of working with grids rather than boards goes back to E. K. Horváth, G. Horváth, Németh, and Szabó [25].

First of all, we rephrase Czédli [8, Lemma 2], which was used practically by all previous approaches dealing with (finitely many) islands. The collection of all subsets of a set U is denoted by $\mathrm{PowSet}\big(U\big)$.

Lemma 4.2 ([8, Lemma 2])**.** *For an arbitrary set $\mathcal{H}$ of cellular rectangles of the board* $\mathrm{Board}(m,n)$*, the following two conditions are equivalent.*

- *$\mathcal{H}$ is the collection of all cellular rectangular islands of $\langle \mathrm{Board}(m,n);h\rangle$ for an appropriate height function h;*
- *$\mathrm{SGrid}(\mathcal{H})$ is a CD-independent subset of $\big\langle \mathrm{PowSet}\big(\mathrm{Grid}(\mathrm{Board}(m,n))\big),\subseteq\big\rangle$ and $\mathrm{Board}(m,n)\in\mathcal{H}$.*

Note that some authors, including Pach, Pluhár, Pongrácz, and Szabó [32], call CD-independent subsets as *laminar systems*.

Proof of Proposition 4.1. We do not deal with $f(m,n)\geq \lfloor (mn+m+n-1)/2\rfloor$ since this inequality is proved by an easy construction without any tool, see [8].

For brevity, let $G=\mathrm{PowSet}\big(\mathrm{Grid}(\mathrm{Board}(m,n))\big)$. By Proposition 1.3, taken from [12], each maximal CD-independent set of $\langle G;\subseteq\rangle$ is of size

$$\mathrm{length}(G)+|\mathrm{Atoms}(G)|=2\cdot|\mathrm{Grid}(\mathrm{Board}(m,n))|=2\cdot(m+1)(n+1).$$

With the notation $\widehat{n}=n+1$ and $\widehat{m}=m+1$,

$$\text{each maximal CD-independent subset of } G \text{ is of size } 2\widehat{m}\widehat{n}. \tag{4.1}$$

Let $\mathcal{H}$ be the collection of all cellular rectangular islands of $\langle \mathrm{Board}(m,n);h\rangle$, and denote $\mathrm{SGrid}(\mathcal{H})$ by $\mathcal{T}$ and $|\mathcal{T}|$ by t. Since $|\mathcal{H}|=|\mathcal{T}|=t$, it suffices to show the inequality in

$$t\leq \lfloor \widehat{m}\widehat{n}/2\rfloor -1=\lfloor (mn+m+n-1)/2\rfloor. \tag{4.2}$$

We know from Lemma 4.2 that $\mathcal{T}$ is a CD-independent subset of G. Since the cellular rectangles of $\mathrm{Board}(m,n)$ are nonempty by definition, each member of $\mathcal{T}$ consists of at least four points. Therefore the set

$$\mathcal{W}=\mathcal{T}\cup\{0_G\}\cup\mathrm{Atoms}(G)=\mathcal{T}\cup\{X : X\subseteq \mathrm{Grid}(\mathrm{Board}(m,n)) \text{ and } |X|\leq 1\}$$

is also CD-independent, and it is of size $t+1+\widehat{m}\widehat{n}$.

A subset X of $\mathrm{Grid}(\mathrm{Board}(m,n))$ will be called *bizarre* if $|X|\geq 2$ and there is no rectangle Y of $\mathrm{Board}(m,n)$ with $X=\mathrm{Grid}(Y)$. We say that a bizarre subset of $\mathrm{Grid}(\mathrm{Board}(m,n))$ is *straight* if all of its points lie on the same vertical or horizontal line. We will only use straight bizarre sets. We claim that there exists a set $\mathcal{B}$ of straight bizarre subsets of $\mathrm{Grid}(\mathrm{Board}(m,n))$ such that

$$\mathcal{W}\cup\mathcal{B} \text{ is CD-independent in } G \text{ and } |\mathcal{B}|\geq \begin{cases} t+1 & \text{if } 2\mid \widehat{m}\widehat{n}\\ t+2 & \text{if } 2\nmid \widehat{m}\widehat{n}\end{cases}. \tag{4.3}$$

Note that the validity of (4.3) will complete the proof as follows. First, let $\widehat{m}\widehat{n}$ be odd. Since $|\mathcal{W}|=t+1+\widehat{m}\widehat{n}$, (4.1) and (4.3) yield $t+1+\widehat{m}\widehat{n}+t+2\leq |\mathcal{W}|+|\mathcal{B}|\leq 2\widehat{m}\widehat{n}$, which clearly implies (4.2). For $\widehat{m}\widehat{n}$ even, we conclude (4.2) from $t+1+\widehat{m}\widehat{n}+t+1\leq|\mathcal{W}|+|\mathcal{B}|\leq 2\widehat{m}\widehat{n}$ even faster.

We prove (4.3) by induction on mn. Assume that $\widehat{m}\widehat{n}$ is even, and let $U_1,\dots,U_k$ be the list of maximal elements of $\mathcal{H}\setminus\{\mathrm{Board}(m,n)\}$. First, assume $k=1$. Clearly, at least one of the four sides of $\mathrm{Board}(m,n)$ is *separated* from U_1 in the sense that no cell on this side belongs to U_1. Note that $|\mathcal{H}\cap{\downarrow}U_1|=t-1$, where ${\downarrow}U_1=\{X\in\mathrm{PowSet}(\mathrm{Board}(m,n)) : X\subseteq U_1\}$ denotes the principal ideal generated by U_1. Applying the induction hypothesis to the subboard U_1, we can add at least $(t-1)+1$ straight bizarre subsets of $\mathrm{Grid}(U_1)$ to $\mathcal{W}$ to obtain a larger CD-independent subset of G. Two neighboring points on the separated side form a straight bizarre set, which we still can add without loosing CD-independence. The set $\mathcal{B}$ of all these bizarre sets is of size at least $(t-1)+1+1=t+1$, as desired.

Second, assume $k \geq 2$. For $i=1,\dots,k$, let $t_i = |\mathcal{H} \cap {\downarrow}U_i|$. Clearly, $t = t_1+\dots+t_k+1$. By the induction hypothesis, we can add at least t_i+1 straight bizarre subsets of $\mathrm{Grid}(U_i)$ to $\mathrm{SGrid}(\mathcal{H}\cap{\downarrow}U_i)$ without spoiling its CD-independence. Since the bizarre subsets we add to $\mathrm{SGrid}(\mathcal{H}\cap{\downarrow}U_i)$ are disjoint from $\mathrm{Grid}(U_j)$ for $j \neq i$, we can add all these bizarre subsets simultaneously to $\mathcal{W}$ without hurting its CD-independence. This way, the set $\mathcal{B}$ of all straight bizarre subsets we add is at least

$$(t_1+1)+\dots+(t_k+1) = t+(k-1) \geq t+1. \tag{4.4}$$

Hence, (4.3) holds in this case again.

Next, we assume that $\widehat{m}\widehat{n}$ is odd. The treatment of this case is more or less the same as that for $2 \mid \widehat{m}\widehat{n}$ but we have to find an appropriate $\mathcal{B}$ of size at least $t+2$. That is, we have to find an extra straight bizarre subset. Hence, it will suffice to compare this case to the case of $2 \mid \widehat{m}\widehat{n}$ wherever it is possible. Observe that $1 \leq m < \widehat{m}$ and $2 \not| \ \widehat{m}$ gives $\widehat{m} \geq 3$, and we also have $\widehat{n} \geq 3$. Therefore, if $k=1$, then we can find two comparable straight bizarre subsets of a separated side of $\mathrm{Board}(m,n)$ rather than just one, and $|\mathcal{B}| \geq t+2$ follows the same way as we obtained $|\mathcal{B}| \geq t+1$ in the previous argument for $\widehat{m}\widehat{n}$ even and $k=1$. If $k \geq 3$, then $|\mathcal{B}| \geq t+2$ comes from (4.4).

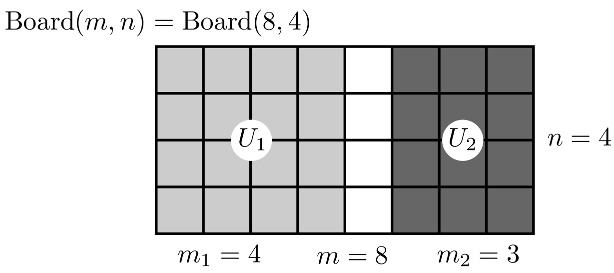

FIGURE 4. The case of $k=2$ and $2 \not| \ \widehat{m}\widehat{n}$

Therefore, we are left with the case $k=2$ such that no side of $\mathrm{Board}(m,n)$ is separated from both U_1 and U_2. The situation, up to rotation by ninety degrees, is exemplified in Figure 4. By the maximality of $\mathcal{H}$, there is no cellular rectangular subset of $\mathrm{Board}(m,n)$ that strictly includes U_i and keeps a positive distance from U_{3-i}, for $i \in \{1,2\}$. It follows that three sides of U_i lie on appropriate sides of $\mathrm{Board}(m,n)$, and the distance between U_1 and U_2 is 1. Let U_i be an m_i-by-n board for $i \in \{1,2\}$. Since $\widehat{m}=m+1$ is odd and $m=m_1+m_2+1$, one of $\widehat{m}_1=m_1+1$ and $\widehat{m}_2=m_2+1$ is odd, and the other is even. Let, say, $\widehat{m}_1$ be odd. Since $\widehat{m}_1\widehat{n}$ is odd, the induction hypothesis allows us to achieve t_1+2 instead of t_1+1 in (4.4), and $|\mathcal{B}| \geq t+2$ follows again. ☐

Remark 4.3. Since we only used straight bizarre subsets rather than arbitrary bizarre ones in the proof above, this method, possibly with non-straight bizarre subsets, will hopefully work for other sorts of boards and islands.

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